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ON CONTROLLABILITY FOR A SYSTEM GOVERNED BY A SEMILINEAR FRACTIONAL FUNCTIONAL DIFFERENTIAL INCLUSION WITH A NONCONVEX-VALUED RIGHT-HAND SIDE IN A BANACH SPACE

VALERI OBUKHOVSKII[1], GARIK PETROSYAN[1], AND JEN-CHIH YAO[2,*]

[1] *Faculty of Physics and Mathematics, Voronezh State Pedagogical University, Voronezh 394043, Russia*
[2] *Research Center for Interneural Computing, China Medical University, Taichung 40447, Taiwan*

Abstract. In this paper we study the controllability for a system governed by a fractional semilinear functional differential inclusion with an almost lower semicontinuous nonconvex-valued nonlinearity and a closed linear operator generating a C_0−semigroup in a separable Banach space. We define the multivalued operator in the space of continuous functions whose fixed points are generating solutions of the problem. By using the fixed point theory for condensing multivalued maps and the methods of fractional analysis we study the properties of this mltioperator, in particular, we demonstrate that under certain conditions it is condensing w.r.t. an appropriate measure of noncompactness. This allows to present the controllability principle as the main result of the paper.

Keywords. Controllability, semilinear differential inclusion, Caputo fractional derivative, almost lower semicontinuous multioperator, condensing map, fixed point, multivalued map, measure of noncompactness.

1. Introduction

In modern mathematics, one of the most important areas is control theory. This is primarily due to numerous applications in physics, chemistry and engineering associated with the modeling of various types of processes and phenomena. From a mathematical point of view, the complexity of such models lies in the need to use the theory of infinite-dimensional spaces. Research in this direction is relevant and is being carried out by a large number of scientists around the world (see, e.g., the surveys [4], [25] and the references therein).

Recently various controllability results were obtained for systems which can be described in terms of semilinear differential and functional differential inclusions in infinite-dimensional Banach spaces (see, among others, [6], [7], [8], [9], [14], [15], [20], [22] and the references therein). It should be mentioned that in the works [7], [8], [20], [22] it was not supposed that the semigroup generated by the linear part of a system is compact. It is known (see [28], [29]) that this compactness condition in the infinite-dimensional case creates some difficulties in the investigation of the controllability problem.

In the last years the study of the controllability problem was extended to systems governed by differential equations and inclusions of a fractional order (see, e.g., [1], [3], [10], [26], [30], [32], [33] and the references therein). In its essential part, it is caused by interesting and important applications which fractional systems find in physics, hydrodynamics, geophysics, engineering, biology, economics and other contemporary branches of natural sciences (see, e.g., [5], [11], [16], [19], [24], [27], [32]).

*Corresponding author.
E-mail address: valerio-ob2000@mail.ru (V. Obukhovskii), garikpetrosyan@yandex.ru (G. Petrosyan), yaojc@mail.cmu.edu.tw (J.-C. Yao)

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In the present paper we consider the control system governed by a fractional semilinear functional differential inclusion in a separable Banach space E of the following form:

$$^{C}D_0^{\alpha} \in Ax(t) + F(t, x(t), x_t) + Bu(t),\ t \in [0, a], \tag{1.1}$$

$$x(s) = \vartheta(s), \qquad s \in [-h, 0], \quad h > 0, \tag{1.2}$$

here $^{C}D^{\alpha},\ 0 < \alpha < 1$, is the Caputo fractional derivative and $F : [0, a] \times E \times C([-h, 0]; E) \multimap E$ is a compact almost lower semicontinuous multivalued map, $A : D(A) \subset E \to E$ is a linear closed (not necessarily bounded) operator in E; $x_t \in C([-h, 0]; E)$ for $t \in [0, a]$ is defined as $x_t(\theta) = x(t+\theta), \theta \in [-h, 0]$; a control function $u \in L^{\infty}([0, a]; U)$, where U is a Banach space of controls; $B : U \to E$ is a bounded linear operator and $\vartheta \in C([-h, 0]; E)$ is an initial function.

The controllability problem which we solve in this paper may be formulated in the following way: for a given initial function $\vartheta(\cdot) \in C([-h, 0]; E)$ and given $x_1 \in E$ we will consider the existence of a mild solution $x \in C([-h, a]; E)$ and a control $u \in L^{\infty}([0, a]; U)$ such that: $x(t) = \vartheta(t), t \in [-h, 0]$ and

$$x(a) = x_1. \tag{1.3}$$

By using the measure of noncompactness theory and the fixed point theory for condensing maps, we present the theorem on the existence of a solution to problem (1.1) - (1.3).

2. Preliminaries

2.1. **Fractional integral and derivative.**

Definition 2.1. [24, 27] The fractional integral of order $\alpha \geq 0$ of a function $g \in L^1([0, a]; E)$ is the function $I_0^{\alpha} g$ of the following form:

$$I_0^{\alpha} g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)\, ds,$$

where Γ is Euler's gamma-function

$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} dx.$$

Definition 2.2. The Caputo fractional derivative of the order $\alpha \in (N-1, N]$ of a function $g \in C^N([0, a]; E)$ is the function $^{C}D_0^{\alpha} g$ of the following form:

$$^{C}D_0^{\alpha} g(t) = \frac{1}{\Gamma(N-\alpha)} \int_0^t (t-s)^{N-\alpha-1} g^{(N)}(s)\, ds.$$

2.2. **Multivalued maps.** [18, 21] We denote by $\mathcal{E}$ is a Banach space and introduce the following notation:

- $P(\mathcal{E}) = \{A \subseteq \mathcal{E} : A \neq \varnothing\}$ is the collection of all non-empty subsets of $\mathcal{E}$;
- $Pb(\mathcal{E}) = \{A \in P(\mathcal{E}) : A \text{ is bounded}\}$;
- $K(\mathcal{E}) = \{A \in P(\mathcal{E}) : A \text{ is compact}\}$;
- $Kv(\mathcal{E}) = \{Pv(\mathcal{E}) \cap K(\mathcal{E})\}$ is the collection of all convex and non-empty compact subsets of $\mathcal{E}$.

Definition 2.3. [2] Let $(\mathcal{A}, \geq)$ is a partially ordered set. A function $\beta : Pb(\mathcal{E}) \to \mathcal{A}$ is called the measure of noncompactness (MNC) in $\mathcal{E}$ if for each set $\Omega \in Pb(\mathcal{E})$ we have:

$$\beta(\overline{\text{co}}\, \Omega) = \beta(\Omega),$$

where $\overline{\text{co}}\, \Omega$ is the closure of the convex hull of Ω.

The example of a real MNC is the Hausdorff MNC $\chi(\Omega)$:

$$\chi(\Omega) = \inf\{\varepsilon > 0, \text{for which } \Omega \text{ has a finite } \varepsilon\text{-net in } \mathcal{E}\ \}.$$

Let X be a metric space and Y is a normed space.

Definition 2.4. A multimap (multivalued map) $\mathcal{F} : X \to P(Y)$ is said to be lower semicontinuous (l.s.c.) at a point $x \in X$, if for each open set $V \subset Y$ such that $\mathcal{F}(x) \cap V \neq \emptyset$ there exists a neighborhood $U(x)$ of the point x such that $\mathcal{F}(x') \cap V \neq \emptyset$ for all $x' \in U(x)$.

A multimap is lower semicontinuous (l.s.c.) if it is lower semicontinuous at every point $x \in X$.

Definition 2.5. A multivalued map $\mathcal{F} : [0, a] \times X \to K(Y)$ is said to be almost lower semicontinuous (a.l.s.c.) if there exists a sequence of disjoint compact sets $I_n \subseteq [0, a]$ such that

(i) $meas([0, a] \setminus I) = 0$, here $I = \cup_n I_n$;
(ii) the restriction of $\mathcal{F}$ on every set $J_n = I_n \times X$ is l.s.c..

Definition 2.6. A continuous map $f : X \subseteq \mathcal{E} \to \mathcal{E}$ is called condensing with respect to a MNC β (or β–condensing) if for each bounded set $\Omega \subseteq X$ which is not relatively compact, we have:

$$\beta(f(\Omega)) \ngeq \beta(\Omega).$$

In the sequel, we will need the following Sadovskii type theorem (see [2, 18]).

Theorem 2.7. *Let $\mathcal{M}$ be a closed convex bounded subset of $\mathcal{E}$ and $f : \mathcal{M} \to \mathcal{M}$ is a β–condensing map, where β is a monotone nonsingular MNC in $\mathcal{E}$. Then the fixed point set* $\mathrm{Fix}\, f$ *is non-empty.*

3. Main Results

Everywhere in the sequel, let $\mathcal{C} := C([-h, 0]; E)$.

Let E be a separable Banach space and a multivalued map $F : [0, a] \times E \times \mathcal{C} \to K(E)$ be such that:

$(F1)$ $F : [0, a] \times E \times \mathcal{C} \to K(E)$ is a.l.s.c.;
$(F2)$ for every $r > 0$ there exists a function $\omega_r \in L^\infty([0, a])$ such that for all $(x, \zeta) \in E \times \mathcal{C}$ with $\|x\|_E \leq r$ and $\|\zeta\|_{\mathcal{C}} \leq r$, we have:

$$\|F(t, x, \zeta)\| \leq \omega_r(t)$$

for a.e. $t \in [0, a]$;
$(F3)$ there exists a function $\mu \in L^\infty([0, a])$ such that for each bounded sets $\Omega \subset E$ and $Q \subset \mathcal{C}$ we have:

$$\chi_E(F(t, \Omega, Q)) \leq \mu(t)\,(\chi_E(\Omega) + \varphi(Q)) \ \text{ a.e. } t \in [0, a],$$

where χ_E is the Hausdorff MNC in E,

$$\varphi(Q) = \sup_{-h \leq \theta \leq 0} \chi_E(Q(\theta)), \quad \text{with } Q(\theta) = \{y(\theta), y \in Q\}\,,\ \theta \in [-h, 0].$$

Let a linear operator A be such that:

(A) $A : D(A) \subset E \to E$ is a closed linear operator in E generating a uniformly bounded C_0–semigroup $\{U(t)\}_{t \geq 0}$.

Denote $M = \sup\{\|U(t)\|\,; t \geq 0\}$.

For $x \in C([0, \tau]; E), 0 < \tau \leq a$, consider the multifunction:

$$\Phi_F : [0, \tau] \to K(E), \qquad \Phi_F(t) = F(t, x(t), x_t).$$

From above conditions $(F1) - (F2)$ and Proposition 1.3.1 from [18] it follows that the multifunction Φ_F is measurable and L^p–integrable for each $p \geq 1$.

Consider a convex closed subset $\mathcal{D} \subset C([0,a];E)$ given as

$$\mathcal{D} = \{\xi \in C([0,a];E), \xi(0) = \vartheta(0)\}$$

and, for a given $\xi \in \mathcal{D}$, define a function $\xi[\vartheta] \in C([-h,a];E)$ by

$$\xi[\vartheta](t) = \begin{cases} \vartheta(t), & t \in [-h,0], \\ \xi(t), & t \in [0,a]. \end{cases}$$

Now we may define the superposition multivalued map $\mathcal{P}_F^\infty \colon \mathcal{D} \to P(L^\infty([0,a];E))$ in the following way:

$$\mathcal{P}_F^\infty(x) = \{\phi \in L^\infty([0,a];E) : \phi(t) \in F(t, x(t), x[\vartheta]_t), \ \text{ a.e. } t \in [0,a]\}.$$

Definition 3.1. A mild solution to problem (1.1) - (1.2) on an interval $[0,a]$ is a function $x \in C([0,a];E)$ such that:

$$x(t) = \begin{cases} \vartheta(t), & t \in [-h,0], \\ \mathcal{K}(t)\vartheta(0) + \int_0^t (t-s)^{\alpha-1}\mathcal{L}(t-s)\phi(s)ds + \\ + \int_0^t (t-s)^{\alpha-1}\mathcal{L}(t-s)Bu(s)ds, & t \in [0,a], \end{cases}$$

here $\phi \in \mathcal{P}_F^\infty(x)$ and

$$\mathcal{K}(t) = \int_0^\infty \xi_\alpha(\theta)U(t^\alpha\theta)d\theta, \qquad \mathcal{L}(t) = \alpha\int_0^\infty \theta\xi_\alpha(\theta)U(t^\alpha\theta)d\theta,$$

$$\xi_\alpha(\theta) = \frac{1}{\alpha}\theta^{-1-\frac{1}{\alpha}}\Psi_\alpha(\theta^{-1/\alpha}),$$

$$\Psi_\alpha(\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-\alpha n-1}\frac{\Gamma(n\alpha+1)}{n!}\sin(n\pi\alpha), \theta \in \mathbb{R}^+.$$

Remark 3.2. (See, e.g. [31]) $\xi_\alpha(\theta) \geq 0, \ \int_0^\infty \xi_\alpha(\theta)\,d\theta = 1, \ \int_0^\infty \theta\xi_\alpha(\theta)\,d\theta = \frac{1}{\Gamma(\alpha+1)}$.

Lemma 3.3. *(See [31], Lemma 3.4.) The operator functions $\mathcal{K}$ and $\mathcal{L}$ possess the following properties:*

1) *for all $t \in [0,a]$, $\mathcal{K}(t)$ and $\mathcal{L}(t)$ are bounded linear operators, more precisely, for every $x \in E$:*

$$\|\mathcal{K}(t)x\|_E \leq M\,\|x\|_E\,; \tag{3.1}$$

$$\|\mathcal{L}(t)x\|_E \leq \frac{M}{\Gamma(\alpha)}\,\|x\|_E\,; \tag{3.2}$$

2) *the operator functions $\mathcal{K}(\cdot)$ and $\mathcal{L}(\cdot)$ are strongly continuous, i.e., functions $t \in [0,a] \to \mathcal{K}(t)x$ and $t \in [0,a] \to \mathcal{L}(t)x$ are continuous for all $x \in E$.*

To sole of problem (1.1) - (1.3) we assume that the linear controllability operator $T : L^\infty([0,a];U) \to E$ given by

$$Tu = \int_0^a (a-s)^{\alpha-1}\mathcal{L}(a-s)Bu(s)ds$$

has a bounded right inverse $T^{-1} : E \to L^\infty([0,a];U)$.

We will assume that the operator T^{-1} satisfies the following regularity condition:

(T) There exists a function $\gamma \in L^\infty([0,a])$ such that for each bounded set $\Omega \subset E$ we have:

$$\chi_U\left(T^{-1}(\Omega)(t)\right) \leq \gamma(t)\chi_E(\Omega) \text{ for a.e. } t \in [0,a],$$

where χ_U is the Hausdorff MNC in U.

Let M_1, $M_2 > 0$ be constants such that

$$\|B\| \leq M_1, \quad \left\|T^{-1}\right\| \leq M_2.$$

To search mild solutions of problem (1.1)-(1.3) we will need the following operator:

$$V : L^{\infty}([0,a];E) \to C([0,a];E),$$

$$V(\phi)(t) = \int_0^t (t-s)^{\alpha-1}\mathcal{L}(t-s)\phi(s)ds +$$

$$+\int_0^t (t-s)^{\alpha-1}\mathcal{L}(t-s)\left[BT^{-1}\left(x_1 - \mathcal{K}(a)\vartheta(0) - \int_0^a (a-\tau)^{\alpha-1}\mathcal{L}(a-\tau)\phi(\tau)d\tau\right)(s)\right]ds.$$

Let us represent the operator V in the form

$$V(\phi) = V'(\phi) + V''(\phi),$$

where

$$V'(\phi)(t) = \int_0^t (t-s)^{\alpha-1}\mathcal{L}(t-s)\phi(s)ds,$$

$$V''(\phi)(t) = \int_0^t (t-s)^{\alpha-1}\mathcal{L}(t-s)\left[BT^{-1}\left(x_1 - \mathcal{K}(a)\vartheta(0) - \int_0^a (a-\tau)^{\alpha-1}\mathcal{L}(a-\tau)\phi(\tau)d\tau\right)(s)\right]ds.$$

Lemma 3.4. *(see [17]) The operator V' obeys the following properties:*

(V_1) if $\frac{1}{\alpha} < p < \infty$, then there exists a constant $C > 0$ such that

$$\left\|V'(\xi)(t) - V'(\eta)(t)\right\|_E^p \leq C^p \int_0^t \|\xi(s) - \eta(s)\|_E^p\, ds, \qquad \xi, \eta \in L^p([0,a];E);$$

(V_2) for every compact set $K \subset E$ and bounded sequence $\{\eta_n\} \subset L^{\infty}([0,a];E)$ such that $\{\eta_n(t)\} \subset K$ for a.e. $t \in [0,a]$, the weak convergence $\eta_n \rightharpoonup \eta_0$ in $L^1([0,a];E)$ implies the convergence $V'(\eta_n) \to V'(\eta_0)$ in $C([0,a];E)$.

The operator V'' may be represented in the form:

$$V''(\phi) = V'(BT^{-1}\left(x_1 - \mathcal{K}(a)\vartheta(0) - \Pi V'(\phi)\right)), \tag{3.3}$$

here $\Pi : C([0,a];E) \to E, \quad \Pi x = x(a)$ is a linear bounded operator. Since T^{-1}, B, and V' are linear bounded operators, we conclude that the assertion of Lemma 3.4 is true for the operator V'' and hence for the operator V.

Now, we consider the multioperator $G : \mathcal{D} \to P(\mathcal{D})$, given as:

$$G(x) = g_0 + V \circ \mathcal{P}_F^{\infty}(x), \qquad t \in [0,T],$$

where the function $g_0(t) = \mathcal{G}(t)\vartheta(0)$.

It is clear that a function $x \in \mathcal{D}$ is a fixed point of the multioperator G if and only if the function $x[\vartheta] \in C([-h,a];E)$ is a mild solution of controllability problem (1.1)-(1.3). So, our problem is reduced to the finding of a fixed point of the multioperator G.

We need the following notion and results.

Definition 3.5. (see [13] and [18], Definition 5.5.1) A nonempty set $\mathcal{M} \subset L^p([0,a];E), p \geq 1$, is said to be decomposable if for all $f, g \in \mathcal{M}$ and every measurable subset m in $[0,a]$:

$$f \cdot \kappa_m + g \cdot \kappa_{[0,a]\setminus m} \in \mathcal{M},$$

where κ is the characteristic function of a set.

It is clear that superposition multivalued map $\mathcal{P}_F^\infty$ has decomposable and closed values, and it is l.s.c. (see [18], Section 5.5 or [21], Theorem 1.5.36).

The set of all closed nonempty decomposable subsets of the space $L^p([0,a];E)$ will be denoted by $D(L^p([0,a];E))$. The following analogue of the Michael selection theorem which is due to Fryszkowski–Bressan–Colombo holds true (see [12]).

Lemma 3.6. *If X is a separable metric space, then every lower semicontinuous multimap $\mathcal{F} : X \to D(L^1([0,a];E))$ admits a continuous selection.*

Taking in account the last result to the multivalued map $\mathcal{P}_F^\infty$, we get that it has a continuous selection $p : \mathcal{D} \to L^\infty([0,a];E)$. Then a map $g : \mathcal{D} \to \mathcal{D}$,

$$g(x)(t) = \mathcal{K}(t)\vartheta(0) + \int_0^t (t-s)^{\alpha-1}\mathcal{L}(t-s)p(x)(s)ds +$$

$$\int_0^t (t-s)^{\alpha-1}\mathcal{L}(t-s)\left[BT^{-1}\left(x_1 - \mathcal{K}(a)\vartheta(0) - \int_0^a (a-\tau)^{\alpha-1}\mathcal{L}(a-\tau)p(x)(\tau)d\tau\right)(s)\right]ds,$$

is a continuous selection of the multivalued map G and its fixed points $Fix\, g \subset Fix\, G$. Therefore, it is sufficient to prove the existence of a fixed point of the map g.

Lemma 3.7. *(see [23]) If $\Omega \subset C([0,a];E)$ is a nonempty set and $\Omega(t)$ is a relatively compact subset of E for every $t \in [0,a]$, then a set of functions*

$$(V' \circ p)(\Omega) = \left\{ y \,|\, y(t) = \int_0^t (t-s)^{\alpha-1}\mathcal{L}(t-s)p(x)(s)ds : x \in \Omega \right\}$$

is equicontinuous.

Now, let us consider conditions under which the operator g is condensing. Consider in the space of all continuous functions $C([0,a];E)$ the following measure of noncompactness

$$\nu : Pb(C([0,a];E)) \to \mathbb{R}_+^2$$

with the values in the cone $\mathbb{R}_+^2$ defined as

$$\nu(\Omega) = (\varphi(\Omega), mod_C(\Omega)),$$

here $\varphi(\Omega)$ is the module of fiber noncompactness

$$\varphi(\Omega) = \sup_{t\in[0,a]} \chi_E(\{y(t) : y \in \Omega\}),$$

and

$$mod_C(\Omega) = \lim_{\delta\to 0} \sup_{y\in\Omega} \max_{|t_1-t_2|\le\delta} \|y(t_1) - y(t_2)\|$$

is the equicontinuity module.

It is known (see [18]) that the MNC ν is monotone, nonsingular, algebraically semiadditive, and regular.

We need the following assertion which follows from [17], Lemma 7.

Lemma 3.8. *Let $\{f_n\}_{n=1}^\infty$ be a bounded sequence in $L^\infty([0,a];E)$ such that*

$$\chi_E(\{f_n(t)\}) \le v(t) \text{ a.e. } t \in [0,a],$$

here $v \in L_+^\infty(0,a)$. Then

$$\chi_E(\{V' f_n(t)\}) \le \frac{2M}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s)ds.$$

Now, let us formulate the following condition under which the operator g is condensing.

Lemma 3.9. *Under conditions* (A), $(F1)$ - $(F3)$, (T) *and condition*

$$(C) \qquad l := \frac{4Ma^\alpha}{\Gamma(1+\alpha)}\|\mu\|_\infty \left(1 + \frac{2MM_1a^\alpha}{\Gamma(1+\alpha)}\|\gamma\|_\infty\right) < 1$$

the operator g *is* ν*-condensing.*

Proof. Let $\Omega \subset \mathcal{D}$ is a non-empty bounded set and

$$\nu(g(\Omega)) \geq \nu(\Omega). \tag{3.4}$$

We will prove that Ω is a relatively compact set. It is sufficient to show the assertion for the map $V \circ p$. Taking in account that the MNC ν is nonsingular we have

$$\nu((V \circ p)(\Omega)) \geq \nu(\Omega). \tag{3.5}$$

From (3.5) it follows that

$$\varphi((V \circ p)(\Omega)) \geq \varphi(\Omega). \tag{3.6}$$

By using regularity condition $(F3)$ for $0 \leq s \leq \tau \leq a$ we will have the following estimate:

$$\chi_E(p(\Omega)(s)) = \chi_E(\{f(s) : f \in p(\Omega)\}) \leq \mu(s)\Big(\chi_E(\{x(s) : x \in \Omega\}) + \varphi(\{x[\vartheta]_s : x \in \Omega\})\Big) =$$

$$= \mu(s) \cdot \Big(\chi_E(\{x(s) : x \in \Omega\}) + \sup_{\tau\in[0,s]} \chi_E(\{x(\tau) : x \in \Omega\})\Big) \leq 2\mu(s)\varphi(\Omega).$$

Now, applyiing this estimate and Lemma 3.8 we get

$$\chi_E\left((V' \circ p)(\Omega)(t)\right) \leq \frac{4M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\mu(s)ds \cdot \varphi(\Omega) \leq \frac{4Ma^\alpha}{\Gamma(1+\alpha)}\|\mu\|_\infty \cdot \varphi(\Omega).$$

Further, we have the estimate

$$\chi_E\left(\left\{BT^{-1}\left(x_1 - \mathcal{K}(a)\vartheta(0) - \int_0^a (a-\tau)^{\alpha-1}\mathcal{L}(a-\tau)f(\tau)d\tau\right)(s) : f \in p(\Omega)\right\}\right) \leq$$

$$M_1\gamma(s)\chi_E\left(\left\{\left(\int_0^a (a-\tau)^{\alpha-1}\mathcal{L}(a-\tau)f(\tau)d\tau\right)(s) : f \in p(\Omega)\right\}\right) \leq$$

$$M_1\,\gamma(s)\,\frac{4Ma^\alpha}{\Gamma(1+\alpha)}\|\mu\|_\infty \cdot \varphi(\Omega) \leq \frac{4MM_1a^\alpha}{\Gamma(1+\alpha)}\|\mu\|_\infty\|\gamma\|_\infty \cdot \varphi(\Omega).$$

Taking in account this estimate and applying Lemma 3.8 we obtain

$$\chi_E\left((V'' \circ p)(\Omega)(t)\right) \leq \frac{2M}{\Gamma(\alpha)}\times$$

$$\int_0^t (t-s)^{\alpha-1}\chi_E\left(\left\{BT^{-1}\left(x_1 - \mathcal{K}(a)\vartheta(0) - \int_0^a (a-\tau)^{\alpha-1}\mathcal{L}(a-\tau)f(\tau)d\tau\right)(s) : f \in p(\Omega)\right\}\right)ds$$

$$\leq \frac{8M^2M_1a^{2\alpha}}{\Gamma^2(1+\alpha)}\,\|\mu\|_\infty\,\|\gamma\|_\infty \cdot \varphi(\Omega).$$

Therefore we obtain

$$\chi_E\left((V \circ p)(\Omega)(t)\right)) \leq \frac{4Ma^\alpha}{\Gamma(1+\alpha)}\|\mu\|_\infty \cdot \varphi(\Omega) + \frac{8M^2M_1a^{2\alpha}}{\Gamma^2(1+\alpha)}\,\|\mu\|_\infty\,\|\gamma\|_\infty \cdot \varphi(\Omega) =$$

$$= \frac{4Ma^\alpha}{\Gamma(1+\alpha)}\|\mu\|_\infty\left(1 + \frac{2MM_1a^\alpha}{\Gamma(1+\alpha)}\|\gamma\|_\infty\right) \cdot \varphi(\Omega) = l\varphi(\Omega).$$

Hence

$$\varphi\left((V \circ p)(\Omega)\right) \leq l\varphi(\Omega), \tag{3.7}$$

where $l < 1$.

Comparing inequalities (3.6) and (3.7) we get

$$\varphi(\Omega) = 0.$$

Now, from (3.5) we have

$$mod_C((V \circ p)(\Omega)) \geq mod_C(\Omega). \tag{3.8}$$

From Lemma 3.7 it is known that the set $(V' \circ p)(\Omega)$ is equicontinuous, hence by using (3.3), we get

$$mod_C((V \circ p)(\Omega)) = 0.$$

So, we obtain

$$mod_C(\Omega) = 0.$$

Thus $\nu(\Omega) = (0, 0)$ and we get that Ω is a relatively compact set, therefore the operator g is condensing w.r.t. the MNC ν. □

Now, we turn to the proof of the maim result.

Theorem 3.10. *Under assumptions (A), $(F1)$, $(F3)$, (T) , (C), suppose that condition $(F2)$ has the following form:*

$(F2')$ *There exists a sequence of functions $\{\omega_n\} \subset L_+^\infty[0, a]$ such that*

$$\sup_{\|x\|_E \leq n, \|\zeta\|_{\mathcal{C}} \leq n} \|F(t, x, \zeta)\| \leq \omega_n(t) \text{ for a.e. } t \in [0, a],\ \ n = 1, 2, ...$$

and

$$\lim_{n\to\infty} \inf \frac{1}{n} \|\omega_n\|_\infty = 0.$$

Then controllability problem (1.1)-(1.3) has a solution.

Proof. We will show that there exists a closed ball $B_R \subset C([0, a]; E)$ centered at the origin of a sufficiently large radius $R > \|\vartheta\|_{\mathcal{C}}$ such that $g(B_R^{\mathcal{D}}) \subset B_R^{\mathcal{D}}$, where $B_R^{\mathcal{D}} = B_R \cap \mathcal{D}$.

Toward this goal, notice that if $n > \|\vartheta\|_{\mathcal{C}}$ then obviously condition $x(\cdot) \in \mathcal{D}$, $\|x\|_C \leq n$ implies $\|x[\vartheta]_t\|_{\mathcal{C}} \leq n$ for all $t \in [0, a]$ and hence for such n we get the following estimate: if $z_n \in g(x_n)$, $\|x_n\|_C \leq n$, then

$$\|z_n\|_C \leq M\|\vartheta(0)\|_E + \frac{Ma^\alpha}{\Gamma(1+\alpha)}\|\omega_n\|_\infty + \frac{MM_1M_2a^\alpha}{\Gamma(1+\alpha)}\Big(\|x_1\|_E + M\|\vartheta(0)\|_E + \frac{Ma^\alpha}{\Gamma(1+\alpha)}\|\omega_n\|_\infty\Big). \tag{3.9}$$

For convenience, rewrite this estimate as

$$\|z_n\|_C \leq M\|\vartheta(0)\|_E + C_1\|\omega_n\|_\infty + C_2\Big(\|x_1\|_E + M\|\vartheta(0)\|_E + C_1\|\omega_n\|_\infty\Big), \tag{3.10}$$

where

$$C_1 = \frac{Ma^\alpha}{\Gamma(1+\alpha)}, \quad C_2 = \frac{MM_1M_2a^\alpha}{\Gamma(1+\alpha)} \tag{3.11}$$

Now, supposing the contrary to our assertion, we will have sequences $\{x_n\}$, $\{z_n\} \subset \mathcal{D}$, such that $z_n \in g(x_n)$, $\|x_n\|_C \leq n$ but $\|z_n\|_C > n$ for all sufficiently large n.

But then, applying (3.10), for all such n we have

$$1 < \frac{\|z_n\|_C}{n} \leq \frac{1}{n}M\|\vartheta(0)\|_E + C_1\frac{\|\omega_n\|_\infty}{n} + C_2\Big(\frac{1}{n}\|x_1\|_E + \frac{1}{n}M\|\vartheta(0)\|_E + C_1\frac{\|\omega_n\|_\infty}{n}\Big),$$

giving the contradiction.

It remains only to apply Theorem 2.7. □

4. Conclusion

In this paper we studied the controllability for a system governed by a fractional semilinear order functional differential inclusion with an almost lower semicontinuous nonconvex valued nonlinearity and a closed linear operator generating a C_0−semigroup in a separable Banach space. We defined the multivalued operator whose fixed points are generating solutions of the problem. By using the methods of fractional analysis and the fixed point theory for condensing multivalued maps we studied the properties of this operator, in particular, we demonstrate that under certain conditions it is condensing w.r.t. an appropriate measure of noncompactness. This allows to present the controllability principle as the main result of the present paper.

Statements and Declarations

The authors declare that they have no conflict of interest, and the manuscript has no associated data.

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