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SOME GENERATING FUNCTIONS OF THE BESSEL AND RELATED ORTHOGONAL POLYNOMIALS

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Abstract. Our main object in this article is to present several families of generating functions for the simple Bessel polynomials $\mathfrak{y}_n(x)$ and the generalized Bessel polynomials $\mathfrak{y}_n(x;\alpha,\beta)$. We also investigate and present various potentially useful generating-function relationships involving such other orthogonal polynomial systems as (for example) the Jacobi, Laguerre and Hermite polynomials, together with their specialized and parametrically-varied forms.

Keywords. Orthogonal polynomial systems; Bessel and generalized Bessel polynomials; Jacobi, Laguerre and Hermite polynomials; Linear, bilinear, bilateral and multilateral generating functions; Lagrange expansion theorem; Hypergeometric functions and hypergeometric polynomials.

1. Introduction, Motivation and Definitions

As usual, throughout this article we use the following standard notations: $\mathbb{N} := \{1, 2, 3, \cdots\}, \mathbb{N}_0 := \{0, 1, 2, 3, \cdots\} = \mathbb{N} \cup \{0\}$ and $\mathbb{Z}^- := \{-1, -2, -3, \cdots\} = \mathbb{Z}_0^- \setminus \{0\}$. We also use the notation $\mathbb{Z}$ for the set of integers, the notation $\mathbb{R}$ for the set of real numbers and the notation $\mathbb{C}$ for the set of complex numbers.

We first introduce the widely-accepted concepts of linear, bilinear (and multilinear) and bilateral (and multilateral) generating functions (see, for details, [12, Chapter 19], [29], [25] and [41]). Suppose that a two variable function $F(x,t)$ possessing a formal power-series expansion in t in the form given by

$$F(x,t) = \sum_{n=0}^{\infty} f_n(x)\, t^n, \tag{1.1}$$

in which each member of the generated set $\{f_n(x)\}_{n\in\mathbb{N}_0}$ is independent of t. Then $F(x,t)$ is called a *linear generating function* (or, simply,a generating function) of the set $\{f_n(x)\}_{n\in\mathbb{N}_0}$. This definition of a linear generating function may be extended to include generating functions of the following type:

$$F^*(x,t) = \sum_{n=0}^{\infty} \alpha_n\, f_n(x)\, t^n, \tag{1.2}$$

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where the coefficient sequence $\{\alpha_n\}_{n\in\mathbb{N}_0}$ may include the parameters of the set $\{f_n(x)\}_{n\in\mathbb{N}_0}$, but is independent of x and t. Similarly, a *bilinear generating function* $G^*(x,t)$ is defined as follows:

$$G^*(x,t) = \sum_{n=0}^{\infty} \beta_n f_n(x) f_n(y) t^n \tag{1.3}$$

and a *bilateral generating function* $H^*(x,t)$ is defined in the following form:

$$H^*(x,t) = \sum_{n=0}^{\infty} \gamma_n \; f_n(x) \; g_n(y) \; t^n, \tag{1.4}$$

for two different function sequences $\{f_n(x)\}_{n\in\mathbb{N}_0}$ and $\{g_n(y)\}_{n\in\mathbb{N}_0}$.

In a manner which is analogous to the above concepts, the definitions in (1.3) and (1.4) can be further extended to multilinear generating functions and to multilateral (and mixed multilateral) generating functions involving products of several functions of the same or different or mixed function classes as the generated sets.

The motivation and interest for the study of various families of generating functions lie in their role in the investigation of various potentially useful properties and characteristics of the sequences which they generate. In the form of Z-transforms, which essentially are the discrete counterparts of the Laplace transform, generating functions are used in converting difference equations of discrete-time signals and systems into algebraic equations, thereby providing simplifications in discrete-time system analysis, and also in a wide variety of problems involving sequential fractional-order difference operators, operations research and other areas of applied sciences (including, for example, queuing theory and related stochastic processes) (see, for details, [36] and the references which are cited therein).

A remarkably effective usage of generating functions involves the determination of the asymptotic behavior of the generated sequence $\{\mathfrak{f}_n\}_{n\in\mathbb{N}_0}$ by suitably adapting Darboux's method. The existence of a generating function for a given sequence $\{\mathfrak{f}_n\}_{n\in\mathbb{N}_0}$ of numbers or functions may be useful in finding the following sum:

$$\sum_{n=0}^{\infty} \mathfrak{f}_n = \mathfrak{f}_0 + \mathfrak{f}_1 + \mathfrak{f}_3 + \cdots$$

by means of such summability methods as those due to Abel and Cesàro.

As it was pointed out and adequately documented by Lando [23], modern combinatorics speaks the language of generating functions, the study of which does not require a bulky knowledge of many parts of mathematics, except for some preliminary acquaintance with calculus and algebra. Moreover, generating functions may prove to be remarkably useful in furthering mathematical education because of their deep involvement in various mathematical activities, including computer science. Furthermore, according to Wilf [50], generating functions provide a bridge between discrete mathematics, on the one hand, and continuous analysis (particularly, the complex variable theory) on the other hand. One can study generating functions solely as tools for solving discrete problems. One can find much in the study of generating functions that is powerful and magical in the way generating functions provide unified methods for handling such problems. The full beauty of the subject of generating functions emerges only from tuning in on both channels: the discrete channel and the continuous channel. One can then see how they make the solution of difference equations into child's play, as also in some of the usages of the above-mentioned Z-transform theory.

With a view to introducing the simple Bessel polynomials $\mathfrak{y}_n(x)$ and the generalized Bessel polynomials $\mathfrak{y}_n(x;\alpha,\beta)$, we recall such relatives of the widely-investigated Bessel function $J_\nu(z)$ as (for example) the modified Bessel functions $I_\nu(z)$ and $K_\nu(z)$ of the first and the second kinds, respectively, which are solutions of the *modified* Bessel's differential equation given by

$$z^2 \frac{\mathrm{d}^2\mathfrak{w}}{\mathrm{d}z^2} + z\frac{\mathrm{d}\mathfrak{w}}{\mathrm{d}z} - (z^2+\nu^2)\mathfrak{w} = 0 \qquad (\nu \in \mathbb{C}). \tag{1.5}$$

In particular, in Macdonald's notation, the modified Bessel function $K_\nu(z)$ of the second kind is defined by (see, for example, [46] and [11, Chapter 7])

$$K_\nu(z) = \frac{1}{2}\,\pi\,[I_{-\nu}(z) - I_\nu(z)]\csc(\nu\pi). \tag{1.6}$$

In terms of the familiar and the most fundamental mathematical function, the (Euler's) Gamma function $\Gamma(z)\ \ (z \in \mathbb{C}\setminus\mathbb{Z}_0^-)$, given by

$$\Gamma(z) := \begin{cases} \displaystyle\int_0^\infty e^{-t}\, t^{z-1}\,\mathrm{d}t & \big(\Re(z) > 0\big) \\[2ex] \dfrac{\Gamma(z+n)}{\prod\limits_{j=0}^{n-1}(z+j)} & \big(z \in \mathbb{C}\setminus\mathbb{Z}_0^-;\ n \in \mathbb{N}\big), \end{cases} \tag{1.7}$$

we have

$$I_\nu(z) := \sum_{n=0}^\infty \frac{\left(\frac{1}{2}\,z\right)^{\nu+2n}}{n!\,\Gamma(\nu+m+1)}. \tag{1.8}$$

A slightly different definition, with $\cot(\nu\pi)$ instead of $\csc(\nu\pi)$ on the right-hand side of the equation (1.6), was used by Basset in 1889 (see, for details, [49, p. 373]).

In the year 1949, by means of a systematic study of a close relationship involving the modified Bessel function $K_\nu(z)$ of the second kind, Krall and Frink [22] introduced what they called the Bessel polynomials $\mathfrak{y}_n(x)$ of degree n in x, defined by

$$\begin{aligned} \mathfrak{y}_n(x) &:= \sum_{k=0}^n \binom{n}{k}\binom{n+k}{k}\,k!\,\left(\frac{x}{2}\right)^k \\ &= \sum_{k=0}^n \frac{(n+k)!}{(n-k)!\,k!}\,\left(\frac{x}{2}\right)^k. \end{aligned} \tag{1.9}$$

More precisely, the above-mentioned relationship resulting essentially in the nomenclature of the simple Bessel polynomials $\mathfrak{y}_n(x)$ is given by (see, for example, [11, p. 10, Eq. 7.2.6 (40)])

$$\mathfrak{y}_n(x) = \sqrt{\frac{2}{\pi x}}\,\exp\left(\frac{1}{x}\right)\,K_{n+\frac{1}{2}}\left(\frac{1}{x}\right). \tag{1.10}$$

The following orthogonality property of the generalized Bessel polynomials $\mathfrak{Y}_n^{(\alpha,\beta)}(x)$ was given by Krall and Frink [22]:

$$\begin{aligned} &\frac{1}{2\pi\mathrm{i}}\int_{|z|=1} \rho^{(\alpha,\beta)}(z)\,\mathfrak{Y}_m^{(\alpha,\beta)}(z)\,\mathfrak{Y}_n^{(\alpha,\beta)}(z)\,\mathrm{d}z \\ &= (-1)^{n+1}\,\frac{n!}{\alpha+2n-1}\,\frac{\beta\Gamma(\alpha)}{\Gamma(\alpha+n-1)}\,\delta_{m,n} \qquad (m,n \in \mathbb{N}_0), \end{aligned} \tag{1.11}$$

where $\delta_{m,n}$ denotes, as usual, the Kronecker symbol and the weight function $\rho^{(\alpha,\beta)}(z)$ is given, in terms of the Kummer's confluent hypergeometric function ${}_1F_1(a;b;z)$, by

$$\rho^{(\alpha,\beta)}(z) = (\alpha-1)\ {}_1F_1\left(1;\alpha-1;-\frac{\beta}{z}\right) \\ := \sum_{n=0}^{\infty} \frac{\Gamma(\alpha)}{\Gamma(\alpha+n-1)} \left(-\frac{\beta}{z}\right), \tag{1.12}$$

the function ${}_1F_1$ being the case $r = s = 1$ of the generalized hypergeometric function ${}_rF_s$ with r numerator parameters and s denominator paramaters for $r, s \in \mathbb{N}_0$.

The following *two-parameter* extension $\mathfrak{Y}_n^{(\alpha,\beta)}(x)$ of the Bessel polynomials $\mathfrak{y}_n(x)$ is referred to as the *generalized Bessel polynomials* (see, for details, [22]). We define $\mathfrak{Y}_n^{(\alpha,\beta)}(x)$ as follows:

$$\mathfrak{Y}_n^{(\alpha,\beta)}(x) := \sum_{k=0}^{n} \binom{n}{k}\binom{\alpha+n+k-2}{k}\, k!\, \left(\frac{x}{\beta}\right)^k \\ = \sum_{k=0}^{n} \binom{n}{k} (n+\alpha-1)_k \left(\frac{x}{\beta}\right)^k \quad \left(n \in \mathbb{N}_0;\ \alpha \notin \mathbb{Z}_0^-;\ \beta \neq 0\right), \tag{1.13}$$

so that, clearly, we have

$$\mathfrak{y}_n(x) = \mathfrak{Y}_n^{(2,2)}(x) = \mathfrak{Y}_n^{(2,\beta)}\left(\frac{\beta x}{2}\right). \tag{1.14}$$

We remark in passing that the parameter β in the definition (1.13) may be viewed as a mere scaling factor. We also find it to be convenient in this article to use the notation $\mathfrak{Y}^{(\alpha,\beta)}(x)$ instead of the relatively more popular notation $\mathfrak{y}_n(x;\alpha,\beta)$, that is,

$$\mathfrak{Y}_n^{(\alpha,\beta)}(x) := \mathfrak{y}_n(x;\alpha,\beta). \tag{1.15}$$

In the definition (1.13), and in the remainder of this paper, we have made use of the general Pochhammer symbol or the *shifted factorial* $(\lambda)_\nu$, since

$$(1)_n = n! \qquad (n \in \mathbb{N}_0),$$

which is defined (for $\lambda, \nu \in \mathbb{C}$), in terms of the Gamma function in (1.7), by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0;\ \lambda \in \mathbb{C} \setminus \{0\}) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\nu = n \in \mathbb{N};\ \lambda \in \mathbb{C}), \end{cases} \tag{1.16}$$

it being understood *conventionally* that $(0)_0 := 1$ and assumed *tacitly* that the Γ-quotient exists.

The simple Bessel polynomials $\mathfrak{y}_n(x)$ and the generalized Bessel polynomials $\mathfrak{Y}_n^{(\alpha,\beta)}(x)$ emerged in the investigation by Krall and Frink [22] of the classical wave equation in spherical polar coordinates. In fact, not only the Bessel polynomials $\mathfrak{y}_n(x)$ and the generalized Bessel polynomials $\mathfrak{Y}_n^{(\alpha,\beta)}(x)$, several different forms of the corresponding reversed Bessel polynomials $\vartheta_n(x)$ and $\vartheta_n^{(\alpha,\beta)}(x)$ have also found applications in many widespread scientific and engineering fields such as (for example) in the design of the so-called *Bessel electronic filters* (see, for details, [18]).

This article is motivated mainly by the theory and multidisciplinary applications of the simple Bessel polynomials $\mathfrak{y}_n(x)$ and the generalized Bessel polynomials $\mathfrak{Y}_n^{(\alpha,\beta)}(x)$ as well as those of their abovementioned reversed forms $\vartheta_n(x)$ and $\vartheta_n^{(\alpha,\beta)}(x)$. We systematically investigate and examine several families of generating functions of not only the Bessel polynomials $\mathfrak{y}_n(x)$ and the generalized Bessel polynomials $\mathfrak{Y}_n^{(\alpha,\beta)}(x)$, but also of their such related orthogonal polynomial systems as (for example) the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, the Laguerre polynomials $L_n^{(\alpha)}(x)$ and the Hermite polynomials $H_n(x)$.

The celebrated Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, which are defined by

$$\begin{aligned} P_n^{(\alpha,\beta)}(x) &:= \sum_{k=0}^{n} \binom{n+\alpha}{n-k}\binom{n+k+\alpha+\beta}{k}\left(\frac{x-1}{2}\right)^k \\ &= \sum_{k=0}^{n} \binom{n+\alpha}{n-k}\binom{n+\beta}{k}\left(\frac{x-1}{2}\right)^k\left(\frac{x+1}{2}\right)^{n-k} \\ &= \binom{\alpha+n}{n}\, {}_2F_1\left(-n, \alpha+\beta+n+1; \alpha+1; \frac{1-x}{2}\right) \end{aligned} \tag{1.17}$$

in terms of the Gauss hypergeometric function ${}_2F_1$, that is, the case $r-1=s=1$ of the generalized hypergeometric function ${}_rF_s$ with r numerator parameters and and s denominator parameters with $r,s \in {}_0$, are known to include such special or limit cases as (for example) the Gegenbauer (or ultraspherical) polynomials $C_n^{\nu}(x)$, the Legendre (or spherical) polynomials $P_n(x)$, and the Chebyshev polynomials $T_n(x)$ and $U_n(x)$ (see, for details, [45]). Moreover, for the Laguerre polynomials $L_n^{(\alpha)}(x)$, we have (see, for details, [45])

$$\begin{aligned} L_n^{(\alpha)}(x) &:= \sum_{k=0}^{n} \binom{n+\alpha}{n-k}\frac{(-x)^k}{k!} \\ &= \binom{\alpha+n}{n}\, {}_1F_1(-n; \alpha+1; x) \\ &= \lim_{|\beta|\to\infty}\left\{P_n^{(\alpha,\beta)}\left(1-\frac{2x}{\beta}\right)\right\}. \end{aligned} \tag{1.18}$$

Also, in the case of the Hermite polynomials $H_n(x)$, we have

$$\begin{aligned} H_n(x) &:= \sum_{k=0}^{[n/2]} \frac{(-1)^k\, n!}{k!\,(n-2k)!}\, x^{n-2k} \\ &= (2x)^n\, {}_2F_0\left[\begin{array}{c} \Delta(2;-n); \\ \underline{\hspace{2em}}\,; \end{array} -\frac{1}{x^2}\right], \end{aligned} \tag{1.19}$$

$[\kappa]$ being the largest integer in $\kappa \in \mathbb{R}$, so that

$$H_{2n}(x) = \lim_{|\epsilon|\to\infty}\left\{(-1)^n\, n!\, 2^{2n}\, P_n^{(\frac{1}{2},-\epsilon)}\left(1+\frac{2x^2}{\epsilon}\right)\right\} \tag{1.20}$$

and

$$H_{2n+1}(x) = \lim_{|\epsilon|\to\infty}\left\{(-1)^n\, n!\, 2^{2n+1}\, x\, P_n^{(-\frac{1}{2},-\epsilon)}\left(1+\frac{2x^2}{\epsilon}\right)\right\}. \tag{1.21}$$

One can indeed make use of these last limit relations in conjunction with the generating functions of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ in order to derive the corresponding generating functions for the

Laguerre polynomials $L_n^{(\alpha)}(x)$ and the Hermite polynomials $H_n(x)$, and also for the above-mentioned special or limit cases of the celebrated Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$.

2. Generating Functions of the Bessel Polynomials

In this section, we first recall Brafman's general form of a known hypergeometric generating function (see [5] and [41, p. 136, Eq. 2.6 (2)]; see also [37, Eq. (21)]):

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \; {}_{m+r}F_s \left[\begin{array}{c} \Delta(m;-n), \alpha_1, \cdots, \alpha_r; \\ \\ \beta_1, \cdots, \beta_s; \end{array} x \right] t^n$$
$$= (1-t)^{-\lambda} \; {}_{m+r}F_s \left[\begin{array}{c} \Delta(m;\lambda), \alpha_1, \cdots, \alpha_r; \\ \\ \beta_1, \cdots, \beta_s; \end{array} x\left(-\frac{t}{1-t}\right)^m \right] (\lambda \in \mathbb{C};\; m \in \mathbb{N};\; |t|<1), \tag{2.1}$$

where $\Delta(m;\lambda)$ denotes the m-parameter sequence:

$$\left\{\frac{\lambda+j-1}{m}\right\}_{j=1}^{m} \qquad (\lambda \in \mathbb{C};\; m \in \mathbb{N}).$$

This last hypergeometric generating function (2.1) not only reduces substantially when $m = 1$, but also applies to the Gould-Hopper generalization $g_n^m(x,h)$ of the Hermite polynomials $H_n(x)$, which is defined by (see, for details, [17])

$$g_n^m(x,h) := \sum_{k=0}^{[n/m]} \frac{n!}{k!\,(n-mk)!}\, h^k\, x^{n-mk}$$
$$= x^n \; {}_mF_0 \left[\begin{array}{c} \Delta(m;-n); \\ \\ \overline{}\,; \end{array} \left(-\frac{m}{x}\right)^m h \right], \tag{2.2}$$

leadng us to the following *divergent* generating function from (2.1):

$$(1-xt)^{-\lambda} \; {}_mF_q \left[\begin{array}{c} \Delta(m;\lambda); \\ \\ \overline{}\,; \end{array} \left(\frac{mt}{1-xt}\right)^m h \right] \cong \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}\, g_n^m(x,h)\, t^n. \tag{2.3}$$

For the orthogonal family of the *two-parameter* Bessel polynomials $\mathfrak{Y}_n^{(\alpha,\beta)}(x)$, it is easily observed from the limit relationship in (1.18), in conjunction with

$$\mathfrak{Y}_n^{(\alpha,\beta)}(x) = n! \left(-\frac{x}{\beta}\right)^n L_n^{(1-\alpha-2n)}\left(\frac{\beta}{x}\right), \tag{2.4}$$

that

$$\mathfrak{Y}_n^{(\alpha,\beta)}(x) = n! \left(-\frac{x}{\beta}\right)^n \lim_{\epsilon\to\infty} \left\{ P_n^{(1-\alpha-2n,\epsilon)}\left(1-\frac{2\beta}{\epsilon x}\right)\right\} \tag{2.5}$$

or, equivalently, that (see, for example, [37, Eq. (35)])

$$\mathfrak{Y}_n^{(\alpha,\beta)}(x) = \lim_{\epsilon\to\infty} \left\{ \frac{n!}{(\epsilon)_n} P_n^{(\epsilon-1,\alpha-\epsilon-1)}\left(\frac{1+2\epsilon x}{\beta}\right)\right\}, \tag{2.6}$$

together with similar relationships for the reversed Bessel polynomials $\vartheta_n^{(\alpha,\beta)}(x)$. Thus, clearly, it follows that generating functions of the generalized Bessel polynomials $\mathfrak{Y}_n^{(\alpha,\beta)}(x)$ can possibly be deduced from the (known or new) generating functions for the relatively more familiar Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ and the (known or new) generating functions for the Laguerre polynomials $L_n^{(\alpha)}(x)$.

The following frequently-cited *divergent* generating function of the simple Bessel polynomials $\mathfrak{y}_n(x)$ was presented by Krall and Frink [22, p. 106, Eq. (25)]:

$$\sum_{n=0}^{\infty} \mathfrak{y}_{n-1}(x)\,\frac{t^n}{n!} = \exp\left(\frac{1-\sqrt{1-2xt}}{x}\right) \qquad \big(y_{-1}(x)=y_0(x)=1\big). \tag{2.7}$$

For detailed descriptions of the success and usefulness of several families of hypergeometric generating functions in the derivation of simpler generating functions for numerous classes of hypergeometric polynomials, including (for example) the simple Bessel polynomials $\mathfrak{y}_n(x)$ and the generalized Bessel polynomials $\mathfrak{Y}_n^{(\alpha,\beta)}(x)$, can be found in the earlier works [36], [37], [38] and [39]. For example, the following general families of generating functions involving an appropriately bounded sequence $\{\Omega(n)\}_{n\in\mathbb{N}_0}$ of essentially arbitrary real or complex numbers (see, for details, [36]):

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}\left(\sum_{k=0}^{\left[\frac{n}{m}\right]} (-n)_{mk}\,\Omega(k)\,\frac{z^k}{k!}\right) t^n$$
$$= (1-t)^{-\lambda}\sum_{k=0}^{\infty}\frac{(\lambda)_{mk}}{k!}\,\Omega(k)\left(\frac{z(-t)^m}{(1-t)^m}\right)^k \ (\lambda\in\mathbb{C};\ m\in\mathbb{N};\ |t|<1), \tag{2.8}$$

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}\left(\sum_{k=0}^{\left[\frac{n}{m}\right]} (-n)_{mk}\,(\lambda+n)_{mk}\,\Omega(k)\,\frac{z^k}{k!}\right) t^n$$
$$= (1-t)^{-\lambda}\sum_{k=0}^{\infty}\frac{(\lambda)_{2mk}}{k!}\,\Omega(k)\left(\frac{z(-t)^m}{(1-t)^{2m}}\right)^k \ (\lambda\in\mathbb{C};\ m\in\mathbb{N};\ |t|<1) \tag{2.9}$$

and

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}\left(\sum_{k=0}^{\left[\frac{n}{m}\right]} \frac{(-n)_{mk}}{(1-\lambda-n)_{mk}}\,\Omega(k)\,\frac{z^k}{k!}\right) t^n$$
$$= (1-t)^{-\lambda}\sum_{k=0}^{\infty}\Omega(k)\,\frac{(zt^m)^k}{k!} \ (\lambda\in\mathbb{C};\ m\in\mathbb{N};\ |t|<1), \tag{2.10}$$

where it is assumed that each member of the generating functions (2.8), (2.9) and (2.10) exists.

A limit case of the hypergeometric generating function (2.9) when $t \mapsto \frac{t}{\lambda}$ and $|\lambda|\to\infty$ leads us to the following companion of the above-mentioned hypergeometric generating functions:

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{\left[\frac{n}{m}\right]} (-n)_{mk}\,\Omega(k)\,\frac{z^k}{k!}\right)\frac{t^n}{n!} = \mathrm{e}^t\sum_{k=0}^{\infty}\,\Omega(k)\,\frac{\left[z(-t)^m\right]^k}{k!} \ (m\in\mathbb{N};\ |t|<\infty). \tag{2.11}$$

In the study of generating functions for the simple Bessel polynomials $\mathfrak{y}_n(x)$ and the generalized Bessel polynomials $\mathfrak{y}_n(x;\alpha,\beta)$. we choose now to recall the following *corrected and modified* form of Burchnall's generating function for the Bessel polynomials $\mathfrak{Y}_n^{(\alpha,\beta)}(x)$ (see [4, p. 67] and [41, p. 84]):

$$\sum_{n=0}^{\infty} \mathfrak{Y}_n^{(\alpha,\beta)}(x)\,\frac{t^n}{n!} = \frac{1}{\sqrt{1-\frac{4xt}{\beta}}}\left(\frac{2}{1+\sqrt{1-\frac{4xt}{\beta}}}\right)^{\alpha-2} \exp\left(\frac{2t}{1+\sqrt{1-\frac{4xt}{\beta}}}\right). \tag{2.12}$$

Various further developments emerging from Burchnall's work [4] can be found in the recent survey-cum-expository review article by Srivastava [37] (see also the references which are cited therein). In this connection, we record here the following development which emerged recently from Burchnall's work [4] (see, for details, [37, Eq. (48)]):

$$\sum_{n=0}^{\infty}(\lambda+n)_n\left(\sum_{k=0}^{\left[\frac{n}{m}\right]}\frac{(-n)_{mk}}{(1-\lambda-2n)_{mk}}\,\Omega(k)\,\frac{z^k}{k!}\right)\frac{t^n}{n!}$$
$$=\frac{1}{\sqrt{1-4t}}\left(\frac{2}{1+\sqrt{1-4t}}\right)^{\lambda-1}\sum_{k=0}^{\infty}\Omega(k)\left(\frac{2t}{1+\sqrt{1-4xt}}\right)^{mk}\frac{z^k}{k!}\quad\left(\lambda\in\mathbb{C};\ m\in\mathbb{N};\ |t|<\tfrac{1}{4}\right). \tag{2.13}$$

Indeed, in each of the assertions (2.8), (2.9), (2.10), (2.11) and (2.13), and elsewhere in this paper, all of the parametric values which would render any member to be invalid or undefined are tacitly excluded.

By applying the Lie algebraic (or group-theoretic) technique of Weisner [47] (see also Miller [26] and [41, Chapter 6]), several interesting and potentially useful generating functions and generating relations for the Bessel polynomials $\mathfrak{y}_n(x)$ were derived by McBride [25, pp. 47–50] including, for example, the following generating relation for the Bessel polynomials $y_n(x)$ [25, p. 50, Eq. (12)]:

$$\sum_{n=0}^{\infty}\mathfrak{y}_{m+n}(x)\,\frac{t^n}{n!}=(1-2xt)^{-\frac{1}{2}(m+1)}\,\exp\left(\frac{1-\sqrt{1-2xt}}{x}\right)\mathfrak{y}_m\left(\frac{x}{\sqrt{1-2xt}}\right)\quad\left(m\in\mathbb{N}_0;\ 2|t|<|x|^{-1}\right), \tag{2.14}$$

which can be applied to established Theorem 2.1 below (see, for details, [31, Part I, p. 229, Corollary 2] and [41, p. 421, Corollary 2]).

Theorem 2.1. *For an identically nonvanishing function $\Omega_\mu(\xi_1,\cdots,\xi_s)$ of s real or complex variables $\xi_1,\cdots,\xi_s$ $(s\in\mathbb{N})$ and of order $\mu\in\mathbb{C}$, if*

$$\Lambda_{m,\mathfrak{p},\mathfrak{q}}[x;\xi_1,\cdots,\xi_s:z]:=\sum_{n=0}^{\infty}a_n\,\mathfrak{y}_{m+\mathfrak{q}n}(x)\,\Omega_{\mu+\mathfrak{p}n}(\xi_1,\cdots,\xi_s)\,\frac{z^n}{(\mathfrak{q}n)!}\ (a_n\neq 0;\ m\in\mathbb{N}_0;\ \mathfrak{p},\mathfrak{q}\in\mathbb{N}) \tag{2.15}$$

and

$$M_{n,\mathfrak{q}}^{\mathfrak{p},\mu}(\xi_1,\cdots,\xi_s;\eta):=\sum_{k=0}^{\left[\frac{n}{\mathfrak{q}}\right]}\binom{n}{\mathfrak{q}k}a_k\,\Omega_{\mu+\mathfrak{p}k}(\xi_1,\cdots,\xi_s)\,\eta^k, \tag{2.16}$$

then the following family of multilinear or mixed multilateral generating functions for the Bessel polynomials $\mathfrak{y}_n(x)$ holds true:

$$\sum_{n=0}^{\infty} \mathfrak{y}_{m+n}(x)\ M_{n,\mathfrak{q}}^{\mathfrak{p},\mu}(\xi_1,\cdots,\xi_s;\eta)\ \frac{t^n}{n!}$$
$$= (1-2xt)^{-\frac{1}{2}(m+1)}\ \exp\left(\tfrac{1-\sqrt{1-2xt}}{x}\right) \Lambda_{m,\mathfrak{p},\mathfrak{q}}\left[\tfrac{x}{\sqrt{1-2xt}};\xi_1,\cdots,\xi_s:\eta\left(\tfrac{t}{\sqrt{1-2xt}}\right)^{\mathfrak{q}}\right]\ \left(|t|<\tfrac{1}{2}|x|^{-1}\right), \tag{2.17}$$

provided that each member of the generating function (2.17) *exists.*

In the case of the generalized Bessel polynomials $\mathfrak{Y}_n^{(\alpha,\beta)}(x)$, we can similarly and appropriately apply each of the following generating functions:

$$\sum_{n=0}^{\infty} \mathfrak{Y}_{m+n}^{(\alpha-n,\beta)}(x)\ \frac{t^n}{n!} = \left(1-\frac{xt}{\beta}\right)^{1-\alpha-m}\ \mathrm{e}^t\ \mathfrak{Y}_m^{(\alpha,\beta)}\left(\frac{\beta x}{\beta-xt}\right)\ \left(m\in\mathbb{N}_0;\ |t|<\left|\tfrac{\beta}{x}\right|\right), \tag{2.18}$$

$$\sum_{n=0}^{\infty} \binom{\alpha+m+n-2}{n}\ \mathfrak{Y}_m^{(\alpha+n;\beta)}(x)\ t^n = (1-t)^{1-\alpha-m}\ \mathfrak{Y}_m^{(\alpha,\beta)}\left(\frac{x}{1-t}\right)\quad \left(m\in\mathbb{N}_0;\ |t|<1\right), \tag{2.19}$$

$$\sum_{n=0}^{\infty} \mathfrak{Y}_m^{(\alpha-n,\beta)}(x)\ \frac{t^n}{n!} = \left(1-\frac{xt}{\beta}\right)^{m}\ \mathrm{e}^t\ \mathfrak{Y}_m^{(\alpha,\beta)}\left(\frac{\beta x}{\beta-xt}\right)\qquad \left(m\in\mathbb{N}_0\right) \tag{2.20}$$

and

$$\sum_{n=0}^{\infty} \mathfrak{Y}_{m+n}^{(\alpha-2n,\beta)}(x)\ \frac{t^n}{n!} = \left(1+\tfrac{xt}{\beta}\right)^{\alpha-2}\ \exp\left(\tfrac{\beta t}{\beta+xt}\right) \mathfrak{Y}_m^{(\alpha,\beta)}(x(1+\tfrac{xt}{\beta}))\qquad \left(m\in\mathbb{N}_0;\ |t|<\left|\tfrac{\beta}{x}\right|\right). \tag{2.21}$$

Such families of multilinear or mixed multilateral generating functions for the generalized Bessel polynomials $\mathfrak{Y}_n^{(\alpha,\beta)}(x)$ as those derivable from (2.18) to (2.21), which are analogous to (2.17), can be found in the works of Chen and Srivastava [9, p. 154], Chen *et al.* [8, pp. 363–364] and Srivastava [35, p. 129] (see also some related developments reported by Lin *et al.* [24]). Some rather obvious special cases of the general families of multilinear or mixed multilateral generating functions for the generalized Bessel polynomials $\mathfrak{Y}_n^{(\alpha,\beta)}(x)$ as those derivable from (2.18) to (2.21) were considered recently by Biswas and Chongdar [3].

We now state the following potentially useful companion of the above generating functions (2.18) to (2.21):

$$\sum_{n=0}^{\infty} \mathfrak{Y}_n^{(\alpha-2n,\beta)}\ \frac{t^n}{n!} = \left(1+\frac{xt}{\beta}\right)^{\alpha-2}\ \exp\left(\frac{\beta t}{\beta+xt}\right)\qquad \left(|t|<\left|\frac{\beta}{x}\right|\right), \tag{2.22}$$

which results from the generating function (2.21) upon setting $m=0$. Furthermore, we shall prove the following generating-function relationship:

$$\sum_{n=0}^{\infty} (\alpha+m-1)_n\ \mathfrak{Y}_m^{(\alpha+n,\beta)}(x)\ \frac{t^n}{n!} = (1-t)^{1-\alpha-m}\ \mathfrak{Y}_m^{(\alpha,\beta)}\left(\frac{x}{1-t}\right)\ (m\in\mathbb{N}_0;\ |t|<1). \tag{2.23}$$

Proof. First of all, in terms of the hypergeometric function ${}_2F_0$, we rewrite the equation (1.13) as follows (see, for example, [41, p. 75, Eq. 1.9 (1)]):

$$\mathfrak{Y}_n^{(\alpha,\beta)}(x) = {}_2F_0\left(-n, \alpha+n-1; \text{——}; -\frac{x}{\beta}\right) = \sum_{k=0}^{n} \frac{(-n)_k\,(\alpha+n-1)_k}{k!}\left(-\frac{x}{\beta}\right)^k \qquad (n \in \mathbb{N}_0). \tag{2.24}$$

By making use of this last result (2.24), we observe for the left-hand side of the generating function (2.23) that

$$\begin{aligned}
&\sum_{n=0}^{\infty} (\alpha+m-1)_n\, \mathfrak{Y}_m^{(\alpha+n,\beta)}(x)\, \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} (\alpha+m-1)_n \sum_{k=0}^{m} \frac{(-m)_k\,(\alpha+n+m-1)_k}{k!}\left(-\frac{x}{\beta}\right)^k \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{m} \frac{(-m)_k\,(\alpha+m-1)_{n+k}}{k!}\left(-\frac{x}{\beta}\right)^k \\
&= \sum_{k=0}^{m} \frac{(-m)_k\,(\alpha+m-1)_k}{k!}\left(-\frac{x}{\beta}\right)^k \sum_{n=0}^{\infty} (\alpha+k+m-1)_n\, \frac{t^n}{n!},
\end{aligned} \tag{2.25}$$

where we have inverted the order of summation and also applied the following consequence of the definition (1.16) of the general Pochhammer symbol $(\lambda)_\nu$:

$$(\lambda)_\nu\,(\lambda+\nu)_\mu = (\lambda)_{\mu+\nu} = (\lambda)_\mu\,(\lambda+\mu)_\nu \qquad (\lambda, \mu, \nu \in \mathbb{C}).$$

Finally, we sum the inner n-sum by applying the binomial expansion in the form:

$$(1-z)^{-\lambda} = \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}\, z^n \qquad (|z| < 1;\ \lambda \in \mathbb{C}). \tag{2.26}$$

Upon interpreting the resulting k-sum by means of (2.24), we are led at once to the right-hand side of the generating function (2.23). □

We now state and prove a bilateral generating function as well as a multilinear or mixed multilateral generating function for the generalized Bessel function $\mathfrak{Y}_n^{(\alpha,\beta)}(x)$, which are asserted by Theorem 2.3 below. In fact, the first assertion (2.31) of Theorem 2.3 is a modified and extended version of a known result which was derived in an earlier work by Chongdar and Alam [10] by using Weisner's group-theoretic method. In our *direct* proof of Theorem 2.3 *without* using Weisner's group-theoretic method, we make use of the following general double-series identities (see [41, p. 101, Lemma 3])

Lemma 2.2. *Let $\{A(k,n)\}_{k,n\in\mathbb{N}_0}$ be a suitably bounded double sequence of essentially arbitrary real or complex numbers. Then*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\left[\frac{n}{m}\right]} A(k, n-mk) \qquad (m \in \mathbb{N}) \tag{2.27}$$

and

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\left[\frac{n}{m}\right]} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k, n+mk) \qquad (m \in \mathbb{N}), \tag{2.28}$$

provided that each member of the double-series identities (2.27) *and* (2.28) *exists,* $[\kappa]$ *being the greatest integer in* $\kappa \in \mathbb{R}$.

We remark in passing that, for relatively more familiar and more widely-used special cases of the above Lemma when $m = 1$ and $m = 2$, the interested reader should refer to [41, p. 100, Lemma 1 and Lemma 2] and [29, p. 56, Lemma 10; p. 57, Lemma 11].

Theorem 2.3. *For a suitably bounded sequence* $\{a_n\}_{n\in\mathbb{N}_0}$, *suppose that there exists a generating relation of the following form*:

$$\mathcal{G}_{m,\mathfrak{p},\mathfrak{q}}(x,t) = \sum_{n=0}^{\infty} a_n\ \mathfrak{Y}_{m+\mathfrak{q}n}^{(\alpha-2\mathfrak{q}n,\beta)}(x)\ t^n \qquad \big(a_n \neq 0;\ m \in \mathbb{N}_0;\ \mathfrak{q} \in \mathbb{N}\big). \tag{2.29}$$

Also let the polynomial sequence $\{\mathcal{P}_n(x;\mathfrak{q})\}_{n\in\mathbb{N}_0}$ *be given by*

$$\mathcal{P}_n(x;\mathfrak{q}) := \sum_{k=0}^{\left[\frac{n}{\mathfrak{q}}\right]} \binom{n}{\mathfrak{q}k} a_k\ x^k \qquad (\mathfrak{q} \in \mathbb{N}) \tag{2.30}$$

Then the following bilateral generating function holds true:

$$\sum_{n=0}^{\infty} \mathfrak{Y}_{m+n}^{(\alpha-2n,\beta)}(x)\ \mathcal{P}_n(y;\mathfrak{q})\ \tfrac{t^n}{n!}$$
$$= \left(1+\tfrac{xt}{\beta}\right)^{\alpha-2} \exp\left(\tfrac{\beta t}{\beta+xt}\right) \mathcal{G}\left(x\left(1+\tfrac{xt}{\beta}\right), \frac{yt}{\left(1+\frac{xt}{\beta}\right)^{2\mathfrak{q}}}\right) \left(|t| < \left|\tfrac{\beta}{x}\right|\right), \tag{2.31}$$

provided that each member of the assertion (2.31) *exists.*

Furthermore, for an identically non-vanishing function $\Omega_\mu(\xi_1,\cdots,\xi_s)$ *of* s *real or complex variables* $\xi_1,\cdots,\xi_s$ $(s \in \mathbb{N})$ *and of order* $\mu \in \mathbb{C}$, *if there exists a generating function of the following form*:

$$\Xi_{m,\mathfrak{p},\mathfrak{q}}[x;\xi_1,\cdots,\xi_s : z] = \sum_{n=0}^{\infty} a_n\ \mathfrak{Y}_{m+\mathfrak{q}n}^{(\alpha-2\mathfrak{q}n,\beta)}(x)\ \Omega_{\mu+\mathfrak{p}n}(\xi_1,\cdots,\xi_s)\ t^n (a_n \neq 0;\ m \in \mathbb{N}_0;\ \mathfrak{p},\mathfrak{q} \in \mathbb{N}), \tag{2.32}$$

and if the polynomial sequence $\mathcal{Q}_n^{\mathfrak{p},\mathfrak{q},\mu}(x;\xi_1,\cdots,\xi_s)$ *is given by*

$$\mathcal{Q}_n^{\mathfrak{p},\mathfrak{q},\mu}(x;\xi_1,\cdots,\xi_s) := \sum_{k=0}^{\left[\frac{n}{\mathfrak{q}}\right]} \binom{n}{\mathfrak{q}k} a_k\ \Omega_{\mu+\mathfrak{p}k}(\xi_1,\cdots,\xi_s)\ x^k, \tag{2.33}$$

then the following family of multilinear or mixed multilateral generating functions for the Bessel polynomials $\mathfrak{Y}_n^{(\alpha,\beta)}(x)$ *holds true*:

$$\sum_{n=0}^{\infty} \mathfrak{Y}_{m+n}^{(\alpha-2n,\beta)}(x)\ \mathcal{Q}_n^{\mathfrak{p},\mathfrak{q},\mu}(y;\xi_1,\cdots,\xi_s)\ \frac{t^n}{n!}$$
$$= \left(1+\frac{xt}{\beta}\right)^{\alpha-2} \exp\left(\frac{\beta t}{\beta+xt}\right) \Xi_{m,\mathfrak{p},\mathfrak{q}}\left[x\left(1+\frac{xt}{\beta}\right), \frac{yt}{(1+\frac{xt}{\beta})^{2\mathfrak{q}}}\right] \qquad \left(|t| < \left|\frac{\beta}{x}\right|\right), \tag{2.34}$$

provided that each member of the generating function (2.34) *exists.*

Proof. First of all, we replace the polynomial $\mathcal{P}_n(y;\mathfrak{q})$ on the left-hand side of the bilateral generating function (2.31) by means of the definition (2.30) and apply the series identity (2.28). Then, upon interpreting the k-series, which results after the inversion of the double sum, we make use of the known generating-function relation (2.21). we are thus led to the right-hand side of the bilateral generating function (2.31) in light of the hypothesis (2.29).

The demonstration of the multilinear (or mixed multilateral) generating function is muck akin to that of the bilateral generating function (2.31). We, therefore, choose to skip the details involved. □

In the proof of Theorem 2.3 above, we have made use of the generating-function relationship (2.21). Each of the other generating-function relationships (2.18), (2.19), (2.20) and (2.23) can also be analogously applied in deriving further families of bilateral and multilinear (or mixed multilateral) generating functions. For example, the generating-function relationship (2.21) in conjunction with the generating-function relationship (2.23) would lead us to Theorem 2.4 below.

Theorem 2.4. *For a suitably bounded sequence $\{a_n\}_{n\in\mathbb{N}_0}$, suppose that there exists a generating relation of the following form:*

$$\mathcal{H}_{m,\mathfrak{p},\mathfrak{q}}^{(\alpha,\beta;\gamma,\delta)}[x,y;t]=\sum_{n=0}^{\infty} a_n\,\mathfrak{Y}_{\mathfrak{q}n}^{(\alpha-2\mathfrak{q}n,\beta)}(x)\,\mathfrak{Y}_{m}^{(\gamma+\mathfrak{p}n,\delta)}(y)\,\frac{t^n}{n!}\big(a_n\neq 0;\ m\in\mathbb{N}_0;\ \mathfrak{p},\mathfrak{q}\in\mathbb{N}\big). \tag{2.35}$$

Then the following bilinear generating function holds true for the generalized Bessel polynomials $\mathfrak{Y}_n^{(\alpha,\beta)}(x)$:

$$\begin{aligned}&\sum_{n,p,q=0}^{\infty} a_n\,(\gamma+m+\mathfrak{p}n-1)_q\,\mathfrak{Y}_{\mathfrak{q}n+p}^{(\alpha-2\mathfrak{q}n-2p,\beta)}(x)\,\mathfrak{Y}_{m}^{(\gamma+\mathfrak{p}n,\delta)}(y)\,\frac{(vw)^n}{n!}\,\frac{(\beta w)^p}{p!}\,\frac{w^q}{q!}\\&=\mathcal{H}_{m,\mathfrak{p},\mathfrak{q}}^{(\alpha,\beta;\gamma,\delta)}\left[x(1+wx),\frac{y}{1-w};\frac{vw}{(1-w)^{\mathfrak{p}}\,(1+wx)^{2\mathfrak{q}}}\right]\left(\left|\frac{vw}{(1-w)^{\mathfrak{p}}\,(1+wx)^{2\mathfrak{q}}}\right|<1\right),\end{aligned} \tag{2.36}$$

provided that each member of the bilinear generating function (2.36) exists.

A multilinear (or mixed multilateral) version of the bilinear generating function (2.36), analogous to the assertion (2.34) of Theorem 2.3 also holds true for the generalized Bessel polynomials $\mathfrak{Y}_n^{(\alpha,\beta)}(x)$.

Proof. For convenience, we denote the left-hand side of the bilinear generating function (2.35) by $\Theta(x,y;t)$. Then we have

$$\begin{aligned}\Theta(u,v,w)&:=\sum_{n,p,q=0}^{\infty} a_n\,(\gamma+m+\mathfrak{p}n-1)_q\,\mathfrak{Y}_{\mathfrak{q}n+p}^{(\alpha-2\mathfrak{q}n-2p,\beta)}(x)\mathfrak{Y}_{m}^{(\gamma+\mathfrak{p}n,\delta)}(y)\frac{(vw)^n}{n!}\,\frac{(\beta w)^p}{p!}\,\frac{w^q}{q!}\\&=\sum_{n,q=0}^{\infty} a_n\,(\gamma+m+\mathfrak{p}n-1)_q\,\mathfrak{Y}_{m}^{(\gamma+\mathfrak{p}n,\delta)}(y)\,\frac{(vw)^n}{n!}\,\frac{w^q}{q!}\sum_{p=0}^{\infty}\mathfrak{Y}_{\mathfrak{q}n+p}^{(\alpha-2\mathfrak{q}n-2p,\beta)}(x)\,\frac{(\beta w)^p}{p!}.\end{aligned} \tag{2.37}$$

Now, upon evaluating the innermost p-sum in (2.37) by means of the equation (2.21), we find that

$$\Theta(u,v,w) = (1+wx)^{\alpha-2}\ \exp\left(\frac{\beta w}{1+wx}\right) \cdot \sum_{n=0}^{\infty} \frac{a_n}{n!} \left(\frac{vw}{(1+wx)^{2\mathfrak{q}}}\right)^n \mathfrak{Y}_{\mathfrak{q}n}^{(\alpha-2\mathfrak{q}n,\beta)}(x(1=wx)) \cdot \sum_{q=0}^{\infty} (\gamma+m+\mathfrak{p}n-1)_q\ \mathfrak{Y}_m^{(\gamma+\mathfrak{p}n,\delta)}(y)\ \frac{w^q}{q!}. \tag{2.38}$$

Finally, we first sum the inner q-series in (2.38) by applying the generating-function relationship (2.23), and then interpret the resulting right-hand side of (2.38) with the definition (2.35) of $\mathcal{H}(x,y;t)$. We thus complete the proof of Theorem 2.4 under the stated hypothesis. □

3. Generating Functions Emerging from the Lagrange Expansion

There are several interesting proofs of Jacobi's generating function for the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ which we introduced in this article by means of the equation (1.17):

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)\ t^n = 2^{\alpha+\beta}\ R^{-1}\ (1-t+R)^{\alpha}\ (1+t+R)^{\beta}\ \left(R := (1-2xt+t^2)^{\frac{1}{2}}\right). \tag{3.1}$$

In addition to the original proof by Jacobi (which was based upon the Lagrange expansion in (3.2) below) and the subsequent second proof by Tchebychef, we cite the recent proofs of Jacobi's generating function (3.1) by Szegö [45, Section 4.4], Rainville [29, Section 140], Carlitz [6], Askey [1], Foata and Leroux [13], and Srivastava [34].

In its more-convenient-to-use form, the Lagrange expansion can be rewritten in the following elegant form: [28, p. 146, Problem 207]:

$$\frac{f(z)}{1-w\varphi'(z)} = \sum_{n=0}^{\infty} \frac{w^n}{n!}\ \frac{\mathrm{d}^n}{\mathrm{d}z^n}\{f(z)[\varphi(z)]^n\}\bigg|_{z=z_0}, \tag{3.2}$$

which, in the special case when $\varphi(z) \equiv 1$, yields the relatively more familiar Taylor-Maclaurin expansion.

From among various corollaries and consequences of the Lagrange expansion theorem (3.2), we recall the following combinatorial identity (see, for example, [28, p. 349, Problem 216]):

$$\sum_{n=0}^{\infty} \binom{\alpha+(\beta+1)n}{n}\ t^n = \frac{(1+\zeta)^{\alpha+1}}{1-\beta\zeta}, \tag{3.3}$$

which, in the special case when $\beta=-1$, corresponds to the binomial expansion (2.26), and the following essentially equivalent version of the combinatorial identity (3.3) [28, p. 348, Problem 212]):

$$\sum_{n=0}^{\infty} \frac{\alpha}{\alpha+(\beta+1)n} \binom{\alpha+(\beta+1)n}{n}\ t^n = (1+\zeta)^{\alpha}. \tag{3.4}$$

Here, and in what follows, the parameters α and β are complex numbers independent of n, and ζ is a function of t defined *implicitly* by

$$\zeta = \zeta(t) = t(1+\zeta)^{\beta+1} \qquad \text{and} \qquad \zeta(0) = 0. \tag{3.5}$$

It should be remarked that a potentially useful combinatorial identity known as Gould's identity, which unifies and extends the above combinatorial identities (3.3) and (3.4), is given by (see [16, p. 196, Eq. (6.1)])

$$\sum_{n=0}^{\infty} \frac{\gamma}{\gamma+(\beta+1)n} \binom{\alpha+(\beta+1)n}{n} t^n = (1+\zeta)^{\alpha} \sum_{n=0}^{\infty} (-1)^n \binom{\alpha-\gamma}{n} \binom{n+\gamma/(\beta+1)}{n}^{-1} \left(\frac{\zeta}{1+\zeta}\right)^n, \tag{3.6}$$

where α, β and γ are complex parameters independent of n, and $\zeta = \zeta(t)$ is given, as in (3.3) and (3.4), by (3.5).

Not only each of the combinatorial identities (3.3), (3.4) and (3.6), the Lagrange expansion theorem (3.2) itself has been applied widely and extensively in the study of generating functions and in other areas. For example, we refer to the widely-cited paper by Srivastava and Singhal [43] who gave the following unification and generalization of a large number of the earlier generating functions of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ defined by (1.17):

$$\sum_{n=0}^{\infty} P_n^{(\alpha+\lambda n,\beta+\mu n)}(x)\ t^n = \frac{(1+\xi)^{\alpha+1}\ (1+\eta)^{\beta+1}}{1-\lambda\xi-\mu\eta-(\lambda+\mu+1)\xi\eta}, \tag{3.7}$$

where the parameters α, β, λ and μ are unrestricted, in general, and ξ and η are functions of x and t defined *implicitly* by

$$\begin{cases} \xi = \xi(x,t) = \frac{1}{2}(x+1)t(1+\xi)^{\lambda+1}\ (1+\eta)^{\mu+1} \\ \\ \eta = \eta(x,t) = \frac{1}{2}(x-1)t(1+\xi)^{\lambda+1}\ (1+\eta)^{\mu+1}. \end{cases} \tag{3.8}$$

Some encouraging developments involving the Srivastava-Singhal generating function (3.7) are worth mentioning here. Strehl [44] gave an interesting combinatorial proof of the Srivastava-Singhal generating function (3.7), and Chen and Ismail [7] made use of Darboux's method in conjunction with the Srivastava-Singhal generating function (3.7) in order to derive the asymptotics of the Jacobi polynomials $P_n^{(\alpha+\lambda n,\beta+\mu n)}(x)$ as $n \to \infty$ when the parameters α, β, λ and μ, as well as the argument x, are fixed. On the other hand, Gawronski and Shawyer [15] used (3.7) to calculate the asymptotic distribution of the zeros of the Jacobi polynomials $P_n^{(\alpha+\lambda n,\beta+\mu n)}(x)$ as $n \to \infty$.

In light of the relationships (2.4), (2.5) and (2.6), generating functions for the Jacobi and Laguerre polynomials can be applied also to derive the corresponding generating functions for the generalized Bessel polynomials $\mathfrak{Y}_n^{(\alpha,\beta)}(x)$. We list below some of such consequences from the generating functions for the Jacobi, Laguerre and other hypergeometric polynomials (see, for details, [8], [9], [24], [27], [32], [35], [40], [41, Chapter 7] and [42]). We choose to present here the following generating function which is derivable by appealing appropriately to Gould's combinatorial identity (3.6).

$$\sum_{n=0}^{\infty} \frac{\gamma}{\gamma-(\sigma+1)n} \mathfrak{Y}_n^{(\alpha+\sigma n,\beta)}(x) \frac{t^n}{n!}$$
$$= (1+w)^{1-\alpha} \sum_{n=0}^{\infty} \Gamma_n(1-\alpha,-\sigma-2,\gamma;w)\, {}_1F_1\left[\begin{array}{c} -\frac{\gamma}{\sigma+1}; \\ 1+n-\frac{\gamma}{\sigma+1}; \end{array} -\frac{\beta w}{x}\right], \tag{3.9}$$

where, and in what follows,

$$\Gamma(\alpha,\beta,\gamma;\zeta) := (-1)^n \binom{\alpha-\gamma}{n} \binom{n+\gamma/(\beta+1)}{n}^{-1} \left(\frac{\zeta}{1+\zeta}\right)^n \tag{3.10}$$

and w is a function of x and t, which is defined *implicitly* by

$$xt = -\beta w(1+w)^{\sigma+1} \quad \text{and} \quad w(x,0)=0. \tag{3.11}$$

Two simpler cases of the generating function (3.9) are given by

$$\sum_{n=0}^{\infty} \mathfrak{Y}_n^{(\alpha+\sigma n,\beta)}(x)\, \frac{t^n}{n!} = \frac{(1+w)^{2-\alpha}}{1+(\sigma+2)w}\, \exp\left(-\frac{bw}{x}\right) \tag{3.12}$$

and

$$\sum_{n=0}^{\infty} \frac{\alpha-1}{\alpha-1+(\sigma+1)n}\, \mathfrak{Y}_n^{(\alpha+\sigma n,\beta)}(x)\, \frac{t^n}{n!} = (1+w)^{1-\alpha}\, {}_1F_1\left[\begin{array}{c} \dfrac{\alpha-1}{\sigma+1}; \\ 1+\dfrac{\alpha-1}{\sigma+1}; \end{array} -\frac{\beta w}{x}\right]. \tag{3.13}$$

In the *further* special case of the generating function (3.12) when $\sigma=-2$, we can readily deduce the generating function (2.22).

Generating functions for the generalized Bessel polynomials $\mathfrak{Y}_n^{(\alpha,\beta)}(x)$ can indeed be found in the above-mentioned and other journal articles and in the treatise on generating functions by Srivastava and Manocha [41] in which the interested reader can find citations of numerous other related works in the literature.

4. Concluding Remarks and Observations

In view of the remarkably close relationship with the modified Bessel function $K_\nu(z)$ of the second kind, which is known also as the Macdonald function (or, with a slightly different definition, the Basset function), the so-called Bessel polynomials $\mathfrak{y}_n(x)$ and their two-parameter version $\mathfrak{Y}_n^{(\alpha,\beta)}$, together with their reversed forms $\vartheta_n(x)$ and $\vartheta_n^{(\alpha,\beta)}(x)$, are widely and extensively investigated, and applied in the existing literature on the subject. Motivated essentially by these developments, herein we have systematically investigated several families of bilinear, bilateral and multilinear (or mixed multilateral) generating functions of the simple Bessel polynomials $\mathfrak{y}_n(x)$ and the generalized Bessel polynomials $\mathfrak{Y}_n^{(\alpha,\beta)}(x)$. Each of the results, which we have presented in this article, is potentially useful in deriving simpler corollaries and consequences by suitably specializing the parameters involved therein.

The targeted reader of this article will also find a systematic introduction and description of many other classes of orthogonal polynomial systems, together with the potentially useful inter-relationships between them. Furthermore, with a view to making this article as comprehensively informative as possible, the reader will have access to an up-to-date listing and citation of the available literature on the subject.

One can appreciate the importance of the Bessel polynomials by the fact that they arise rather naturally in several seemingly diverse contexts including (for example) in connection with the solution of the wave equation in spherical polar coordinates (see [22]), in network synthesis and design (see [14]), in the analysis of the student t-distribution (see, for example, [19] and [2]), in a representation of the energy spectral functions for a family of isotropic turbulence fields (see [48] and [33]), in developing a matrix technique applicable in solving some multi-order pantograph differential equations of fractional order (see [20]), and so on (see, for example, [21]). In many recent and forthcoming publications, the Bessel polynomials and the reversed Bessel polynomials continue to be useful in developing various numerical and approximation techniques, and other collocation and quasi-linearization approaches, in successfully handling a wide variety of problems which stem from several diverse areas of the mathematical, physical, chemical, biological and engineering sciences. The familiarization of these and other recent publications will surely lead to further researches requiring the usefulness of the Bessel polynomials and the reversed Bessel polynomials.

Statements and Declarations

The author declares that he has no conflict of interest and that the manuscript has no associated data.

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