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EXISTENCE OF SOLUTIONS FOR SECOND-ORDER DIFFERENTIAL INCLUSIONS ASSOCIATED WITH HIGHLY NON-SEMICONTINUOUS MULTIFUNCTIONS

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Abstract. In this paper we deal with the existence of generalized solutions for the Cauchy problem associated with a second-order differential inclusion, both in explicit and in implicit form. We firstly prove an existence result for an inclusion of the type $u'' \in F(t, u, u')$, where $F : [0, T] \times \mathbf{R}^n \times \mathbf{R}^n \to 2^{\mathbf{R}^n}$ is a given closed-valued multifunction. The main peculiarity of this latter result is as follows: our assumptions do not imply any kind of semicontinuity for the multifunction $F(t, \cdot, \cdot)$. That is, a multifunction F can satisfy all the assumptions and, at the same time, for every $t \in [0, T]$ the multifunction $F(t, \cdot, \cdot)$ can be neither upper nor lower semicontinuous even at each point $(x, z) \in \mathbf{R}^n \times \mathbf{R}^n$. A viable version of this result is also proved. Furtherly, as an application, an analogous result is proved for an inclusion of the type $u'' \in Q(t, u, u') + S(t, u, u')$, where $Q : [0, T] \times \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ has convex values, and $S : [0, T] \times \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ has closed values. Again, our assumptions do not imply any kind of semicontinuity for the multifunctions $Q(t, \cdot, \cdot)$ and $S(t, \cdot, \cdot)$. Then we consider an application to the implicit differential inclusion $\psi(u'') \in F(t, u, u') + G(t, u, u')$, where F is convex-valued and G is closed-valued. As regards the function ψ, we only assume that it is continuous and locally nonconstant. Finally, we present a further application to the Cauchy problem associated with a Sturm-Liouville type differential inclusion.

Keywords. Cauchy problem, differential inclusions, discontinuous selections, lower semicontinuity.

1. Introduction

Let $T > 0$, $n \in \mathbf{N}$, and let $F : [0, T] \times \mathbf{R}^n \times \mathbf{R}^n \to 2^{\mathbf{R}^n}$ be a multifunction. In this paper we are mainly interested in the existence of generalized solutions in $[0, T]$ for the second-order Cauchy problem

$$\begin{cases} u'' \in F(t, u, u') \\ u(0) = u'(0) = 0_{\mathbf{R}^n}. \end{cases} \tag{1.1}$$

As usual, a generalized solution of problem (1.1) in $[0, T]$ is a function $u \in C^1([0, T]), \mathbf{R}^n)$ such that u' is absolutely continuous in $[0, T]$, $u(0) = u'(0) = 0_{\mathbf{R}^n}$, and

$$u''(t) \in F(t, u(t), u'(t)) \quad \text{for a.e.} \quad t \in [0, T].$$

In the paper [4], some existence results were proved for problem (1.1), which ensure the existence of solutions belonging to the space $W^{2,\infty}([0, T], \mathbf{R}^n)$. Such results require, in particular, that F is

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bounded, jointly measurable, closed valued, and that for each $t \in [0,T]$ the multifunction $F(t,\cdot,\cdot)$ is lower semicontinuous (Theorems 1.2 and Theorem 3.1 of [4]).

Recently, by means of the results of [4] and of a selection theorem (Theorem 3.2 of [10]), some existence results has been proved in [2] for the more general Cauchy problem

$$\begin{cases} u'' \in Q(t,u,u') + S(t,u,u') \\ u(0) = u'(0) = 0_{\mathbf{R}^n}, \end{cases} \tag{1.2}$$

where Q has nonempty convex values and S has nonempty closed values.

It is worth noticing that, in the main result of [2] (see Theorem 2.1 of [2]), the assumptions on the multifunctions $Q : [0,T] \times \mathbf{R}^n \times \mathbf{R}^n \to 2^{\mathbf{R}^n}$ and $S : [0,T] \times \mathbf{R}^n \times \mathbf{R}^n \to 2^{\mathbf{R}^n}$ are as follows:

- (a_1) for every $t \in [0,T]$, the multifuntions $Q(t,\cdot,\cdot)$ and $S(t,\cdot,\cdot)$ are lower semicontinuous on $\mathbf{R}^n \times \mathbf{R}^n$.
- (a_2) Q and S are jointly weakly-measurable with respect to the product σ-algebra $\mathcal{L}([0,T]) \otimes \mathcal{B}(\mathbf{R}^n \times \mathbf{R}^n)$, where $\mathcal{L}([0,T])$ is the family of all Lebesgue-measurable subsets of $[0,T]$, and $\mathcal{B}(\mathbf{R}^n \times \mathbf{R}^n)$ is the Borel family of $\mathbf{R}^n \times \mathbf{R}^n$;
- (a_3) the multifunction $Q + S$ is bounded.

Of course, assumption (a_3) implies that both Q and S are bounded. Furtherly, in the paper [2], the authors study the implicit Cauchy problem

$$\begin{cases} \psi(u'') \in F(t,u,u') + G(t,u,u') \\ u(0) = u'(0) = 0_{\mathbf{R}^n}, \end{cases} \tag{1.3}$$

where $F : [0,T] \times \mathbf{R}^n \times \mathbf{R}^n \to 2^{\mathbf{R}}$ has nonempty convex values and $G : [0,T] \times \mathbf{R}^n \times \mathbf{R}^n \to 2^{\mathbf{R}}$ has nonempty closed values. Again, it is worth noticing that the following basic assumptions are made on F and G (see Theorem 3.1 of [2]):

- (b_1) for every $t \in T$, the multifuntions $F(t,\cdot,\cdot)$ and $G(t,\cdot,\cdot)$ are lower semicontinuous on $\mathbf{R}^n \times \mathbf{R}^n$.
- (b_2) F and G are jointly weakly-measurable with respect to the product σ-algebra $\mathcal{L}([0,T]) \otimes \mathcal{B}(\mathbf{R}^n \times \mathbf{R}^n)$;

As regards the real function ψ, it is required to be continuous and locally non constant on a compact, connected and locally connected set $Y \subseteq \mathbf{R}^n$. Again, we observe that the assumptions on F, G and ψ made in Theorem 3.1 of [2] imply, in particular, that both F and G are bounded.

The aim of this paper is to prove some existence results for problems (1.1), (1.2) and (1.3) where, with respect to the results of [2] and [4], the lower semicontinuity assumption on the involved multifunctions Q, S, F and G is drastically weakened. As a matter of fact, a multifunction which satisfies our assumptions could be neither lower nor upper semicontinuous, with respect to the variable $(x,z) \in \mathbf{R}^n \times \mathbf{R}^n$, even at all points $(x,z) \in \mathbf{R}^n \times \mathbf{R}^n$. On the other side, we pay such a generality by requiring that the ranges of the multifunctions F and $Q + S$, and the set Y in problems (1.1), (1.2) and (1.3), respectively, are (roughly speaking) "well localized" in $\mathbf{R}^n$. We also point out that no requirement of boundedness is made neither on the multifunctions Q, S, F and G, nor on the set Y. As regards the function ψ in problem (1.3), we only assume that it is continuous and locally nonconstant on a closed, connected and locally connected (possibly unbounded) set $Y \subseteq \mathbf{R}^n$.

Finally, an application to the Cauchy problem associated with a Sturm-Liouville-type differential inclusion is given, where the involved multifunction $H : [0,T] \times \mathbf{R}^n \times \mathbf{R}^n \to 2^{\mathbf{R}}$ does not need to be lower semicontinuous with respect to the variable $(x,z) \in \mathbf{R}^n \times \mathbf{R}^n$.

We shall give all the accurate definitions and also some examples in the following sections. Here, just in order to show the nature of our results, we only point out the following very special case of our

Theorem 4.1 below, concerning problem (1.2) (for the definitions and notations not explicitly recalled before, we refer to the next Section 2).

Theorem 1.1. Let a, T be positive real numbers, $p \in [1, +\infty]$, and let $Q : [0,T] \times \mathbf{R} \times \mathbf{R} \to 2^{\mathbf{R}}$ and $S : [0,T] \times \mathbf{R} \times \mathbf{R} \to 2^{\mathbf{R}}$ be two multifunctions. Assume that there exist a Lebesgue measurable set V, with null Lebesgue measure, and a positive function $\beta \in L^p([0,T])$ such that:

(i) for a.e. $t \in [0,T]$, the multifunction $Q(t,\cdot,\cdot)|_{(\mathbf{R}\setminus V)\times(\mathbf{R}\setminus V)}$ is lower semicontinuous with nonempty convex values;
(ii) for a.e. $t \in [0,T]$, the multifunction $S(t,\cdot,\cdot)|_{(\mathbf{R}\setminus V)\times(\mathbf{R}\setminus V)}$ is lower semicontinuous with nonempty closed values;
(iii) the multifunctions $Q|_{[0,T]\times(\mathbf{R}\setminus V)\times(\mathbf{R}\setminus V)}$ and $S|_{[0,T]\times(\mathbf{R}\setminus V)\times(\mathbf{R}\setminus V)}$ are $\mathcal{L}([0,T]) \otimes \mathcal{B}(\mathbf{R}\setminus V) \otimes \mathcal{B}(\mathbf{R}\setminus V)$ - weakly measurable;
(iv) for a.e. $t \in [0,T]$, one has
$$Q(\,t,\,(\mathbf{R}\setminus V)\times(\mathbf{R}\setminus V)) + S(\,t,\,(\mathbf{R}\setminus V)\times(\mathbf{R}\setminus V)) \subseteq [a, \beta(t)];$$

Then, there exists a function $u \in W^{2,p}([0,T])$ such that
$$\begin{cases} (u(t), u'(t)) \in (\mathbf{R}\setminus V)\times(\mathbf{R}\setminus V) & \text{for a.e. } t \in [0,T], \\ u''(t) \in Q(t, u(t), u'(t)) + S(t, u(t), u'(t)) & \text{for a.e. } t \in [0,T], \\ u(0) = u'(0) = 0. \end{cases}$$

It is immediate to check that the assumptions of Theorem 1.1 do not imply the lower semicontinuity of the multifunctions $Q(t,\cdot,\cdot)$ and $S(t,\cdot,\cdot)$. To see this, one can consider the following simple example.

Example 1.1. Let $T > 0$, and let $\mathbf{Q}$ denote the set of all rational real numbers. Let $Q : [0,T]\times\mathbf{R}\times\mathbf{R} \to 2^{\mathbf{R}}$ and $S : [0,T] \times \mathbf{R} \times \mathbf{R} \to 2^{\mathbf{R}}$ be the multifunctions defined by putting
$$Q(t,x,z) = \begin{cases}]2,4[& \text{if } t \in [0,T] \text{ and } (x,z) \in (\mathbf{R}\setminus\mathbf{Q})\times(\mathbf{R}\setminus\mathbf{Q}) \\ \{\arctan(t+x+z)\} & \text{otherwise,} \end{cases}$$
$$S(t,x,z) = \begin{cases} \{10 + \cos^2(x+z),\; 20 + \sin^2 z\} & \text{if } t \in [0,T] \text{ and } (x,z) \in (\mathbf{R}\setminus\mathbf{Q})\times(\mathbf{R}\setminus\mathbf{Q}) \\ \{0\} & \text{otherwise.} \end{cases}$$
It is routine matter to check that all the assumptions of Theorem 1.1 are satisfied with $n = 1$, $V = \mathbf{Q}$, $p = +\infty$, $a = 12$ and $\beta(t) \equiv 25$. In particular, the lower semicontinuity of the multifunction $Q(t,\cdot,\cdot)|_{(\mathbf{R}\setminus\mathbf{Q})\times(\mathbf{R}\setminus\mathbf{Q})}$ is trivial, while the lower semicontinuity of the multifunction $S(t,\cdot,\cdot)|_{(\mathbf{R}\setminus\mathbf{Q})\times(\mathbf{R}\setminus\mathbf{Q})}$ follows by Theorem 7.3.8 of [11]. Hence, since both $Q|_{[0,T]\times(\mathbf{R}\setminus\mathbf{Q})\times(\mathbf{R}\setminus\mathbf{Q})}$ and $S|_{[0,T]\times(\mathbf{R}\setminus\mathbf{Q})\times(\mathbf{R}\setminus\mathbf{Q})}$ do not depend on t explicitly, they are $\mathcal{L}([0,T]) \otimes \mathcal{B}(\mathbf{R}\setminus\mathbf{Q}) \otimes \mathcal{B}(\mathbf{R}\setminus\mathbf{Q})$ - weakly measurable. Consequently, all the assumptions of Theorem 1.1 are satisfied, as claimed. However, it is easy to check that, for each $t \in [0,T]$, the multifunctions $Q(t,\cdot,\cdot)$ and $S(t,\cdot,\cdot)$ are neither lower semicontinuous nor upper semicontinuous at each point $(x,z) \in \mathbf{R}\times\mathbf{R}$. It is also worth noticing that, for every $(t,x,z) \in [0,T]\times(\mathbf{R}\setminus\mathbf{Q})\times(\mathbf{R}\setminus\mathbf{Q})$, the set $Q(t,x,z) + S(t,x,z)$ is neither closed nor convex.

The paper is organized as follows: firstly, in Section 2, we give some notations and definitions, and we prove some preliminary results. Then, in Section 3, we prove our main result (Theorem 3.1) concerning problem (1.1), as well as an application to the viable case. In the same section, we discuss and characterize the class of multifunctions that we are considering, and provide some counter-example to possible improvements of our results. In Section 4, we present some applications of the results of Section 3. More specifically, by means of a selection result, we first prove an existence theorem concerning problem (1.2). Then, we consider further applications to the implicit Cauchy problem (1.3), and to a Sturm-Liouville type differential inclusion.

2. Preliminaries

For each $k \in \mathbf{N}$, we denote by m_k the k-dimensional Lebesgue measure in $\mathbf{R}^k$. In what follows, a measurable set (resp., a measurable function) will mean a Lebeasgue measurable set (resp., a Lebesgue measurable function).

Let $n \in \mathbf{N}$. For each $i \in \{1, \dots, n\}$, we denote by $P_{n,i} : \mathbf{R}^n \to \mathbf{R}$ the projection over the i-th axis. Moreover, we denote by $\mathcal{F}_n$ the family of all subsets $V \subseteq \mathbf{R}^n$ such that there exist sets $V_1, \dots, V_n \subseteq \mathbf{R}^n$, with $m_1(P_{n,i}(V_i)) = 0$ for all $i =, 1 \dots, n$, such that $V = \bigcup_{i=1}^n V_i$. Of course, any set $V \in \mathcal{F}_n$ satisfies $m_n(V) = 0$. If $U \subseteq \mathbf{R}^n$, we denote by $\overline{\mathrm{conv}}\,(U)$ the closed convex hull of the set U. Moreover, we denote by $\mathcal{G}_n$ the family of all subsets $U \subseteq \mathbf{R}^n$ such that, for every $i = 1, \dots, n$, the supremum and the infimum of the projection of $\overline{\mathrm{conv}}\,(U)$ on the i-th axis are both positive or both negative.

We denote by $\|\cdot\|_n$ the Euclidean norm of $\mathbf{R}^n$. Moreover, we denote by $\|\cdot\|_n^*$ the norm

$$\|(x_1, \dots, , x_n)\|_n^* = \max_{i=,1\dots,n} |x_i|$$

of $\mathbf{R}^n$. If $x = (x_1, \dots, , x_n) \in \mathbf{R}^n$ and $r > 0$, we shall denote by $\overline{B}_n(x, r)$ (resp., $\overline{B}^*_n(x, r)$) the closed ball in $\mathbf{R}^n$ centered at x with radius r, with respect to the norm $\|\cdot\|_n$ (resp., to the norm $\|\cdot\|_n^*$).

If X is a topological space, we denote by $\mathcal{B}(X)$ the Borel family of X. Moreover, if $A \subseteq Y \subseteq X$, we denote by $\mathrm{int}_Y(A)$ the interior of A in Y. The interior of A in the whole space X will be denoted by $\mathrm{int}(A)$. If S is a Polish space (that is, a separable complete metric space) endowed with a positive regular Borel measure μ, we shall denote by $\mathcal{S}_\mu$ the completion of the σ-algebra $\mathcal{B}(S)$ with respect to the measure μ. For the definition of Souslin spaces, Souslin sets and their properties, we refer to [1].

Let $T > 0$, $k \in \mathbf{N}$ and $p \in [1, +\infty]$. As usual, we denote by $W^{k,p}([0, T], \mathbf{R}^n)$ the space of all functions $u \in C^{k-1}([0, T], \mathbf{R}^n)$ such that $u^{(k-1)}$ is absolutely continuous in $[0, T]$ and $u^{(k)} \in L^p([0, T], \mathbf{R}^n)$. The space $L^p([0, T], \mathbf{R}^n)$ is endowed with the norm

$$\|u\|_{L^p([0,T],\mathbf{R}^n)} = \begin{cases} \Big(\displaystyle\int_{[0,T]} \|u(t)\|_n^p \, dt \Big)^{1/p} & \text{if } p < +\infty, \\ \operatorname{ess\,sup}_{t\in[0,T]} \|u(t)\|_n & \text{if } p = +\infty. \end{cases}$$

As usual, we put $L^p([0, T]) := L^p([0, T], \mathbf{R})$ and $W^{k,p}([0, T]) := W^{k,p}([0, T], \mathbf{R})$.

For the basic definitions and properties on multifunctions, we refer to [6] and [11]. We only recall that, if $(X, \mathcal{A})$ is a measurable space and Y is a topological space, a multifunction $F : X \to 2^Y$ is said to be $\mathcal{A}$-measurable (resp., $\mathcal{A}$-weakly measurable) if for every closed (resp., open) set $\Omega \subseteq Y$ one has

$$F^-(\Omega) := \{x \in X : \ F(x) \cap \Omega \neq \emptyset\} \in \mathcal{A}.$$

For what concerns measurable multifunctions, we also refer to the paper [9].

If Z is a separable Banach space, we shall consider in the sequel the family $\mathcal{D}(Z)$ of nonempty convex subsets of Z defined at p. 372 of the seminal paper [12]. We recall that, in particular, the family $\mathcal{D}(Z)$ contains all nonempty convex subsets of Z which are either finite-dimensional, or closed, or have an interior point.

The following selection result will be useful in the sequel.

Theorem 2.1. Let S and X be Polish spaces, and let μ be a finite positive regular Borel measure over S. Let Z be a separable Banach space, $W \subseteq X$ a Souslin set, and let $F : S \times W \to 2^Z$ be a multifunction whose values belongs to the family $\mathcal{D}(Z)$. Assume that:

(i) the multifunction F is $\mathcal{S}_\mu \otimes \mathcal{B}(W)$-weakly measurable;
(ii) for every $t \in S$, the multifunction $F(t, \cdot\,)$ is lower semicontinuous.

Then, there exist a function $\phi : S \times W \to Z$ such that:

(a) $\phi(t, x) \in F(t, x)$ for all $(t, x) \in S \times W$;

(b) for all $x \in W$, the function $\phi(\,\cdot\,, x)$ is $\mathcal{S}_\mu$-measurable over S;
(c) for every $t \in S$, the function $\phi(t, \cdot\,)$ is continuous.

We do not think that Theorem 2.1 is unknown. However, we were not able to find an appropriate reference. For instance, it can be checked that the results of [7] cannot be applied since the multifunction F must be defined and non-empty valued on the whole space $S \times X$ (with X complete). Similarly, in Theorem 3.2 of [10] (which is, in turn, Theorem 5.2 of [2]), even if the multifunction F is allowed to have some empty values, the lower semicontinuity of $F(t, \cdot)$ is required *on the whole space* X (which is assumed to be complete). Hence, this latter results cannot be applied for our purposes. Consequently, we now provide a short proof of Theorem 2.1. In order to do this, we firstly recall the following lemma.

Lemma 2.2. (Lemma 2.3 of [5]) Let S, X be two Polish spaces, and let μ be a finite positive regular Borel measure on S. Let $W \subseteq X$ be a Souslin set, $E \subseteq W$ another set. Let Z be a separable metric space, and let $F : S \times W \to 2^Z$ be a multifunction with nonempty values. Assume that:

(i) F is $\mathcal{S}_\mu \otimes \mathcal{B}(W)$- weakly measurable;
(ii) for all $t \in S$, one has

$$\{x \in W : F(t, \cdot) \text{ is not lower semicontinuous at } x\} \subseteq E.$$

Then, for each $\varepsilon > 0$ there exists a compact set $K \subseteq S$ such that $\mu(S \setminus K) \leq \varepsilon$ and the multifunction $F|_{K \times W}$ is lower semicontinuous at each point $(t, x) \in K \times (W \setminus E)$.

Proof of Theorem 2.1. By Lemma 2.2, for each $n \in \mathbf{N}$ there exists a compact set $K_n \subseteq S$ such that $\mu(S \setminus K_n) \leq \frac{1}{n}$ and the multifunction $F|_{K_n \times W}$ is lower semicontinuous at each point $(t, x) \in K_n \times W$. Let us put $D_1 := K_1, \quad D_n := K_n \setminus \bigcup_{j=1}^{n-1} K_j, \quad n \geq 2$. Of course, the sets $\{D_n\}$ are pairwise disjoint and one has $\bigcup_{n \in \mathbf{N}} K_n = \bigcup_{n \in \mathbf{N}} D_n$. Moreover, for every $n \in \mathbf{N}$ the multifunction $F|_{D_n \times W}$ is lower semicontinuous at each point $(t, x) \in D_n \times W$. Let

$$C := S \setminus \bigcup_{n \in \mathbf{N}} D_n.$$

Of course, we have $C \in \mathcal{B}(S)$. Moreover, for every $j \in \mathbf{N}$ one has

$$\mu(C) = \mu\big(S \setminus \bigcup_{n \in \mathbf{N}} D_n\big) = \mu\big(S \setminus \bigcup_{n \in \mathbf{N}} K_n\big) = \mu\big(\bigcap_{n \in \mathbf{N}} (S \setminus K_n)\big) \leq \mu(S \setminus K_j) \leq \frac{1}{j}\,,$$

hence $\mu(C) = 0$. By Theorem 3.1$'''$ of [12], for each $n \in \mathbf{N}$ there exists a continuous function $g_n : D_n \times W \to Z$ such that

$$g_n(t, x) \in F(t, x) \quad \forall\, (t, x) \in D_n \times W.$$

Moreover, again by Theorem 3.1$'''$ of [12], for each $t \in C$ there exists a continuous function $h_t : W \to Z$ such that

$$h_t(x) \in F(t, x) \quad \text{for all} \quad x \in W.$$

Now, let $\phi : S \times W \to Z$ be defined by

$$\phi(t, x) = \begin{cases} g_n(t, x) & \text{if } t \in D_n \\ h_t(x) & \text{if } t \in C. \end{cases}$$

By construction, we immediately have that $\phi(t, x) \in F(t, x)$ for all $(t, x) \in S \times W$ and, for every $t \in S$, the function $\phi(t, \cdot)$ is continuous.

Finally, if we fix $x \in W$, we have that for all $n \in \mathbf{N}$ the function $\phi(\cdot, x)|_{D_n} = g_n(\cdot\,, x)$ is continuous, hence it is $\mathcal{B}(D_n)$-measurable. Since $\mu(C) = 0$, it follows at once that the function $\phi(\,\cdot\,, x)$ is $\mathcal{S}_\mu$-measurable over S. $\square$

For the sake of a better reading, we now explicitly recall two results that will be fundamental in the sequel. Firstly, we recall the following selection result.

Theorem 2.3. (Theorem 2.1 of [5]). Let S and $X_1, X_2, \dots X_k$ be complete separable metric spaces, with $k \in \mathbf{N}$, and let $X := \prod_{j=1}^{k} X_j$ (endowed with the product topology). Let $\mu, \psi_1, \dots, \psi_k$ be positive regular Borel measures over $S, X_1, X_2, \dots X_k$, respectively, with μ finite and $\psi_1, \dots, \psi_k$ σ-finite.

Let Z be a separable metric space, $W \subseteq X$ a Souslin set, and let $F : S \times W \to 2^Z$ be a multifunction with nonempty complete values. Let $E \subseteq W$ be a given set. Finally, for all $i \in \{1, \dots, k\}$, let $P_{*,i} : X \to X_i$ be the projection over X_i. Assume that:

(i) the multifunction F is $\mathcal{S}_\mu \otimes \mathcal{B}(W)$-weakly measurable;
(ii) for a.e. $t \in S$, one has

$$\big\{x = (x_1, \dots, x_k) \in W : F(t, \cdot) \text{ is not lower semicontinuous at } x\big\} \subseteq E.$$

Then, there exist sets $Q_1, \dots, Q_k$, with $Q_i \in \mathcal{B}(X_i)$ and $\psi_i(Q_i) = 0$ for all $i = 1, \dots, k$, and a function $\phi : S \times W \to Z$ such that:

(a) $\phi(t, x) \in F(t, x)$ for all $(t, x) \in S \times W$;
(b) for all $x := (x_1, x_2, \dots, x_k) \in W \setminus \Big[\big(\bigcup_{i=1}^{k} P_{*,i}^{-1}(Q_i)\big) \cup E\Big]$, the function $\phi(\,\cdot\,, x)$ is $\mathcal{S}_\mu$-measurable over S;
(c) for a.e. $t \in S$, one has

$$\big\{x = (x_1, x_2, \dots, x_k) \in W : \phi(\,t, \cdot\,) \text{ is discontinuous at } x\big\} \subseteq$$
$$\subseteq\ E \cup \Big[W \cap \Big(\bigcup_{i=1}^{k} P_{*,i}^{-1}(Q_i)\Big)\Big].$$

Finally, we recall the following proposition.

Proposition 2.4. (Proposition 2.4 of [3]) Let $A \subseteq \mathbf{R}^n$ be a measurable set, $\varphi : A \times \mathbf{R}^h \to \mathbf{R}^k$ be a given function, $H^* \subseteq \mathbf{R}^h$ a Lebesgue measurable set, with $m_h(H^*) = 0$, and let D^* be a countable dense subset of $\mathbf{R}^h$, with $D^* \cap H^* = \emptyset$. Assume that:

(i) for all $x \in A$, the function $\varphi(x, \cdot\,)$ is bounded;
(ii) for all $z \in D^*$, the function $\varphi(\,\cdot\,, z)$ is measurable.

Let $G : A \times \mathbf{R}^h \to 2^{\mathbf{R}^k}$ be the multifunction defined by setting, for each $(x, z) \in A \times \mathbf{R}^h$,

$$G(x, z) := \bigcap_{m \in \mathbf{N}} \overline{\text{conv}}\,\overline{\Big(\bigcup_{\substack{v \in D^* \\ \|v - z\|_h \le \frac{1}{m}}} \{\varphi(x, y)\}\Big)}.$$

Then, one has:

(a) G has nonempty closed convex values;
(b) for all $z \in \mathbf{R}^h$, the multifunction $G(\,\cdot\,, z)$ is $\mathcal{L}(A)$-measurable;
(c) for all $x \in A$, the multifunction $G(\,x,\,\cdot\,)$ has closed graph;
(d) if $x \in A$, and $\varphi(x, \cdot)|_{\mathbf{R}^h \setminus H^*}$ is continuous at $z \in \mathbf{R}^h \setminus H^*$, then one has

$$G(x, z) = \{\varphi(x, z)\}.$$

3. The main result

The following is our main result.

Theorem 3.1. Let $T > 0$, let $F : [0, T] \times \mathbf{R}^n \times \mathbf{R}^n \to 2^{\mathbf{R}^n}$ be a multifunction, and let $p \in [1, +\infty]$. Assume that there exist two sets $V, E \in \mathcal{F}_n$ and a positive function $\beta \in L^p([0, T])$ such that:

(i) for a.e. $t \in [0, T]$, the multifunction $F(t, \cdot, \cdot)|_{(\mathbf{R}^n \setminus V) \times (\mathbf{R}^n \setminus E)}$ is lower semicontinuous with nonempty closed values;

(ii) the multifunction $F|_{[0,T]\times(\mathbf{R}^n\setminus V)\times(\mathbf{R}^n\setminus E)}$ is $\mathcal{L}([0,T])\otimes\mathcal{B}(\mathbf{R}^n\setminus V)\otimes\mathcal{B}(\mathbf{R}^n\setminus E)$ - weakly measurable;

(iii) for a.e. $t\in[0,T]$, one has

$$F(t,\,(\mathbf{R}^n\setminus V)\times(\mathbf{R}^n\setminus E))\subseteq\overline{B}_n(0_{\mathbf{R}^n},\beta(t));$$

(iv) there exists $U_0\subseteq[0,T]$, with $m_1(U_0)=0$, such that

$$F(([0,T]\setminus U_0)\times(\mathbf{R}^n\setminus V)\times(\mathbf{R}^n\setminus E))\in\mathcal{G}_n.$$

Then, there exists $u\in W^{2,p}([0,T],\mathbf{R}^n)$ such that

$$\begin{cases} u''(t)\in F(t,u(t),u'(t)) & \text{for a.e. } t\in[0,T],\\ u(0)=u'(0)=0_{\mathbf{R}^n}, & \\ (u(t),u'(t))\in(\mathbf{R}^n\setminus V)\times(\mathbf{R}^n\setminus E) & \text{for a.e. } t\in[0,T]. \end{cases}$$

Proof. Without loss of generality, we can assume that assumptions (i) and (iii) are satisfied for all $t\in[0,T]$, and that $U_0=\emptyset$. It is routine matter to check that it is not restrictive to do this. Put

$$H:=F([0,T]\times(\mathbf{R}^n\setminus V)\times(\mathbf{R}^n\setminus E)).$$

By the definition of $\mathcal{F}_n$, there exist $2n$ subsets

$$V_1,\ldots,V_n,\quad E_1,\ldots,E_n$$

of $\mathbf{R}^n$ such that

$$V=\bigcup_{i=1}^n V_i,\qquad E=\bigcup_{i=1}^n E_i,$$

and

$$m_1(P_{n,i}(V_i))=m_1(P_{n,i}(E_i))=0\quad\text{for all}\quad i=1,\ldots,n.$$

For each $i=1,\ldots,n$, let $C_i,U_i\in\mathcal{B}(\mathbf{R})$ be such that

$$P_{n,i}(V_i)\subseteq C_i,\qquad P_{n,i}(E_i)\subseteq U_i$$

and $m_1(C_i)=m_1(U_i)=0$. Let

$$W:=\prod_{i=1}^n(\mathbf{R}\setminus C_i)\times\prod_{i=1}^n(\mathbf{R}\setminus U_i).$$

Of course, we have $W\in\mathcal{B}(\mathbf{R}^{2n})$ and

$$W\subseteq(\mathbf{R}^n\setminus V)\times(\mathbf{R}^n\setminus E). \tag{3.1}$$

By assumptions (i) and (ii), taking into account (3.1), we have that $F|_{[0,T]\times W}$ is $\mathcal{L}([0,T])\otimes\mathcal{B}(W)$-weakly measurable, and for all $t\in[0,T]$, the multifunction $F(t,\cdot,\cdot)|_W$ is lower semicontinuous in W with nonempty closed values. Since $W\in\mathcal{B}(\mathbf{R}^{2n})$, it is a Souslin set by Corollary 6.6.7 of [1]. By Theorem 2.3, there exists a set $Q_0\in\mathcal{L}([0,T])$ and $2n$ sets

$$Q_1,\ldots,Q_n,\quad H_1,\ldots,H_n$$

in $\mathcal{B}(\mathbf{R})$, with $m_1(Q_0)=0$ and

$$m_1(Q_i)=m_1(H_i)=0\quad\text{for all}\quad i=1,\ldots,n,$$

and a function $\phi:[0,T]\times W\to\mathbf{R}^n$, such that:

(a) $\phi(t,x,z)\in F(t,x,z)$ for all $(t,x,z)\in[0,T]\times W$ (hence, in particular, the function ϕ takes its values in H);

(b) for all $(x,z)\in W\setminus[\bigcup_{i=1}^n(P_{2n,i}^{-1}(Q_i)\cup P_{2n,n+i}^{-1}(H_i))]$, the function $\phi(\cdot,x,z)$ is $\mathcal{L}([0,T])$-measurable;

(c) for all $t \in [0,T] \setminus Q_0$, one has

$$\{(x,z) \in W : \phi(t,\cdot,\cdot) \text{ is discontinuous at } (x,z)\} \subseteq W \cap \Big[\bigcup_{i=1}^{n}(P_{2n,i}^{-1}(Q_i) \cup P_{2n,n+i}^{-1}(H_i))\Big].$$

Put

$$Z := \bigcup_{i=1}^{n}(P_{2n,i}^{-1}(Q_i) \cup P_{2n,n+i}^{-1}(H_i)),$$

and let $\phi^* : [0,T] \times \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ be defined by putting

$$\phi^*(t,x,z) = \begin{cases} \phi(t,x,z) & \text{if } t \in [0,T] \text{ and } (x,z) \in W, \\ 0_{\mathbf{R}^n} & \text{if } t \in [0,T] \text{ and } (x,z) \in (\mathbf{R}^n \times \mathbf{R}^n) \setminus W. \end{cases}$$

By (b), for all $(x,z) \in W \setminus Z$, the function $\phi^*(\,\cdot, x, z)$ is $\mathcal{L}([0,T])$-measurable. Moreover, by the above construction and by assumption (iii), taking into accont (3.1), we have that

$$\|\phi^*(t,x,z)\|_n \leq \beta(t) \quad \text{for all} \quad (t,x,z) \in [0,T] \times \mathbf{R}^n \times \mathbf{R}^n. \tag{3.2}$$

Consequently, for all $t \in [0,T]$, the function $\phi^*(t,\cdot,\cdot)$ is bounded. Now, observe that

$$W \setminus Z = (\mathbf{R}^n \times \mathbf{R}^n) \setminus \Big[\bigcup_{i=1}^{n}(P_{2n,i}^{-1}(C_i \cup Q_i) \cup P_{2n,n+i}^{-1}(U_i \cup H_i))\Big]. \tag{3.3}$$

Since $m_{2n}\Big(\bigcup_{i=1}^{n}(P_{2n,i}^{-1}(C_i \cup Q_i) \cup P_{2n,n+i}^{-1}(U_i \cup H_i))\Big) = 0$, there exists a countable set $D \subseteq W \setminus Z$ such that D is dense in $\mathbf{R}^n \times \mathbf{R}^n$.

Let $G : [0,T] \times \mathbf{R}^n \times \mathbf{R}^n \to 2^{\mathbf{R}^n}$ be the multifunction defined by setting, for all $(t,x,z) \in [0,T] \times \mathbf{R}^n \times \mathbf{R}^n$,

$$G(t,x,z) = \bigcap_{m\in\mathbf{N}} \overline{\operatorname{conv}\Big(\bigcup_{\substack{(v,w)\in D \\ \|(v,w)-(x,z)\|_{2n} \leq \frac{1}{m}}} \{\phi^*(t,v,w)\}\Big)} = \bigcap_{m\in\mathbf{N}} \overline{\operatorname{conv}\Big(\bigcup_{\substack{(v,w)\in D \\ \|(v,w)-(x,z)\|_{2n} \leq \frac{1}{m}}} \{\phi(t,v,w)\}\Big)}.$$

If we apply Proposition 2.4, with $H^* = \mathbf{R}^{2n} \setminus W$, we get that:

(a)′ G has nonempty closed convex values;
(b)′ for all $(x,z) \in \mathbf{R}^n \times \mathbf{R}^n$, the multifunction $G(\,\cdot, x, z)$ is $\mathcal{L}([0,T])$-measurable;
(c)′ for all $t \in [0,T]$, the multifunction $G(\,t,\,\cdot\,,\cdot\,)$ has closed graph;
(d)′ if $t \in [0,T]$, and the function $\phi^*(t,\cdot,\cdot)|_W = \phi(t,\cdot,\cdot)$ is continuous at $(x,z) \in W$, then one has

$$G(t,x,z) = \{\phi^*(t,x,z)\} = \{\phi(t,x,z)\}.$$

Moreover, observe that by (3.2) and by the above construction we have that

$$G(t,x,z) \subseteq \overline{B}_n(0_{\mathbf{R}^n}, \beta(t)) \cap \overline{\operatorname{conv}}(H) \quad \text{for all} \quad (t,x,z) \in [0,T] \times \mathbf{R}^n \times \mathbf{R}^n. \tag{3.4}$$

Now, let $g_1 : L^p([0,T],\mathbf{R}^n) \to W^{1,p}([0,T],\mathbf{R}^n)$ be defined by putting, for all $v \in L^p([0,T],\mathbf{R}^n)$,

$$g_1(v)(t) = \int_0^t v(s)\,ds \quad \text{for all} \quad t \in [0,T].$$

Let $g_2 : L^p([0,T],\mathbf{R}^n) \to C^1([0,T],\mathbf{R}^n)$ be defined by putting, for all $v \in L^p([0,T],\mathbf{R}^n)$,

$$g_2(v)(t) = \int_0^t g_1(v)(\tau)\,d\tau = \int_0^t d\tau \int_0^\tau v(s)\,ds \quad \text{for all} \quad t \in [0,T].$$

Let $\Phi : L^p([0,T],\mathbf{R}^n) \to W^{1,p}([0,T],\mathbf{R}^n \times \mathbf{R}^n)$ be defined by putting, for all $v \in L^p([0,T],\mathbf{R}^n)$,

$$\Phi(v) := (g_1(v), g_2(v)).$$

Now we want to apply Theorem 1 of [13] to the multifunction G, by choosing $s = p$, $q = 1$, $X = \mathbf{R}^n \times \mathbf{R}^n$, $Y = \mathbf{R}^n$, $V = L^p([0,T],\mathbf{R}^n)$, $\Psi(v) = v$, $\varphi \equiv +\infty$, $r = \|\beta\|_{L^p([0,T])}$, and Φ defined as above. To this aim, we observe what follows.

(a)$''$ For every $v \in L^p([0,T],\mathbf{R}^n)$, and every sequence $\{v_m\}$ in $L^p([0,T],\mathbf{R}^n)$, with $\{v_m\}$ weakly converging to v in $L^1([0,T],\mathbf{R}^n)$, the sequence $\{\Phi(v_m)\}$ converges strongly to $\Phi(v)$ in $L^1([0,T],\mathbf{R}^n \times \mathbf{R}^n)$. To see this, let the sequence $\{v_m\}$ and v in $L^p([0,T],\mathbf{R}^n)$ be fixed, with $\{v_m\}$ weakly convergent to v in $L^1([0,T],\mathbf{R}^n)$. It is routine matter to see that the sequence $\{g_1(v_m)\}$ converges pointwise in $[0,T]$ to $g_1(v)$. That is, one has

$$\lim_{m\to\infty} \|g_1(v_m)(t) - g_1(v)(t)\|_n = 0 \quad \text{for all} \quad t \in [0,T].$$

Since $\{v_m\}$ is weakly convergent in $L^1([0,T],\mathbf{R}^n)$, it is bounded in $L^1([0,T],\mathbf{R}^n)$. Hence, for all $m \in \mathbf{N}$ and $t \in [0,T]$ we have

$$\|g_1(v_m)(t) - g_1(v)(t)\|_n \le \| \int_0^t v(s)\,ds\|_n + \sup_{k\in\mathbf{N}} \|v_k\|_{L^1([0,T],\mathbf{R}^n)}.$$

By applying the Dominated Convergence Theorem we have

$$\lim_{m\to+\infty} \int_0^T \|g_1(v_m)(t) - g_1(v)(t)\|_n\,dt = 0,$$

hence $\{g_1(v_m)\}$ converges strongly to $g_1(v)$ in $L^1([0,T],\mathbf{R}^n)$. In particular, we get that the sequence $\{g_1(v_m)\}$ also converges weakly to $g_1(v)$ in $L^1([0,T],\mathbf{R}^n)$. Thus, if we now apply the same argument to the sequence $\{g_1(v_m)\}$, we have that the sequence $\{g_1(g_1((v_m))\} = \{g_2(v_m)\}$ also converges strongly to $g_1(g_1(v)) = g_2(v)$ in $L^1([0,T],\mathbf{R}^n)$. Hence, the sequence $\{\Phi(v_m)\}$ converges strongly to $\Phi(v)$ in $L^1([0,T],\mathbf{R}^n \times \mathbf{R}^n)$, as desired.

(b)$''$ If we consider the function

$$\omega : t \in [0,T] \to \sup_{(x,z)\in\mathbf{R}^n\times\mathbf{R}^n} \inf_{y\in G(t,x,z)} \|y\|_n,$$

by (3.4) we have that $\omega(t) \le \beta(t)$ for all $t \in [0,T]$. Hence, we have that $\omega \in L^p([0,T])$ and $\|\omega\|_{L^p([0,T])} \le \|\beta\|_{L^p([0,T])}$ (as regards the measurability of ω, we refer to p. 262 of [13]).

Therefore, all the assumptions of Theorem 1 of [13] are satisfied. Hence, there exists a function $\widetilde{v} \in L^p([0,T],\mathbf{R}^n)$ and a set $\Omega_0 \in \mathcal{L}([0,T])$, with $m_1(\Omega_0) = 0$, such that

$$\widetilde{v}(t) \in G(t, \Phi(\widetilde{v})(t)) \quad \text{for all} \quad t \in [0,T] \setminus \Omega_0. \tag{3.5}$$

In particular, by (3.4) we get

$$\widetilde{v}(t) \in \overline{\text{conv}}(H) \quad \text{and} \quad \|\widetilde{v}(t)\|_n \le \beta(t) \quad \text{for all} \quad t \in [0,T] \setminus \Omega_0. \tag{3.6}$$

Now, fix $i \in \{1,\ldots,n\}$, and let $\widetilde{v}_i$ denote the i-th component of the function $\widetilde{v}$. Since by assumption (iv) we have $H \in \mathcal{G}_n$, it follows by (3.6) and by the definition of $\mathcal{G}_n$ that the function $\widetilde{v}_i(t)$ has constant sign for all $t \in [0,T] \setminus \Omega_0$. Assume that

$$\widetilde{v}_i(t) > 0 \quad \text{for all} \quad t \in [0,T] \setminus \Omega_0$$

(if, conversely, $\widetilde{v}_i(t) < 0$ for all $t \in [0,T] \setminus \Omega_0$, then the argument is analogous). We then get

$$g_1(\widetilde{v})_i'(t) = \widetilde{v}_i(t) > 0 \quad \text{for a.e.} \quad t \in [0,T]$$

(as before, $g_1(\widetilde{v})_i$ denotes the i-th component of the function $g_1(\widetilde{v})$). Hence, the absolutely continuous function $g_1(\widetilde{v})_i$ is strictly increasing in $[0,T]$. By Theorem 2 of [16], the function $(g_1(\widetilde{v})_i)^{-1}$ is absolutely continuous in $[0, g_1(\widetilde{v})_i(T)]$. If we put

$$M_i := (g_1(\widetilde{v})_i)^{-1}((U_i \cup H_i) \cap [0, g_1(\widetilde{v})_i(T)]) = \{t \in [0,T] : g_1(\widetilde{v})_i(t) \in U_i \cup H_i\},$$

by Theorem 18.25 of [8] we get $m_1(M_i) = 0$.

Now, we have

$$g_2(\widetilde{v})_i'(t) = g_1(\widetilde{v})_i(t) \quad \text{for all} \quad t \in [0,T],$$

hence

$$g_2(\widetilde{v})_i'(t) > 0 \quad \text{for all} \quad t \in \,]\,0,T].$$

Therefore, the function $g_2(\widetilde{v})_i$ is strictly increasing in $[0,T]$. Again by Theorem 2 of [16], the function $(g_2(\widetilde{v})_i)^{-1}$ is absolutely continuous in $[0, g_2(\widetilde{v})_i(T)]$. If we put

$$N_i := (g_2(\widetilde{v})_i)^{-1}((C_i \cup Q_i) \cap [0, g_2(\widetilde{v})_i(T)]) = \{t \in [0,T] : g_2(\widetilde{v})_i(t) \in C_i \cup Q_i\},$$

by Theorem 18.25 of [8] we get $m_1(N_i) = 0$.

Now, let

$$\Omega := Q_0 \cup \Omega_0 \cup \Big(\bigcup_{i=1}^{n} (M_i \cup N_i) \Big).$$

By the above construction we have $m_1(\Omega) = 0$. Choose $\hat{t} \in [0,T] \setminus \Omega$. Since $\hat{t} \notin \bigcup_{i=1}^{n} (M_i \cup N_i)$, we have that

$$\Phi(\widetilde{v})(\hat{t}) \notin \bigcup_{i=1}^{n} (P_{2n,i}^{-1}(C_i \cup Q_i) \cup P_{2n,n+i}^{-1}(U_i \cup H_i)).$$

Hence, by (3.3) we get $\Phi(\widetilde{v})(\hat{t}) \in W \setminus Z$. In particular, by (3.1), we get

$$\Phi(\widetilde{v})(\hat{t}) \in (\mathbf{R}^n \setminus V) \times (\mathbf{R}^n \setminus E).$$

Since $\hat{t} \notin Q_0$, by (c) we have that $\phi(\hat{t}, \cdot, \cdot)$ is continuous at $\Phi(\widetilde{v})(\hat{t}) = (g_2(\widetilde{v})(\hat{t}), g_1(\widetilde{v})(\hat{t}))$. Hence, by the property (d)$'$, we get

$$G(\hat{t}, \Phi(\widetilde{v})(\hat{t})) = \big\{\phi(\hat{t}, \Phi(\widetilde{v})(\hat{t}))\big\}.$$

By (3.5) and by the property (a), we then get

$$\widetilde{v}(\hat{t}) = \phi(\hat{t}, \Phi(\widetilde{v})(\hat{t})) \in F(\hat{t}, \Phi(\widetilde{v})(\hat{t})).$$

Resuming, we have proved that, for every $t \in [0,T] \setminus \Omega$, one has

$$\widetilde{v}(t) \in F(t, \Phi(\widetilde{v})(t)) = F(t, g_2(\widetilde{v})(t), g_1(\widetilde{v})(t))$$

and

$$\Phi(\widetilde{v})(t) = (g_2(\widetilde{v})(t), g_1(\widetilde{v})(t)) \in (\mathbf{R}^n \setminus V) \times (\mathbf{R}^n \setminus E).$$

If we take $\widetilde{u} := g_2(\widetilde{v})$, it is immediate to check that the function $\widetilde{u} \in W^{2,p}([0,T], \mathbf{R}^n)$ satisfies the conclusion. □

Remark 3.1. It is immediate to check that a multifunction $F : [0,T] \times \mathbf{R}^n \times \mathbf{R}^n \to 2^{\mathbf{R}^n}$ satisfying the assumptions of Theorem 3.1 can be neither lower nor upper semicontinuous, with respect to the variable $(x,z) \in \mathbf{R}^n \times \mathbf{R}^n$, even at each point $(x,z) \in \mathbf{R}^n \times \mathbf{R}^n$. To see this, one can take $n = 1$, and $F : [0,T] \times \mathbf{R} \times \mathbf{R} \to 2^{\mathbf{R}}$ defined by

$$F(t,x,z) = \begin{cases} [3,4] & \text{if } (t,x,z) \in [0,T] \times (\mathbf{R} \setminus \mathbf{Q}) \times (\mathbf{R} \setminus \mathbf{Q}) \\ \{1\} & \text{otherwise.} \end{cases}$$

All the assumptions of Theorem 3.1 are satisfied with $V = E = \mathbf{Q}$, $p = +\infty$, $\beta(t) \equiv 4$. However, for each $t \in [0,T]$ the multifunction $F(t,\cdot,\cdot)$ is neither lower semicontinuous nor upper semicontinuous at each point $(x,z) \in \mathbf{R} \times \mathbf{R}$. In this connection, it is worth noticing that, in Theorem 3.1, the multifunction F could be defined only over the set $[0,T] \times (\mathbf{R}^n \setminus V) \times (\mathbf{R}^n \setminus E)$. Indeed, the behaviour of the multifunction F over the set $[0,T] \times ((V \times \mathbf{R}^n) \cup (\mathbf{R}^n \times E))$ plays no role at all.

In addition to the above remark, it is also useful to observe that assumption (i) of Theorem 3.1 can be formulated in other different equivalent ways, as the following simple proposition shows.

Proposition 3.2. Let $T > 0$, and let $F : [0,T] \times \mathbf{R}^n \times \mathbf{R}^n \to 2^{\mathbf{R}^n}$ be a multifunction. Then, the following conditions are equivalent:

1. there exist two sets $V, E \in \mathcal{F}_n$ such that for a.e. $t \in [0,T]$ the multifunction $F(t,\cdot,\cdot)|_{(\mathbf{R}^n \setminus V)\times(\mathbf{R}^n \setminus E)}$ is lower semicontinuous with nonempty closed (resp., convex) values;
2. there exist a set $B \in \mathcal{B}(\mathbf{R})$, with $m_1(B) = 0$, such that for a.e. $t \in [0,T]$ the multifunction $F(t,\cdot,\cdot)|_{(\mathbf{R}\setminus B)^{2n}}$ is lower semicontinuous with nonempty closed (resp., convex) values;
3. there exist a set $H \in \mathcal{F}_{2n}$ such that for a.e. $t \in [0,T]$, the multifunction
$$F(t,\cdot,\cdot)|_{(\mathbf{R}^n \times \mathbf{R}^n)\setminus H}$$
is lower semicontinuous with nonempty closed (resp., convex) values.

Proof. (1) $\Rightarrow$ (2) By the definition of $\mathcal{F}_n$, there exist $2n$ subsets $V_1, \ldots, V_n$, $E_1, \ldots, E_n$ of $\mathbf{R}^n$ such that $V = \bigcup_{i=1}^n V_i$, $E = \bigcup_{i=1}^n E_i$, and
$$m_1(P_{n,i}(V_i)) = m_1(P_{n,i}(E_i)) = 0 \quad \text{for all} \quad i = 1, \ldots, n.$$
Let $B \in \mathcal{B}(\mathbf{R})$ be such that $m_1(B) = 0$ and
$$\bigcup_{i=1}^n (P_{n,i}(V_i) \cup P_{n,i}(E_i)) \subseteq B.$$
Since
$$(\mathbf{R} \setminus B)^n \times (\mathbf{R} \setminus B)^n \subseteq (\mathbf{R}^n \setminus V) \times (\mathbf{R}^n \setminus E),$$
our claim follows at once.

(2) $\Rightarrow$ (3) Put
$$H := \bigcup_{j=1}^{2n} P_{2n,j}^{-1}(B).$$
Since $m_1(B) = 0$, we have $H \in \mathcal{F}_{2n}$. Moreover, we have
$$(\mathbf{R}^n \times \mathbf{R}^n) \setminus H = (\mathbf{R} \setminus B)^{2n},$$
and thus our claim follows.

(3) $\Rightarrow$ (1) By assumption, there exist $2n$ subsets $H_1, \ldots, H_{2n}$ of $\mathbf{R}^{2n}$ such that $H = \bigcup_{j=1}^{2n} H_j$ and $m_1(P_{2n,j}(H_j)) = 0$ for all $j = 1, \ldots, 2n$. Let $C \in \mathcal{B}(\mathbf{R})$ be such that $m_1(C) = 0$ and
$$\bigcup_{j=1}^{2n} P_{2n,j}(H_j) \subseteq C.$$
It is routine matter to check that
$$(\mathbf{R} \setminus C)^n \times (\mathbf{R} \setminus C)^n \subseteq (\mathbf{R}^n \times \mathbf{R}^n) \setminus H.$$
Put
$$V^* := \bigcup_{i=1}^n P_{n,i}^{-1}(C).$$

Of course, we have $V^* \in \mathcal{F}_n$. Moreover, we have

$$(\mathbf{R} \setminus C)^n = (\mathbf{R}^n \setminus \bigcup_{i=1}^{n} P_{n,i}^{-1}(C)) = \mathbf{R}^n \setminus V^*,$$

hence

$$(\mathbf{R}^n \setminus V^*) \times (\mathbf{R}^n \setminus V^*) \subseteq (\mathbf{R}^n \times \mathbf{R}^n) \setminus H.$$

At this point, the conclusion follows at once by choosing $V = E = V^*$. □

By the proof of the above proposition, it is clear that, together with assumption (i), even assumptions (ii), (iii) and (iv) of Theorem 3.1 can be reformulated in the corresponding equivalent ways, by replacing the set $(\mathbf{R}^n \setminus V) \times (\mathbf{R}^n \setminus E)$ by either $(\mathbf{R} \setminus B)^{2n}$ or $(\mathbf{R}^n \times \mathbf{R}^n) \setminus H$. The same considerations applies to the other results of this paper. We have chosen to formulate the assumptions in the present form in order to emphasize the separation between the variables x and z.

Remark 3.2. Theorem 3.1 does not hold without assumption (iv). In order to see this, let $T > 0$, and let $f : [0, T] \times \mathbf{R} \to \mathbf{R}$ be defined by putting

$$f(t, z) = \begin{cases} 0 & \text{if } t \in [0, T] \text{ and } z \neq 0, \\ 1 & \text{if } t \in [0, T] \text{ and } z = 0. \end{cases}$$

It was proved in Example 1 of [15] that the first-order Cauchy problem

$$\begin{cases} v' = f(t, v) \\ v(0) = 0 \end{cases} \tag{3.7}$$

has no generalized solutions in $[0, T]$. That is, there exists no absolutely continuous function $v : [0, T] \to \mathbf{R}$ such that $v(0) = 0$ and $v'(t) = f(t, v(t))$ for a.e. $t \in [0, T]$. Now, consider the second-order Cauchy problem

$$\begin{cases} u'' \in F(t, u, u') \\ u(0) = u'(0) = 0, \end{cases} \tag{3.8}$$

where the multifunction $F : [0, T] \times \mathbf{R} \times \mathbf{R} \to 2^{\mathbf{R}}$ is defined by

$$F(t, x, z) = \{f(t, z)\}.$$

It is immediate to see that problem (3.8) admits no generalized solutions. Indeed, assume that exists a generalized solution $u \in W^{2,1}([0, T])$ of problem (3.8). This implies that the absolutely continuous function $v := u'$ is a generalized solution of the Cauchy problem (3.7), and this is absurd by what precedes. However, all the assumptions of Theorem 3.1, with the exception of assumption (iv), are satisfied by taking $n = 1$, $p = +\infty$, $V = \emptyset$ and $E = \{0\}$.

The following result gives a local (viable) version of Theorem 3.1, which will be useful in the sequel.

Corollary 3.3. Let T, r be positive real numbers, and let $X := \overline{B}_n^*(0, r)$. Let $F : [0, T] \times X \times X \to 2^{\mathbf{R}^n}$ be a multifunction. Assume that there exist two sets $V, E \in \mathcal{F}_n$ such that:

(i) for a.e. $t \in [0, T]$, the multifunction $F(t, \cdot, \cdot)|_{(X \setminus V) \times (X \setminus E)}$ is lower semicontinuous with nonempty closed values;

(ii) the multifunction $F|_{[0,T] \times (X \setminus V) \times (X \setminus E)}$ is $\mathcal{L}([0, T]) \otimes \mathcal{B}(X \setminus V) \otimes \mathcal{B}(X \setminus E)$ - weakly measurable;

(iii) there exists $M > 0$, with $M \cdot \max\{T, \frac{T^2}{2}\} \leq r$, such that for a.e. $t \in [0, T]$, one has

$$F(t,\ (X \setminus V) \times (X \setminus E)) \subseteq \overline{B}_n(0_{\mathbf{R}^n}, M).$$

(iv) there exists $U_0 \subseteq [0, T]$, with $m_1(U_0) = 0$, such that

$$F(([0, T] \setminus U_0) \times (X \setminus V) \times (X \setminus E)) \in \mathcal{G}_n.$$

Then, there exists $u \in W^{2,\infty}([0,T],\mathbf{R}^n)$ such that

$$(u(t),u'(t)) \in (X\setminus V)\times(X\setminus E) \quad \text{for a.e.} \quad t\in[0,T],$$

and also

$$\begin{cases} u''(t) \in F(t,u(t),u'(t)) & \text{for a.e. } t\in[0,T],\\ u(0)=u'(0)=0_{\mathbf{R}^n}, & \\ \|u''(t)\|_n \le M & \text{for a.e. } t\in[0,T]. \end{cases}$$

Proof. It is not restrictive to assume that assumptions (i) and (iii) are satisfied for every $t\in[0,T]$, and that $U_0=\emptyset$. Choose any point $y^*\in F([0,T]\times(X\setminus V)\times(X\setminus E))$. Let $F^*:[0,T]\times\mathbf{R}^n\times\mathbf{R}^n\to 2^{\mathbf{R}^n}$ be the multifunction defined be setting, for each $(t,x,z)\in[0,T]\times\mathbf{R}^n\times\mathbf{R}^n$,

$$F^*(t,x,z)=\begin{cases} F(t,x,z) & \text{if } (x,z)\in X\times X,\\ \{y^*\} & \text{if } (x,z)\notin X\times X. \end{cases}$$

Put

$$H^* := V\cup E\cup\Big(\bigcup_{i=1}^{n} P_{n,i}^{-1}(\{-r,r\})\Big), \tag{3.9}$$

It is immediate to check that $H^*\in\mathcal{F}_n$. We observe what follows.

(a) the multifunction $F^*|_{[0,T]\times(\mathbf{R}^n\setminus H^*)\times(\mathbf{R}^n\setminus H^*)}$ is $\mathcal{L}([0,T])\otimes\mathcal{B}(\mathbf{R}^n\setminus H^*)\otimes\mathcal{B}(\mathbf{R}^n\setminus H^*)$ - weakly measurable (it follows at once by assumption (ii));

(b) for every $t\in[0,T]$, the multifunction $F^*(t,\cdot,\cdot)|_{(\mathbf{R}^n\setminus H^*)\times(\mathbf{R}^n\setminus H^*)}$ is lower semicontinuous with nonempty closed values. To see this, fix $t\in[0,T]$. By assumption (i) and by the above construction, it follow immediately that the multifunction $F^*(t,\cdot,\cdot)|_{(\mathbf{R}^n\setminus H^*)\times(\mathbf{R}^n\setminus H^*)}$ hash nonempty closed values. Choose $(x_0,z_0)\in(\mathbf{R}^n\setminus H^*)\times(\mathbf{R}^n\setminus H^*)$. In order to prove that $F^*(t,\cdot,\cdot)|_{(\mathbf{R}^n\setminus H^*)\times(\mathbf{R}^n\setminus H^*)}$ is lower semicontinuous at (x_0,z_0), we distinguish two cases. Firstly, we assume that $(x_0,z_0)\in X\times X$. Consequently, by the definition of H^*, we get $(x_0,z_0)\in\operatorname{int}(X\times X)$. By assumption (i), the multifunction

$$F^*(t,\cdot,\cdot)|_{((X\setminus V)\times(X\setminus E))\cap\operatorname{int}(X\times X)} = F(t,\cdot,\cdot)|_{((X\setminus V)\times(X\setminus E))\cap\operatorname{int}(X\times X)}$$

is lower semicontinuous. This immediately implies that

$$F^*(t,\cdot,\cdot)|_{((X\setminus H^*)\times(X\setminus H^*))\cap\operatorname{int}(X\times X)} = F^*(t,\cdot,\cdot)|_{((\mathbf{R}^n\setminus H^*)\times(\mathbf{R}^n\setminus H^*))\cap\operatorname{int}(X\times X)}$$

is lower semicontinuous. In particular, the multifunction

$$F^*(t,\cdot,\cdot)|_{((\mathbf{R}^n\setminus H^*)\times(\mathbf{R}^n\setminus H^*))\cap\operatorname{int}(X\times X)}$$

is lower semicontinuous at (x_0,z_0). Since the set

$$((\mathbf{R}^n\setminus H^*)\times(\mathbf{R}^n\setminus H^*))\cap\operatorname{int}(X\times X)$$

is open in $(\mathbf{R}^n\setminus H^*)\times(\mathbf{R}^n\setminus H^*)$, it follows that the multifunction $F^*(t,\cdot,\cdot)|_{(\mathbf{R}^n\setminus H^*)\times(\mathbf{R}^n\setminus H^*)}$ is lower semicontinuous at (x_0,z_0), as desired. Conversely, assume that $(x_0,z_0)\in(\mathbf{R}^n\times\mathbf{R}^n)\setminus(X\times X)$. Since $F^*(t,\cdot,\cdot)$ is constant in $(\mathbf{R}^n\times\mathbf{R}^n)\setminus(X\times X)$ and the last set is open in $\mathbf{R}^n\times\mathbf{R}^n$, it follows that the multifunction $F^*(t,\cdot,\cdot)$ is lower semicontinuous at each point $(x,z)\in(\mathbf{R}^n\times\mathbf{R}^n)\setminus(X\times X)$. In particular, the multifunction $F^*(t,\cdot,\cdot)$ is lower semicontinuous at (x_0,z_0), hence our claim follows at once.

(c) for every $t\in[0,T]$, we have

$$F^*(\,t,\,(\mathbf{R}^n\setminus H^*)\times(\mathbf{R}^n\setminus H^*))\subseteq\overline{B}_n(0_{\mathbf{R}^n},M). \tag{3.10}$$

Indeed, fix $t \in [0,T]$ and $(x,z) \in (\mathbf{R}^n \setminus H^*) \times (\mathbf{R}^n \setminus H^*)$. If $(x,z) \in X \times X$, it follows that $(x,z) \in (X \setminus V) \times (X \setminus E)$, hence by assumption (iii) we get

$$F^*(t,x,z) = F(t,x,z) \subseteq \overline{B}_n(0_{\mathbf{R}^n}, M).$$

If, conversely, $(x,z) \notin X \times X$, again by assumption (iii) we get

$$F^*(t,x,z) = \{y^*\} \subseteq \overline{B}_n(0_{\mathbf{R}^n}, M),$$

as claimed.

(d) One has

$$F^*([0,T] \times (\mathbf{R}^n \setminus H^*) \times (\mathbf{R}^n \setminus H^*)) \in \mathcal{G}_n.$$

Indeed, by the definition of F^* and H^* we have

$$F^*([0,T] \times (\mathbf{R}^n \setminus H^*) \times (\mathbf{R}^n \setminus H^*)) \subseteq F([0,T] \times (X \setminus V) \times (X \setminus E)),$$

hence our claim follows by assumption (iv).

Hence, all the assumptions of Theorem 3.1 are satisfied with $\beta(t) \equiv M$ and $p = +\infty$. Consequently, there exists $u \in W^{2,\infty}([0,T], \mathbf{R}^n)$ such that

$$\begin{cases} u''(t) \in F^*(t, u(t), u'(t)) & \text{for a.e. } t \in [0,T], \\ u(0) = u'(0) = 0_{\mathbf{R}^n}, \\ (u(t), u'(t)) \in (\mathbf{R}^n \setminus H^*) \times (\mathbf{R}^n \setminus H^*) & \text{for a.e. } t \in [0,T]. \end{cases} \tag{3.11}$$

In particular, by (3.10) and (3.11) we get

$$\|u''(t)\|_n \leq M \quad \text{for a.e.} \quad t \in [0,T]. \tag{3.12}$$

Now, fix $i \in \{1, \dots, n\}$, and let u_i be the i-th component of the function u. Since u_i' is absolutely continuous in $[0,T]$, by (3.11), (3.12) and assumption (iii) we have, for all $t \in [0,T]$,

$$|u_i'(t)| = \left| \int_0^t u_i''(\tau)\, d\tau \right| \leq \int_0^t |u_i''(\tau)|\, d\tau \leq Mt \leq MT \leq r. \tag{3.13}$$

Analogously, since

$u_i \in C^1([0,T])$, by (3.13) and assumption (iii) we have, for all $t \in [0,T]$,

$$|u_i(t)| = \left| \int_0^t u_i'(\tau)\, d\tau \right| \leq \int_0^t |u_i'(\tau)|\, d\tau \leq \int_0^t M\tau\, d\tau = M\frac{t^2}{2} \leq M\frac{T^2}{2} \leq r.$$

Consequently, taking into account (3.9) and (3.11), we have that

$$(u(t), u'(t)) \in (X \setminus V) \times (X \setminus E) \quad \text{for a.e.} \quad t \in [0,T]. \tag{3.14}$$

In particular, by (3.11) and (3.14), and by the definition of F^*, we have that

$$u''(t) \in F(t, u(t), u'(t)) \quad \text{for a.e.} \quad t \in [0,T],$$

and this completes the proof. □

Remark 3.3. As it happens for Theorem 3.1, the assumptions of Corollary 3.3 do not imply any kind of semicontinuity for the multifunction F with respect to the variable $(x,z) \in \overline{B}^*(0_{\mathbf{R}^n}, r) \times \overline{B}^*(0_{\mathbf{R}^n}, r)$. That is, a multifunction $F : [0,T] \times \overline{B}^*(0_{\mathbf{R}^n}, r) \times \overline{B}^*(0_{\mathbf{R}^n}, r) \to 2^{\mathbf{R}^n}$ satisfying the assumptions of Corollary 3.3 could be neither lower nor upper semicontinuous, with respect to the variable (x,z), even at each point $(x,z) \in \overline{B}^*(0_{\mathbf{R}^n}, r) \times \overline{B}^*(0_{\mathbf{R}^n}, r)$. To see this, take $n = 1$, $T = 1$, $M = r = 2$, $V = E = \mathbf{Q}$, and $F : [0,1] \times [-2,2] \times [-2,2] \to 2^{\mathbf{R}}$ defined by

$$F(t,x,z) = \begin{cases} [1,2] & \text{if } (t,x,z) \in [0,1] \times ([-2,2] \setminus \mathbf{Q}) \times ([-2,2] \setminus \mathbf{Q}) \\ \{\frac{1}{2}\} & \text{otherwise.} \end{cases}$$

It is immediate to verify that all the assumptions of Corollary 3.3 are satisfied. However, for each $t \in [0,1]$ the multifunction $F(t,\cdot,\cdot)$ is neither lower semicontinuous nor upper semicontinuous at each point $(x,z) \in [-2,2] \times [-2,2]$.

We also observe that Corollary 3.3 does not hold without assumption (iv). To see this, we argue as in the Remark 3.2. Take $T = M = r = 1$, and let $f : [0,1] \times [-1,1] \to \mathbf{R}$ be defined by

$$f(t,z) = \begin{cases} 0 & \text{if } t \in [0,1] \text{ and } z \neq 0, \\ 1 & \text{if } t \in [0,1] \text{ and } z = 0. \end{cases}$$

Let $F : [0,1] \times [-1,1] \times [-1,1] \to 2^{\mathbf{R}}$ be defined by $F(t,x,z) = \{f(t,z)\}$. All the assumptions of Corollary 3.3 (with the exception of assumption (iv)), are satisfied by taking $n = 1$, $p = +\infty$, $V = \emptyset$ and $E = \{0\}$. However, the Cauchy problem (3.8) admits no generalized solutions in $[0,1]$. Indeed, assume that, with this choiche of F, there exists a generalized solution $u \in W^{2,1}([0,1])$ of problem (3.8). Again, the absolutely continuous function $v := u'$ is a generalized solution of the Cauchy problem (3.7) in $[0,1]$, and this is absurd by Example 1 of [15].

4. Applications

Firstly, we apply Theorem 3.1 to the existence of generalized solutions of the Cauchy problem (1.2).

Theorem 4.1. Let $T > 0$, $p \in [1,+\infty]$, and let $Q : [0,T] \times \mathbf{R}^n \times \mathbf{R}^n \to 2^{\mathbf{R}^n}$ and $S : [0,T] \times \mathbf{R}^n \times \mathbf{R}^n \to 2^{\mathbf{R}^n}$ be two multifunctions. Assume that there exist sets $V, E, U, C \in \mathcal{F}_n$ and a positive function $\beta \in L^p([0,T])$ such that:

(i) for a.e. $t \in [0,T]$, the multifunction $Q(t,\cdot,\cdot)|_{(\mathbf{R}^n \setminus V) \times (\mathbf{R}^n \setminus E)}$ is lower semicontinuous with nonempty convex values;

(ii) the multifunction $Q|_{[0,T] \times (\mathbf{R}^n \setminus V) \times (\mathbf{R}^n \setminus E)}$ is $\mathcal{L}([0,T]) \otimes \mathcal{B}(\mathbf{R}^n \setminus V) \otimes \mathcal{B}(\mathbf{R}^n \setminus E)$ - weakly measurable;

(iii) for a.e. $t \in [0,T]$, the multifunction $S(t,\cdot,\cdot)|_{(\mathbf{R}^n \setminus U) \times (\mathbf{R}^n \setminus C)}$ is lower semicontinuous with nonempty closed values;

(iv) the multifunction $S|_{[0,T] \times (\mathbf{R}^n \setminus U) \times (\mathbf{R}^n \setminus C)}$ is $\mathcal{L}([0,T]) \otimes \mathcal{B}(\mathbf{R}^n \setminus U) \otimes \mathcal{B}(\mathbf{R}^n \setminus C)$ - weakly measurable;

(v) for a.e. $t \in [0,T]$, one has

$$Q(t, (\mathbf{R}^n \setminus V) \times (\mathbf{R}^n \setminus E)) + S(t, (\mathbf{R}^n \setminus U) \times (\mathbf{R}^n \setminus C)) \subseteq \overline{B}_n(0_{\mathbf{R}^n}, \beta(t));$$

(vi) there exists $\Omega \subseteq [0,T]$, with $m_1(\Omega) = 0$, such that

$$Q(([0,T] \setminus \Omega) \times (\mathbf{R}^n \setminus V) \times (\mathbf{R}^n \setminus E)) + S(([0,T] \setminus \Omega) \times (\mathbf{R}^n \setminus U) \times (\mathbf{R}^n \setminus C)) \in \mathcal{G}_n.$$

Then, there exists $u \in W^{2,p}([0,T],\mathbf{R}^n)$ such that

$$(u(t),u'(t)) \in (\mathbf{R}^n \setminus (V \cup U)) \times (\mathbf{R}^n \setminus (E \cup C)) \quad \text{for a.e.} \quad t \in [0,T],$$

and

$$\begin{cases} u''(t) \in Q(t,u(t),u'(t)) + S(t,u(t),u'(t)) & \text{for a.e. } t \in [0,T], \\ u(0) = u'(0) = 0_{\mathbf{R}^n}, \\ \|u''(t)\|_n \leq \beta(t) & \text{for a.e. } t \in [0,T]. \end{cases}$$

Proof. Without loss of generality we can suppose that assumptions (i), (iii) and (v) are satisfied for all $t \in [0,T]$, and that $\Omega = \emptyset$. By the definition of the family $\mathcal{F}_n$, there exist $4n$ measurable subsets

$$V_1,\ldots,V_n, \quad E_1,\ldots,E_n,$$

$$U_1,\ldots,U_n, \quad C_1,\ldots,C_n,$$

of $\mathbf{R}^n$ such that

$$V = \bigcup_{i=1}^n V_i, \qquad E = \bigcup_{i=1}^n E_i, \qquad U = \bigcup_{i=1}^n U_i, \qquad C = \bigcup_{i=1}^n C_i,$$

and

$$m_1(P_{n,i}(V_i)) = m_1(P_{n,i}(E_i)) = m_1(P_{n,i}(U_i)) = m_1(P_{n,i}(C_i)) = 0$$

for all $i = 1, \dots, n$. Let $B \in \mathcal{B}(\mathbf{R})$ be such that $m_1(B) = 0$ and

$$\bigcup_{i=1}^n \Big(P_{n,i}(V_i) \cup P_{n,i}(E_i) \cup P_{n,i}(U_i) \cup P_{n,i}(C_i) \Big) \subseteq B.$$

Consequently, we have

$$W := (\mathbf{R} \setminus B)^{2n} \subseteq (\mathbf{R}^n \setminus (V \cup U)) \times (\mathbf{R}^n \setminus (E \cup C)). \tag{4.1}$$

By assumptions (i) and (ii), we have that the multifunction $Q|_{[0,T]\times W}$ is $\mathcal{L}([0,T]) \otimes \mathcal{B}(W)$ - weakly measurable, and for all $t \in [0,T]$, the multifunction $Q(t,\cdot,\cdot)|_W$ is lower semicontinuous and its values belongs to the family $\mathcal{D}(\mathbf{R}^n)$ (since they are nonempty, convex and finite-dimensional). Since $W \in \mathcal{B}(\mathbf{R}^{2n})$, it is a Souslin set by Corollary 6.6.7 of [1]. By Theorem 2.1, there exists a function $\phi : [0,T] \times W \to \mathbf{R}^n$ such that:

(a) $\phi(t,x,z) \in Q(t,x,z)$ for all $(t,x,z) \in [0,T] \times W$;
(b) for all $(x,z) \in W$, the function $\phi(\cdot,x,z)$ is $\mathcal{L}([0,T])$-measurable;
(c) for every $t \in [0,T]$, the function $\phi(t,\cdot,\cdot)$ is continuous over W.

Let $\phi^* : [0,T] \times \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ be defined by

$$\phi^*(t,x,z) = \begin{cases} \phi(t,x,z) & \text{if } (x,z) \in W \\ 0_{\mathbf{R}_n} & \text{if } (x,z) \in (\mathbf{R}^n \times \mathbf{R}^n) \setminus W. \end{cases}$$

Let $V^* := \bigcup_{i=1}^n P_{n,i}^{-1}(B)$. Of course, we have $V^* \in \mathcal{F}_n$ and

$$(\mathbf{R}^n \setminus V^*) \times (\mathbf{R}^n \setminus V^*) = (\mathbf{R} \setminus B)^n \times (\mathbf{R} \setminus B)^n = W. \tag{4.2}$$

Let $F : [0,T] \times \mathbf{R}^n \times \mathbf{R}^n \to 2^{\mathbf{R}^n}$ be defined by putting, for each $(t,x,z) \in [0,T] \times \mathbf{R}^n \times \mathbf{R}^n$,

$$F(t,x,z) := \phi^*(t,x,z) + S(t,x,z).$$

We observe what follows.

(a)′ For every $t \in [0,T]$, the multifunction $F(t,\cdot,\cdot)|_{(\mathbf{R}^n\setminus V^*)\times(\mathbf{R}^n\setminus V^*)}$ is lower semicontinuous with nonempty closed values. This follows at once by property (c), assumption (iii) and Theorem 7.3.15 of [11], taking into account (4.1) and (4.2).

(b)′ The multifunction F is $\mathcal{L}([0,T]) \otimes \mathcal{B}(\mathbf{R}^n \setminus V^*) \otimes \mathcal{B}(\mathbf{R}^n \setminus V^*)$ - weakly measurable. To see this, observe that by (4.1), by assumptions (iii) and (iv), and by Theorem 3.5 of [9], the multifunction $S|_{[0,T]\times W}$ is $\mathcal{L}([0,T]) \otimes \mathcal{B}(W)$-measurable, with nonempty closed values. Moreover, by the properties (b) and (c), and by Lemma 13.2.3 of [11], the function $\phi^*|_{[0,T]\times W} = \phi$ is $\mathcal{L}([0,T]) \times \mathcal{B}(W)$-measurable. Let

$$f : [0,T] \times (\mathbf{R}^n \setminus V^*) \times (\mathbf{R}^n \setminus V^*) \times \mathbf{R}^n \to \mathbf{R}^n$$

be defined by putting, for each $(t,x,z,y) \in [0,T] \times (\mathbf{R}^n \setminus V^*) \times (\mathbf{R}^n \setminus V^*) \times \mathbf{R}^n$,

$$f(t,x,z,y) := \phi^*(t,x,z) + y = \phi(t,x,z) + y.$$

By Theorem 6.5 of [9], taking into account (4.2), it follows that the multifunction

$$(t,x,z) \in [0,T] \times W \to f(\{(t,x,z)\} \times S(t,x,z)) = \phi^*(t,x,z) + S(t,x,z)$$

is $\mathcal{L}([0,T]) \otimes \mathcal{B}(W)$-weakly measurable, that is our claim.

(c)′ For all $t \in [0, T]$, one has

$$F(t, (\mathbf{R}^n \setminus V^*) \times (\mathbf{R}^n \setminus V^*)) \subseteq \overline{B}_n(0_{\mathbf{R}^n}, \beta(t)). \tag{4.3}$$

This follows at once by assumption (v) and property (a), taking into account (4.1) and (4.2).

(d)′ One has

$$F([0, T] \times (\mathbf{R}^n \setminus V^*) \times (\mathbf{R}^n \setminus V^*)) \in \mathcal{G}_n.$$

This follows easily by assumption (vi) and property (a), taking into account (4.1), (4.2) and the definition of the family $\mathcal{G}_n$.

Hence, all the assumptions of Theorem 3.1 are satisfied. Consequently, there exist a function $u \in W^{2,p}([0, T], \mathbf{R}^n)$, and a set $K_0 \subseteq [0, T]$, with $m_1(K_0) = 0$, such that $u(0) = u'(0) = 0_{\mathbf{R}^n}$, and also

$$(u(t), u'(t)) \in (\mathbf{R}^n \setminus V^*) \times (\mathbf{R}^n \setminus V^*) \quad \text{and} \quad u''(t) \in F(t, u(t), u'(t)) \tag{4.4}$$

for all $t \in [0, T] \setminus K_0$. In particular, by (4.3) and (4.4) we get

$$\|u''(t)\|_n \leq \beta(t) \quad \text{for all} \quad t \in [0, T] \setminus K_0.$$

Moreover, for every fixed $t \in [0, T] \setminus K_0$, by (4.2), (4.4) and property (a), we get

$$\begin{aligned} u''(t) \in F(t, u(t), u'(t)) &= \\ &= \phi^*(t, u(t), u'(t)) + S(t, u(t), u'(t)) = \\ &= \phi(t, u(t), u'(t)) + S(t, u(t), u'(t)) \in \\ &\in Q(t, u(t), u'(t)) + S(t, u(t), u'(t)). \end{aligned}$$

Finally, we observe that by (4.1), (4.2) and (4.4) we have

$$(u(t), u'(t)) \in (\mathbf{R}^n \setminus (V \cup U)) \times (\mathbf{R}^n \setminus (E \cup C)) \quad \text{for all} \quad t \in [0, T] \setminus K_0.$$

The proof is now complete. □

Remark 4.1. We observe that Theorem 1.1 is an immediate consequence of Theorem 4.1. Moreover, we point out that the assumptions of Theorem 4.1 do not imply any kind of semicontinuity for the multifunctions $Q(t, \cdot, \cdot)$ and $S(t, \cdot, \cdot)$, which are defined on the whole $\mathbf{R}^n \times \mathbf{R}^n$. As a matter of fact, it may happen that S and Q satisfy all the assumptions of Theorem 4.1, and, for all $t \in [0, T]$, the multifunctions $Q(t, \cdot, \cdot)$ and $S(t, \cdot, \cdot)$ are neither lower nor upper semicontinuous at each point $(x, z) \in \mathbf{R}^n \times \mathbf{R}^n$. The Example 1.2 clearly illustrates such a circumstance. At the same time, Example 1.2 shows how, under the assumptions of Theorem 4.1, the multifunction $Q + S$ may have values that are neither convex nor closed, even if Q is convex-valued and S is closed-valued.

Remark 4.2. The example in Remark 3.4 shows (taking $Q(t, x, z) \equiv \{0_{\mathbf{R}^n}\}$ and $S = F$) that Theorem 4.1 does not hold without assumption (vi).

As a further application of Theorem 3.1, we obtain the following existence results for the generalized solutions of the implicit Cauchy problem (1.3).

Theorem 4.2. Let $Y \in \mathcal{G}_n$ be a closed, connected and locally connected subset of $\mathbf{R}^n$, and let $T > 0$ and $p \in [1, +\infty]$. Let $\psi : Y \to \mathbf{R}$ be a given function, and let $F : [0, T] \times \mathbf{R}^n \times \mathbf{R}^n \to 2^{\mathbf{R}}$ and $G : [0, T] \times \mathbf{R}^n \times \mathbf{R}^n \to 2^{\mathbf{R}}$ be two multifunctions. Moreover, let $\beta \in L^p([0, T])$ be a positive function, and let $E, V, U, C \in \mathcal{F}_n$. Assume that:

(i) the function ψ is continuous in Y, and $\operatorname{int}_Y(\psi^{-1}(r)) = \emptyset$ for every $r \in \operatorname{int}_{\mathbf{R}}(\psi(Y))$.
(ii) for a.e. $t \in [0, T]$, the multifunction $F(t, \cdot, \cdot)|_{(\mathbf{R}^n \setminus V) \times (\mathbf{R}^n \setminus E)}$ is lower semicontinuous with nonempty convex values;
(iii) the multifunction $F|_{[0,T] \times (\mathbf{R}^n \setminus V) \times (\mathbf{R}^n \setminus E)}$ is $\mathcal{L}([0, T]) \otimes \mathcal{B}(\mathbf{R}^n \setminus V) \otimes \mathcal{B}(\mathbf{R}^n \setminus E)$ - weakly measurable;

(iv) for a.e. $t \in [0,T]$, the multifunction $G(t,\cdot,\cdot)|_{(\mathbf{R}^n\setminus U)\times(\mathbf{R}^n\setminus C)}$ is lower semicontinuous with nonempty closed values;

(v) the multifunction $G|_{[0,T]\times(\mathbf{R}^n\setminus U)\times(\mathbf{R}^n\setminus C)}$ is $\mathcal{L}([0,T]) \otimes \mathcal{B}(\mathbf{R}^n \setminus U) \otimes \mathcal{B}(\mathbf{R}^n \setminus C)$ - weakly measurable;

(vi) for a.e. $t \in [0,T]$, one has

$$F(t, (\mathbf{R}^n \setminus E) \times (\mathbf{R}^n \setminus V)) + G(t, (\mathbf{R}^n \setminus U) \times (\mathbf{R}^n \setminus C)) \subseteq \psi(Y);$$

(vii) for a.e. $t \in [0,T]$, and for all $(x,z) \in (\mathbf{R}^n \setminus (V \cup U)) \times (\mathbf{R}^n \setminus (E \cup C))$, one has

$$\sup \{\|y\|_n : \ y \in Y \text{ and } \ \psi(y) \in F(t,x,z) + G(t,x,z) \ \} \ \leq \ \beta(t).$$

Then, there exists $u \in W^{2,p}([0,T], \mathbf{R}^n)$ such that

$$u''(t) \in Y \quad \text{and} \quad (u(t), u'(t)) \in (\mathbf{R}^n \setminus (V \cup U)) \times (\mathbf{R}^n \setminus (E \cup C))$$

for a.e. $t \in [0,T]$, and

$$\begin{cases} \psi(u''(t)) \in F(t,u(t),u'(t)) + G(t,u(t),u'(t)) & \text{for a.e. } t \in [0,T], \\ u(0) = u'(0) = 0_{\mathbf{R}^n}, \\ \|u''(t)\|_n \leq \beta(t) & \text{for a.e. } t \in [0,T]. \end{cases}$$

Proof. Without loss of generality, we can assume that assumptions (ii), (iv), (vi) and (vii) are satisfied for all $t \in [0,T]$. The first part of the proof follows a construction similar to the beginning of the proof of Theorem 4.1. By assumption, we have that $V, E, U, C \in \mathcal{F}_n$. Therefore, reasoning exactly as in the first part of the proof of Theorem 4.1, it is easily seen that there exists a set $N \in \mathcal{B}(\mathbf{R})$, with $m_1(N) = 0$, such that

$$W := (\mathbf{R} \setminus N)^n \times (\mathbf{R} \setminus N)^n \subseteq [\mathbf{R}^n \setminus (U \cup V)] \times [\mathbf{R}^n \setminus (E \cup C)]. \tag{4.5}$$

By assumption (iii), we have that $F|_{[0,T]\times W}$ is $\mathcal{L}([0,T]) \otimes \mathcal{B}(W)$ - weakly measurable. Moreover, by assumption (ii), for every $t \in [0,T]$ the multifunction $F(t,\cdot,\cdot)|_W$ is lower semicontinuous with nonempty convex values. Since $W \in \mathcal{B}(\mathbf{R}^n \times \mathbf{R}^n)$, it is a Souslin set by Corollary 6.6.7 of [1]. Therefore, By Theorem 2.1, there exists function $f : [0,T] \times W \to \mathbf{R}^n$ such that:

(a) $f(t,x,z) \in F(t,x,z)$ for all $(t,x,z) \in T \times W$;

(b) for all $(x,z) \in W$, the function $f(\,\cdot\,,x,z)$ is $\mathcal{L}([0,T])$-measurable;

(c) for every $t \in [0,T]$, the function $f(t,\cdot,\cdot)$ is continuous in W.

We observe that the two following properties hold.

(a)′ For every $t \in [0,T]$, the multifunction

$$(x,z) \in W \to f(t,x,z) + G(t,x,z)$$

is lower semicontinuous in W with nonempty closed values. This follows at once by (c), assumption (iv) and Theorem 7.3.15 of [11], taking into account (4.5).

(b)′ The multifunction

$$(t,x,z) \in [0,T] \times W \to f(t,x,z) + G(t,x,z)$$

is $\mathcal{L}([0,T]) \otimes \mathcal{B}(W)$ - weakly measurable. To see this, observe that by (4.5), by assumptions (iv) and (v), and by Theorem 3.5 of [9], the multifunction $G|_{[0,T]\times W}$ is $\mathcal{L}([0,T]) \otimes \mathcal{B}(W)$-measurable, with nonempty closed values. By (b), (c), and Lemma 13.2.3 of [11], the function f is $\mathcal{L}([0,T]) \times \mathcal{B}(W)$-measurable. Let

$$h : [0,T] \times (\mathbf{R} \setminus N)^n \times (\mathbf{R} \setminus N)^n \times \mathbf{R}^n \to \mathbf{R}^n$$

be defined by setting, for each $(t,x,z,y) \in [0,T] \times (\mathbf{R} \setminus N)^n \times (\mathbf{R} \setminus N)^n \times \mathbf{R}^n$,

$$h(t,x,z,y) := f(t,x,z) + y.$$

By Theorem 6.5 of [9], taking into account (4.5), it follows that the multifunction

$$(t,x,z)\in[0,T]\times W\to h(\{(t,x,z)\}\times G(t,x,z))=f(t,x,z)+G(t,x,z)$$

is $\mathcal{L}([0,T])\otimes\mathcal{B}(W)$-weakly measurable, that is our claim.

Now, observe that, by assumption (i) and Theorem 2.4 of [14], there exists a set $X\subseteq Y$ such that $\psi(X)=\psi(Y)$ and the function $\psi|_X : X\to\psi(Y)$ is open (it maps open subsets of X onto open subsets of $\psi(Y)=\psi(X)$). Hence, it follows easily that the multifunction $\Phi:\psi(Y)\to 2^X$ defined by setting, for each $t\in\psi(Y)$,

$$\Phi(t):=\psi^{-1}(t)\cap X,$$

is lower semicontinuous in $\psi(Y)$ with nonempty values. Let $M:[0,T]\times W\to 2^{\mathbf{R}^n}$ be the multifunction defined by putting, for each $(t,x,z)\in[0,T]\times W$,

$$M(t,x,z)=\Phi(f(t,x,z)+G(t,x,z))=\psi^{-1}(f(t,x,z)+G(t,x,z))\cap X.$$

By property (a), assumption (vi) and (4.5), we have that the multifunction M is well-defined and has nonempty values. Moreover, by the property (a)′ and by the lower semicontinuity of Φ, taking into account Theorem 7.3.11 of [11], for each $t\in[0,T]$ the multifunction $M(t,\cdot,\cdot)$ is lower semicontinuous in W (with nonempty values). Finally, by the lower semicontinuity of Φ, by the property (b)′ and by Theorem 7.1.7 of [11], we have that M is $\mathcal{L}([0,T])\otimes\mathcal{B}(W)$ - weakly measurable (see also Proposition 2.5 of [9]).

Now, let $\overline{M}:[0,T]\times W\to 2^{\mathbf{R}^n}$ be the multifunction defined by putting, for each $(t,x,z)\in[0,T]\times W$,

$$\overline{M}(t,x,z):=\overline{M(t,x,z)}.$$

By Proposition 2.6 of [9], the multifunction $\overline{M}$ is $\mathcal{L}([0,T])\otimes\mathcal{B}(W)$ - weakly measurable. Moreover, by Proposition 7.3.3 of [11], for each fixed $t\in[0,T]$ the multifunction $\overline{M}(t,\cdot,\cdot)$ is lower semicontinuous in W, with nonempty closed (in $\mathbf{R}^n$) values.

Let $H:[0,T]\times\mathbf{R}^n\times\mathbf{R}^n\to 2^{\mathbf{R}^n}$ be the multifunction defined by putting, for each $(t,x,z)\in[0,T]\times\mathbf{R}^n\times\mathbf{R}^n$,

$$H(t,x,z)=\begin{cases}\overline{M}(t,x,z) & \text{if } t\in[0,T] \text{ and } (x,z)\in W,\\ \{0_{\mathbf{R}^n}\} & \text{if } t\in[0,T] \text{ and } (x,z)\in(\mathbf{R}^n\times\mathbf{R}^n)\setminus W.\end{cases}$$

Of course, by what precedes, we have that the multifunction $H|_{[0,T]\times W}$ is $\mathcal{L}([0,T])\otimes\mathcal{B}(W)$ - weakly measurable, and for each fixed $t\in[0,T]$ the multifunction $H(t,\cdot,\cdot)|_W$ is lower semicontinuous in W, with nonempty closed (in $\mathbf{R}^n$) values.

Now, observe that

$$W=(\mathbf{R}^n\setminus\bigcup_{i=1}^n P_{n,i}^{-1}(N))\times(\mathbf{R}^n\setminus\bigcup_{i=1}^n P_{n,i}^{-1}(N))$$

and $\bigcup_{i=1}^n P_{n,i}^{-1}(N)\in\mathcal{F}_n$. Moreover, by assumption (vii), taking into account (4.5) and property (a), for every $(t,x,z)\in[0,T]\times W$ we have

$$H(t,x,z)=\overline{(\psi^{-1}(f(t,x,z)+G(t,x,z))\cap X)}\subseteq\overline{B}(0_{\mathbf{R}^n},\beta(t)). \tag{4.6}$$

Finally, for every $(t,x,z)\in[0,T]\times W$ we have

$$H(t,x,z)=\overline{(\psi^{-1}(f(t,x,z)+G(t,x,z))\cap X)}\subseteq Y.$$

Since $Y\in\mathcal{G}_n$, by the definition of $\mathcal{G}_n$ we immediately get that $H([0,T]\times W)\in\mathcal{G}_n$.

Therefore, all the assumptions of Theorem 3.1 are satisfied. Thus, there exist $u\in W^{2,p}([0,T],\mathbf{R}^n)$ and a set $K_0\subseteq[0,T]$, with $m_1(K_0)=0$, such that $u(0)=u'(0)=0_{\mathbf{R}^n}$, and

$$u''(t)\in H(t,u(t),u'(t)) \quad\text{and}\quad (u(t),u'(t))\in W \quad\text{for all}\quad t\in[0,T]\setminus K_0. \tag{4.7}$$

By (4.6) and (4.7) we get

$$\|u''(t)\|_n \leq \beta(t) \quad \text{for all} \quad t \in [0,T] \setminus K_0.$$

Moreover, by (4.5) and (4.7) we immediately have

$$(u(t), u'(t)) \in (\mathbf{R}^n \setminus (V \cup U)) \times (\mathbf{R}^n \setminus (E \cup C)) \quad \text{for all} \quad t \in [0,T] \setminus K_0.$$

Since, by the above construction, the multifunction $H|_{[0,T]\times W} = \overline{M}$ takes its values in Y, by (4.7) we also get

$$u''(t) \in Y \quad \text{for all} \quad t \in [0,T] \setminus K_0.$$

Now, fix $t \in [0,T] \setminus K_0$. By (4.7), by the continuity of ψ, and by the closedness of the sets Y and $G(t, u(t), u'(t))$, we get

$$\begin{aligned} u''(t) &\in H(t, u(t), u'(t)) = \\ &= \overline{M}(t, u(t), u'(t)) = \\ &= \overline{(\psi^{-1}(f(t, u(t), u'(t)) + G(t, u(t), u'(t))) \cap X)} \subseteq \\ &\subseteq \overline{\psi^{-1}(f(t, u(t), u'(t)) + G(t, u(t), u'(t)))} = \\ &= \psi^{-1}(f(t, u(t), u'(t)) + G(t, u(t), u'(t))). \end{aligned}$$

Hence, taking into account (4.7) and the property (a), we have

$$\psi(u''(t)) \in f(t, u(t), u'(t)) + G(t, u(t), u'(t)) \subseteq F(t, u(t), u'(t)) + G(t, u(t), u'(t)).$$

Thus, the function u satisfies the conclusion. □

Remark 4.3. As in the preceding results, it is immediately seen that the assumption on F and G in Theorem 4.2 do not imply any kind of semicontinuity for the multifunctions $F(t, \cdot, \cdot)$ and $G(t, \cdot, \cdot)$ (which are defined on the whole $\mathbf{R}^n \times \mathbf{R}^n$). That is, it may happen that two multifunctions $F : [0,T] \times \mathbf{R}^n \times \mathbf{R}^n \to 2^{\mathbf{R}^n}$ and $G : [0,T] \times \mathbf{R}^n \times \mathbf{R}^n \to 2^{\mathbf{R}^n}$ satisfy the assumptions of Theorem 4.2, and, simultaneously, for each $t \in [0,T]$ the multifunctions $F(t, \cdot, \cdot)$ and $G(t, \cdot, \cdot)$ are neither upper nor lower semicontinuous at each point $(x, z) \in \mathbf{R}^n \times \mathbf{R}^n$. The Example 1.2 and the Remark 3.2 illustrate such a circumstance. Moreover, the example in Remark 3.4 shows that Theorem 4.2 does not hold without the assumption $Y \in \mathcal{G}_n$.

We now present an example of application of Theorem 4.2.

Example 4.4. Let $T > 0$, and let $F : [0,T] \times \mathbf{R} \times \mathbf{R} \to 2^{\mathbf{R}}$ and $G : [0,T] \times \mathbf{R} \times \mathbf{R} \to 2^{\mathbf{R}}$ be defined as the multifunctions Q and S of Example 1.2, respectively. That is, we put

$$F(t,x,z) = \begin{cases}]2,4[& \text{if } t \in [0,T] \text{ and } (x,z) \in (\mathbf{R} \setminus \mathbf{Q}) \times (\mathbf{R} \setminus \mathbf{Q}), \\ \{\arctan(t+x+z)\} & \text{otherwise,} \end{cases}$$

$$G(t,x,z) = \begin{cases} \{10 + \cos^2(x+z),\, 20 + \sin^2 z\} & \text{if } t \in [0,T] \text{ and } (x,z) \in (\mathbf{R} \setminus \mathbf{Q}) \times (\mathbf{R} \setminus \mathbf{Q}), \\ \{0\} & \text{otherwise} \end{cases}$$

(as in what precedes, $\mathbf{Q}$ denotes the set of all rational real numbers). We have already checked in Example 1.2 that:

(a) for every $t \in [0,T]$, the multifunctions $F(t, \cdot, \cdot)|_{(\mathbf{R}\setminus\mathbf{Q})\times(\mathbf{R}\setminus\mathbf{Q})}$ and $G(t, \cdot, \cdot)|_{(\mathbf{R}\setminus\mathbf{Q})\times(\mathbf{R}\setminus\mathbf{Q})}$ are lower semicontinuous;

(b) the multifunctions $F|_{[0,T]\times(\mathbf{R}\setminus\mathbf{Q})\times(\mathbf{R}\setminus\mathbf{Q})}$ and $G|_{[0,T]\times(\mathbf{R}\setminus\mathbf{Q})\times(\mathbf{R}\setminus\mathbf{Q})}$ are $\mathcal{L}([0,T]) \otimes \mathcal{B}(\mathbf{R} \setminus \mathbf{Q}) \otimes \mathcal{B}(\mathbf{R} \setminus \mathbf{Q})$ - weakly measurable;

(c) for every $t \in [0,T]$, one has

$$F(t,(\mathbf{R}\setminus\mathbf{Q})\times(\mathbf{R}\setminus\mathbf{Q}))+G(t,(\mathbf{R}\setminus\mathbf{Q})\times(\mathbf{R}\setminus\mathbf{Q}))\subseteq[12,25].$$

Now, let $\alpha > 0$ and $\gamma \in \mathbf{R}$ be fixed, and let $\psi : \mathbf{R} \to \mathbf{R}$ be the function

$$\psi(y) = \alpha y - \gamma \sin y.$$

Since $\lim_{y\to 0^+}\psi(y) = 0$, there exists $y^* > 0$ such that $\psi(y^*) < 12$. Moreover, since $\lim_{y\to+\infty}\psi(y) = +\infty$, there exists $y^{**} > y^*$ such that $\psi(y^{**}) > 25$. Now we apply Theorem 4.2, with $n = 1$, $Y = [y^*, y^{**}]$, $p = +\infty$, $\beta(t) \equiv y^{**}$, and $E = V = U = C = \mathbf{Q}$. To this aim, observe that $Y \in \mathcal{G}_1$, and assumptions (ii)–(v) of Theorem 4.2 are satified by properties (a) and (b). Moreover, observe that assumption (i) of Theorem 4.2 is satified since the derivative $\psi'(y) = \alpha - \gamma\cos y$ never vanishes identically over an interval. As regards assumption (vi), observe that, by the above construction and by (c), for every $t \in [0,T]$ we have

$$F(t,(\mathbf{R}\setminus\mathbf{Q})\times(\mathbf{R}\setminus\mathbf{Q}))+G(t,(\mathbf{R}\setminus\mathbf{Q})\times(\mathbf{R}\setminus\mathbf{Q}))\subseteq[12,25]\subseteq\psi([y^*,y^{**}]).$$

Finally, assumption (vii) of Theorem 4.2 is trivially satified.

Thus, all the assumptions of Theorem 4.2 are satisfied. Hence, there exists $u \in W^{2,+\infty}([0,T])$ such that $u''(t) \in [y^*, y^{**}]$ for a.e. $t \in [0,T]$, and

$$\begin{cases} \alpha u''(t) - \gamma\sin(u''(t)) \in F(t,u(t),u'(t)) + G(t,u(t),u'(t)) & \text{for a.e. } t\in[0,T],\\ u(0) = u'(0) = 0, & \\ (u(t),u'(t)) \in (\mathbf{R}\setminus\mathbf{Q})\times(\mathbf{R}\setminus\mathbf{Q}) & \text{for a.e. } t\in[0,T]. \end{cases}$$

We have already pointed out in the Example 1.2 that, for each $t \in [0,T]$, the multifunctions $F(t,\cdot,\cdot)$ and $G(t,\cdot,\cdot)$ are neither lower semicontinuous nor upper semicontinuous at each point $(x,z) \in \mathbf{R}\times\mathbf{R}$.

We finally present an application to the Cauchy problem associated with a Sturm-Liouville type differential inclusion.

Theorem 4.3. Let T, b, λ be positive real numbers, and let $X := [-b,b]$. Let $H : [0,T]\times X\times X \to 2^{\mathbf{R}}$ be a given multifunction. Assume that there exist two sets $V, E \in \mathcal{L}(X)$, with $m_1(V) = m_1(E) = 0$, such that:

(i) the multifunction $H|_{[0,T]\times(X\setminus V)\times(X\setminus E)}$ is $\mathcal{L}([0,T])\otimes\mathcal{B}(X\setminus V)\otimes\mathcal{B}(X\setminus E)$ - weakly measurable;
(ii) for every $t \in [0,T]$, the multifunction $H(t,\cdot,\cdot)|_{(X\setminus V)\times(X\setminus E)}$ is lower semicontinuous with nonempty closed values.

Let $\varphi \in L^\infty([0,T])$ and $\psi \in W^{1,\infty}([0,T])$ be two given functions, with $\psi(t) > 0$ for all $t \in [0,T]$. Assume that there exists three constants $M_1, M_2, \alpha > 0$, with

$$M_1 + M_2 \le \frac{b}{\max\left\{T, \dfrac{T^2}{2}\right\}}$$

and

$$\frac{M_1}{\lambda}\cdot\max_{s\in[0,T]}\psi(s) < \alpha < \frac{M_2}{\lambda}\cdot\min_{s\in[0,T]}\psi(s), \tag{4.8}$$

such that:

(iii) for a.e. $t \in [0,T]$, one has

$$\frac{|\psi'(t)|}{\psi(t)} + \frac{|\varphi(t)|}{\psi(t)} \le \frac{M_1}{b}. \tag{4.9}$$

(iv) for a.e. $t \in [0,T]$, one has

$$H(t,\ (X\setminus V)\times(X\setminus E)) \subseteq [\ \alpha,\ \frac{M_2}{\lambda}\cdot\min_{s\in[0,T]}\psi(s)\ ]. \tag{4.10}$$

Then, there exists $u \in W^{2,\infty}([0,T])$ such that

$$(u(t), u'(t)) \in (X \setminus V) \times (X \setminus E) \quad \text{for a.e.} \quad t \in [0,T],$$

and

$$\begin{cases} -(\psi(t)\, u'(t))' + \varphi(t)\, u(t) \in \lambda\, H(t, u(t), u'(t)) & \text{for a.e. } t \in [0,T], \\ u(0) = u'(0) = 0, \\ |u''(t)| \le M_1 + M_2 & \text{for a.e. } t \in [0,T]. \end{cases}$$

Proof. Let $K_1 \subseteq [0,T]$ be such that $m_1(K_1) = 0$, and, for every $t \in [0,T] \setminus K_1$, the derivative $\psi'(t)$ exists and (4.9) and (4.10) hold.

Let $F : [0,T] \times X \times X \to 2^{\mathbf{R}}$ be the multifunction defined by putting, for each $(t,x,z) \in [0,T] \times X \times X$,

$$F(t,x,z) = \begin{cases} \dfrac{\varphi(t)}{\psi(t)}\, x - \dfrac{\psi'(t)}{\psi(t)}\, z - \dfrac{\lambda}{\psi(t)}\, H(t,x,z) & \text{if } t \in [0,T] \setminus K_1, \\ \\ \{0\} & \text{if } t \in K_1. \end{cases}$$

We claim that the multifunction $F|_{[0,T]\times(X\setminus V)\times(X\setminus E)}$ is $\mathcal{L}([0,T]) \otimes \mathcal{B}(X \setminus V) \otimes \mathcal{B}(X \setminus E)$ - weakly measurable. Indeed, by Lemma 13.2.3 of [11], the function $g : [0,T] \times (X \setminus V) \times (X \setminus E) \to \mathbf{R}$ defined by

$$g(t,x,z) := \begin{cases} \dfrac{\varphi(t)}{\psi(t)}\, x - \dfrac{\psi'(t)}{\psi(t)}\, z & \text{if } t \in [0,T] \setminus K_1 \text{ and } (x,z) \in (X \setminus V) \times (X \setminus E), \\ \\ 0 & \text{if } t \in K_1 \text{ and } (x,z) \in (X \setminus V) \times (X \setminus E) \end{cases}$$

is $\mathcal{L}([0,T]) \otimes \mathcal{B}(X \setminus V) \otimes \mathcal{B}(X \setminus E)$ - measurable. By Theorem 3.5 of [9], the multifunction $H|_{[0,T]\times(X\setminus V)\times(X\setminus E)}$ is $\mathcal{L}([0,T]) \otimes \mathcal{B}(X \setminus V) \otimes \mathcal{B}(X \setminus E)$ - measurable (with nonempty closed values). Let $h : [0,T] \to \mathbf{R}$ be defined by

$$h(t) := \begin{cases} -\dfrac{\lambda}{\psi(t)} & \text{if } t \in [0,T] \setminus K_1, \\ 0 & \text{if } t \in K_1. \end{cases}$$

Of course, h is measurable in $[0,T]$. Let $j : [0,T] \times (X \setminus V) \times (X \setminus E) \times \mathbf{R} \to \mathbf{R}$ be defined by

$$j(t,x,z,y) := g(t,x,z) + h(t) \cdot y.$$

By Theorem 6.5 of [9], the multifunction

$$(t,x,z) \in [0,T] \times (X \setminus V) \times (X \setminus E) \to j(\{(t,x,z)\} \times H(t,x,z)) =$$

$$= g(t,x,z) + h(t) \cdot H(t,x,z) =$$

$$= \begin{cases} \dfrac{\varphi(t)}{\psi(t)}\, x - \dfrac{\psi'(t)}{\psi(t)}\, z - \dfrac{\lambda}{\psi(t)}\, H(t,x,z) & \text{if } t \in [0,T] \setminus K_1, \\ \\ \{0\} & \text{if } t \in K_1. \end{cases}$$

is $\mathcal{L}([0,T]) \otimes \mathcal{B}(X \setminus V) \otimes \mathcal{B}(X \setminus E)$ - weakly measurable, that is our claim.

Now, observe that for every $t \in [0,T]$, the multifunction

$$F(t,\cdot,\cdot)|_{(X\setminus V)\times(X\setminus E)}$$

is lower semicontinuous with nonempty closed values. Indeed, by assumption (ii) and by Theorem 7.3.11 of [11], for every $t \in [0,T]$ the multifunction

$$\big(h(t)\cdot H(t,\cdot,\cdot)\big)|_{(X\setminus V)\times(X\setminus E)}$$

is lower semicontinuous with nonempty closed values. Hence, by Theorem 7.3.15 of [11], for every $t \in [0,T]$ the multifunction

$$(x,z)\in (X\setminus V)\times(X\setminus E)\to g(t,x,z)+h(t)\cdot H(t,x,z)$$

is lower semicontinuous in $(X\setminus V)\times(X\setminus E)$ with nonempty closed values, that is our claim.

Now, put

$$\gamma := M_1 - \alpha\,\frac{\lambda}{\max_{s\in[0,T]}\psi(s)}. \tag{4.11}$$

Assumption (4.8) imply $\gamma < 0$. Let $t \in [0,T]\setminus K_1$ and $(x,z)\in (X\setminus V)\times(X\setminus E)$ be fixed, and let $v \in H(t,x,z)$. By (4.9) and (4.10) we get

$$\left|\frac{\varphi(t)}{\psi(t)}\,x-\frac{\psi'(t)}{\psi(t)}\,z-\frac{\lambda}{\psi(t)}\,v\right| \le \frac{|\varphi(t)|}{\psi(t)}\,b+\frac{|\psi'(t)|}{\psi(t)}\,b+\frac{\lambda}{\psi(t)}\,|v| \le M_1+M_2.$$

Moreover, by (4.9), (4.10) and (4.11) we have

$$\frac{\varphi(t)}{\psi(t)}\,x-\frac{\psi'(t)}{\psi(t)}\,z-\frac{\lambda}{\psi(t)}\,v \le \left|\frac{\varphi(t)}{\psi(t)}\,x\right|+\left|\frac{\psi'(t)}{\psi(t)}\,z\right|-\frac{\lambda}{\psi(t)}\,\alpha \le \gamma,$$

hence $F(t,x,z)\subseteq[-(M_1+M_2),\gamma]$. This implies that

$$F(([0,T]\setminus K_1)\times(X\setminus V)\times(X\setminus E))\subseteq[-(M_1+M_2),\gamma],$$

hence

$$F(([0,T]\setminus K_1)\times(X\setminus V)\times(X\setminus E))\in\mathcal{G}_1.$$

Therefore, all the assumptions of Corollary 3.3 are satisfied. Consequently, there exists $u \in W^{2,\infty}([0,T])$ such that

$$(u(t),u'(t))\in(X\setminus V)\times(X\setminus E)\quad \text{for a.e.}\quad t\in[0,T],$$

and

$$\begin{cases} u''(t)\in F(t,u(t),u'(t)) & \text{for a.e. } t\in[0,T],\\ u(0)=u'(0)=0, & \\ |u''(t)|\le M_1+M_2 & \text{for a.e. } t\in[0,T]. \end{cases}$$

It is routine matter to check that the function u satisfies the conclusion. □

Remark 4.5. It can be easily checked that the assumptions of Theorem 4.3 do not imply any kind of semicontinuity for the multifunction $H(t,\cdot,\cdot)$ (which is defined on the whole $[-b,b]\times[-b,b]$). Indeed, it is enough to modify slightly the preceding examples, in order to construct a multifunction $H : [0,T]\times[-b,b]\times[-b,b]\to 2^{\mathbf{R}}$ (for fixed $b>0$) such that H satisfies assumptions (i) and (ii) of Theorem 4.3, and, simultaneously, for each $t\in[0,T]$ the multifunction $H(t,\cdot,\cdot)$ is neither upper nor lower semicontinuous at each point $(x,z)\in[-b,b]\times[-b,b]$.

Remark 4.6. Theorem 4.3 can be usefully compared with Theorem 4.1 of [2]. In this latter result, the multifunction H cannot depend on t explicitly, and it is assumed, in particular, to be lower semicontinuous in $[-b,b]\times[-b,b]$, with nonempty closed convex values. Moreover, it is assumed that $\varphi\in C([0,T])$ and $\psi\in C^1([0,T])$.

In Theorem 4.3, conversely, in addition to the weaker regularity discussed in Remark 4.8, the multifunction H can depend on t explicitly, and the convexity of its values is not required. Moreover, a weaker regularity is required on φ and ψ. That is, we only assume that $\varphi\in L^{\infty}([0,T])$ and $\psi\in W^{1,\infty}([0,T])$.

We also point out that the two results are formally independent.

Statements and Declarations

The authors declare that they have no conflict of interest, and the manuscript has no associated data.

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