APPLICABLE NONLINEAR ANALYSIS
Volume 1 (2024), No. 1, 44–63
https://doi.org/10.69829/apna-024-0101-ta03
Tulipa Opera Scholarum

STABILITY AND OPTIMAL CONTROL FOR DIFFERENTIAL CONSTRAINED VARIATIONAL-HEMIVARIATIONAL INEQUALITIES

STANISŁAW MIGÓRSKI [1,*], YUNRU BAI [2], AND SYLWIA DUDEK [3]

[1] *Jagiellonian University in Krakow, Chair of Optimization and Control, ul. Lojasiewicza 6, 30348 Krakow, Poland*
[2] *School of Science, Guangxi University of Science and Technology, Liuzhou 545006, Guangxi Province, P.R. China*
[3] *Department of Applied Mathematics, Faculty of Computer Science and Telecommunications, Krakow University of Technology, ul. Warszawska 24, 31155 Krakow, Poland*

Abstract. In this paper we analyze a differential variational-hemivariational inequality which consists of an evolution equation of first order and a time-dependent constrained variational-hemivariational inequality. First, we present a new stability result for the solution set with respect to a control parameter. Then, we derive an existence result for a general optimal control problem for the differential variational-hemivariational inequality. We provide an appliction of the results to a weak formulation of a quasistatic frictional elastic contact problem. A stability result of a set of weak solutions with respect to the densities of volume forces, tractions and heat sources, and the initial conditions for the temperature is examined. Finally, an existence of solutions for an optimal control problem for the contact model is discussed.

Keywords. Differential variational-hemivariational inequality, thermoelasticity, unilateral constraint, contact problem.

1. Introduction

In this paper we study a differential variational-hemivariational inequality. The problem consists of the system of a Cauchy problem for an evolution equation of first order and a time-dependent variational-hemivariational inequality with a constraint set.

Problem 1.1. Find $w\colon I \to E$ and $u\colon I \to V$ such that

$$\begin{cases} w'(t) + A(t, w(t)) = f(\lambda, t, w(t)) + G(t)\, u(t) \ \text{ a.e. } \ t \in I, \\ w(0) = w_0(\lambda), \end{cases} \tag{1.1}$$

and for a.e. $t \in I$, $u(t) \in K$ satisfies the inequality

$$\begin{aligned} &\langle B(t, w(t), u(t)) - g(\lambda, t), v - u(t)\rangle + J^0(t, \delta w(t), \gamma u(t); \gamma(v - u(t))) \\ &\quad + \varphi(t, \delta w(t), v) - \varphi(t, \delta w(t), u(t)) \ge 0 \ \text{ for all } \ v \in K. \end{aligned} \tag{1.2}$$

In Problem 1.1, $I = [0, T]$ is a finite time interval, (E, H, E^*) is an evolution triple of spaces with compact embeddings, X and Z are Banach spaces, V is a separable, reflexive Banach space, K is a closed convex subsets of V, $\lambda \in \Lambda$ represents a control parameter, $A\colon I \times E \to E^*$ is a monotone coercive map, $f\colon \Lambda \times I \times H \to H$ is a nonlinear mapping, $G(t)\colon V \to H$ is a linear bounded map, $B\colon I \times H \times V \to V^*$ is a nonlinear operator, J^0 is the generalized directional derivative of a locally

*Corresponding author.
E-mail addresses: stanislaw.migorski@uj.edu.pl (S. Migórski), yunrubai@163.com (Y. Bai), and sylwia.dudek@pk.edu.pl (S. Dudek)
2020 Mathematics Subject Classification: 35J87, 35M86, 47J20, 35R45, 74M15
Accepted: April 17, 2024.

Lipschitz function $J\colon I\times Z\times X\to\mathbb{R}$, $\varphi\colon I\times Z\times V\to\mathbb{R}$ is a convex, lower semicontinuous function, and $\delta\colon E\to Z$ and $\gamma\colon V\to X$ are prescribed linear bounded operators.

The paper is a continuation of [14]. There, an existence result for Problem 1.1 is established. The purpose and novelties of the current paper are three-fold. First, we present a new stability result for the solution set of a differential variational inequality. To this aim we study the upper semicontinuity of the set-valued solution map $\Lambda\ni\lambda\mapsto\mathcal{S}(\lambda)\in 2^{\mathbb{W}\times L^2(I;V)}\setminus\{\emptyset\}$, where

$$\mathcal{S}(\lambda)=\{(w,u)\in\mathbb{W}\times L^2(I;V)\mid (w,u) \text{ is a solution to (1.1), (1.2) corresponding to } \lambda\} \tag{1.3}$$

denotes the set of solutions to Problem 1.1. Second, we derive an existence result for a general optimal control problem for the differential variational-hemivariational inequality. The optimal control under consideration reads as follows: Find $\lambda^*\in\Lambda_{ad}$ such that

$$m(\lambda^*)=\inf\{m(\lambda)\mid\lambda\in\Lambda_{ad}\}\quad\text{with}\quad m(\lambda):=\inf\{\mathcal{F}(\lambda,w,u)\mid(w,u)\in\mathcal{S}(\lambda)\},$$

where $\Lambda_{ad}\subset\Lambda$ represents an admissible set of control parameters in a metric space Λ, and $\mathcal{F}$ denotes a cost functional. Third, we turn to an appliction and examine a quasistatic frictional contact problem in thermoelasticity. We provide a stability result a set of weak solutions to the contact problem with respect to the densities of volume forces, tractions and heat sources, the heat flux between the body surface and the foundation, and the initial conditions for the temperature. Finally, we are concerned with an optimal control problem for a contact model and discuss the issue of its solvability. To the best of the authors' knowledge, none of the problems studied in this paper have been treated in the literature up to now.

Differential variational inequalities have been investigated for the first time in [22] in finite dimension. The evolution equation supplemented by a variational inequality was treated in [11], and supplemented by a variational-hemivariational inequality was studied in [9, 12, 28]. In all these papers the solution of (1.1) is coupled with (1.2) through only the operator B. Moreover, in these papers the notion of measure of noncompactness and a fixed point theorem for condensing multivalued maps have been used. The Rothe method for related differential hemivariational inequalities was used in [5, 21]. Results on optimal control problems for various variational-hemivariational inequalities can be found in [3, 15, 25, 31]. Other closely related interesting results in this area can be found in [4, 6, 7, 14, 16, 17, 18] and the references therein.

Notation and preliminaries. In this article we use the concepts from nonlinear analysis which we shortly recall below. Let Y_τ be a Hausdorff topological space and $\{C_n\}\subset 2^Y\setminus\{\emptyset\}$ for $n\in\mathbb{N}$. We define the sequential Kuratowski lower and upper limits by

$$K(Y_\tau)\text{-}\liminf C_n=\{\,y\in Y\mid y=\tau\text{-}\lim y_n,\ y_n\in C_n,\ n\in\mathbb{N}\,\},$$
$$K(Y_\tau)\text{-}\limsup C_n=\{\,y\in Y\mid y=\tau\text{-}\lim y_{n_k},\ y_{n_k}\in C_{n_k},\ n_1<n_2<\ldots<n_k<\ldots\}.$$

If $K(Y_\tau)\text{-}\liminf C_n=K(Y_\tau)\text{-}\limsup C_n=C$, then we write $C=K(Y_\tau)\text{-}\lim C_n$ to denote the τ-Kuratowski limit of C_n in Y_τ. Let Y_w and Y denote the weak and strong topologies, respectively, on a real Banach space Y.

Let X be a Banach space. An operator $A\colon X\to X^*$ is said to be monotone, if for all $u,v\in X$, it holds $\langle Au-Av,u-v\rangle\ge 0$. Operator A is bounded, if A maps bounded sets of X into bounded sets of X^*. Operator A is called pseudomonotone, if it is bounded and $u_n\rightharpoonup u$ in X with $\limsup\langle Au_n,u_n-u\rangle\le 0$ imply $\langle Au,u-v\rangle\le\liminf\langle Au_n,u_n-v\rangle$ for all $v\in X$. It is known that if X is a reflexive Banach space, then $A\colon X\to X^*$ is pseudomonotone, if and only if it is bounded and $u_n\rightharpoonup u$ in X with $\limsup\langle Au_n,u_n-u\rangle\le 0$ imply $\lim\langle Au_n,u_n-u\rangle=0$ and $Au_n\rightharpoonup Au$ in X^*.

Let $h\colon X \to \mathbb{R}$ be a locally Lipschitz function. The generalized (Clarke) directional derivative of h at the point $x \in X$ in the direction $v \in X$ is defined by

$$h^0(x;v) = \limsup_{\lambda \downarrow 0,\, w \to x} \frac{h(w+\lambda v) - h(w)}{\lambda}.$$

The generalized subgradient of $h\colon X \to \mathbb{R}$ at $x \in X$ is given by

$$\partial h(x) = \{\, x^* \in X^* \mid h^0(x;v) \geq \langle x^*, v\rangle_{X^* \times X} \ \text{ for all } \ v \in X \,\}.$$

The function h is (Clarke) regular, if for all $x, v \in X$, the directional derivative

$$h'(x;v) = \lim_{\lambda \downarrow 0} \frac{h(x+\lambda v) - h(x)}{\lambda}$$

exists and $h'(x,v) = h^0(x;v)$.

2. Main stability result

Let (E, H, E^*) be an evolution triple of spaces, that is, E is a separable, reflexive Banach space and H is a separable Hilbert space such that the embedding $E \subset H$ is continuous and dense, and, additionally, compact. The duality brackets $\langle \cdot, \cdot \rangle_E$ for the pair (E^*, E) and the inner product $\langle \cdot, \cdot \rangle_H$ on H coincide on $H \times E$. In what follows we denote by $\| \cdot \|_E$ the norm in E. Let

$$\mathbb{W} = \{\, w \in L^2(I;E) \mid w' \in L^2(I;E^*) \,\},$$

where the time derivative w' is understood in the distributional sense. It is known that $\mathbb{W}$ endowed with the norm $\|w\|_{\mathbb{W}} = \|w\|_{L^2(I;E)} + \|w'\|_{L^2(I;E^*)}$ is a separable reflexive Banach space.

Let X and Z be Banach spaces, V be a separable, reflexive Banach space, and Λ be a metric space. The duality pairing between X^* and X is denoted by $\langle \cdot, \cdot \rangle_X$.

We need the following hypotheses on the data of Problem 1.1.

$\underline{H(A)}$: $A\colon I \times E \to E^*$ is such that
- (a) for all $v \in E$, $t \mapsto A(t,v)$ is measurable,
- (b) for a.e. $t \in I$, $v \mapsto A(t,v)$ is hemicontinuous and monotone,
- (c) for all $v \in E$, a.e. $t \in I$, $\|A(t,v)\|_{E^*} \leq a_0(t) + c_0\|v\|_E$ with $a_0 \in L^2(I)$, $c_0 > 0$,
- (d) for all $v \in E$, a.e. $t \in I$, $\langle A(t,v), v\rangle \geq a_2\|v\|_E^2 - a_1(t)$ with $a_1 \in L^1(I)$, $a_2 > 0$.

$\underline{H(f)}$: $f\colon \Lambda \times I \times H \to H$ is such that
- (a) for all $v \in H$, $\lambda \in \Lambda$, $t \mapsto f(\lambda, t, v)$ is measurable,
- (b) for all $v \in H$, a.e. $t \in I$, $\lambda \mapsto f(\lambda, t, v)$ is continuous,
- (c) for $v_1, v_2 \in H$, $\lambda \in \Lambda$, a.e. $t \in I$, $\|f(\lambda,t,v_1) - f(\lambda,t,v_2)\|_H \leq k(t)\|v_1 - v_2\|_H$, where $k \in L^\infty(I)_+$,
- (d) for all $v \in H$, $\lambda \in \Lambda$, a.e. $t \in I$, $\|f(\lambda,t,v)\|_H \leq a_4(t)(1 + \|v\|_H)$ with $a_4 \in L^2(I)_+$.

$\underline{H(B)}$: $B\colon I \times H \times V \to V^*$ is such that
- (a) for a.e. $t \in I$, all $\eta \in H$, $v \mapsto B(t,\eta,v)$ is pseudomonotone,
- (b) there exist $u_0 \in K$, $\alpha > 0$, $\beta \geq 0$ and $b \in L^\infty(I)$ such that for a.e. $t \in I$, all $\eta \in H$, $z \in Z$, $v \in V$, we have $\langle B(t,\eta,v), v - u_0\rangle + \inf_{\xi \in \partial J(t,z,\gamma v)} \langle \xi, \gamma(v - u_0)\rangle_{X^* \times X} \geq \alpha\|v\|^2 - \beta\|v\| - b(t)$,
- (c) for a.e. $t \in I$, all $\eta \in H$, $z \in Z$, $v \mapsto B(t,\eta,v) + \gamma^*\partial J(t,z,\gamma v)$ is monotone,
- (d) $(t,\eta,v) \mapsto B(t,\eta,v)$ is continuous,
- (e) for a.e. $t \in I$, all $\eta \in H$, $v \in V$, $\|B(t,\eta,v)\|_{V^*} \leq c_0(t) + c_1\|\eta\|_H + c_2\|v\|$ with $c_0 \in L^2(I)$, $c_0, c_1, c_2 > 0$.

$\underline{H(G)}:$ $\quad G \in L^\infty(I; \mathcal{L}(V, H))$.

$\underline{H(\delta,\gamma)}:$ $\quad \gamma \in \mathcal{L}(V, X)$ is compact, $\delta \in \mathcal{L}(E, Z)$ and its Nemitsky operator $\widetilde{\delta}\colon \mathbb{W} \to L^2(I; Z)$ is compact.

$\underline{H(J)}:$ $\quad J\colon I \times Z \times X \to \mathbb{R}$ is such that

(a) for all $z \in Z$, $v \in X$, $t \mapsto J(t, z, v)$ is measurable,
(b) for a.e. $t \in I$, all $z \in Z$, $v \mapsto J(t, z, v)$ is a locally Lipschitz function,
(c) for all $z \in Z$, $v \in X$, a.e. $t \in I$, $\|\partial J(t, z, v)\|_{X^*} \le d_J (1 + \|z\|_Z + \|v\|_X)$ with $d_J \ge 0$,
(d) for all $\{t_n\} \subset I$, $t_n \to t$, $\{z_n\} \subset Z$, $z_n \to z$ in Z, $\{v_n\} \subset X$, $v_n \to v$ in X, all $x \in X$, we have $\limsup J^0(t_n, z_n, x; v_n) \le J^0(t, z, x; v)$.
(e) for all $z \in L^2(I; V)$, $\{\zeta_n\} \subset L^2(I; Z)$, $\zeta_n \to \zeta$ in $L^2(I; Z)$, $\{u_n\} \subset L^2(I; V)$, $u_n \rightharpoonup u$ in $L^2(I; V)$, we have
$$\limsup \int_I J^0(t, \zeta_n(t), \gamma z(t); \gamma(z(t) - u_n(t)))\, dt \le \int_I J^0(t, \zeta(t), \gamma z(t); \gamma(z(t) - u(t)))\, dt.$$

$\underline{H(K)}$: $\quad K$ is a nonvoid, closed and convex subset of V.

$\underline{H(\varphi)}:$ $\quad \varphi\colon I \times Z \times V \to \mathbb{R}$ is such that

(a) for all $v \in V$, $z \in Z$, $t \mapsto \varphi(t, z, v)$ is measurable,
(b) for a.e. $t \in I$, all $z \in Z$, $v \mapsto \varphi(t, z, v)$ is convex and lower semicontinuous,
(c) there exists $c_\varphi \in L^\infty(I)_+$ such that $|\varphi(t, z, u_0)| \le c_\varphi(t)$ for a.e. $t \in I$, all $z \in Z$, where $u_0 \in K$ is as in $H(B)$(b),
(d) there are $a_3 \in L^\infty(I)$ and $b_3 > -\alpha$ such that $\varphi(t, z, v) \ge a_3(t) + b_3 \|v\|^2$ for all $z \in Z$, $v \in V$, a.e. $t \in I$,
(e) for all $\{t_n\} \subset I$, $t_n \to t$, $\{z_n\} \subset Z$, $z_n \to z$ in Z, $\{u_n\} \subset V$, $u_n \rightharpoonup u$ in V, all $v \in V$, we have
$$\limsup(\varphi(t_n, z_n, v) - \varphi(t_n, z_n, u_n)) \le \varphi(t, z, v) - \varphi(t, z, u).$$
(f) for all $z \in L^2(I; V)$, $\{\zeta_n\} \subset L^2(I; Z)$, $\zeta_n \to \zeta$ in $L^2(I; Z)$, $\{u_n\} \subset L^2(I; V)$, $u_n \rightharpoonup u$ in $L^2(I; V)$, we have
$$\limsup \int_I \big(\varphi(t, \zeta_n(t), z(t)) - \varphi(t, \zeta_n(t), u_n(t))\big)\, dt \le \int_I \big(\varphi(t, \zeta(t), z(t)) - \varphi(t, \zeta(t), u(t))\big)\, dt.$$

$\underline{H(g, w_0)}:$ $\quad g\colon \Lambda \times I \to V^*$, $g(\cdot, t)$ is continuous for a.e. $t \in I$, $w_0\colon \Lambda \to H$ is continuous, there is $c_1 > 0$ such that for all $\lambda \in \Lambda$, a.e. $t \in I$, $\|g(\lambda, t)\|_{V^*} \le c_1$, $\|w_0(\lambda)\|_H \le c_1$.

Recall that in the hypothesis $H(\delta, \gamma)$, the Nemitsky (superposition) operator $\widetilde{\delta}\colon L^2(I; E) \to L^2(I; Z)$ is defined by $(\widetilde{\delta}\eta)(t) := \delta(\eta(t))$ for $\eta \in L^2(I; E)$ and a.e. $t \in I$, see, e.g., [20, (2.2)].

We begin with the following two auxiliary results for the Cauchy problem for evolution equation, and the variational-hemivariational inequality, respectively.

Lemma 2.1. *Under the hypotheses $H(A)$, $H(f)$, $h \in L^2(I; H)$, fixed $\lambda \in \Lambda$, and $w_0(\lambda) \in H$, there is a unique $w \in \mathbb{W}$ solution to the problem*
$$\begin{cases} w'(t) + A(t, w(t)) = f(\lambda, t, w(t)) + h(t) \ \text{ a.e. } \ t \in I, \\ w(0) = w_0(\lambda). \end{cases} \tag{2.1}$$
The solution satisfies the following estimate
$$\|w\|_{\mathbb{W}} \le c\,(1 + \|h\|_{L^2(I;H)} + \|w_0(\lambda)\|_H) \ \text{ with } \ c > 0. \tag{2.2}$$
Moreover, the solution map $p\colon H \times L^2(I, H) \to \mathbb{W} \subset C(I; H)$ defined by $p(w_0, h) = w$, where $w \in \mathbb{W}$ is the unique solution to (2.1) has the property: if $\{w_0^n\} \subset H$, $w_0^n \to w_0$ in H_w, $\{h_n\} \subset L^2(I; H)$, $h_n \to h$ in $L^2(I; H)_w$, then $p(w_0^n, h_n) \to p(w_0, h)$ in $\mathbb{W}_w$ and strongly in $C(I; H)$.

Proof. It is an easy modification of [14, Lemma 3] and is omitted. □

Lemma 2.2. *Assume the hypotheses $H(B)$, $H(J)$, $H(K)$, $H(\varphi)$, $H(\delta,\gamma)$, and $H(g,w_0)$. Let $\lambda\in\Lambda$, $g(\lambda)\in L^\infty(I;V^*)$ and $(w,z)\in L^2(I;H)\times L^2(I;Z)$ be fixed. Then, the following two formulations are equivalent.*

$$\begin{cases} \textit{Find } u\in L^2(I;V), u(t)\in K \textit{ for a.e. } t\in I \textit{ such that} \\ \langle B(t,w(t),u(t))-g(\lambda,t),v-u(t)\rangle + J^0(t,z(t),\gamma u(t);\gamma(v-u(t))) \\ \quad +\varphi(t,z(t),v)-\varphi(t,z(t),u(t))\geq 0 \ \text{ for all } \ v\in K, \ \text{a.e. } t\in I, \end{cases} \tag{2.3}$$

$$\begin{cases} \textit{find } u\in L^2(I;V), u(t)\in K \textit{ for a.e. } t\in I \textit{ such that} \\ \displaystyle\int_I \Big(\langle B(t,w(t),\eta(t))-g(\lambda,t),\eta(t)-u(t)\rangle + J^0(t,z(t),\gamma\eta(t);\gamma(\eta(t)-u(t))) \\ \qquad +\varphi(t,z(t),\eta(t))-\varphi(t,z(t),u(t))\Big)\,dt\geq 0 \\ \qquad \text{for all } \ \eta\in L^2(I;V), \ \eta(t)\in K \text{ a.e. } t\in I. \end{cases} \tag{2.4}$$

Proof. Let $\lambda\in\Lambda$, $g(\lambda)\in L^\infty(I;V^*)$ and $(w,z)\in L^2(I;H)\times L^2(I;Z)$. We apply [14, Proposition 10] to get that the problem (2.3) is equivalent to its Minty formulation

$$\begin{cases} \text{find } u\in L^2(I;V), u(t)\in K \text{ for a.e. } t\in I \text{ such that} \\ \langle B(t,w(t),v)-g(\lambda,t),v-u(t)\rangle + J^0(t,z(t),\gamma v;\gamma(v-u(t))) \\ \quad +\varphi(t,z(t),v)-\varphi(t,z(t),u(t))\geq 0 \ \text{ for all } \ v\in K, \ \text{a.e. } t\in I, \end{cases} \tag{2.5}$$

In what follows, we will prove that the formulations (2.4) and (2.5) are equivalent.

Let $u\in L^2(I;V)$, $u(t)\in K$ for a.e. $t\in I$ be a solution to the inequality (2.5). Let $\eta\in L^2(I;V)$ with $\eta(t)\in K$ for a.e. $t\in I$. We test (2.5) with $v=\eta(t)\in K$ and get

$$\begin{aligned} &\langle B(t,w(t),\eta(t))-g(\lambda,t),\eta(t)-u(t)\rangle + J^0(t,z(t),\gamma\eta(t);\gamma(\eta(t)-u(t))) \\ &\quad +\varphi(t,z(t),\eta(t))-\varphi(t,z(t),u(t))\geq 0 \ \text{ for a.e. } t\in I. \end{aligned}$$

Integrating the last inequality over I, we infer that $u\in L^2(I;V)$ with $u(t)\in K$ for a.e. $t\in I$ is a solution to (2.4).

Conversely, let $u\in L^2(I;V)$ with $u(t)\in K$ for a.e. $t\in I$ be a solution to the inequality (2.4). We have

$$\begin{aligned} \int_0^T &\Big(\langle B(t,w(t),\eta(t))-g(\lambda,t),\eta(t)-u(t)\rangle + J^0(t,z(t),\gamma\eta(t);\gamma(\eta(t)-u(t))) \\ &\qquad +\varphi(t,z(t),\eta(t))-\varphi(t,z(t),u(t))\Big)\,dt\geq 0 \end{aligned} \tag{2.6}$$

for all $\eta\in L^2(I;V)$ with $\eta(t)\in K$ for a.e. $t\in I$. We will establish (2.5). By contradiction, we suppose (2.5) does not hold. Hence

$$\begin{aligned} &\exists\,\mathcal{O}\subset I \text{ with } |\mathcal{O}|>0,\ \exists\,\eta_0\in K \text{ such that} \\ &\langle B(t,w(t),\eta_0)-g(\lambda,t),\eta_0-u(t)\rangle + J^0(t,z(t),\gamma\eta_0;\gamma(\eta_0-u(t))) \\ &\qquad +\varphi(t,z(t),\eta_0)-\varphi(t,z(t),u(t))<0 \ \text{ for all } \ t\in\mathcal{O}. \end{aligned}$$

From the last condition, we have

$$\begin{aligned} \int_{\mathcal{O}} &\Big(\langle B(t,w(t),\eta_0)-g(\lambda,t),\eta_0-u(t)\rangle + J^0(t,z(t),\gamma\eta_0;\gamma(\eta_0-u(t))) \\ &\qquad +\varphi(t,z(t),\eta_0)-\varphi(t,z(t),u(t))\Big)\,dt<0. \end{aligned}$$

Next, we choose a suitable test function in (2.6). Let $\eta(t) := \eta_0$ if $t \in \mathcal{O}$, and $\eta(t) := u(t)$ if $t \notin \mathcal{O}$. Since $\eta_0 \in K$ and $u \in L^2(I;V)$ with $u(t) \in K$ for a.e. $t \in I$, we have $\eta \in L^2(I;V)$ with $\eta(t) \in K$ for a.e. $t \in I$. Using this test function $\eta \in L^2(I;V)$ in (2.6), we obtain

$$\begin{aligned}
0 \le \int_I \Big(&\langle B(t, w(t), \eta(t)) - g(\lambda, t), \eta(t) - u(t)\rangle + J^0(t, z(t), \gamma\eta(t); \gamma(\eta(t) - u(t))) \\
&+ \varphi(t, z(t), \eta(t)) - \varphi(t, z(t), u(t))\Big)\, dt \\
= \int_{\mathcal{O}} \Big(&\langle B(t, w(t), \eta_0) - g(\lambda, t), \eta_0 - u(t)\rangle + J^0(t, z(t), \gamma\eta_0; \gamma(\eta_0 - u(t))) \\
&+ \varphi(t, z(t), \eta_0) - \varphi(t, z(t), u(t))\Big)\, dt \\
+ \int_{I\setminus\mathcal{O}} \Big(&\langle B(t, w(t), u(t)) - g(\lambda, t), u(t) - u(t)\rangle + J^0(t, z(t), \gamma u(t); \gamma(u(t) - u(t))) \\
&+ \varphi(t, z(t), u(t)) - \varphi(t, z(t), u(t))\Big)\, dt < 0,
\end{aligned}$$

which is a contradiction. Hence $u \in L^2(I;V)$ with $u(t) \in K$ for a.e. $t \in I$ is a solution to problem (2.5). This completes the proof of the lemma. □

Now we are in the position to state the main result of this section on the stability of the solution set.

Theorem 2.3. *Under hypotheses $H(A)$, $H(f)$, $H(B)$, $H(G)$, $H(J)$, $H(K)$, $H(\varphi)$, $H(\delta,\gamma)$ and $H(g, w_0)$, the solution set $\mathcal{S}(\lambda)$ of Problem 1.1 is a nonempty and compact subset of $\mathbb{W}_w \times L^2(I;V)_w$ for each fixed $\lambda \in \Lambda$, and*

$$K(\mathbb{W}_w \times L^2(I;V)_w)\text{–}\limsup \mathcal{S}(\lambda_n) \subset \mathcal{S}(\lambda) \ \text{ for all } \ \lambda_n \to \lambda \ \text{ in } \ \Lambda. \tag{2.7}$$

Proof. The proof of nonemptiness of $\mathcal{S}(\lambda)$ for each fixed λ can be found in [14, Theorem 10].

Now, we will show the following estimate: there are positive constants r_1, r_2 such that for all $\lambda \in \Lambda$, all $(w, u) \in \mathcal{S}(\lambda)$, we have

$$\|w\|_{\mathbb{W}} \le r_1, \quad \|u\|_{L^2(I;V)} \le r_2.$$

We have

$$\begin{cases} w'(t) + A(t, w(t)) = f(\lambda, t, w(t)) + G(t)\, u(t) \ \text{ a.e. } \ t \in I, \\ w(0) = w_0(\lambda), \end{cases} \tag{2.8}$$

and for a.e. $t \in I$, $u(t) \in K$ satisfies the inequality

$$\begin{aligned}
&\langle B(t, w(t), u(t)) - g(\lambda, t), v - u(t)\rangle + J^0(t, \delta w(t), \gamma u(t); \gamma(v - u(t))) \\
&\quad + \varphi(t, \delta w(t), v) - \varphi(t, \delta w(t), u(t)) \ge 0 \ \text{ for all } \ v \in K.
\end{aligned} \tag{2.9}$$

We test (2.9) with $v = u_0 \in K$ as in $H(B)$(b). From [19, Proposition 3.23 (iii)] we can find $\xi = \xi_{t,w(t),u(t)} \in \partial J(t, \delta w(t), \gamma u(t))$ such that

$$J^0(t, \delta w(t), \gamma u(t); \gamma(u_0 - u(t))) = \langle \xi, \gamma(u_0 - u(t))\rangle.$$

Hence

$$\begin{aligned}
&\langle B(t, w(t), u(t)), u(t) - u_0\rangle + \langle \xi, \gamma(u(t) - u_0)\rangle_{X^*\times X} \\
&\le \langle g(\lambda, t), u(t) - u_0\rangle + \varphi(t, \delta w(t), u_0) - \varphi(t, \delta w(t), u(t)).
\end{aligned}$$

We exploit $H(B)$(b) and $H(\varphi)$(c) to get

$$(\alpha + b_3)\, \|u(t)\|^2 \le (\beta + \|g(\lambda, t)\|_{V^*})\|u(t)\| + \|g(\lambda, t)\|_{V^*}\|u_0\| + b(t) + c_\varphi(t) - a_3(t)$$

for a.e. $t \in I$. We use the elementary property: for all a, b, $x \geq 0$, if $x^2 \leq ax + b$, then $x^2 \leq a^2 + 2b$. We obtain $\|u(t)\| \leq c\,(1 + \|g(\lambda, t)\|_{V^*} + c_\varphi(t))$ for a.e. $t \in I$ with $c > 0$, and subsequently

$$\|u\|_{L^2(I;V)} \leq c\,(1 + \|g(\lambda, \cdot)\|_{L^2(I;V^*)} + \|c_\varphi\|_{L^2(I)}).$$

From $H(g, w_0)$, there is $c_1 > 0$ such that for all $\lambda \in \Lambda$, $\|g(\lambda, \cdot)\|_{L^2(I;V^*)} \leq c_1$. Hence, we infer $\|u\|_{L^2(I;V)} \leq r_2$ with some $r_2 > 0$.

Next, from Lemma 2.1 applied to $h(\cdot) = G(\cdot)u(\cdot)$, and $H(g, w_0)$, we immediately get

$$\|w\|_{\mathbb{W}} \leq c\,(1 + \|G(\cdot)u(\cdot)\|_{L^2(I;H)} + \|w_0(\lambda)\|_H) \leq c\,(1 + \|u\|_{L^2(I;V)} + \|w_0(\lambda)\|_H) \leq r_1$$

with $r_1 > 0$. This proves the desired estimate. Hence, the solution set $\mathcal{S}(\lambda)$ remains in a bounded subset of $\mathbb{W} \times L^2(I; V)$. From the reflexivity of this space, we deduce that, for any $\lambda \in \Lambda$, the solution set of Problem 1.1 is a compact subset of $\mathbb{W}_w \times L^2(I; V)_w$.

We will show the inclusion (2.7). Let $\lambda_n \to \lambda$ in Λ and

$$(w, u) \in K(\mathbb{W}_w \times L^2(I; V)_w)\text{–}\limsup \mathcal{S}(\lambda_n).$$

Then we can find a subsequence of $\{n\}$, denoted in the same way, and $(w_n, u_n) \in \mathcal{S}(\lambda_n)$ such that

$$(w_n, u_n) \to (w, u) \text{ in } \mathbb{W}_w \times L^2(I; V)_w. \tag{2.10}$$

We have $w_n \in \mathbb{W}$, $u_n \in L^2(I; V)$ with $u_n(t) \in K$ for a.e. $t \in I$ and

$$\begin{cases} w_n'(t) + A(t, w_n(t)) = f(\lambda_n, t, w_n(t)) + G(t)\, u_n(t) \text{ a.e. } t \in I, \\ w_n(0) = w_0(\lambda_n), \end{cases} \tag{2.11}$$

and for a.e. $t \in I$, it holds

$$\begin{aligned} &\langle B(t, w_n(t), u_n(t)) - g(\lambda_n, t), v - u_n(t)\rangle + J^0(t, \delta w_n(t), \gamma u_n(t); \gamma(v - u_n(t))) \\ &\quad + \varphi(t, \delta w_n(t), v) - \varphi(t, \delta w_n(t), u_n(t)) \geq 0 \text{ for all } v \in K. \end{aligned} \tag{2.12}$$

Next, we pass to the limit in (2.11) and (2.12). First, we establish the convergence of solution to (2.11) to a limit problem. Consider $\mathcal{G}\colon L^2(I; V) \to L^2(I; H)$ the Nemitsky operator corresponding to G defined by $(\mathcal{G}v)(t) := G(t)v(t)$ for $v \in L^2(I; V)$, a.e. $t \in I$. The operator $\mathcal{G}$ is linear and bounded, by $H(G)$, and $G(\cdot)v(\cdot) \in L^2(I; H)$ for any $v \in L^2(I; V)$. Hence, $\mathcal{G}$ preserves the weak convergences, which means that $u_n \to u$ in $L^2(I; V)_w$ entails

$$\mathcal{G}\, u_n \to \mathcal{G}\, u \text{ in } L^2(I; H)_w. \tag{2.13}$$

On the other hand, let $h_n\colon I \to H$ be defined by

$$h_n(t) := f(\lambda_n, t, w_n(t)) + G(t)u_n(t) \text{ for a.e. } t \in I.$$

We claim that $h_n \to h$ in $L^2(I; H)_w$ with $h(t) := f(\lambda, t, w(t)) + G(t)u(t)$. Indeed, from the compact embedding of $\mathbb{W}$ into $L^2(I; H)$, we have $w_n \to w$ in $L^2(I; H)$. We use the inequality

$$\begin{aligned} &\Big|\int_I \langle h_n(t) - h(t), \eta(t)\rangle\, dt\Big| \leq \int_I \|f(\lambda_n, t, w_n(t)) - f(\lambda_n, t, w(t))\|_H\, \|\eta(t)\|_H\, dt \\ &+ \int_I \|f(\lambda_n, t, w(t)) - f(\lambda, t, w(t))\|_H\, \|\eta(t)\|_H\, dt + \int_I \langle G(t)(u_n(t) - u(t)), \eta(t)\rangle\, dt \\ &\leq \int_I k(t)\|w_n(t) - w(t)\|_H \|\eta(t)\|_H\, dt + \int_I \|f(\lambda_n, t, w(t)) - f(\lambda, t, w(t))\|_H\, \|\eta(t)\|_H\, dt \\ &+ \int_I \langle G(t)(u_n(t) - u(t)), \eta(t)\rangle\, dt \end{aligned}$$

for all $\eta \in L^2(I; H)$, hypothesis $H(f)$(b), (c), Hölder's inequality, and (2.13) to deduce the claim.

Next, we use $H(g, w_0)$ and apply Lemma 2.1 to deduce that $w \in \mathbb{W}$ is the unique solution of the problem

$$\begin{cases} w'(t) + A(t, w(t)) = f(\lambda, t, w(t)) + G(t)\, u(t) \ \text{ a.e. } \ t \in I, \\ w(0) = w_0(\lambda), \end{cases} \tag{2.14}$$

corresponding to λ, w_0 and u.

Second, we pass to the limit in the inequality (2.12). The problem (2.12) by Lemma 2.2 is equivalent to the following inequality: Find $u_n \in L^2(I; V)$ with $u_n(t) \in K$ for a.e. $t \in I$ such that

$$\begin{aligned} \int_I \Big(&\langle B(t, w_n(t), \eta(t)) - g(\lambda_n, t), \eta(t) - u_n(t)\rangle + J^0(t, \delta w_n(t), \gamma\eta(t); \gamma(\eta(t) - u_n(t))) \\ &+ \varphi(t, \delta w_n(t), \eta(t)) - \varphi(t, \delta w_n(t), u_n(t))\Big)\, dt \geq 0 \end{aligned} \tag{2.15}$$

for all $\eta \in L^2(I; V)$, $\eta(t) \in K$ for a.e. $t \in I$.

Let us fix $\eta \in L^2(I; V)$ and define the operator $B_\eta \colon I \times H \to V^*$ by

$$B_\eta(t, v) := B(t, v, \eta(t)) \ \text{ for } \ v \in H, \ \text{a.e. } t \in I.$$

From hypotheses $H(B)$(d),(e), we know that: for all $v \in H$, $t \mapsto B_\eta(t, v)$ is measurable; for a.e. $t \in I$, $v \mapsto B_\eta(t, v)$ is continuous, and for all $(t, v) \in I \times H$, $\|B_\eta(t, v)\|_{V^*} \leq \alpha(t) + c\,\|v\|_H$, where $\alpha \in L^2(I)$ and $c > 0$. Because V^* is a separable Banach space, we use [23, Proposition 1.1.28(a)] to deduce that the Nemitsky operator $\mathcal{B}_\eta \colon L^2(I; H) \to L^2(I; V^*)$ corresponding to B_η defined by $(\mathcal{B}_\eta v)(t) := B_\eta(t, v(t))$ for $v \in L^2(I; H)$ is continuous and bounded from $L^2(I; H)$ to $L^2(I; V^*)$. Hence, for any sequence $v_n \to v$ in $L^2(I; H)$, we have

$$\mathcal{B}_\eta v_n \to \mathcal{B}_\eta v \ \text{ in } \ L^2(I; V^*) \ \text{ for any fixed } \ \eta \in L^2(I; V). \tag{2.16}$$

Next, we will pass to the limit in the problem (2.15). Since $u_n \in L^2(I; K)$ with $u_n \to u$ in $L^2(I; V)_w$ and the set $L^2(I; K)$ is weakly closed in $L^2(I; V)$, it is obvious that $u \in L^2(I; K)$. Let $\widetilde{\gamma} \colon L^2(I; V) \to L^2(I; X)$ be the Nemitsky operator corresponding to γ. Since $\widetilde{\gamma}$ is linear and bounded, we have $\widetilde{\gamma} u_n \rightharpoonup \widetilde{\gamma} u$ in $L^2(I; X)$. On the other hand, by the compact embedding $\mathbb{W} \subset L^2(I; H)$, we have $\eta_n \to \eta$ in $L^2(I; H)$. We use the compactness of the operator $\widetilde{\delta} \colon \mathbb{W} \to L^2(I, Z)$, see hypothesis $H(\delta, \gamma)$, and obtain $\widetilde{\delta}\eta_n \to \widetilde{\delta}\eta$ in $L^2(I; Z)$. From $H(J)$(e), we infer that

$$\begin{aligned} \limsup \int_I & J^0(t, \delta w_n(t), \gamma\eta(t); \gamma(\eta(t) - u_n(t)))\, dt \\ &\leq \int_I J^0(t, \delta w(t), \gamma\eta(t); \gamma(\eta(t) - u(t)))\, dt \end{aligned} \tag{2.17}$$

for all $\eta \in L^2(I; V)$. From hypothesis $H(\varphi)$(f), we have

$$\limsup \int_I \varphi(t, \delta w_n(t), \eta(t)) - \varphi(t, \delta w_n(t), u_n(t))\, dt \leq \int_I \varphi(t, \delta w(t), \eta(t)) - \varphi(t, \delta w(t), u(t))\, dt \tag{2.18}$$

for all $\eta \in L^2(I; V)$.

We use the convergences $\widetilde{\delta}\eta_n \to \widetilde{\delta}\eta$ in $L^2(I;Z)$, $u_n \to u$ in $L^2(I;V)_w$, (2.16), (2.17), (2.18), $H(g,w_0)$ and pass to the upper limit in (2.15) to get

$$\begin{aligned} 0 \le &\limsup \int_I \langle B(t,w_n(t),\eta(t)) - g(\lambda_n,t), \eta(t) - u_n(t)\rangle\, dt \\ &+ \limsup \int_I J^0(t,\delta w_n(t),\gamma\eta(t);\gamma(\eta(t)-u_n(t)))\, dt \\ &+ \limsup \int_I \varphi(t,\delta w_n(t),\eta(t)) - \varphi(t,\delta w_n(t),u_n(t))\, dt \\ \le &\int_I \Big(\langle B(t,w(t),\eta(t)) - g(\lambda,t), \eta(t) - u(t)\rangle\, dt + J^0(t,\delta w(t),\gamma\eta(t);\gamma(\eta(t)-u(t)))\, dt \\ &+ \varphi(t,\delta w(t),\eta(t)) - \varphi(t,\delta w(t),u(t)) \Big) dt \end{aligned}$$

for all $\eta \in L^2(I;K)$. By applying Lemma 2.2 again, we conclude that $u \in L^2(I;K)$ is a solution to (1.2) corresponding to $w \in \mathbb{W}$. Thus, $(w,u) \in \mathcal{S}(\lambda)$ which completes the proof of (2.7). $\square$

Remark 2.4. Note that if the function $X \ni v \mapsto J(t,z,v) \in \mathbb{R}$ is convex for all $z \in Z$, a.e. $t \in I$, then the variational-hemivariational inequality (1.2) reduces to the following variational inequality: Find $u \in L^2(I;V)$, $u(t) \in K$ for a.e. $t \in I$ such that

$$\begin{aligned} &\langle B(t,w(t),u(t)) - g(\lambda,t), v - u(t)\rangle + \alpha(t,\delta w(t),v) - \alpha(t,\delta w(t),u(t)) \ge 0 \\ &\qquad \text{for all } \ v \in K, \ \text{ for a.e. } t \in I, \end{aligned}$$

where the potential function $\alpha\colon I \times Z \times V \to \mathbb{R}$ is defined by $\alpha(t,z,v) := J(t,z,\gamma v) + \varphi(t,z,v)$ for $z \in Z$, $v \in V$, a.e. $t \in I$. In this case, Problem 1.1 reduces to the corresponding differential variational inequality.

3. Optimal control problem

In this section we apply the stability result of the previous section to study an optimal control problem for the differential variational-hemivariational inequality formulated in Problem 1.1.

Let $\mathcal{S}(\lambda)$, for $\lambda \in \Lambda$, denote the solution set to Problem 1.1 defined by (1.3). Consider the following optimization problem: Find $\lambda^* \in \Lambda_{ad}$ such that

$$m(\lambda^*) = \inf\{m(\lambda) \mid \lambda \in \Lambda_{ad}\} \quad \text{with} \quad m(\lambda) := \inf\{\mathcal{F}(\lambda,w,u) \mid (w,u) \in \mathcal{S}(\lambda)\}. \tag{3.1}$$

We need the additional hypotheses on the cost functional and the admissible set of control parameters.

$\underline{H(\mathcal{F})}$: The functional $\mathcal{F}\colon \Lambda \times \mathbb{W} \times L^2(I;V) \to \mathbb{R}$ is bounded from below and lower semicontinuous on $\Lambda \times \mathbb{W}_w \times L^2(I;V)_w$,

$\underline{H(\Lambda_{ad})}$: The set Λ_{ad} is a compact subset of a metric space Λ.

Theorem 3.1. *Under the hypotheses of Theorem 2.3 and $H(\mathcal{F})$, for any $\lambda \in \Lambda$, the problem $m(\lambda) := \inf\{\mathcal{F}(\lambda,w,u) \mid (w,u) \in \mathcal{S}(\lambda)\}$ has a solution.*

Proof. Let $\lambda \in \Lambda$ be fixed. Let $\{(w_n,u_n)\} \subset \mathcal{S}(\lambda)$ be a minimizing sequence, that is, $m(\lambda) = \lim \mathcal{F}(\lambda,w_n,u_n)$. From the compactness of the set $\mathcal{S}(\lambda)$ in $\mathbb{W}_w \times L^2(I;V)_w$, guaranteed by Theorem 2.3, we can find a subsequence of $\{(w_n,u_n)\}$ denoted in the same way such that $(w_n,u_n) \to (w,u)$ in $\mathbb{W}_w \times L^2(I;V)_w$ with $(w,u) \in \mathcal{S}(\lambda)$. From $H(\mathcal{F})$ we obtain

$$\mathcal{F}(\lambda,w,u) \le \liminf \mathcal{F}(\lambda,w_n,u_n) = \lim \mathcal{F}(\lambda,w_n,u_n) = m(\lambda) \le \mathcal{F}(\lambda,w,u),$$

i.e., $\mathcal{F}(\lambda,w,u) = m(\lambda)$. This completes the proof of the theorem. $\square$

Theorem 3.2. *Under the hypotheses of Theorem 2.3 and $H(\mathcal{F})$, the value function $m\colon \Lambda \to \mathbb{R}$ is lower semicontinuous on Λ.*

Proof. It is sufficient to prove that $m(\lambda) \le \liminf m(\lambda_n)$ for all $\lambda_n \to \lambda$ in Λ. Let $\lambda_n \to \lambda$ in Λ with some $\lambda \in \Lambda$. For every $n \in \mathbb{N}$, from Theorem 3.1, we are able to find $(w_n^*, u_n^*) \in \mathcal{S}(\lambda_n)$ such that $\mathcal{F}(\lambda_n, w_n^*, u_n^*) = m(\lambda_n)$. From Theorem 2.3 to deduce that $(w_n^*, u_n^*) \in \mathcal{S}(\lambda_n)$ is uniformly bounded in $\mathbb{W} \times L^2(I; V)$. From the reflexivity of the latter, we may assume, at least for a subsequence, that

$$(w_n^*, u_n^*) \to (w^*, u^*) \text{ in } \mathbb{W}_w \times L^2(I; V)_w \text{ with some } (w^*, u^*) \in \mathbb{W} \times L^2(I; V).$$

From the inclusion (2.7), we have $(w^*, u^*) \in \mathcal{S}(\lambda)$ which implies $m(\lambda) \le \mathcal{F}(\lambda, w^*, u^*)$. Finally, we use $H(\mathcal{F})$ to get

$$m(\lambda) \le \mathcal{F}(\lambda, w^*, u^*) \le \liminf \mathcal{F}(\lambda_n, w_n^*, u_n^*) = \liminf m(\lambda_n).$$

We deduce that m is lower semicontinuous on Λ which completes the proof. □

Using the hypothesis $H(\Lambda_{ad})$ and Theorem 3.2, it is immediate to obtain the following result.

Theorem 3.3. *Under the hypotheses of Theorem 2.3, $H(\mathcal{F})$, and $H(\Lambda_{ad})$, the problem $m(\lambda^*) = \inf\{m(\lambda) \mid \lambda \in \Lambda_{ad}\}$ has a solution.*

We will comment on an example of the cost functional which satisfies $H(\mathcal{F})$. Let $\mathcal{F} \colon \Lambda \times \mathbb{W} \times L^2(I; V) \to \mathbb{R}$ be defined by

$$\mathcal{F}(\lambda, w, u) = l_1(\lambda) + l_2(w(T_0)) + \int_I L(t, w(t), u(t))\, dt \tag{3.2}$$

for $\lambda \in \Lambda$, $w \in \mathbb{W}$, $u \in L^2(I; V)$, where T_0 is any time moment in $(0, T]$. We need the following hypotheses on the cost.

$\underline{H(l, L)}$: $\quad \mathcal{F} \colon \Lambda \times \mathbb{W} \times L^2(I; V) \to \mathbb{R}$ is such that

(a) $l_1 \colon \Lambda \to \mathbb{R}$ is lower semicontinuous, and $l_2 \colon H \to \mathbb{R}$ is weakly lower semicontinuous,

(b) $L(t, \cdot, \cdot)$ is lower semicontinuous on $H \times V$, a.e. $t \in I$,

(c) $L(t, w, \cdot)$ is convex on V, for all $w \in H$, a.e. $t \in I$,

(d) there exists $M > 0$ and $\psi \in L^1(I)$ such that for all $w \in H$, $v \in V$, a.e. $t \in I$, we have

$$L(t, w, v) \ge \psi(t) - M(\|w\|_H + \|v\|_V).$$

Example 3.4. Under hypotheses $H(l, L)$, the cost functional defined by (3.2) satisfies $H(\mathcal{F})$.

Proof. From [1, Theorem 2.1] we obtain that the functional $\Phi(w, u) = \int_I L(t, w(t), u(t))\, dt$ is lower semicontinuous on $L^1(I; H) \times L^1(I; V)_w$. Hence, using the compact embedding of $\mathbb{W}$ into $L^2(I; H)$, we infer that Φ is lower semicontinuous on $\mathbb{W}_w \times L^2(I; V)_w$. Next, by [19, Lemma 2.55(ii)], we know that $w_n, w \in \mathbb{W}$ and $w_n \to w$ in $\mathbb{W}_w$ imply $w_n(t) \to w(t)$ in H_w for all $t \in I$. We combine this fact with $H(l, L)$(a) to get $H(\mathcal{F})$. □

4. Application to quasistatic frictional contact problem

In this section we illustrate the results of the previous sections by a quasistatic frictional contact problem coupled with the heat equation, which, in a weak form, is governed by a differential variational-hemivariational inequality.

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, represent a bounded domain occupied by an elastic body. The boundary Γ is Lipschitz and is divided into three mutually disjoint measurable parts Γ_D, Γ_N and Γ_C such that the measure of Γ_D is positive. The unit outward normal on Γ is denoted by $\boldsymbol{\nu}$. Let $Q := \Omega \times I$ with $I := [0, T]$. The part Γ_C represents the contact surface between the body and the foundation. The body is clamped on Γ_D, and is subjected to a volume force of density $\boldsymbol{g}_0(\lambda)$ and a heat source $\rho_1(\lambda)$ in Ω. Surface tractions of density $\boldsymbol{g}_N(\lambda)$ act on the part Γ_N and ρ_2 stands for the heat flux through Γ_C. The parameter $\lambda \in \Lambda$ represents the control variable from a metric space Λ. In what follows we often do not indicate explicitly the dependence of various functions on $\boldsymbol{x}$ and t.

The classical formulation of the elastic contact model reads as follows.

Problem 4.1. Find a displacement field $\boldsymbol{u}\colon \Omega \to \mathbb{R}^d$, a stress field $\boldsymbol{\sigma}\colon \Omega \to \mathbb{S}^d$, a temperature $\theta\colon \Omega \to \mathbb{R}$, and a heat flux $\boldsymbol{q}\colon \Omega \to \mathbb{R}^d$ such that

$$-\operatorname{Div} \boldsymbol{\sigma}(t) = \boldsymbol{g}_0(\lambda, t) \quad \text{in } \Omega, \tag{4.1}$$

$$\theta'(t) + \operatorname{div} \boldsymbol{q}(t) + \psi(t, \theta(t)) + \mathcal{R}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) = \rho_1(\lambda, t) \quad \text{in } \Omega, \tag{4.2}$$

$$\boldsymbol{\sigma}(t) = \mathcal{E}(t, \boldsymbol{\varepsilon}(\boldsymbol{u}(t))) \quad \text{in } \Omega, \tag{4.3}$$

$$\boldsymbol{q}(t) = -\mathcal{K}(t, \nabla\theta(t)) \quad \text{in } \Omega, \tag{4.4}$$

$$\boldsymbol{u}(t) = \mathbf{0} \quad \text{on } \Gamma_D, \tag{4.5}$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{g}_N(\lambda, t) \quad \text{on } \Gamma_N, \tag{4.6}$$

$$\theta(t) = 0 \quad \text{on } \Gamma_D \cup \Gamma_N, \tag{4.7}$$

$$-\frac{\partial\theta(t)}{\partial\nu_{\mathcal{K}}} = \rho_2(t) \quad \text{on } \Gamma_C, \tag{4.8}$$

$$\sigma_\nu(t) = \sigma_\nu^1(t) + \sigma_\nu^2(t) \quad \text{on } \Gamma_C, \tag{4.9}$$

$$-\sigma_\nu^1(t) \in p_\nu(t, \theta(t))\, \partial j(u_\nu(t)) \quad \text{on } \Gamma_C, \tag{4.10}$$

$$\begin{cases} u_\nu(t) \le g,\ \sigma_\nu^2(t) + p(t, \theta(t)) \le 0, \\ (u_\nu(t) - g)(\sigma_\nu^2(t) + p(t, \theta(t))) = 0 \end{cases} \quad \text{on } \Gamma_C, \tag{4.11}$$

$$\begin{cases} \|\boldsymbol{\sigma}_\tau(t)\| \le \mu\, p(t, \theta(t)) \\ -\boldsymbol{\sigma}_\tau(t) = \mu\, p(t, \theta(t)) \dfrac{\boldsymbol{u}_\tau(t)}{\|\boldsymbol{u}_\tau(t)\|} \quad \text{if } \boldsymbol{u}_\tau(t) \neq \mathbf{0} \end{cases} \quad \text{on } \Gamma_C, \tag{4.12}$$

$$\theta(0) = \theta_0(\lambda) \quad \text{in } \Omega. \tag{4.13}$$

The standard notation is used. The normal and tangential components on the boundary of a vector $\mathbf{v}$ are defined by $v_\nu = \mathbf{v}\cdot\boldsymbol{\nu}$ and $\mathbf{v}_\tau = \mathbf{v} - v_\nu\boldsymbol{\nu}$, respectively. Given a tensor $\boldsymbol{\sigma}$, the symbols σ_ν and $\boldsymbol{\sigma}_\tau$ stand for its normal and tangential components on the boundary, that is, $\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu})\cdot\boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu\boldsymbol{\nu}$. The linearized strain tensor is defined by $\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u})), \quad \varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}(u_{i,j} + u_{j,i})$ in Ω.

The relation (4.1) is the equilibrium equation for the stress. The heat equation (4.2) represents the law of conservation of energy, where $\boldsymbol{q}$ is the heat flux vector, and the function $\mathcal{R}$ describes the influence of the displacement field on the temperature. It is well known that thermal effects often accompany the friction phenomena. By Div and div we denote the divergence operators for tensor and vector valued functions, respectively. The elastic constitutive relation (4.3) involves the elasticity operator $\mathcal{E}$. For the thermal diffusion, we use the law (4.4) with a nonlinear thermal conductivity operator $\mathcal{K}$ which in a linear case reduces to the Fourier law $\boldsymbol{q}(t) = -k(\boldsymbol{x}, t)\nabla\theta(t)$ in Ω, $k = k(\boldsymbol{x}, t)$ being the conductivity tensor. Conditions (4.5), (4.6) and (4.7) are the displacement and the traction boundary conditions, and the thermal boundary condition. Moreover, the conormal derivative $\frac{\partial\theta}{\partial\nu_{\mathcal{K}}} = \mathcal{K}(\boldsymbol{x}, t, \nabla\theta)\cdot\boldsymbol{\nu}$ in condition (4.8) specifies the heat flux between the body surface Γ_C and the foundation. The normal stress has an additive decomposition in (4.9). One part of the normal stress in (4.10) represents a multivalued nonmonotone version of the normal compliance condition in which the stiffness coefficient function p_ν is temperature dependent, and ∂j stands for the Clarke generalized gradient of a locally Lipschitz potential, see [19, Section 6.3]. The relation (4.11) represents the frictional Signorini unilateral contact condition with a gap $g > 0$ for the normal displacement associated to the Coulomb's law of dry friction (4.12), where μ is a positive bounded function which describes the coefficient of friction. The initial temperature θ_0 is prescribed in (4.13). For more details on the mathematical modeling of contact problems, we refer to [19, 24, 26].

For the weak formulation of Problem 4.1, we introduce the following spaces

$$\begin{cases} E = \{\eta \in H^1(\Omega) \mid \eta = 0 \text{ on } \Gamma_D \cup \Gamma_N\}, \quad H = L^2(\Omega), \\ V = \{\mathbf{v} \in H^1(\Omega;\mathbb{R}^d) \mid \mathbf{v} = \mathbf{0} \text{ on } \Gamma_D\}, \quad \mathcal{H} = L^2(\Omega;\mathbb{S}^d). \end{cases} \tag{4.14}$$

The triplet (E, H, E^*) forms an evolution triple of spaces with dense, continuous and compact embeddings. On V we use the norm $\|\mathbf{v}\|_V = \|\boldsymbol{\varepsilon}(\mathbf{v})\|_{\mathcal{H}}$ for $\mathbf{v} \in V$ which is, by the Korn inequality equivalent to the usual norm $\|\cdot\|_{H^1(\Omega;\mathbb{R}^d)}$. Recall that the normal trace operator $\gamma\colon V \to L^2(\Gamma)$ is linear and compact, i.e., $\|\gamma(\mathbf{v})\|_{L^2(\Gamma)} \le \|\gamma\|\|\mathbf{v}\|_V$ for $\mathbf{v} \in V$, where $\|\gamma\|$ denotes the operator norm.

We need the following hypotheses on the data to Problem 4.1.

$\underline{H(\mathcal{E})}$: The elasticity operator $\mathcal{E}\colon Q \times \mathbb{S}^d \to \mathbb{S}^d$ is such that

(1) for all $\boldsymbol{\varepsilon} \in \mathbb{S}^d$, a.e. $t \in I$, $\mathcal{E}(\cdot, t, \boldsymbol{\varepsilon})$ is measurable on Ω,

(2) for a.e. $\boldsymbol{x} \in \Omega$, $\mathcal{E}(\boldsymbol{x}, \cdot, \cdot)$ is continuous on $I \times \mathbb{S}^d$.

(3) for all $\boldsymbol{\varepsilon} \in \mathbb{S}^d$, a.e. $(\boldsymbol{x}, t) \in Q$, $\|\mathcal{E}(\boldsymbol{x}, t, \boldsymbol{\varepsilon})\| \le \widetilde{a}_0(\boldsymbol{x}, t) + \widetilde{a}_2 \|\boldsymbol{\varepsilon}\|$
with $\widetilde{a}_0, \widetilde{a}_2 \ge 0$, $\widetilde{a}_0 \in L^\infty(I; L^2(\Omega))$,

(4) there exists $m_{\mathcal{E}} > 0$ such that $(\mathcal{E}(\boldsymbol{x}, t, \boldsymbol{\varepsilon}_1) - \mathcal{E}(\boldsymbol{x}, t, \boldsymbol{\varepsilon}_2)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \ge m_{\mathcal{E}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2$
for all $\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d$, a.e. $(\boldsymbol{x}, t) \in Q$,

(5) for a.e. $(\boldsymbol{x}, t) \in Q$, $\mathcal{E}(\boldsymbol{x}, t, \mathbf{0}) = \mathbf{0}$.

$\underline{H(\mathcal{R})}$: $\mathcal{R}\colon \Omega \times \mathbb{S}^d \to \mathbb{R}$ is such that

(1) for all $\boldsymbol{\varepsilon} \in \mathbb{S}^d$, a.e. $\boldsymbol{x} \in \Omega$, $\mathcal{R}(\boldsymbol{x}, \boldsymbol{\varepsilon}) = R(\boldsymbol{x})\boldsymbol{\varepsilon}$.

(2) $R(\boldsymbol{x}) = (R_{ij}(\boldsymbol{x}))$ with $R_{ij} \in L^\infty(\Omega)$.

$\underline{H(\mathcal{K})}$: The thermal conductivity operator $\mathcal{K}\colon Q \times \mathbb{R}^d \to \mathbb{R}^d$ is such that

(1) for all $\boldsymbol{\xi} \in \mathbb{R}^d$, $\mathcal{K}(\cdot, \cdot, \boldsymbol{\xi})$ is measurable on Q,

(2) for a.e. $(\boldsymbol{x}, t) \in Q$, $\mathcal{K}(\boldsymbol{x}, t, \cdot)$ is continuous on $\mathbb{R}^d$,

(3) for all $\boldsymbol{\xi} \in \mathbb{R}^d$, a.e. $(\boldsymbol{x}, t) \in Q$, $\|\mathcal{K}(\boldsymbol{x}, t, \boldsymbol{\xi})\| \le k_0(\boldsymbol{x}, t) + k_1 \|\boldsymbol{\xi}\|$
with $k_0 \in L^2(Q)$, $k_0 \ge 0$, $k_1 > 0$,

(4) for all $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{R}^d$, a.e. $(\boldsymbol{x}, t) \in Q$, $(\mathcal{K}(\boldsymbol{x}, t, \boldsymbol{\xi}_1) - \mathcal{K}(\boldsymbol{x}, t, \boldsymbol{\xi}_2)) \cdot (\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2) \ge 0$,

(5) for all $\boldsymbol{\xi} \in \mathbb{R}^d$, a.e. $(\boldsymbol{x}, t) \in Q$, $\mathcal{K}(\boldsymbol{x}, t, \boldsymbol{\xi}) \cdot \boldsymbol{\xi} \ge \alpha_{\mathcal{K}} \|\boldsymbol{\xi}\|^2$ with $\alpha_{\mathcal{K}} > 0$.

$\underline{H(\psi)}$: $\psi\colon Q \times \mathbb{R} \to \mathbb{R}$ is such that

(1) for all $r \in \mathbb{R}$, $(\boldsymbol{x}, t) \mapsto \psi(\boldsymbol{x}, t, r)$ is measurable,

(2) there exists $k \in L^1(I; L^\infty(\Omega))_+$ such that $|\psi(\boldsymbol{x}, t, r_1) - \psi(\boldsymbol{x}, t, r_2)| \le k(\boldsymbol{x}, t)|r_1 - r_2|$
for all $r_1, r_2 \in \mathbb{R}$ and a.e. $(\boldsymbol{x}, t) \in Q$,

(3) there are $a \in L^2(Q)$ and $b > 0$ such that $|\psi(\boldsymbol{x}, t, r)| \le a(\boldsymbol{x}, t) + b\,|r|$ for all $r \in \mathbb{R}$, a.e. $(\boldsymbol{x}, t) \in Q$.

$\underline{H(p_\nu)}$: $p_\nu\colon \Gamma_C \times I \times \mathbb{R} \to \mathbb{R}$ is such that

(1) $p_\nu(\cdot, t, r)$ is measurable on Γ_C for all $r \in \mathbb{R}$, $t \in I$.

(2) $p_\nu(\boldsymbol{x}, \cdot, r)$ is continuous on I for all $r \in \mathbb{R}$, a.e. $\boldsymbol{x} \in \Gamma_C$.

(3) there is $L_{p\nu} > 0$ such that $|p_\nu(\boldsymbol{x}, t, r_1) - p_\nu(\boldsymbol{x}, t, r_2)| \le L_{p\nu}\,|r_1 - r_2|$ for all $r_1, r_2 \in \mathbb{R}$,
a.e. $(\boldsymbol{x}, t) \in \Gamma_C \times I$.

(4) there is $p_\nu^* > 0$ such that $0 \le p_\nu(\boldsymbol{x}, t, r) \le p_\nu^*$ for $r \in \mathbb{R}$, a.e. $(\boldsymbol{x}, t) \in \Gamma_C \times I$.

$\underline{H(p)}$: $p\colon \Gamma_C \times I \times \mathbb{R} \to \mathbb{R}$ is such that

(1) $p(\cdot, t, r)$ is measurable on Γ_C for all $r \in \mathbb{R}$, a.e. $t \in I$.

(2) there is $L_p > 0$ such that $|p(\boldsymbol{x}, t_1, r_1) - p(\boldsymbol{x}, t_2, r_2)| \le L_p\,(|t_1 - t_2| + |r_1 - r_2|)$ for all $t_1, t_2 \in I$, $r_1, r_2 \in \mathbb{R}$, a.e. $\boldsymbol{x} \in \Gamma_C$.

(3) there is $p^* > 0$ such that $0 \le p(\boldsymbol{x}, t, r) \le p^*$ for $r \in \mathbb{R}$, a.e. $(\boldsymbol{x}, t) \in \Gamma_C \times I$.

$\underline{H(\mu)}$: $\mu \in L^\infty(\Gamma_C)$, $0 \le \mu(x) \le \mu^*$ a.e. $\boldsymbol{x} \in \Gamma_C$.

$\underline{H(j)}$: $j \colon \mathbb{R} \to \mathbb{R}$, $j(r) = \int_0^r \varrho(s)\,ds$ for all $r \in \mathbb{R}$, where $\varrho \colon \mathbb{R} \to \mathbb{R}$ is such that

(1) there are constants ϱ_0, ϱ_1 such that $0 < \varrho_0 \le \varrho(r) \le \varrho_1$ for all $r > 0$,

(2) there exists $L_\varrho > 0$ such that $|\varrho(r_1) - \varrho(r_2)| \le L_\varrho\,|r_1 - r_2|$ for all $r_1, r_2 \in \mathbb{R}$,

(3) $\varrho(r) = 0$ if and only if $r \le 0$.

$\underline{(H_0)}$: $\boldsymbol{g}_0, \boldsymbol{g}_N \colon \Lambda \times I \to \mathbb{R}^d$ are such that $\boldsymbol{g}_0(\lambda, \cdot) \in L^2(\Omega; \mathbb{R}^d)$, $\boldsymbol{g}_N(\lambda, \cdot) \in L^2(\Gamma_N; \mathbb{R}^d)$, and there is $c_1 > 0$ such that for all $\lambda \in \Lambda$, a.e. $t \in I$, $\|\boldsymbol{g}_0(\lambda, t)\| + \|\boldsymbol{g}_N(\lambda, t)\| \le c_1$, $\theta_0 \in L^2(\Omega)$ and $\|\theta_0(\lambda)\|_H \le c_1$ for some $c_1 > 0$, $\rho_1(\cdot) \in L^2(\Omega)$, $\rho_2(\lambda, \cdot) \in L^2(\Gamma_C)$ and $\rho_2(\lambda_n, \cdot) \to \rho_2(\lambda, \cdot)$ in $L^2(\Gamma_C)$ for $\lambda_n \to \lambda$ in Λ.

Next, we derive the variational formulation of Problem 4.1. Let $(\boldsymbol{u}, \boldsymbol{\sigma}, \theta)$ be a triple of smooth functions which satisfies (4.1)–(4.13). Let $\mathbf{v} \in V$ and $t \in I$. We multiply (4.1) by $\mathbf{v} - \boldsymbol{u}(t)$, exploit the Green formula, see [19, Theorem 2.25], and apply the boundary conditions (4.5)-(4.6) to get

$$\int_\Omega \boldsymbol{\sigma}(t) \cdot \big(\boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t))\big)\,dx = \int_\Omega \boldsymbol{g}_0(\lambda, t) \cdot (\mathbf{v} - \boldsymbol{u}(t))\,dx$$
$$+ \int_{\Gamma_N} \boldsymbol{g}_N(\lambda, t) \cdot (\mathbf{v} - \boldsymbol{u}(t))\,d\Gamma + \int_{\Gamma_C} \boldsymbol{\sigma}(t)\boldsymbol{\nu} \cdot (\mathbf{v} - \boldsymbol{u}(t))\,d\Gamma.$$

By the decomposition formula $\boldsymbol{\sigma\nu} \cdot \mathbf{v} = (\sigma_\nu\,\boldsymbol{\nu} + \boldsymbol{\sigma}_\tau) \cdot (v_\nu\,\boldsymbol{\nu} + \mathbf{v}_\tau) = \sigma_\nu\,v_\nu + \boldsymbol{\sigma}_\tau \cdot \mathbf{v}_\tau$, we have

$$\langle \boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)) \rangle_{\mathcal{H}} = \langle \boldsymbol{g}(t), \mathbf{v} - \boldsymbol{u}(t) \rangle$$
$$+ \int_{\Gamma_C} \sigma_\nu(t)(v_\nu - u_\nu(t)) + \boldsymbol{\sigma}_\tau(t) \cdot (\mathbf{v}_\tau - \boldsymbol{u}_\tau(t))\,d\Gamma, \tag{4.15}$$

where $\boldsymbol{g} \in L^\infty(\Lambda \times I; V^*)$ is defined by

$$\langle \boldsymbol{g}(\lambda, t), \mathbf{v} \rangle = \langle \boldsymbol{g}_0(\lambda, t), \mathbf{v} \rangle_{L^2(\Omega;\mathbb{R}^d)} + \langle \boldsymbol{g}_N(\lambda, t), \mathbf{v} \rangle_{L^2(\Gamma_N;\mathbb{R}^d)} \tag{4.16}$$

for $\mathbf{v} \in V$, $t \in I$. From (4.10) and (4.12), for all $\mathbf{v} \in V$, we have

$$-\sigma_\nu^1(t)(v_\nu - u_\nu(t)) \le p_\nu(t, \theta(t))\,j^0(u_\nu(t); v_\nu - u_\nu(t)), \tag{4.17}$$

$$-\boldsymbol{\sigma}_\tau(t) \cdot (\mathbf{v}_\tau - \boldsymbol{u}_\tau(t)) \le \mu\,p(t, \theta(t))\,(\|\mathbf{v}_\tau\| - \|\boldsymbol{u}_\tau(t)\|), \tag{4.18}$$

respectively. Next, we introduce the set K of admissible velocity fields

$$K = \{\, \mathbf{v} \in V \mid v_\nu \le g \ \text{ on } \ \Gamma_C \,\}. \tag{4.19}$$

We use (4.11), and for $\mathbf{v} \in K$, we obtain

$$-\sigma_\nu^2(t)(v_\nu - u_\nu(t)) = p(t, \theta(t))(v_\nu - u_\nu(t)) - \ (\sigma_\nu^2(t) + p(t, \theta(t)))(v_\nu - g)$$
$$- (\sigma_\nu^2(t) + p(t, \theta(t)))(g - u_\nu(t)) \le p(t, \theta(t))(v_\nu - u_\nu(t)). \tag{4.20}$$

By the constitutive law (4.3), we have

$$\langle \boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)) \rangle_{\mathcal{H}} \le \langle \mathcal{E}(t, \boldsymbol{\varepsilon}(\boldsymbol{u}(t))), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)) \rangle_{\mathcal{H}} \tag{4.21}$$

for all $\mathbf{v} \in V$. We combine (4.13), (4.15), (4.16), (4.17), (4.18), (4.20), and (4.21) to obtain the variational-hemivariational inequality for the displacement field. Next, we use (4.2), (4.4), (4.7), (4.8) and (4.13) to derive a variational equation for the temperature. As a result we obtain the following weak formulation of Problem 4.1.

Problem 4.2. Find $\theta\colon I \to E$ and $\boldsymbol{u}\colon I \to K$ such that $\theta(0) = \theta_0(\lambda)$, and

$$\int_\Omega \theta'(t)\,\eta\,dx + \int_\Omega \mathcal{K}(t, \nabla\theta(t))\cdot\nabla\eta\,dx + \int_{\Gamma_C} \rho_2(t)\,\eta\,d\Gamma + \int_\Omega \mathcal{R}\boldsymbol{\varepsilon}(\boldsymbol{u}(t))\,\eta\,dx$$
$$+ \int_\Omega \psi(t, \theta(t))\,\eta\,dx = \int_\Omega \rho_1(\lambda, t)\,\eta\,dx \ \text{ for all } \ \eta \in E, \ \text{a.e. } t \in I,$$
$$\int_\Omega \mathcal{E}(t, \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))\cdot(\boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))\,dx + \int_{\Gamma_C} p_\nu(t, \theta(t))\, j^0(u_\nu(t); v_\nu - u_\nu(t))\,d\Gamma$$
$$+ \int_{\Gamma_C} p(t, \theta(t))(v_\nu - u_\nu(t))\,d\Gamma + \int_{\Gamma_C} \mu\, p(t, \theta(t))\,(\|\mathbf{v}_\tau\| - \|\boldsymbol{u}_\tau(t)\|)\,d\Gamma$$
$$\geq \langle \boldsymbol{g}(\lambda, t), \mathbf{v} - \boldsymbol{u}(t)\rangle \text{ for all } \mathbf{v} \in K, \ \text{a.e. } t \in I.$$

Problem 4.2 couples the parabolic differential equation for the temperature with a time dependent variational-hemivariational inequality for the displacement field.

Theorem 4.3. *Assume hypotheses $H(\mathcal{E})$, $H(\mathcal{R})$, $H(\mathcal{K})$, $H(\psi)$, $H(p_\nu)$, $H(p)$, $H(\mu)$, $H(j)$, (H_0) and the smallness conditions*

$$m_{\mathcal{E}} > \max\{L_\varrho\, p_\nu^*\, \|\gamma\|^2, \tfrac{1}{2}\} \tag{4.22}$$

Then, for any $\lambda \in \Lambda$, Problem 4.2 has a nonempty and compact set of solutions $\mathcal{S}(\lambda)$ in $\mathbb{W}_w \times L^2(I; V)_w$ and

$$K(\mathbb{W}_w \times L^2(I; V)_w) - \limsup \mathcal{S}(\lambda_n) \subset \mathcal{S}(\lambda) \ \text{ for all } \ \lambda_n \to \lambda \ \text{ in } \ \Lambda. \tag{4.23}$$

Proof. We will use the following functional framework: E, V, H and $\mathcal{H}$ are defined by (4.14), $Z = X = L^2(\Gamma_C)$ and K is given by (4.19). Let the operators $A\colon I \times E \to E^*$, $f\colon \Lambda \times I \times H \to H$, $G\colon I \to \mathcal{L}(V, H)$, $B\colon I \times H \times V \to V^*$ and functions $J\colon I \times Z \times X \to \mathbb{R}$, $\varphi\colon I \times Z \times V \to \mathbb{R}$ be defined by

$$\langle A(t, v), \eta\rangle = \langle \mathcal{K}(\boldsymbol{x}, t, \nabla v), \nabla\eta\rangle_{L^2(\Omega;\mathbb{R}^d)} + \langle \rho_2(t), \eta\rangle_X \text{ for } v, \eta \in E, \text{a.e. } t \in I, \tag{4.24}$$

$$f(\lambda, t, v)(\boldsymbol{x}) = -\,\psi(\boldsymbol{x}, t, v(\boldsymbol{x})) + \rho_1(\lambda, t) \ \text{ for } \ v \in H, \ \text{a.e. } t \in I, \tag{4.25}$$

$$G(t)\mathbf{v} = -\,\mathcal{R}(\boldsymbol{\varepsilon}(\mathbf{v})) \ \text{ for } \ \mathbf{v} \in V, \ \text{a.e. } t \in I, \tag{4.26}$$

$$\langle B(t, w, \boldsymbol{u}), \mathbf{v}\rangle = \langle \mathcal{E}(t, \boldsymbol{\varepsilon}(\boldsymbol{u})), \boldsymbol{\varepsilon}(\mathbf{v})\rangle_{\mathcal{H}} \ \text{ for } \ w \in H, \boldsymbol{u}, \mathbf{v} \in V, \ \text{a.e. } t \in I, \tag{4.27}$$

$$J(t, z, v) = \int_{\Gamma_C} p_\nu(t, z)\, j(v)\,d\Gamma \ \text{ for } \ z \in Z, \ v \in X, \ \text{a.e. } t \in I, \tag{4.28}$$

$$\varphi(t, z, \mathbf{v}) = \int_{\Gamma_C} (p(t, z)\, v_\nu + \mu\, p(t, z)\, \|\mathbf{v}_\tau\|)\,d\Gamma \ \text{ for } \ z \in Z, \ \mathbf{v} \in V, \ \text{a.e. } t \in I. \tag{4.29}$$

Moreover, let $\delta\colon E \to Z$ and $\gamma\colon V \to X$ be the trace and the normal trace operators, respectively, i.e., $\delta(\theta) = \theta$ for $\theta \in E$, and $\gamma\mathbf{v} = v_\nu$ for $\mathbf{v} \in V$, and $\langle\cdot,\cdot\rangle_X$ denote the inner product in X.

With the above notation, we consider the inequality problem associated with Problem 4.2.

Problem 4.4. Find $\theta \in \mathbb{W}$ and $\boldsymbol{u} \in L^2(I; K)$ such that

$$\begin{cases} \theta'(t) + A(t, \theta(t)) = f(\lambda, t, \theta(t)) + G(t)\boldsymbol{u}(t) \ \text{ a.e. } \ t \in I, \\ \langle B(t, \theta(t), \boldsymbol{u}(t)) - \boldsymbol{g}(\lambda, t), \mathbf{v} - \boldsymbol{u}(t)\rangle + J^0(t, \delta\theta(t), \gamma\boldsymbol{u}(t); \gamma(\mathbf{v} - \boldsymbol{u}(t))) \\ \quad + \varphi(t, \delta\theta(t), \mathbf{v}) - \varphi(t, \delta\theta(t), \boldsymbol{u}(t)) \geq 0 \ \text{ for all } \ \mathbf{v} \in K, \ \text{a.e. } t \in I, \\ \theta(0) = \theta_0(\lambda). \end{cases}$$

We will verify hypotheses $H(A)$, $H(f)$, $H(B)$, $H(G)$, $H(J)$, $H(K)$, $H(\varphi)$, $H(\delta,\gamma)$ and $H(g,w_0)$ of Theorem 2.3.

For the hypotheses $H(A)$ we follow the lines of the proof in [14, Theorem 15] to deduce that $H(A)$(c) holds with $a_0(t) = c\,\|k_0(t,\cdot)\|_H + \|\rho_2(t,\cdot)\|_X$, $a_0 \in L^2(I)$ and $c, c_0 > 0$, and $H(A)$(d) is satisfied with $a_2 := \alpha_{\mathcal{K}}/2$ and $a_1(t) := (c\,\|\rho_2(t,\cdot)\|_X)^2/(2\alpha_{\mathcal{K}})$, $a_1 \in L^1(I)$, $c > 0$.

Next, we will show that the operator f defined by (4.25) satisfies $H(f)$. By the definition of f, we know that for all $v, z \in H$, $\lambda \in \Lambda$, a.e. $t \in I$, we have

$$\langle f(\lambda,t,v), z\rangle_H = -\int_\Omega (\psi(\boldsymbol{x},t,v(\boldsymbol{x})) + \rho_1(\lambda,\boldsymbol{x},t))\,z\,dx.$$

By the Fubini theorem $t \mapsto \langle f(\lambda,t,v), z\rangle_H$ is measurable for all $v, z \in H$, $\lambda \in \Lambda$. In a consequence, $t \mapsto f(\lambda,t,v)$ is weakly measurable for all $v \in H$. Due to the separability of H, from the Pettis measurability theorem, the function $t \mapsto f(\lambda,t,v)$ is also measurable for all $v \in H$. Therefore, $H(f)$(a) holds. $H(f)$(b) holds by H_0. The conditions $H(f)$(c) and (d) follow easily from hypotheses $H(\psi)$(2) and (3), respectively.

Now, we will verify condition $H(B)$. We choose $\boldsymbol{u}_0 = \mathbf{0} \in K$. By $H(\mathcal{E})$(4) and (5), we have $\mathcal{E}(\boldsymbol{x},t,\boldsymbol{\varepsilon})\cdot\boldsymbol{\varepsilon} \geq m_{\mathcal{E}}\,\|\boldsymbol{\varepsilon}\|^2$ for all $\boldsymbol{\varepsilon} \in \mathbb{S}^d$, a.e. $(\boldsymbol{x},t) \in Q$. From this condition, we obtain

$$\langle B(t,\eta,\mathbf{v}),\mathbf{v}\rangle \geq m_{\mathcal{E}}\|\mathbf{v}\|^2 \tag{4.30}$$

for a.e. $t \in I$, all $\eta \in H$, $\mathbf{v} \in V$. Moreover, we apply [19, Theorem 7.3] to deduce that $v \mapsto B(t,\eta,v)$ is bounded, strongly monotone and continuous for all $\eta \in H$, a.e. $t \in I$. Hence, it is pseudomonotone, see [19, Theorem 3.69]. Thus, $H(B)$(a) is satisfied.

We will check $H(B)$(b). From $H(j)$, it follows that j is a C^1 function. Thus, j is (Clarke) regular which implies that $\partial j(r) = j'(r) = \varrho(r)$ for $r \in \mathbb{R}$ and

$$j^0(r;s) = \max\{\xi s \mid \xi \in \partial j(r)\} = \partial j(r)s = \varrho(r)\,s \ \text{ for all } \ r,s \in \mathbb{R}.$$

We are in a position to apply [19, Theorem 3.47(i)-(iii)] to infer that $H(J)$(b) is satisfied, and since j is regular, $J(t,z,\cdot)$ is also regular and

$$J^0(t,z,v;w) = \int_{\Gamma_C} p_\nu(t,z)j^0(v;w)\,d\Gamma, \quad \partial J(t,z,v) = \int_{\Gamma_C} p_\nu(t,z)\partial j(v)\,d\Gamma \tag{4.31}$$

for all $z \in Z$, $v, w \in X$, a.e. $t \in I$. By [19, Theorem 3.47(v)] for all $u^* \in \partial J(t,z,\gamma\boldsymbol{u})$, $\boldsymbol{u} \in V$, we can find $\zeta\colon \Gamma_C \to \mathbb{R}$ such that $\zeta(\boldsymbol{x}) \in p_\nu(t,z(\boldsymbol{x}))\partial j(\gamma\boldsymbol{u}(\boldsymbol{x}))$ for a.e. $\boldsymbol{x} \in \Gamma_C$ and

$$\langle u^*, v\rangle_{X^*\times X} = \int_{\Gamma_C} \zeta v\,d\Gamma \ \text{ for } \ v \in X.$$

Recall that $\gamma\colon V \to X$ is the normal trace operator defined by $\gamma\mathbf{v} = v_\nu$ for $\mathbf{v} \in V$. Hence, we deduce $u^*(\boldsymbol{x}) = p_\nu(t,z(\boldsymbol{x}))\varrho(\gamma\boldsymbol{u}(\boldsymbol{x}))$ a.e. $\boldsymbol{x} \in \Gamma_C$, and

$$|\langle u^*,\gamma\mathbf{v}\rangle_{X^*\times X}| \leq p_\nu^*\,\varrho_1\sqrt{|\Gamma_C|}\,\|\mathbf{v}\|_{L^2(\Gamma_C;\mathbb{R}^d)} \leq p_\nu^*\,\varrho_1\,c\sqrt{|\Gamma_C|}\,\|\mathbf{v}\| \tag{4.32}$$

for all $\mathbf{v} \in V$, where $c > 0$. From (4.30) and (4.32), we have

$$\langle B(t,\eta,\mathbf{v}),\mathbf{v}\rangle + \inf_{\xi\in\partial J(t,z,\gamma v)}\langle \xi,\gamma\mathbf{v}\rangle_{X^*\times X} \geq m_{\mathcal{E}}\|\mathbf{v}\|^2 - p_\nu^*\,\varrho_1\,c\sqrt{|\Gamma_C|}\,\|\mathbf{v}\| \tag{4.33}$$

for a.e. $t \in I$, all $\eta \in H$, $\mathbf{v} \in V$, with $c > 0$. Hence, we deduce $H(B)$(b).

Subsequently, from the strong monotonicity of $\mathbf{v} \to B(t, \eta, \mathbf{v})$ and the inequality

$$\begin{aligned}&|\langle \partial J(t, z, \gamma \mathbf{v}_1) - \partial J(t, z, \gamma \mathbf{v}_2), \gamma(\mathbf{v}_1 - \mathbf{v}_2)\rangle_{X^* \times X}| \\ &\leq \int_{\Gamma_C} p_\nu(t, z(x))(\varrho(v_{1\nu}) - \varrho(v_{2\nu}))(v_{1\nu} - v_{2\nu})\, d\Gamma \\ &\leq p_\nu^* L_\varrho \int_{\Gamma_C} |v_{1\nu} - v_{2\nu}|^2\, d\Gamma \leq p_\nu^* L_\varrho \|\gamma\|^2 \|\mathbf{v}_1 - \mathbf{v}_2\|^2\end{aligned}$$

for all $\mathbf{v}_1$, $\mathbf{v}_2 \in V$, $z \in Z$, a.e. $t \in I$, we have

$$\begin{aligned}&\langle B(t, \eta, \mathbf{v}_1) - B(t, \eta, \mathbf{v}_2), \mathbf{v}_1 - \mathbf{v}_2\rangle + \langle \gamma^* \partial J(t, z, \gamma \mathbf{v}_1) - \gamma^* \partial J(t, z, \gamma \mathbf{v}_2), \mathbf{v}_1 - \mathbf{v}_2\rangle \\ &\geq (m_{\mathcal{E}} - p_\nu^* L_\varrho \|\gamma\|^2)\|\mathbf{v}_1 - \mathbf{v}_2\|^2.\end{aligned}$$

Hence, by (4.22), we obtain $H(B)$(c).

Next, let $(t, \eta, \boldsymbol{u}) \in I \times H \times V$, $\{t_n\} \subset I$, $t_n \to t$, $\{\eta_n\} \subset H$, $\eta_n \to \eta$ in H, $\{\boldsymbol{u}_n\} \subset V$, $\boldsymbol{u}_n \to \boldsymbol{u}$ in V. Then, for any $\mathbf{v} \in V$, we have

$$\begin{aligned}&|\langle B(t_n, \eta_n, \boldsymbol{u}_n) - B(t, \eta, \boldsymbol{u}), \mathbf{v}\rangle| = |\langle \mathcal{E}(t_n, \boldsymbol{\varepsilon}(\boldsymbol{u}_n)) - \mathcal{E}(t, \boldsymbol{\varepsilon}(\boldsymbol{u})), \boldsymbol{\varepsilon}(\mathbf{v})\rangle_{\mathcal{H}}| \\ &\leq \|\mathcal{E}(t_n, \boldsymbol{\varepsilon}(\boldsymbol{u}_n)) - \mathcal{E}(t, \boldsymbol{\varepsilon}(\boldsymbol{u}))\|_{\mathcal{H}} \|\boldsymbol{\varepsilon}(\mathbf{v})\|_{\mathcal{H}} = \|\mathcal{E}(t_n, \boldsymbol{\varepsilon}(\boldsymbol{u}_n)) - \mathcal{E}(t, \boldsymbol{\varepsilon}(\boldsymbol{u}))\|_{\mathcal{H}} \|\mathbf{v}\|,\end{aligned}$$

which together with $H(\mathcal{E})$(2) implies

$$\|B(t_n, \eta_n, \boldsymbol{u}_n) - B(t, \eta, \boldsymbol{u})\|_{V^*} \leq \|\mathcal{E}(t_n, \boldsymbol{\varepsilon}(\boldsymbol{u}_n)) - \mathcal{E}(t, \boldsymbol{\varepsilon}(\boldsymbol{u}))\|_{\mathcal{H}} \to 0.$$

Hence, since $(t, \eta, \boldsymbol{u}) \in I \times H \times V$ is arbitrary, we deduce $H(B)(d)$. Then, from condition $H(\mathcal{E})$(3), by the Hölder inequality, it is easy to find that

$$\|B(t, \eta, \boldsymbol{u})\|_{V^*} \leq \sqrt{2}\|\widetilde{a_0}(t, \cdot)\|_H + \sqrt{2}\widetilde{a_2}\|\boldsymbol{u}\|$$

for a.e. $t \in I$, all $\eta \in H$, $\boldsymbol{u} \in V$, Hence, $H(B)$(e) is verified with $c_0(t) = \sqrt{2}\|\widetilde{a_0}(t, \cdot)\|_H$, $c_1 = 0$, and $c_2 = \sqrt{2}\widetilde{a_2}$.

Subsequently, we use hypothesis $H(\mathcal{R})$ to deduce that

$$\|G(t)(\mathbf{v})\|_H^2 \leq \int_\Omega |R(\boldsymbol{x})\boldsymbol{\varepsilon}(\mathbf{v}(\boldsymbol{x}))|^2\, dx \leq c \int_\Omega \|\boldsymbol{\varepsilon}(\mathbf{v})\|_{\mathbb{S}^d}^2\, dx = c\,\|\boldsymbol{\varepsilon}(\mathbf{v})\|_{\mathcal{H}}^2 = c\,\|\mathbf{v}\|^2$$

for all $\mathbf{v} \in V$, a.e. $t \in I$ with a constant $c > 0$. Thus, it is clear that $H(G)$ holds.

Condition $H(J)$(a) is a consequence of the Fubini theorem. Also, similarly as in (4.32), one can easily obtain $H(J)$(c) while conditions $H(J)$(d)-(e) are a consequence of calculations proved in [14, Theorem 15]. It is clear that the set K of unilateral constraints defined by (4.19) satisfies $H(K)$.

We shall verify that the functional φ defined by (4.29) satisfies $H(\varphi)$. We can use the Fubini theorem again to deduce that $t \mapsto \varphi(t, z, \mathbf{v})$ is measurable for all $z \in Z$, $\mathbf{v} \in V$, i.e., $H(\varphi)$(a) holds. The condition $H(\varphi)$(b) is clearly satisfied. Since $\varphi(t, z, \mathbf{0}) = 0$ for a.e. $t \in I$, all $z \in Z$, the condition $H(\varphi)$(c) holds. From $H(p)$(3), $H(\mu)$ and Hölder's inequality, we get

$$|\varphi(t, z, v)| \leq p^* \int_{\Gamma_C} |v_\nu|\, d\Gamma + \mu^* p^* \int_{\Gamma_C} \|\mathbf{v}_\tau\|\, d\Gamma \leq p^* \sqrt{|\Gamma_C|}\|\mathbf{v}\|_{L^2(\Gamma_C)} + \mu^* p^* \sqrt{|\Gamma_C|}\|\mathbf{v}\|_{L^2(\Gamma_C)}.$$

Moreover, by Young's inequality, we have

$$|\varphi(t, z, v)| \leq \tfrac{1}{2}\,|\Gamma_C|\,\big(p^*\|\gamma\|(1 + \mu^*)\big)^2 + \tfrac{1}{2}\,\|\mathbf{v}\|_V^2.$$

Hence, $\varphi(t, z, v) \geq a_3(t) + b_3\|\mathbf{v}\|_V^2$ with $a_3 = -\frac{1}{2}\,|\Gamma_C|\,\big(p^*\|\gamma\|(1 + \mu^*)\big)^2$ and $b_3 = -\frac{1}{2}$. By the smallness condition (4.22), we deduce $H(\varphi)$(d). To verify $H(\varphi)$(e), let $\{t_n\} \subset I$, $t_n \to t$, $\{z_n\} \subset Z$,

$z_n \to z$ in Z, $\{\boldsymbol{u}_n\} \subset V$, $\boldsymbol{u}_n \rightharpoonup \boldsymbol{u}$ in V, and $\mathbf{v} \in V$. From $H(p)$ and $H(\mu)$, we have

$$\begin{aligned}
&\limsup\left(\varphi(t_n, z_n, \mathbf{v}) - \varphi(t_n, z_n, \boldsymbol{u}_n)\right) \le L_p \limsup \|z_n - z\|_Z(\|\mathbf{v}\|_X + \|\boldsymbol{u}_n\|_X) \\
&\quad + \limsup\Big(\int_{\Gamma_C} p(t_n, z)v_\nu \, d\Gamma\Big) + \limsup\Big(-\int_{\Gamma_C} p(t_n, z)u_{n\nu} \, d\Gamma\Big) \\
&+\mu^* L_p \limsup \|z_n - z\|_Z(\|\mathbf{v}\|_X + \|\boldsymbol{u}_n\|_X) \\
&\quad + \limsup\Big(\int_{\Gamma_C} \mu p(t_n, z)\|\mathbf{v}_\tau\| \, d\Gamma\Big) + \limsup\Big(-\int_{\Gamma_C} \mu p(t_n, z)\|\boldsymbol{u}_{n\tau}\| \, d\Gamma\Big) \\
&\qquad \le \limsup \varphi(t_n, z, \mathbf{v}) - \liminf \varphi(t_n, z, \boldsymbol{u}_n) \le \varphi(t, z, \mathbf{v}) - \varphi(t, z, \boldsymbol{u}),
\end{aligned}$$

which entails $H(\varphi)$(e). We will show $H(\varphi)$(f). Let $\boldsymbol{z} \in L^2(I;V)$, $\{\eta_n\} \subset L^2(I;Z)$, $\eta_n \to \eta$ in $L^2(I;Z)$, $\{\boldsymbol{u}_n\} \subset L^2(I;V)$ with $\boldsymbol{u}_n \rightharpoonup \boldsymbol{u}$ in $L^2(I;V)$. Then, from $H(p)$, $H(\mu)$ and Hölder's inequality, we have

$$\begin{aligned}
&\int_0^T \Big(\varphi(t, \eta_n(t), \boldsymbol{z}(t)) - \varphi(t, \eta_n(t), \boldsymbol{u}_n(t))\Big) \, dt \\
&= \int_0^T \int_{\Gamma_C} \Big([p(t, \eta_n(t)) - p(t, \eta(t))]z_\nu + p(t, \eta(t))z_\nu\Big) d\Gamma dt \\
&+ \int_0^T \int_{\Gamma_C} [p(t, \eta(t)) - p(t, \eta_n(t))]u_{n\nu} - p(t, \eta(t))u_{n\nu} \, d\Gamma dt \\
&\quad + \int_0^T \int_{\Gamma_C} \Big(\mu[p(t, \eta_n(t)) - p(t, \eta(t))]\|\boldsymbol{z}_\tau(t)\| + \mu p(t, \eta(t))\|\boldsymbol{z}_\tau(t)\|\Big) \, d\Gamma dt \\
&\qquad + \int_0^T \int_{\Gamma_C} \Big(\mu[p(t, \eta(t)) - p(t, \eta_n(t))]\|\boldsymbol{u}_{n\tau}(t)\| - \mu p(t, \eta(t))\|\boldsymbol{u}_{n\tau}(t)\|\Big) \, d\Gamma dt \\
&\le L_p \int_0^T \int_{\Gamma_C} |\eta_n(t) - \eta(t)| \, z_\nu \, d\Gamma dt + \int_0^T \int_{\Gamma_C} p(t, \eta(t))z_\nu \, d\Gamma dt \\
&\quad + L_p \int_0^T \int_{\Gamma_C} |\eta_n(t) - \eta(t)| \, u_{n\nu} \, d\Gamma dt - \int_0^T \int_{\Gamma_C} p(t, \eta(t))u_{n\nu} \, d\Gamma dt \\
&\quad + L_p\mu^* \int_0^T \int_{\Gamma_C} |\eta_n(t) - \eta(t)| \, \|\boldsymbol{z}_\tau(t)\| \, d\Gamma dt + \int_0^T \int_{\Gamma_C} \mu p(t, \eta(t))\|\boldsymbol{z}_\tau(t)\| \, d\Gamma dt \\
&\qquad + \mu^* L_p \int_0^T \int_{\Gamma_C} |\eta(t) - \eta_n(t)|\|\boldsymbol{u}_{n\tau}(t)\| \, d\Gamma dt - \int_0^T \int_{\Gamma_C} \mu p(t, \eta(t))\|\boldsymbol{u}_{n\tau}(t)\| \, d\Gamma dt \\
&\le L_p\|\eta_n - \eta\|_{L^2(I;Z)}\big(\|z_\nu\|_{L^2(I;L^2(\Gamma_C;\mathbb{R}^d))} + \|u_{n\nu}\|_{L^2(I;L^2(\Gamma_C;\mathbb{R}^d))}\big) \\
&\quad + L_p\mu^*\|\eta_n - \eta\|_{L^2(I;Z)}\big(\|\boldsymbol{z}\|_{L^2(I;L^2(\Gamma_C;\mathbb{R}^d))} + \|\boldsymbol{u}_n\|_{L^2(I;L^2(\Gamma_C;\mathbb{R}^d))}\big) + \Phi(\eta, \boldsymbol{z}) - \Phi(\eta, \boldsymbol{u}_n) \\
&\le c\,\|\eta_n - \eta\|_{L^2(I;Z)} + \Phi(\eta, \boldsymbol{z}) - \Phi(\eta, \boldsymbol{u}_n),
\end{aligned}$$

where $c > 0$ is a constant and the functional $\Phi\colon L^2(I;Z) \times L^2(I;V) \to \mathbb{R}$ is defined by

$$\Phi(z, \mathbf{v}) = \int_0^T \varphi(t, z(t), \mathbf{v}(t)) \, dt \;\text{ for }\; (z, \mathbf{v}) \in L^2(I;Z) \times L^2(I;V).$$

Note that, by Fatou's lemma, we know that $\Phi(z,\cdot)$ is convex and weakly lower semicontinuous on $L^2(I;V)$ for all $z \in L^2(I;Z)$. Using this property and passing to the upper limit, we have

$$\limsup \int_0^T \big(\varphi(t,\eta_n(t),\boldsymbol{z}(t)) - \varphi(t,\eta_n(t),\boldsymbol{u}_n(t))\big)\,dt \\ \le \int_0^T \big(\varphi(t,\eta(t),\boldsymbol{z}(t)) - \varphi(t,\eta(t),\boldsymbol{u}(t))\big)\,dt$$

which proves $H(\varphi)$(f).

It is clear that the normal trace $\gamma \in \mathcal{L}(V,X)$ is a compact operator. We refer to [20, Theorem 2.18, p.59] for the proof that the Nemitsky operator $\widetilde{\delta}\colon \mathbb{W} \to L^2(I;Z)$ corresponding to the trace operator satisfies $H(\delta,\gamma)$. The condition $H(g,w_0)$ is a consequence of definition (4.16) and hypothesis (H_0). Finally, since $J(t,z,\cdot)$ is Clarke regular for all $z \in Z$, a.e. $t \in I$, see (4.31), we conclude that Problems 4.4 and 4.2 are equivalent. Therefore, invoking Theorem 2.3, we know that Problem 4.2 has a solution $(\theta,\boldsymbol{u}) \in \mathbb{W} \times L^2(I;K)$. This completes the proof of the theorem. □

To conclude, we say that a quadruple of functions $(\boldsymbol{u},\boldsymbol{\sigma},\theta,\boldsymbol{q})$ which satisfies (4.3) and (4.4), and the equation and inequality in Problem 4.2 is called a weak solution to Problem 4.1. Under the hypotheses of Theorem 4.3, we deduce the following regularity of the solution of Problem 4.1:

$$\boldsymbol{u} \in L^2(I;K),\ \boldsymbol{\sigma} \in L^2(I;L^2(\Omega;\mathbb{S}^d)),\ \mathrm{Div}\,\boldsymbol{\sigma} \in L^\infty(I;L^2(\Omega;\mathbb{R}^d)),$$
$$\theta \in L^2(I;E),\ \theta' \in L^2(I;E^*),\ \boldsymbol{q} \in L^2(I;L^2(\Omega;\mathbb{R}^d)),\ \mathrm{div}\,\boldsymbol{q} \in L^2(I;L^2(\Omega)).$$

In what follows we deal with a class of optimal control problems for the differential variational inequality formulated in Problem 4.2. In the optimal control problem, one seeks to determine the parameters describing the density of volume force $\boldsymbol{g}_0(\lambda)$, the density of heat sources $\rho_1(\lambda)$ in Ω, the density of surface tractions $\boldsymbol{g}_N(\lambda)$ on the part Γ_N, and the initial temperature $\theta_0(\lambda)$ in Ω, in such a way that a given cost functional is minimized. In this way we are lead to a distributed optimal control, to a boundary control, and to a control in initial conditions, see the classical monographs [2, 10, 29, 30].

We apply Theorem 3.3 to deduce the following result.

Corollary 4.5. *Under the hypotheses of Theorem 4.3, $H(\mathcal{F})$ and $H(\Lambda_{ad})$, the control problem: Find $\lambda^* \in \Lambda_{ad}$ such that*

$$m(\lambda^*) = \inf\{m(\lambda) \mid \lambda \in \Lambda_{ad}\} \quad \textit{with} \quad m(\lambda) := \inf\{\mathcal{F}(\lambda,\theta,\boldsymbol{u}) \mid (\theta,\boldsymbol{u}) \in \mathcal{S}(\lambda)\}, \tag{4.34}$$

where $\mathcal{S}(\lambda) \subset \mathbb{W} \times L^2(I;V)$, for $\lambda \in \Lambda$, denotes the solution set to Problem 4.2, is solvable.

A variety of optimal control problems for the contact model in Problem 4.2 can be formulated as in (4.34). The cost functionals can be provided based on Example 3.4. We restrict to some particular simple choices met in applications.

The typical examples of the cost, met in applications, are based on the output least-squares formulation and are the following.

$$\mathcal{F}(\lambda,\theta,\boldsymbol{u}) = \int_I \Big(\rho_0 \int_{\Gamma_C} |u_\nu(t) - d_1(t)|^2\,d\Gamma + \rho_1 \int_{\Gamma_C} |\theta(t) - d_2(t)|^2\,d\Gamma\Big)\,dt,$$

where elements d_1, $d_2 \in L^2(I;L^2(\Gamma_C))$ are prescribed. With this choice of the objective functional, we look for a parameter $\lambda \in \Lambda_{ad}$ such that the corresponding penetration (displacement u_ν in the normal direction) of the elastic body over the total time interval is as close as possible to the "desired penetration" $d_1 = d_1(t)$, and the temperature θ lies as close as possible to the value $d_2 = d_2(t)$.

$$\mathcal{F}(\lambda,\theta,\boldsymbol{u}) = \int_I \Big(\rho_2 \int_\Omega \|\boldsymbol{\varepsilon}(\boldsymbol{u})\|^2_{\mathbb{S}^d}\,dx + \rho_3 \int_{\Gamma_N} |g_N(\lambda,t) - d_3(t)|^2\,d\Gamma\Big)\,dt,$$

where $d_3 \in L^2(I; L^2(\Gamma_N))$ is given. The first term corresponds to the minimization of the deformation of the body over the time interval, while in the second term we require that the surface traction density $g_N = g_N(\lambda, t)$ is close to an available element $d_3 = d_3(t)$.

$$\mathcal{F}(\lambda, \theta, \boldsymbol{u}) = \rho_4 \int_I \int_\Omega \|\boldsymbol{u}(\boldsymbol{x}, t) - \boldsymbol{u}_d\|^2_{\mathbb{R}^d} \, dx \, dt + \rho_5 \, \|\theta(T) - d_4\|^2_H$$

where $\boldsymbol{u}_d \in L^2(I; L^2(\Omega; \mathbb{R}^d))$ is given. With this cost functional we would like to minimize two components, $\boldsymbol{u}_d$ is the desired displacement profile, d_4 is the desired temperature one wishes to achieve at the final time $t = T$. In the above examples, a compromise policy between the various goals has to be found and the relative importance of each criterion with respect to the other is expressed by the choice of the weights $\rho_i \in L^\infty(I)_+$, $i = 0, \ldots, 5$.

5. Comments on further research

The research of this paper can be continued in several directions. First, we have touched perturbations neither in the constraint set nor the differential operators of the differential variational-hemivariational inequality. It would be interesting to study the stability result and optimal control when A, B and K depend on the control parameter as well. Second, another open research direction is to examine the stability with respect to the constraint set to differential quasi-variational-hemivariational problems, when the constraint set is solution dependent. Third, other optimal control problems which are of practical importance are worth to be investigated, for instance, time optimal control problems, maximum stay control problems, etc., see [3] for instance, as well as the variational-hemivariational inequalities involving history-dependent operators, see [27], and the references therein.

Finally, since the set of measured data in an optimal control problem comes, in general, from the experiments, it naturally fraught with errors and contain perturbations. Therefore, it would be interesting to study the variational sensitivity of control problems answering the question what happens to the set of optimal solutions to the control problem when the cost functional, involving noisy or contaminated data, is subjected to perturbations. For this issue some ideas of [9, 13] can be used.

Statements and Declarations

The authors declare that they have no conflict of interest, and the manuscript has no associated data.

Acknowledgments.

The project is supported by the European Union's Horizon 2020 Research and Innovation Programme under the Marie Skłodowska-Curie grant agreement No. 823731 CONMECH, the Ministry of Science and Higher Education of Republic of Poland under Grants Nos. 4004/GGPJII/H2020/2018/0 and 440328/PnH2/2019, and the National Science Centre of Poland under Project No. 2021/41/B/ST1/01636.

References

[1] E. J. Balder, Necessary and sufficient conditions for L^1-strong-weak lower semicontinuity of integral functionals, Nonlinear Analysis: Theory, Methods & Applications, 11(12):1399–1404, 1987.

[2] V. Barbu, Optimal Control of Variational Inequalities, Pitman Advanced Publishers, London, 1984.

[3] D. -L. Cai, S. Migórski, and Y. -B. Xiao, Optimal control of differential quasi-variational-hemivariational inequalities with applications, Science China-Mathematics (2023), in press.

[4] D. L. Cai and Y. B. Xiao, Regularization for a class of quasi-variational-hemivariational inequalities, Nonlinear Analysis: Real World Applications, 69: 103709, 2023.

[5] L. C. Ceng, C. F. Wen, Y. C. Liou, and J. C. Yao, A general class of differential hemivariational inequalities systems in reflexive Banach spaces, Mathematics, 9(24):3173, 2021.

[6] J. Gwinner, Three-field modelling of nonlinear nonsmooth boundary problems and stability of differential mixed variational inequalities, Abstract and Applied Analysis, 2013:108043, 2013.

[7] J. Gwinner, On a new class of differential variational inequalities and a stability result, Mathematical Programming, 139:205–221, 2013.

[8] W. Han, S. Migórski, and M. Sofonea, A class of variational-hemivariational inequalities with applications to frictional contact problems, SIAM Journal on Mathematical Analysis, 46(6):3891–3912, 2014.

[9] X. Li and Z. Liu, Sensitivity analysis of optimal control problems described by differential hemivariational inequalities, SIAM Journal on Control and Optimization, 56(5):3569–3597, 2018.

[10] J.-L. Lions, Optimal Control of Systems Governed by Partial Differential Equations, Springer-Verlag, Berlin, 1971.

[11] Z. Liu and S. Zeng, Differential variational inequalities in infinite Banach spaces, Acta Mathematica Scientia, 37(1):26–32, 2017.

[12] Z. Liu, S. Zeng, and D. Motreanu, Partial differential hemivariational inequalities, Advances in Nonlinear Analysis, 7:571–586, 2018.

[13] S. Migórski, Evolution hemivariational inequalities in infinite dimension and their control, Nonlinear Analysis: Theory, Methods & Applications, 47(1):101–112, 2001.

[14] S. Migórski, D.-L. Cai, and S. Dudek, Differential variational-hemivariational inequalities with application to contact mechanics, Nonlinear Analysis: Real World Applications, 71:103816, 2023.

[15] S. Migórski, D.-L. Cai, and Y.-B. Xiao, Inverse problems for constrained parabolic variational-hemivariational inequalities, Inverse Problems, 39(8):085012, 2023.

[16] S. Migórski and S. Dudek, Constrained evolutionary variational-hemivariational inequalities with application to fluid flow model, Communications in Nonlinear Science and Numerical Simulation, 127:107555, 2023.

[17] S. Migórski, W. Han, and S.D. Zeng, A new class of hyperbolic variational-hemivariational inequalities driven by non-linear evolution equations, European Journal of Applied Mathematics, 32(1):59–88, 2021.

[18] S. Migórski, Z.H. Liu, and S.D. Zeng, A class of history-dependent differential variational inequalities with application to contact problems, Optimization, 69(4):743–775, 2019.

[19] S. Migórski, A. Ochal, and M. Sofonea, Nonlinear Inclusions and Hemivariational Inequalities. Models and Analysis of Contact Problems, Advances in Mechanics and Mathematics, 26 Springer, New York, 2013.

[20] S. Migórski, A. Ochal, and M. Sofonea, Evolutionary inclusions and hemivariational inequalities, Chapter 2 in: Advances in Variational and Hemivariational Inequalities, W. Han et al. (eds), Advances in Mechanics and Mathematics,33 Springer, New York, 2015.

[21] S. Migórski and S. Zeng, A class of differential hemivariational inequalities in Banach spaces, Journal of Global Optimization, 72:761–779, 2018.

[22] J.S. Pang and D.E. Stewart, Differential variational inequalities, Mathematical Programming, 113(2):345–424, 2008.

[23] N. S. Papageorgiou and S.Th. Kyritsi-Yiallourou, Handbook of Applied Analysis, Springer, Dordrecht, Heidelberg, London, New York, 2009.

[24] M. Shillor, M. Sofonea, and J.J. Telega, Models and Analysis of Quasistatic Contact, Lect. Notes Physics 655, Springer, Berlin, Heidelberg, 2004.

[25] M. Sofonea, J. Bollati, and D. A. Tarzia, Optimal control of differential quasivariational inequalities with applications in contact mechanics, Journal of Mathematical Analysis and Applications, 493(2):124567, 2021.

[26] M. Sofonea and A. Matei, Mathematical Models in Contact Mechanics, London Mathematical Society Lecture Notes Series, 398, Cambridge University Press, Cambridge, 2012.

[27] M. Sofonea and S. Migórski, Variational–Hemivariational Inequalities with Applications, Chapman & Hall, CRC, Monographs and Research Notes in Math., Boca Raton, 2018.

[28] G. Tang, J. Cen, V. T. Nguyen, and S. D. Zeng, Differential variational-hemivariational inequalities: existence, uniqueness, stability, and convergence, Journal of Fixed Point Theory and Application, 22:83, 2020.

[29] D. Tiba, Optimal Control of Nonsmooth Distributed Parameter Systems, Lecture Notes in Mathematics, 1459, Springer, Berlin, 1990.

[30] F. Tröltzsch, Optimal Control of Partial Differential Equations: Theory, Methods and Applications, Graduate Studies in Mathematics, 112, American Mathematical Society, Providence, 2010.

[31] Y. -B. Xiao and M. Sofonea, On the optimal control of variational-hemivariational inequalities, Journal of Mathematical Analysis and Applications, 475(1):364–384. 2019