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TWO-STEP INERTIAL BREGMAN PROJECTION ITERATIVE ALGORITHM FOR SOLVING THE SPLIT FEASIBILITY PROBLEM

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Abstract. In this paper, we propose a new two-step inertial Bregman projection iterative algorithm by Halphern iterative method to solve the split feasibility problem in real Hilbert spaces. We give two selection strategies of stepsizes and prove that the proposed iterative sequence convergents strongly to solution of the split feasibility problem. Our results extend and improve the corresponding results.

Keywords. The split feasibility problem, Bregman projection, two-step inertial iterative algorithm, strong convergence, self-adaptive stepsize.

1. Introduction

The split feasibility problem (SFP) was firstly introduced by Censor and Elfving [5] for modelling some inverse problems. Since then, it has played an important role in many real-world application problems, such as signal processing, image reconstruction [4] and radiation therapy [6, 7]. Let H_1 and H_2 be real Hilbert spaces and let $A : H_1 \to H_2$ be a bounded linear operator. The SFP can mathematically be formulated as the problem of finding a point $\hat{\phi}$ with the property

$$\hat{\phi} \in C \text{ and } A\hat{\phi} \in Q, \tag{1.1}$$

where C and Q are nonempty closed convex subsets of H_1 and H_2, respectively. In particular, when $Q = \{b\}$, the SFP (1.1) becomes the following convex constrained linear inverse problem:

$$\hat{\phi} \in C \text{ and } A\hat{\phi} = b.$$

For solving the SFP (1.1), Byrne [4] introduced the following well-known CQ algorithm which generates iterative sequence $\{\phi_m\}$ by

$$\phi_{m+1} = P_C(I - \sigma_m A^*(I - P_Q)A)\phi_m, \tag{1.2}$$

where $\sigma_m \in (0, \frac{2}{\sigma})$ with σ being the spectral radius of the operator A^*A, P_C and P_Q are the projections onto C and Q, respectively.

We assume that the SFP (1.1) is consistent(i.e., problem (1.1) at least has a solution) and use Θ to denote the solution set of the SFP (1.1), i.e.,

$$\Theta = \{\hat{\phi} \in C : A\hat{\phi} \in Q\}.$$

We know that Θ is nonempty, closed and convex subset. And $\hat{\phi} \in \Theta$ if and only if $\hat{\phi}$ is the solution of the following fixed point equation:

$$\hat{\phi} = P_C(I - \sigma A^*(I - P_Q)A)\hat{\phi},$$

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where $\sigma > 0$. This implies that we can use fixed point algorithms (see [1, 8, 11, 21, 22]) to solve the SFP (1.1).

Let C be a nonempty closed convex subsets of H and let $f : C \to C$ be a contractive mapping. In [19], Moudafi proposed the following viscous iterative algorithm to solve the fixed point of the nonexpansive operator:

$$\phi_{m+1} = t_m f(\phi_m) + (1 - t_m) T\phi_m, \tag{1.3}$$

where $T : C \to C$ is a nonexpansive operator and $\{t_m\} \subset [0, 1]$. Moreover, we got the strong convergence result of the algorithm (1.3).

It is observed that, in the CQ algorithm (1.2), the stepsize σ_m depends on the bounded linear operator (matrix) norm $\|A\|$ (or the largest eigenvalue of A^*A). It is not always easy in practice to compute the matrix norm of A. To avoid this difficulty, there have been many self-adaptive algorithms where the stepsize dose not depend on the norm of the bounded linear operator A. In [16], Lopez et al. improved CQ algorithm (1.2) by selecting the following stepsize:

$$\sigma_m = \frac{\rho_m h(\phi_m)}{\|\nabla h(\phi_m)\|^2},$$

where $\inf_m \rho_m(4 - \rho_m) > 0$ and $h(\phi) = \frac{1}{2}\|(I - P_Q)A\phi\|^2$.

In optimization theory, the inertial technique is an important method to speed up the convergence rate. In [10], Dang proposed the inertial relaxed CQ algorithm for solving the SFP (1.1) in Hilbert spaces, which is formulated as

$$\begin{cases} \varphi_m = \phi_m + \varpi_m(\phi_m - \phi_{m-1}), \\ \phi_{m+1} = P_{C_m}(\varphi_m - \sigma_m A^*(I - P_{Q_m})A\varphi_m), \end{cases}$$

where $\sigma_m \in (0, \frac{2}{\sigma})$ for all $m \geq 1$ and σ is the spectral radius of the operator A^*A, $0 \leq \varpi_m \leq \bar{\varpi}_m$, and,

$$\bar{\varpi}_m = \min\{\varpi, \frac{1}{\max\{m^2\| \phi_{m-1} - \phi_m\|^2, m^2\| \phi_m - \phi_{m-1}\|\}}\},$$

In [15], Wang et al. gave the adaptive inertial relaxed CQ algorithm for solving the problem the SFP (1.1) in Hilbert spaces, which is generated as follows:

$$\begin{cases} \varphi_m = \phi_m + \varpi_m(\phi_m - \phi_{m-1}), \\ \phi_{m+1} = P_{\tilde{C}_m}(\varphi_m - \sigma_m A^*(I - P_{\tilde{Q}_m})A\varphi_m), \end{cases}$$

where

$$\tilde{C}_m = \{u \in H_1 : c(\varphi_m) + \langle \vartheta_m, u - \varphi_m \rangle + \frac{\kappa}{2}\|u - \varphi_m\|^2 \leq 0\},$$

$$\tilde{Q}_m = \{v \in H_2 : q(A\varphi_m) + \langle \eta_m, v - A\varphi_m \rangle + \frac{\beta}{2}\|v - A\varphi_m\|^2 \leq 0\},$$

and $\vartheta_m \in \partial c(\varphi_m)$, $\eta_m \in \partial q(A\varphi_m)$. Besides, $c : H_1 \to R$ and $q : H_2 \to R$ are convex and lower semi-continuous functions, $0 \leq \varpi_m \leq \bar{\varpi}_m$, $\bar{\varpi}_m$ is chosen by

$$\bar{\varpi}_m = \begin{cases} \min\{\varpi, \frac{\vartheta_m}{\max\{\| \phi_{m-1} - \phi_m\|^2, \| \phi_m - \phi_{m-1}\|\}}\}, & \phi_{m-1} \neq \phi_m, \\ \varpi, & \text{otherwise.} \end{cases}$$

The σ_m can be selected as follows:

$$\sigma_m = \begin{cases} \frac{\varsigma_m}{\|A^*(I - P_{Q_m})A\varphi_m)\|}, & (I - P_{Q_m})A\varphi_m \neq 0, \\ 0, & (I - P_{Q_m})A\varphi_m = 0, \end{cases}$$

where $\sum_{m=1}^{\infty} \varsigma_m = \infty$, $\sum_{m=1}^{\infty} \varsigma_m^2 < \infty$.

As we all know, the SFP (1.1) can be converted to the variational inequality problem. The variational inequality problem (VIP) is to find a point $\hat{\phi} \in C$ such that

$$\langle F\hat{\phi}, z - \hat{\phi}\rangle \geq 0, \ \forall z \in C, \tag{1.4}$$

where C is a nonempty closed and convex subset of H and $F : C \to H$ is an operator. Indeed, let

$$h(\phi) = \frac{1}{2}\|(I - P_Q)A\phi\|^2.$$

Then for the following convex minimization problem

$$\min_{\phi \in C} h(\phi),$$

the objective function h is differentiable and its gradient

$$\nabla h(\phi) = A^*(I - P_Q)A\phi.$$

So, the SFP (1.1) becomes the following VIP:

$$\langle A^*(I - P_Q)A\hat{\phi}, z - \hat{\phi}\rangle \geq 0, \ \forall z \in C.$$

It is known that the application of the Bregman distance is more flexibility than selection of projections. Many authors introduced Bregman projection methods to solve optimization problems. Let the function $e : H \to \mathbb{R}$ be α-strongly convex, Fréchet differentiable and bounded on bounded subsets of H. In [20], Sunthrayuth et al. proposed the following Bregman projection algorithm for solving the VIP (1.4):

$$\begin{cases} \varphi_m = \Pi_c^e(\nabla e)^{-1}(\nabla e(\phi_m) - \sigma_m F\phi_m), \\ \phi_{m+1} = (\nabla e)^{-1}(\nabla e(\varphi_m) - \sigma_m(F\varphi_m - F\phi_m)), \end{cases}$$

where $F : H \to H$ is pseudo-monotone, $\Pi_C^e(\phi)$ is the Bregman projection with respect to e of $\phi \in int(dom\ e)$, σ_{m+1} is chosen by

$$\sigma_{m+1} = \begin{cases} \min\{\mu\frac{\|\phi_m - \varphi_m\|^2 + \|\phi_{m+1} - \varphi_m\|^2}{2\langle F\phi_m - F\varphi_m, \phi_{m+1} - \varphi_m\rangle}, \sigma_m\}, & \text{if } \langle F\phi_m - F\varphi_m, \phi_{m+1} - \varphi_m\rangle > 0, \\ \sigma_m, & \text{otherwise} \end{cases}$$

and $\mu \in (0, \alpha)$. In [14], Sunthrayuth et al. proposed the following Bregman projection relaxed inertial subgradient extragradient algorithm for solving the VIP (1.4):

$$\begin{cases} u_m = (\nabla e)^*(\nabla e(\phi_m) + \varpi_m(\nabla e(\phi_{m-1}) - \nabla e(\phi_m))), \\ \varphi_m = \Pi_c^e(\nabla e)^*(\nabla e(u_m) - \sigma_m F u_m), \\ T_m = \{\phi \in H : \langle \nabla e(u_m) - \sigma_m F u_m - \nabla e(\varphi_m), \phi - \varphi_m\rangle \leq 0\}, \\ z_m = \Pi_{T_m}^e(\nabla e)^*(\nabla e(u_m) - \sigma_m F u_m), \\ \phi_{m+1} = (\nabla e)^*(\kappa_m \nabla e(\phi_1) + (1 - \kappa_m)(\nabla e(z_m)), \end{cases}$$

where $m \geq 1$, $\{\kappa_m\} \subset (0, 1)$, $0 \leq \varpi_m \leq \bar{\varpi}_m$, and,

$$\bar{\varpi}_m = \begin{cases} \min\{\varpi, \frac{\vartheta_m}{\| \nabla e(\phi_{m-1}) - \nabla e(\phi_m)\|}\}, & \text{if } \nabla e(\phi_{m-1}) \neq \nabla e(\phi_m), \\ \varpi, & \text{otherwise.} \end{cases}$$

Besides, $\sigma_m = \tau l^{s_m}$, with s_m is the smallest nonnegative integer satisfying

$$\tau l^{s_m}\|Fu_m - F\varphi_m\| \leq \mu\|u_m - \varphi_m\|,$$

and $F : H \to H$ is pseudo-monotone and uniformly continuous on H.

In [13], Hao and Zhao proposed the following Bregman projection algorithm for solving the SFP (1.1):

$$\phi_{m+1} = \Pi_c^e (\nabla e)^{-1}(\nabla e(\phi_m) - \sigma_m A^*(I - P_Q)A\phi_m), \tag{1.5}$$

where $\sigma_m > 0$. If $0 < \liminf_{m\to\infty} \sigma_m \le \limsup_{m\to\infty} \sigma_m < \frac{2\sigma}{\|A\|^2}$, then the sequence $\{\phi_m\}$ generated by algorithm (1.5) converges weakly to a solution of the SFP (1.1).

In this paper, two-step inertial Bregman projection iterative algorithms are given to solve the split feasibility problem. The paper is organized as follows. Some definitions and notions are presented in Section 2. New two-step inertial Bregman projection iterative algorithms are proposed in Section 3 and Section 4. Two selection strategies of stepsizes are constructed and the convergence results are obtained.

2. Preliminaries

Let H be a real Hilbert space, C be a nonempty closed convex subset of H. Throughout this paper, $\langle \cdot, \cdot \rangle$ denotes the inner product and $\| \cdot \|$ denotes the norm. $\to$ and $\rightharpoonup$ denote the strong convergence and weak convergence, respectively. $\omega_w(\phi_m)$ denotes the weak limit set of $\{\phi_m\}$. For each $\phi \in H$, the projection $P_C\phi$ from H on to C is the unique point in C such that

$$P_C\phi = \arg\min\{\|\phi - \varphi\| : \varphi \in C\}.$$

Lemma 2.1. *Given $\phi \in H$ and $z \in C$. Then $z = P_C\phi$ if and only if, for all $\varphi \in C$*

$$\langle \phi - z, \varphi - z \rangle \le 0. \tag{2.1}$$

Definition 2.2. An operator $T : H \to H$ is said to be

(i) *nonexpansive* if

$$\|T\phi - T\varphi\| \le \|\phi - \varphi\|,$$

for all $\phi,\ \varphi \in H$;

(ii) *firmly nonexpansive* if $2T - I$ is nonexpansive or, equivalently,

$$\langle \phi - \varphi, T\phi - T\varphi \rangle \ge \|T\phi - T\varphi\|^2,$$

for all $\phi,\ \varphi \in H$.

It is well known that both P_C and $I - P_C$ are firmly nonexpansive.

Definition 2.3. The Bregman bifunction $B_e : dom\ e \times int(dom\ e) \to [0, \infty)$ corresponding to the convex and differentiable function e with its gradient ∇e is defined by

$$B_e(\phi, \varphi) = e(\phi) - e(\varphi) - \langle \nabla e(\varphi), \phi - \varphi \rangle.$$

The Bregman distance has the following important property called the three-point identity: for any $\phi \in dom\ e$ and $\varphi, z \in int(dom\ e)$

$$B_e(\phi, \varphi) = B_e(\phi, z) - B_e(\varphi, z) + \langle \nabla e(z) - \nabla e(\varphi), \phi - \varphi \rangle. \tag{2.2}$$

The Bregman projection with respect to e of $\phi \in int(dom\ e)$ is denoted by $\Pi_c^e(\phi)$ and

$$\Pi_c^e(\phi) = \arg\min\{B_e(\varphi, \phi) : \varphi \in C\}.$$

Moreover, Π_c^e has the following property ([14]): for each $\varphi \in H$,

$$B_e(\phi, \Pi_c^e(\varphi)) + B_e(\Pi_c^e(\varphi), \varphi) \le B_e(\phi, \varphi),\ \forall \phi \in C. \tag{2.3}$$

Definition 2.4. The conjugate function of e is the function e^* on H defined by

$$e^*(\phi^*) := \sup_{\phi \in H}\{\langle \phi^*, \phi\rangle - e(\phi)\},\ \forall(\phi, \phi^*) \in H \times H.$$

Definition 2.5. Let $e : H \to \mathbb{R}$ be a Legendre function. Let $G_e : H \times H \to [0, \infty)$ associated with e be defined by

$$G_e(\phi, \phi^*) := e(\phi) - \langle \phi^*, \phi\rangle + e^*(\phi^*),\ \forall(\phi, \phi^*) \in H \times H.$$

We know the following properties (see [18]):

(1) G_e is nonnegative and convex in the second variable;
(2) $G_e(\phi, \phi^*) = B_e(\phi, \nabla e^*(\phi^*)),\ \forall(\phi, \phi^*) \in H \times H$;
(3) $G_e(\phi, \phi^*) + \langle \varphi^*, \nabla e^*(\phi^*) - \phi\rangle \leq G_e(\phi^* + \varphi^*, \phi),\ \forall(\phi, \phi^*) \in H \times H$ and $\varphi^* \in H$.

Since G_e is convex in the second variable, it follows that, for all $z \in H$,

$$B_e(z, \nabla e^*(\Sigma_{i=1}^N t_i \nabla e(\phi_i)) \leq \Sigma_{i=1}^N t_i B_e(z, \phi_i), \tag{2.4}$$

where $\{\phi_i\}_{i=1}^N \subset H$ and $\{t_i\}_{i=1}^N \subset [0, 1]$ with $\sum_{i=1}^N t_i = 1$.

Lemma 2.6. *([3]) For each $\phi \in H$, $z = \Pi_c^e(\phi)$ if and only if, for all $\varphi \in C$,*

$$\langle \nabla e(z) - \nabla e(\phi), \varphi - z\rangle \geq 0. \tag{2.5}$$

Definition 2.7. A convex and differentiable function e is said to be α-strongly convex if there exists a constant $\alpha > 0$ such that

$$e(\phi) \geq e(\varphi) + \langle \nabla e(\varphi), \phi - \varphi\rangle + \frac{\alpha}{2}\|\phi - \varphi\|^2.$$

for any $\phi \in dom\ e$ and $\varphi \in int(dom\ e)$.

If the function e is α-strongly convex, from the definition of the Bregman distance, we get the following inequality:

$$B_e(\phi, \varphi) \geq \frac{\alpha}{2}\|\phi - \varphi\|^2. \tag{2.6}$$

Moreover, if e is assumed to be α-strongly convex, Fréchet differentiable and bounded on bounded subsets of H, then for any two sequences $\{\phi_m\}$ and $\{\varphi_m\}$ in H, we have

$$\lim_{m\to\infty} B_e(\phi_m, \varphi_m) = 0 \Rightarrow \lim_{m\to\infty}\|\phi_m - \varphi_m\| = 0 \Rightarrow \lim_{m\to\infty}\|\nabla e(\phi_m) - \nabla e(\varphi_m)\| = 0. \tag{2.7}$$

Definition 2.8. An operator $T : H \to H$ is said to be demiclosed at origin if, for any sequence $\{\phi_m\}$ which weakly converges to ϕ, the sequence $\{T\phi_m\}$ strongly converges to 0, then $T\phi = 0$.

Lemma 2.9. *Let E be a uniformly convex Banach space, K be a nonempty closed convex subset of E and $T : K \to K$ be a nonexpansive mapping. Then $I - T$ is demiclosed at origin.*

Lemma 2.10. *([17]) Let $\{a_m\}$ be a nonnegative real sequence such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in N$, Then, there exists an increasing sequence $\{m_k\} \subset \mathbb{N}$ such that $\lim_{k\to\infty} m_k = \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:*

$$\kappa_{m_k} \leq \kappa_{m_k+1} \text{ and } \kappa_k \leq \kappa_{m_k+1}.$$

In fact, $m_k = \max\{j \leq k : a_j \leq a_{j+1}\}$.

Lemma 2.11. *([9]) Let* $\{a_m\}$, $\{\tau_m\}$, $\{\delta_m\}$ *and* $\{t_m\}$ *be nonnegative real sequences such that* $\tau_m \subset [0, 1/2]$, $\limsup_{m\to\infty} s_m \leq 0$, $\sum_{n=n_0}^{\infty} \delta_m < \infty$, $\sum_{n=n_0}^{\infty} t_m = \infty$ *and, for each* $m \geq n_0$, *(where* n_0 *is a positive integer,*

$$a_{m+1} \leq (1 - t_m - \tau_m)a_m + \tau_m a_{m-1} + t_m s_m + \delta_m.$$

Then, $\lim_{m\to\infty} a_m = 0$.

3. Main Results: TWO-STEP INERTIAL BREGMAN PROJECTION ITERATIVE ALGORITHM

In this paper, the following assumptions are holds:

(A1) The function $e : H_1 \to \mathbb{R}$ is α-strongly convex, Legendre, uniformly Fréchet differentiable, and bounded on bounded of H_1;
(A2) Bounded linear operator $A \neq \mathbf{0}$;
(A3) The solution set Θ of the SFP (1.1) is nonempty.

Algorithm 3.1. *(two-step inertial Bregman projection iterative algorithm for solving the SFP (1.1)) Choose two sequences* $\{\kappa_m\} \subset (0, 1)$ *and* $\{\vartheta_m\} \subset (0, +\infty)$*, where*$\{\vartheta_m\}$ *and* $\{\kappa_m\}$ *satisfy*

$$\sum_{m=1}^{\infty} \vartheta_m < \infty, \quad \lim_{m\to\infty} \kappa_m = 0, \quad \sum_{m=1}^{\infty} \kappa_m = \infty \tag{3.1}$$

and

$$\lim_{m\to\infty} \frac{\vartheta_m}{\kappa_m} = 0.$$

Given $\varpi \in [0, 1/2]$. *Select arbitrary starting points* ϕ_0, ϕ_1, $\phi_2 \in H_1$.

Iterative step: *For* $m \geq 2$, *choose* ϖ_m *and* τ_m *such that* $0 < \tau_m \leq \varpi_m < 1, 0 \leq \max\{\varpi_m, \tau_m\} \leq \bar{\varpi}_m$, *where*

$$\bar{\varpi}_m = \begin{cases} \min\{\varpi, \frac{\vartheta_m}{\|\nabla e(\phi_{m-1}) - \nabla e(\phi_m)\| + \|\nabla e(\phi_{m-2}) - \nabla e(\phi_{m-1})\|}\}, & \nabla e(\phi_{m-1}) \neq \nabla e(\phi_m) \\ & \text{or } \nabla e(\phi_{m-2}) \neq \nabla e(\phi_{m-1}), \\ \varpi, & \text{otherwise.} \end{cases} \tag{3.2}$$

Compute

$$\begin{cases} \varphi_m = (\nabla e)^*(\nabla e(\phi_m) + \varpi_m(\nabla e(\phi_{m-1}) - \nabla e(\phi_m)) + \tau_m(\nabla e(\phi_{m-2}) - \nabla e(\phi_{m-1}))), \\ \psi_m = \Pi_c^e(\nabla e)^*(\nabla e(\varphi_m) - \sigma_m A^*(I - P_Q)A\varphi_m), \\ \phi_{m+1} = (\nabla e)^*(\kappa_m \nabla e(\phi_1) + (1 - \kappa_m)(\nabla e(\psi_m))), \end{cases} \tag{3.3}$$

where $\sigma_m > 0$.

Remark 3.2. From (3.2), it is easy to see that $\max\{\varpi_m, \tau_m\}(\|\nabla e(\phi_{m-1}) - \nabla e(\phi_m)\| + \|\nabla e(\phi_{m-2}) - \nabla e(\phi_{m-1})\|) \leq \vartheta_m$ for all $m \in \mathbb{N}$. Since $\lim_{m\to\infty} \frac{\vartheta_m}{\kappa_m} = 0$, it follows that

$$\lim_{m\to\infty} \frac{\max\{\varpi_m, \tau_m\}(\|\nabla e(\phi_{m-1}) - \nabla e(\phi_m)\| + \|\nabla e(\phi_{m-2}) - \nabla e(\phi_{m-1})\|)}{\kappa_m} \leq \lim_{m\to\infty} \frac{\vartheta_m}{\kappa_m} = 0.$$

Theorem 3.3. *If* $0 < \liminf_{m\to\infty} \sigma_m \leq \limsup_{m\to\infty} \sigma_m < \frac{2\alpha}{\|A\|^2}$, *then the sequence* $\{\phi_m\}$ *generated by Algorithm 3.1 converges strongly to* $z \in \Theta$, *where* $z = \Pi_\Theta^e(\phi_1)$.

Proof. First, we show that $\{\phi_m\}$ is bounded. Let $r \in \Theta$ and from (2.3)we can obtain,

$$\begin{aligned}
B_e(r,\psi_m) =& B_e(r,\Pi_c^e(\nabla e)^*(\nabla e(\varphi_m)-\sigma_m A^*(I-P_Q)A\varphi_m))\\
\le& B_e(r,(\nabla e)^*(\nabla e(\varphi_m)-\sigma_m A^*(I-P_Q)A\varphi_m))\\
&- B_e(\psi_m,(\nabla e)^*(\nabla e(\varphi_m)-\sigma_m A^*(I-P_Q)A\varphi_m))\\
=& G_e(r,(\nabla e)^*(\nabla e(\varphi_m)-\sigma_m A^*(I-P_Q)A\varphi_m))\\
&- G_e(\psi_m,(\nabla e)^*(\nabla e(\varphi_m)-\sigma_m A^*(I-P_Q)A\varphi_m))\\
=& e(r)-e(\psi_m)-\langle\nabla e(\varphi_m)-\sigma_m A^*(I-P_Q)A\varphi_m, r\rangle\\
&+ e^*(\nabla e(\varphi_m)-\sigma_m A^*(I-P_Q)A\varphi_m))+\langle\nabla e(\varphi_m)-\sigma_m A^*(I-P_Q)A\varphi_m),\psi_m\rangle\\
&- e^*(\nabla e(\varphi_m)-\sigma_m A^*(I-P_Q)A\varphi_m))\\
=& e(r)-\langle\nabla e(\varphi_m),r\rangle+\sigma_m\langle A^*(I-P_Q)A\varphi_m,r\rangle-e(\psi_m)+\langle\nabla e(\varphi_m),\psi_m\rangle\\
&- \sigma_m\langle A^*(I-P_Q)A\varphi_m,\psi_m\rangle\\
=& e(r)-e(\psi_m)-e(\varphi_m)+e(\varphi_m)-\langle\nabla e(\varphi_m),r-\varphi_m\rangle-\langle\nabla e(\varphi_m),\varphi_m-\psi_m\rangle\\
&+ \sigma_m\langle\, A^*(I-P_Q)A\varphi_m,r-\psi_m\rangle\\
=& B_e(r,\varphi_m)-B_e(\psi_m,\varphi_m)+\sigma_m\langle\, A^*(I-P_Q)A\varphi_m,r-\psi_m\rangle\\
=& B_e(r,\varphi_m)-B_e(\psi_m,\varphi_m)+\sigma_m\langle\, A^*(I-P_Q)A\varphi_m,r-\varphi_m\rangle\\
&+ \sigma_m\langle\, A^*(I-P_Q)A\varphi_m,\varphi_m-\psi_m\rangle.
\end{aligned} \tag{3.4}$$

Since $Ar \in Q$, from (2.1), we have

$$\langle A\varphi_m - P_Q A\varphi_m, Ar - P_Q A\varphi_m\rangle \le 0.$$

So

$$\begin{aligned}
&\sigma_m\langle\, A^*(I-P_Q)A\varphi_m, r-\varphi_m\rangle\\
=&\sigma_m\langle(I-P_Q)A\varphi_m, Ar-A\varphi_m\rangle\\
=&\sigma_m\langle(I-P_Q)A\varphi_m, Ar-P_QA\varphi_m\rangle+\sigma_m\langle(I-P_Q)A\varphi_m, R_QA\varphi_m-A\varphi_m\rangle\\
\le& -\sigma_m\|(I-P_Q)A\varphi_m\|^2.
\end{aligned} \tag{3.5}$$

For all $\mu > 0$, we have

$$\begin{aligned}
\sigma_m\langle\, A^*(I-P_Q)A\varphi_m,\varphi_m-\psi_m\rangle &\le \sigma_m\|A^*(I-P_Q)A\varphi_m\|\cdot\|\varphi_m-\psi_m\|\\
&\le \frac{\mu\sigma_m}{2}\|A^*(I-P_Q)A\varphi_m\|^2+\frac{\sigma_m}{2\mu}\|\varphi_m-\psi_m\|^2.
\end{aligned} \tag{3.6}$$

Substituting (3.5) and (3.6) into (3.4), the following result is hold:

$$\begin{aligned}
B_e(r,\psi_m) \le& B_e(r,\varphi_m)-B_e(\psi_m,\varphi_m)-\sigma_m\|(I-P_Q)A\varphi_m\|^2\\
&+\frac{\mu\sigma_m}{2}\|A^*(I-P_Q)A\varphi_m\|^2+\frac{\sigma_m}{2\mu}\|\varphi_m-\psi_m\|^2.
\end{aligned}$$

Using (2.6), it is easy to see

$$\begin{aligned}
B_e(r,\psi_m) \le& B_e(r,\varphi_m)-B_e(\psi_m,\varphi_m)-\sigma_m(1-\frac{\mu}{2}\|A\|^2)\|(I-P_Q)A\varphi_m\|^2+\frac{\sigma_m}{2\mu}\frac{2}{\alpha}B_e(\psi_m,\varphi_m)\\
=& B_e(r,\varphi_m)-(1-\frac{\sigma_m}{\mu\alpha})B_e(\psi_m,\varphi_m)-\sigma_m(1-\frac{\mu}{2}\|A\|^2)\|(I-P_Q)A\varphi_m\|^2.
\end{aligned} \tag{3.7}$$

Since $0 < \liminf_{m\to\infty}\sigma_m \le \limsup_{m\to\infty}\sigma_m < \frac{2\alpha}{\|A\|^2}$, we can take $\mu > 0$ such that

$$\frac{1}{\alpha}\limsup_{m\to\infty}\sigma_m < \mu < \frac{2}{\|A\|^2}.$$

Then

$$\liminf_{m\to\infty} \sigma_m(1-\frac{\mu}{2}\|A\|^2)>0 \tag{3.8}$$

and

$$\liminf_{m\to\infty}(1-\frac{\sigma_m}{\mu\alpha})>0. \tag{3.9}$$

From (3.8) and (3.9), we obtain

$$B_e(r,\psi_m)\leq B_e(r,\varphi_m). \tag{3.10}$$

From (2.4) and (3.3), it follows that

$$\begin{aligned} B_e(r,\varphi_m) &= B_e(r,(\nabla e)^*(\nabla e(\phi_m)+\varpi_m(\nabla e(\phi_{m-1})-\nabla e(\phi_m))+\tau_m(\nabla e(\phi_{m-2})-\nabla e(\phi_{m-1}))) \\ &= B_e(r,(\nabla e)^*((1-\varpi_m)\nabla e(\phi_m)+(\varpi_m-\tau_m)\nabla e(\phi_{m-1})+\tau_m\nabla e(\phi_{m-2}))) \\ &\leq (1-\varpi_m)B_e(r,\phi_m)+(\varpi_m-\tau_m)B_e(r,\phi_{m-1})+\tau_m B_e(r,\phi_{m-2}) \end{aligned} \tag{3.11}$$

and so, from (3.10) and (3.11),

$$\begin{aligned} &B_e(r,\phi_{m+1}) \\ =&B_e(r,(\nabla e)^*(\kappa_m\nabla e(\phi_1)+(1-\kappa_m)(\nabla e(\psi_m))) \\ \leq&\kappa_m B_e(r,\phi_1)+(1-\kappa_m)B_e(r,\psi_m) \\ \leq&\kappa_m B_e(r,\phi_1)+(1-\kappa_m)B_e(r,\varphi_m) \\ \leq&\kappa_m B_e(r,\phi_1)+(1-\kappa_m)(1-\varpi_m)B_e(r,\phi_m)+(1-\kappa_m)(\varpi_m-\tau_m)B_e(r,\phi_{m-1}) \\ &+(1-\kappa_m)\tau_m B_e(r,\phi_{m-2}) \\ \leq&\kappa_m B_e(r,\phi_1)+(1-\kappa_m)\max\{(1-\varpi_m)B_e(r,\phi_m),(\varpi_m-\tau_m)B_e(r,\phi_{m-1}),\tau_m B_e(r,\phi_{m-2})\} \\ \leq&\ldots\leq\max\{B_e(r,\phi_1),B_e(r,\phi_0)\}. \end{aligned} \tag{3.12}$$

Hence $\{B_e(r,\phi_m)\}$ is bounded. Applying (2.6), we have $\{\phi_m\}$ is bounded. So, $\{\varphi_m\}$ and $\{\psi_m\}$ are bounded. Assume that $z=\Pi^e_\Theta(\phi_1)$. From (3.7) and (3.11), we have

$$\begin{aligned} B_e(z,\phi_{m+1}) =&B_e(z,(\nabla e)^*(\kappa_m\nabla e(\phi_1)+(1-\kappa_m)(\nabla e(\psi_m))) \\ \leq&\kappa_m B_e(z,\phi_1)+(1-\kappa_m)B_e(z,\psi_m) \\ \leq&\kappa_m B_e(z,\phi_1)+(1-\kappa_m)B_e(z,\varphi_m)-(1-\kappa_m)(1-\frac{\sigma_m}{\mu\alpha})B_e(\psi_m,\varphi_m) \\ &-\sigma_m(1-\kappa_m)(1-\frac{\mu}{2}\|A\|^2)\|(I-P_Q)A\varphi_m\|^2 \\ \leq&\kappa_m B_e(z,\phi_1)+(1-\kappa_m)(1-\varpi_m)B_e(z,\phi_m)+(1-\kappa_m)(\varpi_m-\tau_m)B_e(z,\phi_{m-1}) \\ &+(1-\kappa_m)\tau_m B_e(z,\phi_{m-2})-(1-\kappa_m)(1-\frac{\sigma_m}{\mu\alpha})B_e(\psi_m,\varphi_m) \\ &-\sigma_m(1-\kappa_m)(1-\frac{\mu}{2}\|A\|^2)\|(I-P_Q)A\varphi_m\|^2. \end{aligned}$$

So

$$
\begin{aligned}
&(1-\kappa_m)(1-\frac{\sigma_m}{\mu\alpha})B_e(\psi_m,\varphi_m)+\sigma_m(1-\kappa_m)(1-\frac{\mu}{2}\|A\|^2)\|(I-P_Q)A\varphi_m\|^2\\
\leq&\kappa_m B_e(z,\phi_1)-B_e(z,\phi_{m+1})+(1-\kappa_m)(1-\varpi_m)B_e(z,\phi_m)+(1-\kappa_m)(\varpi_m-\tau_m)B_e(z,\phi_{m-1})\\
&+(1-\kappa_m)\tau_m B_e(z,\phi_{m-2})\\
=&\kappa_m B_e(z,\phi_1)-B_e(z,\phi_{m+1})+(1-\kappa_m)B_e(z,\phi_m)-(1-\kappa_m)\varpi_m B_e(z,\phi_m)\\
&+(1-\kappa_m)(\varpi_m-\tau_m)B_e(z,\phi_{m-1})+(1-\kappa_m)\tau_m B_e(z,\phi_{m-2})\\
=&B_e(z,\phi_m)-B_e(z,\phi_{m+1})+(1-\kappa_m)\varpi_m(B_e(z,\phi_{m-1})-B_e(z,\phi_m))\\
&+(1-\kappa_m)\tau_m(B_e(z,\phi_{m-2})-B_e(z,\phi_{m-1}))+\kappa_m K,
\end{aligned}
\tag{3.13}
$$

where $K=\sup_{m\geq 1}\{|B_e(z,\phi_1)-B_e(z,\phi_m)|\}$.

Now, two possible cases are considered to prove $\lim_{m\to\infty}B_e(z,\phi_m)=0$.

Case1. There exists $N\in\mathbb{N}$ such that $B_e(z,\phi_{m+1})\leq B_e(z,\phi_m)$ for all $m\geq N$. Then the sequence $\{B_e(z,\phi_m)\}$ is convergent and

$$
\begin{aligned}
&\lim_{m\to\infty}(B_e(z,\phi_m)-B_e(z,\phi_{m+1}))\\
=&\lim_{m\to\infty}(B_e(z,\phi_{m-1})-B_e(z,\phi_m))=\lim_{m\to\infty}(B_e(z,\phi_{m-2})-B_e(z,\phi_{m-1}))=0.
\end{aligned}
$$

It follows from (3.13) that $\lim_{m\to\infty}B_e(\psi_m,\varphi_m)=\lim_{m\to\infty}\|(I-P_Q)A\varphi_m\|=0$. From (2.7) we have

$$
\lim_{m\to\infty}\|\nabla e(\psi_m)-\nabla e(\varphi_m)\|=0. \tag{3.14}
$$

Note that

$$
\begin{aligned}
\|\nabla e(\phi_{m+1})-\nabla e(\varphi_m)\|&\leq\|\nabla e(\phi_{m+1})-\nabla e(\psi_m)\|+\|\nabla e(\psi_m)-\nabla e(\varphi_m)\|\\
&=\kappa_m\|\nabla e(\phi_1)-\nabla e(\psi_m)\|+\|\nabla e(\psi_m)-\nabla e(\varphi_m)\|.
\end{aligned}
$$

By (3.14), it can be obtained that

$$
\lim_{m\to\infty}\|\nabla e(\phi_{m+1})-\nabla e(\varphi_m)\|=0. \tag{3.15}
$$

Since $\kappa_m\in(0,1)$, so

$$
\begin{aligned}
&\max\{\varpi_m,\tau_m\}(\|\nabla e(\phi_{m-1})-\nabla e(\phi_m)\|+\|\nabla e(\phi_{m-2})-\nabla e(\phi_{m-1})\|)\\
\leq&\frac{\max\{\varpi_m,\tau_m\}(\|\nabla e(\phi_{m-1})-\nabla e(\phi_m)\|+\|\nabla e(\phi_{m-2})-\nabla e(\phi_{m-1})\|)}{\kappa_m},
\end{aligned}
$$

which implies that

$$
\lim_{m\to\infty}\max\{\varpi_m,\tau_m\}(\|\nabla e(\phi_{m-1})-\nabla e(\phi_m)\|+\|\nabla e(\phi_{m-2})-\nabla e(\phi_{m-1})\|)=0.
$$

Since

$$
\begin{aligned}
&\|\nabla e(\varphi_m)-\nabla e(\phi_m)\|\\
\leq&\varpi_m\|\nabla e(\phi_{m-1})-\nabla e(\phi_m)\|+\tau_m\|\nabla e(\phi_{m-2})-\nabla e(\phi_{m-1})\|\\
\leq&\max\{\varpi_m,\tau_m\}(\|\nabla e(\phi_{m-1})-\nabla e(\phi_m)\|+\|\nabla e(\phi_{m-2})-\nabla e(\phi_{m-1})\|),
\end{aligned}
$$

we have

$$
\lim_{m\to\infty}\|\nabla e(\varphi_m)-\nabla e(\phi_m)\|=0. \tag{3.16}
$$

It follows from (3.15), (3) and

$$
\|\nabla e(\phi_{m+1})-\nabla e(\phi_m)\|\leq\|\nabla e(\phi_{m+1})-\nabla e(\varphi_m)\|+\|\nabla e(\varphi_m)-\nabla e(\phi_m)\|
$$

that $\lim_{m\to\infty}\|\nabla e(\phi_{m+1})-\nabla e(\phi_m)\|=0$. We can obtain that

$$
\lim_{m\to\infty}\|\phi_{m+1}-\phi_m\|=0. \tag{3.17}
$$

In fact, $\omega_w(\phi_m) \neq \emptyset$ since the sequence $\{\phi_m\}$ is bounded. Taking $\hat{\phi} \in \omega_w(\phi_m)$, we know that there exists a subsequence $\{\phi_{n_k}\}$ of $\{\phi_m\}$ and $\phi_{n_k} \rightharpoonup \hat{\phi} \in C$,

$$\limsup_{m\to\infty} \langle \nabla e(\phi_1) - \nabla e(z), \phi_m - z \rangle = \limsup_{k\to\infty} \langle \nabla e(\phi_1) - \nabla e(z), \phi_{n_k} - z \rangle.$$

From (3.16), we have $\varphi_{n_k} \rightharpoonup \hat{\phi}$. Then $A\varphi_{n_k} \rightharpoonup A\hat{\phi}$ as $k \to \infty$. We know P_Q is nonexpansive, by Lemma 2.9 and $\lim_{m\to\infty} \|(I - P_Q)A\varphi_m\| = 0$, $A\hat{\phi}$ is the fixed point of P_Q, so $A\hat{\phi} \in Q$ and $\hat{\phi} \in \Theta$. Then, from (2.5), we obtain

$$\limsup_{m\to\infty} \langle \nabla e(\phi_1) - \nabla e(z), \phi_m - z \rangle = \langle \nabla e(\phi_1) - \nabla e(z), \hat{\phi} - z \rangle \leq 0.$$

It follows from (3.17) that

$$\limsup_{m\to\infty} \langle \nabla e(\phi_1) - \nabla e(z), \phi_{m+1} - z \rangle \leq 0. \tag{3.18}$$

By the properties of G_e, we get

$$\begin{aligned} B_e(z, \phi_{m+1}) =& G_e(z, \kappa_m \nabla e(\phi_1) + (1-\kappa_m)(\nabla e(\psi_m))) \\ \leq& G_e(z, \kappa_m \nabla e(\phi_1) + (1-\kappa_m)\nabla e(\psi_m) - \kappa_m(\nabla e(\phi_1) - \nabla e(z))) \\ &+ \kappa_m \langle \nabla e(\phi_1) - \nabla e(z), \phi_{m+1} - z \rangle \\ =& G_e(z, \kappa_m \nabla e(z) + (1-\kappa_m)(\nabla e(\psi_m)) + \kappa_m \langle \nabla e(\phi_1) - \nabla e(z), \phi_{m+1} - z \rangle \\ =& B_e(z, (\nabla e)^*(\kappa_m \nabla e(z) + (1-\kappa_m)(\nabla e(\psi_m))) + \kappa_m \langle \nabla e(\phi_1) - \nabla e(z), \phi_{m+1} - z \rangle \\ \leq& \kappa_m B_e(z, z) + (1-\kappa_m) B_e(z, \psi_m) + \kappa_m \langle \nabla e(\phi_1) - \nabla e(z), \phi_{m+1} - z \rangle \\ \leq& (1-\kappa_m)((1-\varpi_m) B_e(z, \phi_m) + (\varpi_m - \tau_m) B_e(z, \phi_{m-1}) + \tau_m B_e(z, \phi_{m-2})) \\ &+ \kappa_m \langle \nabla e(\phi_1) - \nabla e(z), \phi_{m+1} - z \rangle \\ =& (1 - \kappa_m - (1-\kappa_m)\varpi_m) B_e(z, \phi_m) + (1-\kappa_m)\varpi_m B_e(z, \phi_{m-1}) \\ &+ (1-\kappa_m)\tau_m (B_e(z, \phi_{m-2}) - B_e(z, \phi_{m-1})) + \kappa_m \langle \nabla e(\phi_1) - \nabla e(z), \phi_{m+1} - z \rangle. \end{aligned} \tag{3.19}$$

Using Lemma 2.11, (3.18) and $\lim_{m\to\infty}(B_e(z, \phi_{m-2}) - B_e(z, \phi_{m-1})) = 0$, we obtain

$$\lim_{m\to\infty} B_e(z, \phi_m) = 0.$$

So $\phi_m \to z (m \to \infty)$.

Case2. There exists a subsequence $\{B_e(z, \phi_{m_i})\}$ of $\{B_e(z, \phi_m)\}$ such that

$$B_e(z, \phi_{m_i}) < B_e(z, \phi_{m_i+1}),\ \forall i \in \mathbb{N}.$$

By Lemma 2.10, we know that there exists an increasing sequence $\{l_k\}$ of $\mathbb{N}$ such that $\lim_{k\to\infty} l_k = \infty$,

$$B_e(z, \phi_{l_k}) \leq B_e(z, \phi_{l_k+1}) \tag{3.20}$$

and

$$B_e(z, \phi_k) \leq B_e(z, \phi_{l_k+1}) \tag{3.21}$$

hold for all $k \in \mathbb{N}$. From (3.13), we can obtain that

$$\begin{aligned} &(1-\kappa_{l_k})(1 - \frac{\sigma_{l_k}}{\mu\alpha}) B_e(h_{l_k}, \varphi_{l_k}) + \sigma_{l_k}(1-\kappa_{l_k})(1 - \frac{\mu}{2}\|A\|^2)\|(I - P_Q)A\varphi_{l_k}\|^2 \\ \leq& B_e(z, \phi_{l_k}) - B_e(z, \phi_{l_k+1}) + (1-\kappa_{l_k})\varpi_{l_k}(B_e(z, \phi_{l_k-1}) - B_e(z, \phi_{l_k})) \\ &+ (1-\kappa_{l_k})\tau_{l_k}(B_e(z, \phi_{l_k-2}) - B_e(z, \phi_{m_k-1})) + \kappa_{l_k} K, \end{aligned} \tag{3.22}$$

where $K > 0$. By the three-point identity (2.2), we have

$$\begin{aligned} &|B_e(z,\phi_{l_k-1}) - B_e(z,\phi_{l_k})| \\ =&| - B_e(\phi_{l_k-1},\phi_{l_k}) + \langle z-\phi_{l_k-1}, \nabla e(\phi_{l_k}) - \nabla e(\phi_{l_k-1})\rangle| \\ \le&|\langle z-\phi_{l_k-1}, \nabla e(\phi_{l_k}) - \nabla e(\phi_{l_k-1})\rangle| \\ \le&\|\nabla e(\phi_{l_k}) - \nabla e(\phi_{l_k-1})\| M, \end{aligned} \tag{3.23}$$

where $M = \sup_{k\ge 1}\{\|z-\phi_{l_k-1}\|\}$. Then from (3.2), we obtain

$$\begin{aligned} \sum_{l_k=1}^{\infty} \varpi_{l_k} |(B_e(z,\phi_{l_k-1}) - B_e(z,\phi_{l_k}))| &\le \sum_{l_k=1}^{\infty} \varpi_{l_k} \|\nabla e(\phi_{l_k}) - \nabla e(\phi_{l_k-1})\| M \\ &\le \sum_{l_k=1}^{\infty} \vartheta_{l_k} \|\nabla e(\phi_{l_k}) - \nabla e(\phi_{l_k-1})\| M \\ &< +\infty. \end{aligned} \tag{3.24}$$

From (3.24),we have

$$\lim_{k\to\infty} (1-\kappa_{l_k})\varpi_{l_k}(B_e(z,\phi_{l_k-1}) - B_e(z,\phi_{l_k})) = 0,$$

similar to (3.23), we have

$$\lim_{k\to\infty} (1-\kappa_{l_k})\tau_{l_k}(B_e(z,\phi_{l_k-2}) - B_e(z,\phi_{m_k-1})) = 0.$$

Since $\lim_{k\to\infty}\kappa_{l_k} = 0$, from (3.22),

$$\lim_{k\to\infty} B_e(h_{l_k},\varphi_{l_k}) = \lim_{k\to\infty} \|(I-P_Q)A\varphi_{l_k}\| = 0.$$

So

$$\lim_{k\to\infty} \|\nabla e(h_{l_k}) - \nabla e(\varphi_{l_k})\| = 0.$$

Similar to Case 1, the following results hold:

$$\begin{aligned} &\lim_{k\to\infty} \|\nabla e(\phi_{l_k+1}) - \nabla e(\varphi_{l_k})\| \\ =&\lim_{k\to\infty} \|\nabla e(\varphi_{l_k}) - \nabla e(\phi_{l_k})\| = \lim_{k\to\infty} \|\nabla e(\phi_{l_k+1}) - \nabla e(\phi_{l_k})\| = 0 \end{aligned} \tag{3.25}$$

and

$$\limsup_{k\to\infty} \langle \nabla e(\phi_1) - \nabla e(z), \phi_{l_k+1} - z\rangle \le 0. \tag{3.26}$$

And, it follows from (3.19) and (3.20) that

$$\begin{aligned} B_e(z,\phi_{l_k+1}) \le&(1-\kappa_{l_k} - (1-\kappa_{l_k})\varpi_{l_k})B_e(z,\phi_{l_k}) + (1-\kappa_{l_k})\varpi_{l_k} B_e(z,\phi_{l_k-1}) \\ &+ (1-\kappa_{l_k})\tau_{l_k}(B_e(z,\phi_{l_k-2}) - B_e(z,\phi_{l_k-1})) \\ &+ \kappa_{l_k}\langle \nabla e(\phi_1) - \nabla e(z), \phi_{l_k+1} - z\rangle \\ \le&(1-\kappa_{l_k})B_e(z,\phi_{l_k}) + \kappa_{l_k}\langle \nabla e(\phi_1) - \nabla e(z), \phi_{l_k+1} - z\rangle \\ \le&(1-\kappa_{l_k})B_e(z,\phi_{l_k+1}) + \kappa_{l_k}\langle \nabla e(\phi_1) - \nabla e(z), \phi_{l_k+1} - z\rangle. \end{aligned}$$

By $\kappa_{l_k} > 0$ and (3.21), we have

$$B_e(z,\phi_k) \le B_e(z,\phi_{l_k+1}) \le \langle \nabla e(\phi_1) - \nabla e(z), \phi_{l_k+1} - z\rangle. \tag{3.27}$$

Combining (3.26) and (3.27),

$$\limsup_{k\to\infty} B_e(z,\phi_k) \le 0$$

holds, which implies that $\limsup_{k\to\infty} B_e(z,\phi_k) = 0$ and $\phi_k \to z (k\to\infty)$. Combing Case 1 and Case 2, the sequence $\{\phi_m\}$ converges strongly to $z = \Pi_{\Theta}^{e}(\phi_1)$.

Remark 3.4. If $\tau_m \equiv 0$, Algorithm 3.1 becomes the following inertial Bregman projection iterative algorithm for solving the SFP (1.1):

$$\begin{cases} \varphi_m = (\nabla e)^*(\nabla e(\phi_m) + \varpi_m(\nabla e(\phi_{m-1}) - \nabla e(\phi_m))), \\ \psi_m = \Pi_c^e(\nabla e)^*(\nabla e(\varphi_m) - \sigma_m A^*(I - P_Q)A\varphi_m), \\ \phi_{m+1} = (\nabla e)^*(\kappa_m \nabla e(\phi_1) + (1 - \kappa_m)(\nabla e(\psi_m))), \end{cases}$$

where $0 < \liminf_{m\to\infty} \sigma_m \le \limsup_{m\to\infty} \sigma_m < \frac{2\alpha}{\|A\|^2}$ and $0 \le \varpi_m \le \bar{\varpi}_m$, $\bar{\varpi}_m$ is chosen by the following way:

$$\bar{\varpi}_m = \begin{cases} \min\{\varpi, \frac{\vartheta_m}{\| \nabla e(\phi_{m-1}) - \nabla e(\phi_m)\|}\}, & \text{if } \nabla e(\phi_{m-1}) \neq \nabla e(\phi_m), \\ \varpi, & \text{otherwise.} \end{cases}$$

□

4. Main Results: TWO-STEP INERTIAL BREGMAN PROJECTION ADAPTIVE ITERATIVE ALGORITHM

In this section, we take self-adaptive stepsize to modify two-step inertial Bregman projection iterative algorithm.

Algorithm 4.1. *(two-step inertial Bregman projection adaptive iterative algorithm for solving the SFP* (1.1) *)*

Choose two sequences $\{\kappa_m\} \subset (0, 1)$ *and* $\{\vartheta_m\} \subset (0, +\infty)$, *where* $\{\vartheta_m\}$ *and* $\{\kappa_m\}$ *satisfy*

$$\sum_{m=1}^{\infty} \vartheta_m < \infty, \quad \lim_{m\to\infty} \kappa_m = 0, \quad \sum_{m=1}^{\infty} \kappa_m = \infty$$

and

$$\lim_{m\to\infty} \frac{\vartheta_m}{\kappa_m} = 0.$$

Given $\varpi \in [0, 1/2]$. *Select arbitrary starting points* $\phi_0, \phi_1, \phi_2 \in H_1$.

Iterative step: *For* $m \ge 2$, *choose* ϖ_m *and* τ_m *such that* $0 < \tau_m \le \varpi_m < 1, 0 \le \max\{\varpi_m, \tau_m\} \le \bar{\varpi}_m$, *where*

$$\bar{\varpi}_m = \begin{cases} \min\{\varpi, \frac{\vartheta_m}{\|\nabla e(\phi_{m-1}) - \nabla e(\phi_m)\| + \|\nabla e(\phi_{m-2}) - \nabla e(\phi_{m-1})\|}\}, & \nabla e(\phi_{m-1}) \neq \nabla e(\phi_m) \\ & \text{or } \nabla e(\phi_{m-2}) \neq \nabla e(\phi_{m-1}), \\ \varpi, & \text{otherwise.} \end{cases} \tag{4.1}$$

Compute

$$\begin{cases} \varphi_m = (\nabla e)^*(\nabla e(\phi_m) + \varpi_m(\nabla e(\phi_{m-1}) - \nabla e(\phi_m)) + \tau_m(\nabla e(\phi_{m-2}) - \nabla e(\phi_{m-1}))), \\ \psi_m = \Pi_c^e(\nabla e)^*(\nabla e(\varphi_m) - \sigma_m A^*(I - P_Q)A\varphi_m), \\ \phi_{m+1} = (\nabla e)^*(\kappa_m \nabla e(\phi_1) + (1 - \kappa_m)(\nabla e(\psi_m))), \end{cases} \tag{4.2}$$

where σ_m *is chosen by*

$$\sigma_m = \begin{cases} \min\{\frac{\rho\alpha\|(I - P_{Q_m})A\varphi_m)\|^2}{\|A^*(I - P_{Q_m})A\varphi_m)\|^2}, \sigma_{m-1}\} & (I - P_{Q_m})A\varphi_m \neq 0, \\ \varpi, & (I - P_{Q_m})A\varphi_m = 0 \end{cases} \tag{4.3}$$

with $\sigma_2 = \frac{\rho\alpha\|(I-P_{Q_2})A\varphi_2\|^2}{\|A^*(I-P_{Q_2})A\varphi_2\|^2}$, $0 < \rho < 2$.

Lemma 4.2. σ_m *defined by* (4.3) *is well-defined.*

Proof. Taking $y \in \Theta$, i.e., $y \in C$, $Ay \in Q$, since $I - P_Q$ is firmly nonexpansive, we have

$$\begin{aligned}\|A^*(I-P_Q)A\varphi_m\| \cdot \|\varphi_m - y\| \geq& \langle A^*(I-P_Q)A\varphi_m, \varphi_m - y\rangle \\ =& \langle (I-P_Q)A\varphi_m, A\varphi_m - Ay\rangle \\ \geq& \|(I-P_Q)A\varphi_m\|^2.\end{aligned}$$

For $\|(I-P_Q)A\varphi_m\| \neq 0$, we have $\|A^*(I-P_Q)A\varphi_m\| > 0$, so σ_m is well-defined. □

Theorem 4.3. *Let the sequence* $\{\phi_m\}$ *is generated by Algorithm 4.1. Then* $\{\phi_m\}$ *converges strongly to* $z \in \Theta$, *where* $z = \Pi^e_\Theta(\phi_1)$.

Proof. First, the sequence $\{\phi_m\}$ is bounded. As proved in Theorem 3.3, for $\mu > 0$, we can deduce that

$$\begin{aligned}B_e(r,\psi_m) \leq& B_e(r,\varphi_m) - B_e(\psi_m,\varphi_m) - \sigma_m\|(I-P_Q)A\varphi_m\|^2 \\ &+ \frac{\mu\sigma_m}{2}\|A^*(I-P_Q)A\varphi_m\|^2 + \frac{\sigma_m}{2\mu}\|\varphi_m - \psi_m\|^2.\end{aligned} \tag{4.4}$$

Using (2.6), we have

$$\begin{aligned}&B_e(r,\psi_m) \\ \leq& B_e(r,\varphi_m) - B_e(\psi_m,\varphi_m) - \sigma_m\|A^*(I-P_Q)A\varphi_m\|^2\Big(\frac{\|(I-P_Q)A\varphi_m\|^2}{\|A^*(I-P_Q)A\varphi_m\|^2} - \frac{\mu}{2}\Big) + \frac{\sigma_m}{2\mu}\frac{2}{\alpha}B_e(\psi_m,\varphi_m) \\ =& B_e(r,\varphi_m) - \Big(1 - \frac{\sigma_m}{\mu\alpha}\Big)B_e(\psi_m,\varphi_m) - \sigma_m\|A^*(I-P_Q)A\varphi_m\|^2\Big(\frac{\|(I-P_Q)A\varphi_m\|^2}{\|A^*(I-P_Q)A\varphi_m\|^2} - \frac{\mu}{2}\Big).\end{aligned} \tag{4.5}$$

Since $\frac{2\|(I-P_Q)A\varphi_m\|^2}{\|A^*(I-P_Q)A\varphi_m\|^2} \geq \frac{2}{\|A\|^2} > 0$, then $\inf_m \frac{2\|(I-P_Q)A\varphi_m\|^2}{\|A^*(I-P_Q)A\varphi_m\|^2} \geq \frac{2}{\|A\|^2} > 0$. By the definition of σ_m and $0 < \rho < 2$ we have

$$\frac{\sigma_m}{\alpha} \leq \inf_{k\leq m} \frac{\rho\|(I-P_Q)A\varphi_k\|^2}{\|A^*(I-P_Q)A\varphi_k\|^2} < \inf_{k\leq m} \frac{2\|(I-P_Q)A\varphi_k\|^2}{\|A^*(I-P_Q)A\varphi_k\|^2}.$$

Since $\{\sigma_m\}$ is non-increasing and $\sigma_m \geq \frac{\rho\alpha}{\|A\|^2}$, we have $\lim_{m\to\infty}\sigma_m$ exists, so

$$\frac{1}{\alpha}\lim_{m\to\infty}\sigma_m < \liminf_{m\to\infty} \frac{2\|(I-P_Q)A\varphi_m\|^2}{\|A^*(I-P_Q)A\varphi_m\|^2}.$$

Take μ with $\frac{1}{\alpha}\lim_{m\to\infty}\sigma_m < \mu < \liminf_{m\to\infty}\frac{2\|(I-P_Q)A\varphi_m\|^2}{\|A^*(I-P_Q)A\varphi_m\|^2}$. Then,

$$\liminf_{m\to\infty}\Big(1 - \frac{\sigma_m}{\mu\alpha}\Big) > 0 \tag{4.6}$$

and

$$\liminf_{m\to\infty}\Big(1 - \frac{\mu\|A^*(I-P_Q)A\phi_m\|^2}{2\|(I-P_Q)A\phi_m\|^2}\Big) > 0. \tag{4.7}$$

From (4.6) and (4.7), we obtain

$$B_e(r,\psi_m) \leq B_e(r,\varphi_m).$$

Similar to Theorem 3.3, we can get that $\{B_e(r,\phi_m)\}$ is bounded. Applying (2.6), we have $\{\phi_m\}$ is bounded and consequently $\{\varphi_m\}$ and $\{\psi_m\}$ are bounded. Let $z = \Pi^e_\Theta(\phi_1)$. From (3.10) and (3.11), we have

$$\begin{aligned} B_e(z,\phi_{m+1}) =& B_e(z,(\nabla e)^*(\kappa_m \nabla e(\phi_1)+(1-\kappa_m)(\nabla e(\psi_m))) \\ \leq& \kappa_m B_e(z,\phi_1)+(1-\kappa_m)B_e(z,\psi_m) \\ \leq& \kappa_m B_e(z,\phi_1)+(1-\kappa_m)B_e(z,\varphi_m) \\ &-(1-\kappa_m)(1-\frac{\rho\|(I-P_Q)A\varphi_m\|^2}{\|A^*(I-P_Q)A\varphi_m\|^2}\frac{1}{\mu})B_e(\psi_m,\varphi_m) \\ &-(1-\kappa_m)\rho\alpha\|(I-P_Q)A\varphi_m\|^2(\frac{\|(I-P_Q)A\varphi_m\|^2}{\|A^*(I-P_Q)A\varphi_m\|^2}-\frac{\mu}{2}). \end{aligned} \tag{4.8}$$

This implies that

$$\begin{aligned} &(1-\kappa_m)(1-\frac{\rho\|(I-P_Q)A\varphi_m\|^2}{\|A^*(I-P_Q)A\varphi_m\|^2}\frac{1}{\mu})B_e(\psi_m,\varphi_m) \\ &+(1-\kappa_m)\rho\alpha\|(I-P_Q)A\varphi_m\|^2(\frac{\|(I-P_Q)A\varphi_m\|^2}{\|A^*(I-P_Q)A\varphi_m\|^2}-\frac{\mu}{2}) \\ \leq& B_e(z,\phi_m)-B_e(z,\phi_{m+1})+(1-\kappa_m)\varpi_m(B_e(z,\phi_{m-1})-B_e(z,\phi_m)) \\ &+(1-\kappa_m)\tau_m(B_e(z,\phi_{m-2})-B_e(z,\phi_{m-1}))+\kappa_m K, \end{aligned} \tag{4.9}$$

where $K=\sup_{m\geq 1}\{|B_e(z,\phi_1)-B_e(z,\phi_m)|\}$.

Using the same arguments as in the proof of Case 1 and Case 2 of Theorem 3.3, we can show that the sequence $\{\phi_m\}$ converges strongly to $z=\Pi^e_\Theta(\phi_1)$. This completes the proof. □

Remark 4.4. If $\tau_m \equiv 0$, Algorithm 3.1 becomes the following inertial Bregman projection iterative algorithm for solving the SFP (1.1):

$$\begin{cases} \varphi_m=(\nabla e)^*(\nabla e(\phi_m)+\varpi_m(\nabla e(\phi_{m-1})-\nabla e(\phi_m))), \\ \psi_m=\Pi^e_c(\nabla e)^*(\nabla e(\varphi_m)-\sigma_m A^*(I-P_Q)A\varphi_m), \\ \phi_{m+1}=(\nabla e)^*(\kappa_m\nabla e(\phi_1)+(1-\kappa_m)(\nabla e(\psi_m))), \end{cases}$$

where σ_m is chosen by (4.3) and $0\leq\varpi_m\leq\bar{\varpi}_m$, $\bar{\varpi}_m$ is chosen by the following way:

$$\bar{\varpi}_m=\begin{cases} \min\{\varpi,\frac{\vartheta_m}{\|\nabla e(\phi_{m-1})-\nabla e(\phi_m)\|}\}, & \text{if } \nabla e(\phi_{m-1})\neq\nabla e(\phi_m), \\ \varpi, & \text{otherwise.} \end{cases}$$

Statements and Declarations

The authors declare that they have no conflict of interest, and the manuscript has no associated data.

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