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ON THE EXISTENCE OF SOLUTIONS FOR SOME KIND OF MIXED EVOLUTIONARY VARIATIONAL-HEMIVARIATIONAL INEQUALITY PROBLEMS

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Abstract. This paper discusses the mixed evolutionary variational-hemivariational inequality problem, which incorporates a set of constraints and history-dependent operators. We utilized a mixed equilibrium formulation with appropriate functions and a fixed-point principle for history-dependent operators to prove the existence and uniqueness of solutions. As an application to demonstrate the peculiar weak solution for a viscoelastic frictional contact problem that involves the unilateral signorini type condition for the normal velocity together with non-monotone normal damped response conditions and the Coulomb law of dry friction that friction bounds depend on the amount of accumulated slip.

Keywords. The mixed evolutionary variational-hemivariational inequality problems, History-dependent operator, Set of constraints, Existence and uniqueness, Frictional contact, $(\alpha_{\mathcal{A}}, \beta_{\mathcal{A}})$-relaxed cocoercivity.

1. Introduction

With the groundbreaking studies of Panagiotopoulos, *see* [1], the theory of hemivariational and variational-hemivariational inequalities was introduced in the early 1980s. Since then, the theory has significantly advanced in pure and applied mathematics owing to innovative and efficient methods incorporating convex and nonsmooth analysis,*see* [2, 3]. Hemivariational inequalities, which include nonconvex, nondifferentiable, and local Lipschitz functions, are variational representations of physical processes. They play a crucial role in the depiction of a wide range of mechanical problems that occur in solid and fluid mechanics, we refer to [4, 5, 6, 7, 8, 9].

Let $\mathbb{V}$ be a reflexive Banach space and $\Omega \subset \mathbb{V}$ be a nonempty, closed and convex set of constraints. Let $\mathcal{A}, \mathcal{R}_1, \mathcal{R}_2, \mathcal{S}$ be the nonlinear operators and $\mathcal{M}$ be the affine operator, which is supplemented by an initial condition. Assume that φ is a convex function and $\jmath$ is a local Lipschitz function. Let $\mathcal{R}_1, \mathcal{R}_2, \mathcal{S}$ be the history-dependent operators. The problem reads as follows: find $w : (0,T) \longrightarrow \mathbb{V}$ such that $w(t) \in \Omega$ for a.e. $t \in (0,T)$ and

$$\begin{aligned}\langle w'(t) + \mathcal{A}(t, w(t)) + (\mathcal{R}_1 w)(t) - f(t), v - w(t)\rangle_{\mathbb{V}^*\times\mathbb{V}} + \jmath^0(t, (\mathcal{S}w)(t), \mathcal{M}w(t); \mathcal{M}v - \mathcal{M}w(t))\\ + \varphi(t, (\mathcal{R}_2 w)(t), \mathcal{M}v) - \varphi(t, (\mathcal{R}_2 w)(t), \mathscr{M}w(t)) \geq 0, \ \forall\, v \in \Omega, \ \text{a.e. } t \in (0,T) \qquad (1.1)\end{aligned}$$

is known as first-order evolutionary variational -hemivariational inequality problems involving together with history-dependent operators and a set of constraints.

Inspired and motivated by the recent works [10, 11, 12, 13, 14, 15, 16, 17], in this paper, we establish the existence, uniqueness and regularity of the solution at (1.1). We also discuss the problem (1.1) in

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the framework of evolution triple of spaces, exploiting the result on the mixed equilibrium inequality and a fixed-point principle history-–dependent operators. The second is to obtain the unique weak solution to a dynamic frictional contact problem for a viscoelastic material with a long memory and unilateral constraints in velocity. The contact condition involves a unilateral Signorini-type condition for the normal velocity combined with the nonmonotone normal damped response condition.

2. Preliminaries

Unless otherwise stated everywhere in this paper $\mathcal{A}(t,\cdot)=\mathcal{A}_t(\cdot), \varphi(t,\cdot,\cdot)=\varphi_t(\cdot,\cdot)$, and $J^0(t,\cdot,\cdot)=J^0_t(\cdot,\cdot)$. Let $(\mathbb{X},\|\cdot\|_{\mathbb{X}})$ be a Banach space, $\mathbb{X}^*$ be its dual space, and $\langle\cdot,\cdot\rangle_{\mathbb{X}^*\times\mathbb{X}}$ be the duality pairing between $\mathbb{X}^*$ and $\mathbb{X}$. The symbols $\to$ and $\rightharpoonup$ represent the strong and weak convergence, *respectively*. For a set $\mathbb{D}\subset\mathbb{X}$, $conv(\mathbb{D})$ is the convex hull of $\mathbb{D}$. The notation $\mathscr{L}(\mathbb{E},\mathbb{F})$ stands for the space of linear bounded operators from a Banach space $\mathbb{E}$ to a Banach space $\mathbb{F}$, and is endowed with the usual norm $\|\cdot\|_{\mathscr{L}(\mathbb{E},\mathbb{F})}$. For a set $\mathbb{S}\subset\mathbb{X}$, we write $\|\mathbb{S}\|_{\mathbb{X}}=\sup\{\|u\|_{\mathbb{X}}\mid u\in\mathbb{S}\}$. Let $J:\mathbb{X}\longrightarrow 2^{\mathbb{X}^*}$ be the set-valued mapping. The duality mapping is defined by

$$Ju=\left\{u^*\in\mathbb{X}^*|\langle u^*,u\rangle_{\mathbb{X}^*\times\mathbb{X}}=\|u\|^2_{\mathbb{X}}=\|u^*\|^2_{\mathbb{X}^*}\right\},\ \forall u\in\mathbb{X}.$$

We review several details pertaining to single-valued operators and bifunctions from the reference [5, 18].

Definition 2.1. A single-valued operator $\mathcal{A}:\mathbb{X}\longrightarrow\mathbb{X}^*$ is said to be

(i) demicontinuous, if $u_n\longrightarrow u\in\mathbb{X}$ implies $\mathcal{A}u_n\rightharpoonup\mathcal{A}u\in\mathbb{X}^*$,

(ii) monotone, if $\langle\mathcal{A}u-\mathcal{A}v,u-v\rangle_{\mathbb{X}^*\times\mathbb{X}}\geq 0,\forall u,v\in\mathbb{X}$,

(iii) strongly monotone if there exists a constant $\alpha_{\mathcal{A}}>0$ such that

$$\langle\mathcal{A}u-\mathcal{A}v,u-v\rangle_{\mathbb{X}^*\times\mathbb{X}}\geq\alpha_{\mathcal{A}}\|u-v\|^2_{\mathbb{X}},\forall u,v\in\mathbb{X},$$

(iv) relaxed monotone if there exists a constant $\alpha_{\mathcal{A}}>0$ such that

$$\langle\mathcal{A}u-\mathcal{A}v,u-v\rangle_{\mathbb{X}^*\times\mathbb{X}}\geq-\alpha_{\mathcal{A}}\|u-v\|^2_{\mathbb{X}},\forall u,v\in\mathbb{X},$$

(v) cocoercive if there exists a constant $\alpha_{\mathcal{A}}>0$ such that

$$\langle\mathcal{A}u-\mathcal{A}v,u-v\rangle_{\mathbb{X}^*\times\mathbb{X}}\geq\alpha_{\mathcal{A}}\|\mathcal{A}u-\mathcal{A}v\|^2_{\mathbb{X}^*},\forall u,v\in\mathbb{X},$$

(vi) relaxed cocoercive if there exists a constant $\alpha_{\mathcal{A}}>0$ such that

$$\langle\mathcal{A}u-\mathcal{A}v,u-v\rangle_{\mathbb{X}^*\times\mathbb{X}}\geq-\alpha_{\mathcal{A}}\|\mathcal{A}u-\mathcal{A}v\|^2_{\mathbb{X}^*},\forall u,v\in\mathbb{X},$$

(vii) $(\alpha_{\mathcal{A}},\beta_{\mathcal{A}})$-relaxed cocoercive if there exist constants $\alpha_{\mathcal{A}}>0,\beta_{\mathcal{A}}>0$ such that

$$\langle\mathcal{A}u-\mathcal{A}v,u-v\rangle_{\mathbb{X}^*\times\mathbb{X}}\geq-\alpha_{\mathcal{A}}\|\mathcal{A}u-\mathcal{A}v\|_{\mathbb{X}^*}+\beta_{\mathcal{A}}\|u-v\|^2_{\mathbb{X}},\forall u,v\in\mathbb{X},$$

(viii) Lipschitz continuous if the exists a constant $\zeta_{\mathcal{A}}>0$ such that

$$\|\mathcal{A}u-\mathcal{A}v\|_{\mathbb{X}^*}\leq\zeta_{\mathcal{A}}\|u-v\|_{\mathbb{X}},\ \forall v\in\mathbb{X}^*,$$

(ix) maximal monotone, if it is monotone and the conditions $(u,u^*)\in\mathbb{X}\times\mathbb{X}^*$ and

$$\langle u^*-\mathcal{A}v,u-v\rangle_{\mathbb{X}^*\times\mathbb{X}}\geq 0,\ \forall v\in\mathbb{X}^*$$

imply $u^*=\mathcal{A}u$,

(x) quasi monotone, if $\limsup\langle\mathcal{A}u_n,u_n-u\rangle_{\mathbb{X}^*\times\mathbb{X}}\geq 0$ for any sequence $\{u_n\}\subset\mathbb{X}$ with $u_n\rightharpoonup u\in\mathbb{X}$,

(xi) pseudomonotone, if for any sequence $\{u_n\}\subset\mathbb{X}$ such that $u_n\rightharpoonup u\in\mathbb{X}$ and

$$\limsup\langle\mathcal{A}u_n,u_n-u\rangle_{\mathbb{X}^*\times\mathbb{X}}\leq 0,$$

we have

$$\liminf\langle\mathcal{A}u_n,u_n-v\rangle_{\mathbb{X}^*\times\mathbb{X}}\geq\langle\mathcal{A}u,u-v\rangle_{\mathbb{X}^*\times\mathbb{X}},\forall v\in\mathbb{X},$$

(xii) bounded, if it maps bounded subsets of $\mathbb{X}$ into bounded subsets of $\mathbb{X}^*$.

Definition 2.2. [19] A function $f : \mathbb{X} \longrightarrow \mathbb{R}$ is said to be

(i) (resp. weakly) upper semicontinuous (usc) at $x_0 \in \mathbb{X}$, if for any sequence $\{x_n\} \subset \mathbb{X}$ with (resp. $x_n \rightharpoonup x_0$) $x_n \longrightarrow x_0$, we have

$$\limsup f(x_n) \leq f(x_0),$$

(ii) (resp. weakly) lower semicontinuous (lsc) at $x_0 \in \mathbb{X}$, if for any sequence $\{x_n\} \subset \mathbb{X}$ with (resp. $x_n \rightharpoonup x_0$) $x_n \longrightarrow x_0$, we have

$$f(x_0) \leq \liminf f(x_n),$$

(iii) f is said to be (resp. weakly) usc (lsc) on $\mathbb{X}$, if f is (resp. weakly) usc (lsc) at x, for all $x \in \mathbb{X}$.

Definition 2.3. Let Ω be a nonempty, closed and convex subset of $\mathbb{X}$. $\Gamma : \Omega \times \Omega \longrightarrow \mathbb{R}$ is said to be

(i) monotone, if $\Gamma(u, v) + \Gamma(v, u) \leq 0, \forall u, v \in \Omega$,

(ii) quasimonotone, if for all $\{u_n\} \subset \Omega$ with $u_n \rightharpoonup u \in \mathbb{X}$, we have

$$\liminf \Gamma(u_n, u) \leq 0,$$

(iii) pseudomonotone, if for all $\{u_n\} \subset \Omega$ with $u_n \rightharpoonup u \in \mathbb{X}$ and

$$\liminf \Gamma(u_n, u) \geq 0,$$

we have

$$\limsup \Gamma(u_n, v) \leq \Gamma(u, v), \forall v \in \Omega.$$

Definition 2.4. [19] Let Ω be a nonempty, closed and convex subset of $\mathbb{X}$. Let $\Gamma : \Omega \times \Omega \longrightarrow \mathbb{R}$ be a bifunction with $\Gamma(u, u) = 0, \forall u \in \Omega$. The bifunction Γ is said to be maximal monotone if for every $u \in \Omega$ and for every convex function $\psi : \Omega \longrightarrow \mathbb{R}$ with $\psi(u) = 0$, we have

$$\psi(v) \geq \Gamma(v, u), \forall v \in \Omega \Rightarrow \psi(v) \geq -\Gamma(u, v), \forall v \in \Omega.$$

We recall the existence of a solution to the mixed equilibrium problems. Let $\mathbb{U}$ be a subset of a reflexive Banach space $\mathbb{X}$. Find $u \in \mathbb{U}$ such that

$$\Gamma(u, v) + \Upsilon(u, v) + \pounds(u, v) \geq 0, \forall v \in \mathbb{U}. \tag{2.1}$$

We require the following presumptions.

$$\emptyset \neq \mathbb{U} \text{ is a closed convex subset of } \mathbb{X}. \tag{2.2}$$

$$\Gamma : \mathbb{U} \times \mathbb{U} \longrightarrow \mathbb{R} \text{ is such that}$$
$$\begin{cases} (a)\ \Gamma \text{ is monotone and maximal monotone,} \\ (b)\ \Gamma(u, \cdot) \text{ is convex and lsc for all } u \in \mathbb{U}, \\ (c)\ \Gamma(u, u) = 0, \text{for all } u \in \mathbb{U}. \end{cases} \tag{2.3}$$

$$\Upsilon : \mathbb{U} \times \mathbb{U} \longrightarrow \mathbb{R} \text{ is such that}$$
$$\begin{cases} (a)\ \Upsilon \text{ is pseudomonotone,} \\ (b) \text{ for each finite subset } \mathcal{D} \text{ of } \mathbb{U}, \Upsilon(\cdot, v) \text{ is usc on } conv(\mathcal{D}), \text{for all } v \in \mathbb{U}, \\ (c)\ \Upsilon(u, \cdot) \text{ is convex for all } u \in \mathbb{U}, \\ (d)\ \Upsilon(u, u) = 0, \text{for all } u \in \mathbb{U}. \end{cases} \tag{2.4}$$

$\pounds : \mathbb{U} \times \mathbb{U} \longrightarrow \mathbb{R}$ is such that

$$\begin{cases} (a)\ \pounds \text{ is quasimonotone,} \\ (b)\ \pounds(\cdot, v) \text{ is usc for all } v \in \mathbb{U}, \\ (c)\ \pounds(u, \cdot) \text{ is convex for all } u \in \mathbb{U}, \\ (d)\ \pounds(u, u) = 0, \text{for all } u \in \mathbb{U}. \end{cases} \tag{2.5}$$

There is a nonempty weakly compact subset $\mathbb{W}$ such that for every $\lambda > 0$ small enough, a weakly compact and convex subset $\mathbb{B}_\lambda$ of $\mathbb{U}$ exists that satisfies the following condition: for all $u \in \mathbb{U}\backslash\mathbb{W}$, there exists $v \in \mathbb{B}_\lambda$ such that

$$\Upsilon(u, v) + \pounds(u, v) + \lambda\langle \jmath u, v - u\rangle_{\mathbb{X}^* \times \mathbb{X}} < \Gamma(v, u). \tag{2.6}$$

Theorem 2.5. [20] *Assume that the hypotheses (2.2)-(2.6) hold. Then, (2.1) has at least one solution $u \in \mathbb{U}$.*

Definition 2.6. [21] Let $\mathbb{X}$ be a Banach space and $\jmath : \mathbb{X} \longrightarrow \mathbb{R}$ be a locally Lipschitz function, that is, for each $x \in \mathbb{X}$, there are a neighbourhood $\mathscr{N} = \mathscr{N}(x)$ and a constant $\kappa_{\mathscr{N}} > 0$ such that

$$|\jmath(w) - \jmath(z)| \leq \kappa_{\mathscr{N}} \|w - z\|_{\mathbb{X}},\ \forall w, z \in \mathscr{N}.$$

The generalized directional derivative of $\jmath$ at $x \in \mathbb{X}$ in the direction $v \in \mathbb{X}$, denoted by $\jmath^0(x; v)$, is defined by

$$\jmath^0(x; v) = \limsup_{y \longrightarrow x, \lambda \downarrow 0} \frac{\jmath(y + \lambda v) - \jmath(y)}{\lambda}.$$

The generalized gradient of $\jmath$ at x, denoted by $\partial\jmath(x)$, is given by

$$\partial\jmath(x) = \{x^* \in \mathbb{X}^* | \langle x^*, v\rangle \leq \jmath^0(x; v),\ \forall\ v \in \mathbb{X}\}.$$

Lemma 2.7. [3] *Let $\mathbb{X}$ be a Banach space, $0 < T < \infty$. Let $\Im : L^2(0, T; \mathbb{X}) \longrightarrow L^2(0, T; \mathbb{X})$ be an operator such that*

$$\|(\Im\eta_1)(t) - (\Im\eta_2)(t)\|^2_{\mathbb{X}} \leq \vartheta \int_0^t \|\eta_1(s) - \eta_2(s)\|^2_{\mathbb{X}} ds,\ \forall \eta_1, \eta_2 \in L^2(0, T; \mathbb{X}), t \in (0, T)$$

where $\vartheta > 0$ is a constant. There exists a unique $\eta^* \in L^2(0, T; \mathbb{X})$ such that

$$\Im\eta^* = \eta^*.$$

3. Main Results

In this section, we discuss the solution of the mixed evolutionary variational–hemivariational inequality problems with a set of constraints and history-dependent operators.
Let $(\mathbb{V}, \mathbb{H}, \mathbb{V}^*)$ be an evolution triple of spaces, where $\mathbb{V}$ is a separable reflexive Banach space, $\mathbb{H}$ is a separable Hilbert space, the embedding $\mathbb{V} \subset \mathbb{H}$ is continuous and compact, and $\mathbb{V}$ is dense in $\mathbb{H}$. Given $0 < T < +\infty$, we give the following Bochner spaces

$$\mathscr{V} = L^2(0, T; \mathbb{V}),\ \mathscr{V}^* = L^2(0, T; \mathbb{V}^*),\ \mathscr{W} = \{v \in \mathscr{V} | v' \in \mathscr{V}^*\},$$

where v' denotes the distributional derivative of v, and $\mathscr{V}$ and $\mathscr{V}^*$ are reflexive Banach spaces, $\mathscr{W} \subset L^2(0, T; \mathbb{H})$ is a separable, reflexive Banach space and compact. It is well known that the embedding $\mathscr{V} \subset L^2(0, T; \mathbb{H}) \subset \mathscr{V}^*$ are continuous. The duality pairing between $\mathscr{V}^*$ and $\mathscr{V}$ is defined by

$$\langle w, v\rangle_{\mathscr{V}^* \times \mathscr{V}} = \int_0^T \langle w(t), v(t)\rangle_{\mathbb{V}^* \times \mathbb{V}}\ dt,\ \forall w \in \mathscr{V}^*, v \in \mathscr{V},$$

and operator $\mathcal{L} : D(\mathcal{L}) \subset \mathscr{V} \longrightarrow \mathscr{V}^*$ is

$$\mathcal{L}v = v'\ \forall v \in D(\mathcal{L}), \tag{3.1}$$

where $D(\mathcal{L}) = \{v \in \mathscr{W} | v(0) = 0\}$ is linear and maximal monotone.

Let $\mathbb{X}$, $\mathbb{Y}$ and $\mathbb{Z}$ be Banach spaces. The mixed evolutionary variational–hemivariational inequality problem for finding $w \in \mathscr{W}$ such that $w(t) \in \Omega$ for a.e. $t \in (0, T)$ and

$$\begin{cases} \langle w'(t) + \mathcal{A}_t(w(t)) + (\mathcal{R}_1 w)(t) - f(t), v - w(t)\rangle_{\mathbb{V}^* \times \mathbb{V}} + \jmath_t^0((\mathcal{S}w)(t), \mathcal{M}w(t); \mathcal{M}v - Mw(t)) \\ +\varphi_t((\mathcal{R}_2 w)(t), \mathcal{M}v) - \varphi_t((\mathcal{R}_2 w)(t), \mathcal{M}w(t)) \geq 0, \forall v \in \Omega, t \in (0, T), \\ w(0) = w_0. \end{cases} \tag{3.2}$$

The assumptions for (3.2) are the following.

$\mathcal{A} : (0, T) \times \mathbb{V} \longrightarrow \mathbb{V}^*$ is such that

$$\begin{cases} (a)\ \mathcal{A}_{(\cdot)}(v) \text{ is measurable on } (0, T) \text{ for all } v \in \mathbb{V}, \\ (b)\ \mathcal{A}_t(\cdot) \text{ is demicontinuous on } \mathbb{V} \text{ for a.e. } t \in (0, T), \\ (c)\ \|\mathcal{A}_t(v)\|_{\mathbb{V}^*} \leq \varrho_0(t) + \varrho_1 \|v\|_{\mathbb{V}}, \forall v \in \mathbb{V}, t \in (0, T) \text{ with a function} \\ \varrho_0 \in L^2(0, T) \text{ satisfying } \varrho_0 \geq 0 \text{ a.e. in } (0, T), \text{ and a constant } \varrho_1 \geq 0, \\ (d)\ \mathcal{A}_t(\cdot) \text{ is relaxed cocoercive for a.e. } t \in (0, T), \text{ i.e., for the constants } \alpha_{\mathcal{A}} > 0, \beta_{\mathcal{A}} > 0 \\ \text{such that} \\ \langle \mathcal{A}_t(v_1) - \mathcal{A}_t(v_2), v_1 - v_2\rangle_{\mathbb{V}^* \times \mathbb{V}} \geq -\alpha_{\mathcal{A}} \|\mathcal{A}_t(v_1) - \mathcal{A}_t(v_2)\|_{\mathbb{V}^*}^2 + \beta_{\mathcal{A}} \|v_1 - v_2\|_{\mathbb{V}}^2, \forall v_1, v_2 \in \mathbb{V}, \\ (e)\ \mathcal{A}_t(\cdot) \text{ is Lipschitz continuous with respect to the constant } \zeta_{\mathcal{A}} > 0 \text{ such that} \\ \|\mathcal{A}_t(v_1) - \mathcal{A}_t(v_2)\|_{\mathbb{V}^*} \leq \zeta_{\mathcal{A}} \|v_1 - v_2\|_{\mathbb{V}}, \forall v_1, v_2 \in \mathbb{V}, t \in (0, T). \end{cases} \tag{3.3}$$

$\jmath : (0, T) \times \mathbb{Z} \times \mathbb{X} \longrightarrow \mathbb{R}$ is such that

$$\begin{cases} (a)\ \jmath_{(\cdot)}(z, v) \text{ is measurable on } (0, T) \text{ for all } z \in \mathbb{Z}, v \in \mathbb{X}, \\ (b)\ \jmath_t(\cdot, v) \text{ is continuous on } Z \text{ for all } v \in \mathbb{X}, \text{ a.e. } t \in (0, T), \\ (c)\ \jmath_t(z, \cdot) \text{ is locally Lipschitz on } \mathbb{X} \text{ for all } z \in \mathbb{Z}, \text{ a.e. } t \in (0, T), \\ (d)\ \|\partial \jmath_t(z, v)\|_{\mathbb{V}^*} \leq \vartheta_0^j(t) + \vartheta_1^j \|z\|_{\mathbb{Z}} + \vartheta_2^j \|v\|_{\mathbb{X}}, \forall z \in \mathbb{Z}, v \in \mathbb{X}, t \in (0, T) \\ \text{with } \vartheta_0^j \in L^2(0, T) \text{ and } \vartheta_0^j, \vartheta_1^j, \vartheta_2^j \geq 0, \\ (e)\ \jmath_t^0(z_1, v_1; v_2 - v_1) + \jmath_t^0(z_2, v_2; v_1 - v_2) \leq \bar{\alpha}_2^j \|z_1 - z_2\|_{\mathbb{Z}} \|v_1 - v_2\|_{\mathbb{X}} + \alpha_2^j \|v_1 - v_2\|_{\mathbb{X}}^2 \\ \text{for all } z_i \in \mathbb{Z}, v_i \in \mathbb{X}, i = 1, 2, \text{ a.e. } t \in (0, T) \text{ with } \bar{\alpha}_2^j \geq 0, \ \alpha_2^j \geq 0. \end{cases} \tag{3.4}$$

$\varphi : (0, T) \times \mathbb{Y} \times \mathbb{X} \longrightarrow \mathbb{R}$ is such that

$$\begin{cases} (a)\ \varphi_{(\cdot)}(y, v) \text{ is measurable on } (0, T) \text{ for all } y \in \mathbb{Y}, v \in \mathbb{X}, \\ (b)\ \varphi_t(\cdot, v) \text{ is continuous on } \mathbb{Y} \text{ for all } v \in \mathbb{X}, \text{ a.e. } t \in (0, T), \\ (c)\ \varphi_t(y, \cdot) \text{ is convex and lower semi continuous on } \mathbb{X} \text{ for all } y \in \mathbb{Y}, \text{ a.e. } t \in (0, T), \\ (d)\ \varphi_t(y_1, v_2) - \varphi_t(y_1, v_1) + \varphi_t(y_2, v_1) - \varphi_t(y_2, v_2) \leq \beta_\varphi \|y_1 - y_2\|_{\mathbb{Y}} \|v_1 - v_2\|_{\mathbb{X}} \\ \text{for all } y_i \in \mathbb{Y}, v_i \in \mathbb{X}, i = 1, 2, \text{ a.e. } t \in (0, T) \text{ with } \beta_\varphi \geq 0, \\ (e)\ \|\partial \varphi_t(y, v)\|_{\mathbb{X}^*} \leq \vartheta_0^\varphi + \vartheta_1^\varphi \|y\|_{\mathbb{Y}} + \vartheta_2^\varphi \|v\|_{\mathbb{V}} \forall y \in \mathbb{Y}, v \in \mathbb{V}, \text{ a.e. } t \in (0, T) \\ \text{with } \vartheta_0^\varphi \in L^2(0, T), \vartheta_0^\varphi, \vartheta_1^\varphi, \vartheta_2^\varphi \geq 0. \end{cases} \tag{3.5}$$

$\mathcal{R}_1 : \mathscr{V} \longrightarrow \mathscr{V}^*, \mathcal{R}_2 : \mathscr{V} \longrightarrow L^2(0,T;\mathbb{Y}),$ and $\mathcal{S} : \mathscr{V} \longrightarrow L^2(0,T;\mathbb{Z})$ are such that

$$\begin{cases} (a)\ \|(\mathcal{R}_1 v_1)(t) - (\mathcal{R}_1 v_2)(t)\|_{\mathbb{V}^*} \leq \vartheta^{\mathcal{R}_1} \int_0^t \|v_1(s) - v_2(s)\|_{\mathbb{V}} ds, \forall v_1, v_2 \in \mathscr{V} \\ \text{ a.e. } t \in (0,T) \text{ with } \vartheta^{\mathcal{R}_1} > 0, \\ (b)\ \|(\mathcal{R}_2 v_1)(t) - (\mathcal{R}_2 v_2)(t)\|_{\mathbb{Y}} \leq \vartheta^{\mathcal{R}_2} \int_0^t \|v_1(s) - v_2(s)\|_{\mathbb{V}} ds,\ \forall v_1, v_2 \in \mathscr{V}, \\ \text{ a.e. } t \in (0,T) \text{ with } \vartheta^{R_2} > 0, \\ (c)\ \|(\mathcal{S} v_1)(t) - (\mathcal{S} v_2)(t)\|_{\mathbb{Z}} \leq \vartheta^{\mathcal{S}} \int_0^t \|v_1(s) - v_2(s)\|_{\mathbb{V}} ds, \forall v_1, v_2 \in \mathscr{V}, \\ \text{ a.e. } t \in (0,T) \text{ with } \vartheta^{\mathcal{S}} > 0. \end{cases} \tag{3.6}$$

$$\emptyset \neq \Omega \text{ is a closed and convex subset of } \mathbb{V}. \tag{3.7}$$

$\mathcal{M} : \mathbb{V} \longrightarrow \mathbb{X}$ is such that

$$\begin{cases} (a)\ \mathcal{M} \text{ is an affine operator.} \\ (b) \text{ the Nemitsky operator } \mathfrak{M} : \mathscr{V} \longrightarrow L^2(0,T;\mathbb{X}) \text{ corresponding to } \mathcal{M} \text{ is compact.} \end{cases} \tag{3.8}$$

$$f \in \mathscr{V}^*, w_0 \in \mathbb{V}. \tag{3.9}$$

$$\begin{cases} \beta_{\mathcal{A}} - \alpha_{\mathcal{A}} \zeta_{\mathcal{A}}^2 > \alpha_2^j \|\mathcal{A}_{\mathcal{M}}\|^2, \\ \text{where } \mathcal{A}_{\mathcal{M}} : \mathbb{V} \longrightarrow \mathbb{X} \text{ is defined by } \mathcal{A}_{\mathcal{M}} v = \mathcal{M}v - \mathcal{M}_0\ \forall v \in \mathbb{V}. \end{cases} \tag{3.10}$$

We prove the existence and uniqueness result of (3.2).

Theorem 3.1. *If* (3.3)-(3.10) *hold with* $w_0 = 0$*, then* (3.2) *has a unique solution.*

Proof. This is achieved in several steps.
Step 1. Let $\xi \in \mathscr{V}, \eta \in L^2(0,T;\mathbb{Y})$ and $\varsigma \in L^2(0,T:\mathbb{Z})$ be fixed and consider the following auxiliary problem.

Find $w = w_{\xi\eta\varsigma} \in \mathscr{W}$ with $w(t) \in \Omega$ for a.e. $t \in (0,T)$ such that

$$\begin{cases} \langle w'(t) + \mathcal{A}_t(w(t)) - f(t) + \xi(t), v - w(t)\rangle_{\mathbb{V}^* \times \mathbb{V}} + j_t^0(\varsigma(t), \mathcal{M}w(t); \mathcal{M}v - \mathcal{M}w(t)) \\ +\varphi_t(\eta(t), \mathcal{M}v) - \varphi_t(\eta(t), \mathcal{M}w(t)) \geq 0,\ \forall v \in \Omega,\ \text{ a.e. } t \in (0,T), \\ w(0) = 0. \end{cases} \tag{3.11}$$

We demonstrate the uniqueness of the solution to (3.11). For the sake of simplicity, we exclude the subscripts ξ, η and ς from the proof of this part. Let $w_i \in \mathscr{W}, i = 1,2$ be solutions to (3.11), i.e., $w_i(t) \in \Omega$ for a.e. $t \in (0,T)$, $w_i(0) = 0$ and

$$\langle w_i'(t) + \mathcal{A}_t(w_i(t)) - f(t) + \xi(t), v - w_i(t)\rangle_{\mathbb{V}^* \times \mathbb{V}} + j_t^0(\varsigma(t), \mathcal{M}w_i(t); \mathcal{M}v - \mathcal{M}w_i(t))$$
$$+\varphi_t(\eta(t), \mathcal{M}v) - \varphi_t(\eta(t), \mathcal{M}w_i(t)) \geq 0, \forall v \in \Omega,\ \text{ a.e. } t \in (0,T),\ i = 1,2.$$

From the aforementioned inequalities, we get

$$\langle w_1'(t) - w_2'(t), w_1(t) - w_2(t)\rangle_{\mathbb{V}^* \times \mathbb{V}} + \langle \mathcal{A}_t(w_1(t)) - \mathcal{A}_t(w_2(t)), w_1(t) - w_2(t)\rangle_{\mathbb{V}^* \times \mathbb{V}}$$
$$\leq j_t^0(\varsigma(t), \mathcal{M}w_1(t); \mathcal{M}w_2(t) - \mathcal{M}w_1(t)) + j_t^0(\varsigma(t), \mathcal{M}w_2(t); \mathcal{M}w_1(t) - \mathcal{M}w_2(t)).$$

Using the integration by parts formula, (3.3)(d)(e) and (3.4)(e), we have

$$\frac{1}{2}\|w_1(t) - w_2(t)\|_{\mathbb{H}}^2 + (-\alpha_{\mathcal{A}} \zeta_{\mathcal{A}}^2 + \beta_{\mathcal{A}}) \int_0^t \|w_1(s) - w_2(s)\|_{\mathbb{V}}^2 ds$$
$$\leq\ \alpha_2^j \int_0^t \|\mathcal{M}w_1(s) - \mathcal{M}w_2(s)\|_{\mathbb{X}}^2 ds, \forall t \in [0,T].$$

Then, from (3.8), we obtain

$$\left(-\alpha_{\mathcal{A}} \zeta_{\mathcal{A}}^2 + \beta_{\mathcal{A}} - \alpha_2^j \|\mathcal{A}_{\mathcal{M}}\|^2\right) \|w_1 - w_2\|_{L^2(0,t;\mathbb{V})}^2 \leq 0, \forall t \in [0,T].$$

Therefore, from (3.10), we have $w_1 = w_2$ and the proof is completed.
Step 2. We are now proof that (3.11) has a solution. For this purpose, we introduce

$$\mho = L^2(0,T;\Omega) = \{v \in \mathscr{V} | v(t) \in \Omega \text{ for a.e. } t \in (0,T)\}, \tag{3.12}$$

and $\mho_1 = D(\mathcal{L}) \cap \mho$, where

$$D(\mathcal{L}) = \{w \in \mathscr{W} \mid w(0) = 0\}.$$

Consider the following problem for finding $w \in \mho_1$ such that

$$\begin{aligned}&\int_0^T \langle w'(t) + \mathcal{A}_t(w(t)) - f(t) + \xi(t), z(t) - w(t)\rangle_{\mathbb{V}^*\times\mathbb{V}} dt \\ &\quad + \int_0^T \jmath_t^0(\varsigma(t), \mathcal{M}w(t); \mathcal{M}z(t) - \mathcal{M}w(t))dt \\ &\quad + \int_0^T (\varphi_t(\eta(t), \mathcal{M}z(t)) - \varphi_t(\eta(t), \mathcal{M}w(t)))dt \geq 0, \forall\, z \in \mho_1.\end{aligned} \tag{3.13}$$

We now prove that (3.11) and (3.13) are equivalent. For this, we assume that $w \in \mathscr{W}$ is a solution of (3.11). This implies that $w \in \mho$ and $w(0) = 0$, which leads to $w \in \mho_1$. Let $z \in \mho_1$. Then, we have

$$\begin{aligned}\langle w'(t) + \mathcal{A}_t(w(t)) - f(t) + \xi(t), z(t) - w(t)\rangle_{\mathbb{V}^*\times\mathbb{V}} + \jmath_t^0(\varsigma(t), \mathcal{M}w(t); \mathcal{M}z(t) - \mathcal{M}w(t)) \\ +\varphi_t(\eta(t), \mathcal{M}z(t)) - \varphi_t(\eta(t), \mathcal{M}w(t)) \geq 0, \text{ for a.e. } t \in (0,T).\end{aligned}$$

By integration, we deduce that $w \in \mho_1$ is a solution to (3.13). Assume that $w \in \mho_1$ solves (3.13). Since $0 \in \Omega$ and $\mho$ is a convex set, by using [[22], Theorem 9.1, p.270], we have

$$D(\mathcal{L}) \bigcap \mho \text{ is dense in } \mho. \tag{3.14}$$

By taking advantage of (3.14), we discover that $w \in \mho_1$ is a solution to the following problem. Find $w \in \mho_1$ such that

$$\begin{aligned}\int_0^T \langle w'(t) + \mathcal{A}_t(w(t)) - f(t) + \xi(t), \bar{z}(t) - w(t)\rangle_{\mathbb{V}^*\times\mathbb{V}} dt + \int_0^T \jmath_t^0(\varsigma(t), \mathcal{M}w(t); \mathcal{M}\bar{z}(t) - \mathcal{M}w(t))dt \\ + \int_0^T (\varphi_t(\eta(t), \mathcal{M}\bar{z}(t)) - \varphi_t(\eta(t), \mathcal{M}w(t)))dt \;\geq 0, \forall \bar{z} \in \mho_1.\end{aligned} \tag{3.15}$$

In fact, let $\bar{z} \in \mho = L^2(0,T;\Omega)$. From (3.14), there exists a sequence $z_n \in D(\mathcal{L}) \cap \mho$ such that

$$z_n \longrightarrow \bar{z} \in \mathscr{V}.$$

From (3.13), we have

$$\begin{aligned}&\int_0^T \langle w'(t) + \mathcal{A}_t(w(t)) - f(t) + \xi(t), z_n(t) - w(t)\rangle_{\mathbb{V}^*\times\mathbb{V}} dt \\ &+ \int_0^T \jmath_t^0(\varsigma(t), \mathcal{M}w(t); \mathcal{M}z_n(t) - \mathcal{M}w(t))dt \\ &+ \int_0^T (\varphi_t(\eta(t), \mathcal{M}z_n(t)) - \varphi_t(\eta(t), \mathcal{M}w(t)))dt \longrightarrow 0 \text{ as } n \longrightarrow \infty,\ \forall n \in \mathbb{N}.\end{aligned} \tag{3.16}$$

From [[5], Theorem 2.39], we have

$$z_n(t) \longrightarrow \bar{z}(t) \in \mathbb{V},\ \text{ for a.e. } t \in (0,T)$$

and there is $g \in L^2(0,T)$ such that

$$\|z_n(t)\|_{\mathbb{V}} \leq g(t) \text{ for a.e. } t \in (0,T).$$

Hence,

$$\mathcal{M}z_n(t) \longrightarrow \mathcal{M}\bar{z}(t) \in \mathbb{X} \text{ for a.e. } t \in (0,T).$$

From the upper semicontinuity of $\jmath_t^0$ with the last variable, we have

$$\limsup \jmath_t^0(\varsigma(t), \mathcal{M}w(t); \mathcal{M}z_n(t) - \mathcal{M}w(t)) \leq \jmath_t^0(\varsigma(t), \mathcal{M}w(t); \mathcal{M}\bar{z} - \mathcal{M}w(t)), \forall t \in (0, T). \tag{3.17}$$

Using (3.4)(d) to estimate (3.17) as following:

$$\begin{aligned} |\jmath_t^0(\varsigma(t), \mathcal{M}w(t); \mathcal{M}z_n(t) - \mathcal{M}w(t))| &\leq \|\partial \jmath_t(\varsigma(t), \mathcal{M}w(t))\|_{\mathbb{X}^*} \|\mathcal{M}z_n(t) - \mathcal{M}w(t)\|_{\mathbb{X}} \\ &\leq \left(\vartheta_0^j(t) + \vartheta_1^j \|\varsigma\|_{\mathbb{Z}} + \vartheta_2^j \|\mathcal{M}w(t)\|_{\mathbb{X}}\right) (\|\mathcal{M}\| g(t) + \|\mathcal{M}w(t)\|_{\mathbb{X}}) \\ &= \Lambda(t), \text{ for a.e. } t \in (0, T) \text{ with } \Lambda \in L^1(0, T). \end{aligned}$$

Using the Fatou Lemma to get

$$\limsup \int_0^T \jmath_t^0(\varsigma(t), \mathcal{M}w(t); \mathcal{M}z_n(t) - \mathcal{M}w(t))dt \leq \int_0^T \jmath_t^0(\varsigma(t), \mathcal{M}w(t); \mathcal{M}\bar{z}(t) - \mathcal{M}w(t))dt. \tag{3.18}$$

Since the function $\varphi_t(\eta(t), \cdot)$ is continuous on $\mathbb{X}$. therefore, from [23], we have

$$\varphi_t(\eta(t), \mathcal{M}z_n(t)) - \varphi_t(\eta(t), \mathcal{M}w(t)) \longrightarrow \varphi_t(\eta(t), \mathcal{M}\bar{z}(t)) - \varphi_t(\eta(t), \mathcal{M}w(t)) \text{ for a.e. } t \in (0, T).$$

It follows from (3.5)(e) that

$$\begin{aligned} |\varphi_t(\eta(t), \mathcal{M}z_n(t)) - \varphi_t(\eta(t), \mathcal{M}w(t))| &\leq \|\partial\varphi_t(\eta(t), \mathcal{M}z_n(t))\|_{\mathbb{X}^*} \|\mathcal{M}z_n(t) - \mathcal{M}w(t)\|_{\mathbb{X}} \\ &\leq (\vartheta_0^\varphi(t) + \vartheta_1^\varphi \|\eta(t)\|_{\mathbb{Y}} + \vartheta_2^\varphi \|\mathcal{M}\| g(t)) (\|\mathcal{M}\| g(t) + \|\mathcal{M}w(t)\|_{\mathbb{X}}) \\ &= \tilde{\Lambda}(t), \text{ for a.e. } t \in (0, T) \text{ with } \tilde{\Lambda} \in L^2(0, T). \end{aligned}$$

Using the Lebesgue-dominated convergence theorem, we have

$$\lim \int_0^T (\varphi_t(\eta(t), \mathcal{M}z_n(t)) - \varphi_t(\eta(t), \mathcal{M}w(t)))dt = \int_0^T (\varphi_t(\eta(t), \mathcal{M}\bar{z}(t)) - \varphi_t(\eta(t), \mathcal{M}w(t)))dt. \tag{3.19}$$

Using (3.18) and (3.19) and taking the upper limit in (3.16), we conclude that $w \in \mho_1$ is a solution to (3.15).

Finally, we show that (3.11) and (3.15) are equivalent. It is clear that (3.11) implies (3.15). The converse implication follows from [[24], Lemma 2.3], we conclude that (3.11) and (3.15) are equivalent and the proof is completed. □

Next to demonstrate the existence of a solution to (3.11), it is sufficient to prove that there exists a solution to (3.15).

We hereby demonstrate that there is at least one solution to the equation (3.15). This can be done in various methods. For our purpose, applying Theorem 2.5 is sufficient. We introduce the following additional notation which allows us to rewrite (3.15) in the form of the mixed equilibrium inequality as stated in (2.1). Let the operators $\mathscr{A} : \mathscr{V} \longrightarrow \mathscr{V}^*$, $\mathscr{M} : \mathscr{V} \longrightarrow L^2(0, T; \mathbb{X})$, and the functions $\psi : \mathscr{V} \longrightarrow \mathbb{R}$, $J : \mathscr{V} \longrightarrow \mathbb{R}$ be defined by

$$\begin{aligned} (\mathscr{A}w)(t) &= \mathcal{A}_t(w(t)), \forall w \in \mathscr{V}, \text{ a.e. } t \in (0, T), \\ (\mathscr{M}w)(t) &= \mathcal{M}w(t), \forall w \in \mathscr{V}, \text{ a.e. } t \in (0, T), \\ \psi(w) &= \int_0^T \varphi_t(\eta(t), \mathcal{M}w(t))dt \text{ for } w \in \mathscr{V}, \\ J(w) &= \int_0^T \jmath_t(\varsigma(t), \mathcal{M}w(t))dt \text{ for } w \in \mathscr{V}. \end{aligned}$$

Next, we establish the bifunctions $\Gamma, \Upsilon, \mathscr{L} : \mho_1 \times \mho_1 \longrightarrow \mathbb{R}$ described by

$$\Gamma(w, z) = \langle \mathcal{L}w, z - w \rangle_{\mathscr{V}^* \times \mathscr{V}} + \psi(z) - \psi(w), \tag{3.20}$$

$$\Upsilon(w, z) = \langle \mathcal{A}w - f + \xi, z - w \rangle_{\mathscr{V}^* \times \mathscr{V}}, \tag{3.21}$$

$$\pounds(w, z) = J^0(w; z - w), \forall w, z \in \mho_1. \tag{3.22}$$

Using (3.15), we can equivalently formulated as follows.

Find $w \in \mho_1$ such that

$$\Gamma(w, z) + \Upsilon(w, z) + \pounds(w, z) \geq 0, \forall\, z \in \mho_1. \tag{3.23}$$

From Theorem 2.5, we prove the existence of a solution to (3.23). We shall verify the assumptions made in this theorem for $\mathbb{U} = \mho_1$.

Claim 1. *The bifunctional* Γ *described by* (3.20) *verifies the condition* (2.3).

First, we prove (2.3)(a). Since

$$\Gamma(w, z) + \Gamma(z, w) = -\langle \mathcal{L}w - \mathcal{L}z, w - z\rangle_{\mathscr{V}^* \times \mathscr{V}} \leq 0, \forall\, w, z \in \mho_1.$$

Therefore, Γ is monotone. Now, we prove that Γ is a maximal monotone. Let $\Gamma_1(w, z) = \langle \mathcal{L}w, z - w\rangle_{\mathscr{V}^* \times \mathscr{V}}$ for $w, z \in \mho_1$, and $w \in \mho_1$, $\phi : \mho_1 \longrightarrow \mathbb{R}$ with $\phi(w) = 0$ be a convex function, then

$$\phi(z) \geq \Gamma(z, w), \forall\, z \in \mho_1 \Rightarrow \phi(v) \geq \Gamma_1(z, w) + \psi(w) - \psi(z),\ \forall\, z \in \mho_1.$$

Hence

$$\phi(z) + \psi(z) - \psi(w) \geq \Gamma_1(z, w), \forall\, z \in \mho_1. \tag{3.24}$$

Since $\mathcal{L} : \mathscr{V} \longrightarrow \mathscr{V}^*$ is a maximal monotone, then Γ_1 is also a maximal monotone. Thus, (3.24) implies that

$$\phi(z) + \psi(z) - \psi(w) \geq -\Gamma_1(w, z), \forall\, z \in \mho_1.$$

Therefore,

$$\phi(z) \geq -\Gamma_1(w, z) - \psi(w) + \psi(z) = -\Gamma(w, z), \forall\, z \in \mho_1,$$

which show that Γ is maximal monotone.

Assumptions (3.8) and (3.5)(c) bring us closer to condition (2.3)(b). We will prove that $\Gamma(w, \cdot)$ is convex and lower semi-continuous for all $w \in \mho_1$. Let $w, z_1, z_2 \in \mho_1$ and $\gamma \in (0, 1)$. According to (3.5)(c), we have

$$\begin{aligned}
\Gamma(w, \gamma z_1 + (1-\gamma)z_2) &= \langle \mathcal{L}w, \gamma z_1 + (1-\gamma)z_2 - w\rangle_{\mathscr{V}^* \times \mathscr{V}} + \psi(\gamma z_1 + (1-\gamma)z_2) - \psi(w) \\
&\leq \gamma\langle \mathcal{L}w, z_1 - w\rangle_{\mathscr{V}^* \times \mathscr{V}} + (1-\gamma)\langle \mathcal{L}w, z_2 - w\rangle_{\mathscr{V}^* \times \mathscr{V}} + \gamma(\psi(z_1) - \psi(w)) \\
&\quad + (1-\gamma)(\psi(z_2) - \psi(w)) \\
&= \gamma\Gamma(w, z_1) + (1-\gamma)\Gamma(w, z_2),
\end{aligned}$$

which suggests that for all $w \in \mho_1$, $\Gamma(w, \cdot)$ is convex. Furthermore, (3.8) gives us

$$\mathscr{M}z_n \longrightarrow \mathscr{M}z \in L^2(0, T; \mathbb{X}),\ \text{with } z_n \longrightarrow z \in \mathscr{V},\ \text{as } n \longrightarrow \infty, \forall z_n, z \in \mho_1.$$

Therefore, we may assume that $(\mathscr{M}z_n)(t) \longrightarrow (\mathscr{M}z)(t) \in \mathbb{X}$ for a.e. $t \in (0, T)$, *i.e.,*

$$\mathcal{M}z_n(t) \longrightarrow \mathcal{M}z(t) \in \mathbb{X} \text{ for a.e. } t \in (0, T).$$

Finally, utilizing Fatou Lemma and (3.5)(c), we arrive at

$$\begin{aligned}
\liminf \psi(z_n) &= \liminf \int_0^T \varphi_t(\eta(t), \mathcal{M}z_n(t))dt \\
&\geq \int_0^T \liminf \varphi_t(\eta(t), \mathcal{M}z_n(t))dt \\
&\geq \int_0^T \varphi_t(\eta(t), \mathcal{M}z(t))dt \\
&= \psi(z).
\end{aligned}$$

Thus,

$$\begin{aligned}\liminf \Gamma(w, z_n) &= \liminf [\langle \mathcal{L}w, z_n - w\rangle_{\mathscr{V}^*\times\mathscr{V}} + \psi(z_n) - \psi(w)] \\ &\geq \liminf \langle \mathcal{L}w, z_n - w\rangle_{\mathscr{V}^*\times\mathscr{V}} + \liminf \psi(z_n) - \psi(w) \\ &\geq \langle \mathcal{L}w, z - w\rangle_{\mathscr{V}^*\times\mathscr{V}} + \psi(z) - \psi(w) \\ &= \Gamma(w, z), \forall w \in \mathbb{V},\end{aligned}$$

implies that for all $w \in \mho_1, \Gamma(w, \cdot)$ is lower semi-continuous. The requisite (2.3)(c) is obvious. Therefore, we deduce that Claim 1 is accurate.

Claim 2. *The bifunction* Υ *described by* (3.21) *verifies the condition* (2.4).

First, we demonstrate that for each finite subset $\mathcal{D} \subset \mho_1$, $\Upsilon(\cdot, z)$ is upper semi-continuous on $conv(\mathcal{D})$ for all $z \in \mho_1$. Suppose $\{w_n\} \subset conv(D)$ is such that

$$w_n \longrightarrow w \in \mathbb{V}.$$

Since $conv(\mathcal{D})$ is a closed and convex set, such that $w \in conv(\mathcal{D})$. From (3.3)(b) and [[25], Proposition 27.7(b)], it follows that $\mathscr{A} : \mathscr{V} \longrightarrow \mathscr{V}^*$ is demicontinuous. So, here we are

$$\mathscr{A}w_n \rightharpoonup \mathscr{A}w \in \mathscr{V}^*$$

and

$$\begin{aligned}\limsup \Upsilon(w_n, z) &= \limsup\left(\langle \mathscr{A}w_n, z - w_n\rangle_{\mathscr{V}^*\times\mathscr{V}} + \langle \xi - f, z - w_n\rangle_{\mathscr{V}^*\times\mathscr{V}}\right) \\ &= \limsup \langle \mathscr{A}w_n, z - w_n\rangle_{\mathscr{V}^*\times\mathscr{V}} + \lim\langle \xi - f, z - w_n\rangle_{\mathscr{V}^*\times\mathscr{V}} \\ &= \langle \mathscr{A}w, z - w\rangle_{\mathscr{V}^*\times\mathscr{V}} + \langle \xi - f, z - w\rangle_{\mathscr{V}^*\times\mathscr{V}} \\ &= \Upsilon(w, z).\end{aligned}$$

Since $\Upsilon(\cdot, z)$ is upper semi-continuous on $conv(\mathcal{D})$ for all $z \in \mho_1$, *that is,* the condition (2.4)(b) is hold. From (3.3)(b),(d), Γ is pseudomonotone on $\mho_1$. The proof is similar to the [20, 26], therefore, for this reason, it is omitted. Hence, (2.4)(a) is hold. The conditions (2.4)(c),(d) is simply verified. Therefore, Claim 2 is true.

Claim 3. *The bifunctional* $\pounds$ *described by* (3.22) *verifies the condition* (2.5).

First, we prove that (2.5)(a). Let $\{w_n\} \subset \mho_1$ with $w_n \rightharpoonup w \in \mathscr{W}$. From (3.10), we obtain that $\mathscr{M}w_n \longrightarrow \mathscr{M}w \in L^2(0, T; \mathbb{X})$. Using the converse Lebesgue-dominated convergence theorem, we can determine that $\eta \in L^2(0, T)$, such that

$$\|\mathcal{M}w_n(t)\|_{\mathbb{X}} \leq \eta(t), \text{for a.e. } t \in (0, T)$$

and

$$\mathcal{M}w_n(t) \longrightarrow \mathcal{M}w(t) \in \mathbb{X} \text{ for a.e. } t \in (0, T).$$

Afterward, we consider the function $\Lambda_n : (0, T) \longrightarrow \mathbb{R}$ defined by

$$\Lambda_n(t) = \jmath_t^0(\varsigma(t), \mathcal{M}w_n(t); \mathcal{M}w(t) - \mathcal{M}w_n(t)) \text{ for a.e. } t \in (0, T).$$

Using (3.4)(d) and [[5], Proposition 3.23(iii)], we get

$$\begin{aligned}|\Lambda_n(t)| &= |\jmath_t^0(\varsigma(t), \mathcal{M}w_n(t); \mathcal{M}w(t) - \mathcal{M}w_n(t))| \\ &\leq \|\partial\jmath_t(\varsigma(t), \mathcal{M}w_n(t))\|_{\mathbb{X}^*}\|\mathcal{M}w(t) - \mathcal{M}w_n(t)\|_{\mathbb{X}} \\ &\leq \left(\vartheta_0^j(t) + \vartheta_1^j\|\varsigma(t)\|_{\mathbb{Z}} + \vartheta_2^j\|\mathcal{M}w_n(t)\|_{\mathbb{X}}\right)(\|\mathcal{M}w(t)\|_{\mathbb{X}} + \|\mathcal{M}w_n(t)\|_{\mathbb{X}}), \text{ for a.e. } t \in (0, T).\end{aligned}$$

Hence, we have

$$|\Lambda_n(t)| \leq \bar{\Lambda}(t) \text{ a.e. } t \in (0, T) \text{ with } \bar{\Lambda} \in L^1(0, T),$$

$$\bar{\Lambda}(t) = \left(\vartheta_0^j(t) + \vartheta_1^j\|\varsigma(t)\|_{\mathbb{Z}} + \vartheta_2^j\eta(t)\right)(\|\mathcal{M}w(t)\|_{\mathbb{X}} + \eta(t)) \text{ for a.e. } t \in (0, T).$$

Additionally, according to [[5], Proposition 3.23(ii)], we see that $\jmath_t^0(z,\cdot,\cdot)$ is usc on $\mathbb{X}\times\mathbb{X}$ for $t\in(0,T)$, $z\in\mathbb{Z}$. Consequently, using Fatou Lemma, we have

$$\begin{aligned}\liminf \mathcal{L}(w_n,w) &= \liminf J^0(w_n;w-w_n)\\ &\le \limsup \int_0^T \jmath_t^0\left(\varsigma(t),\mathcal{M}w_n(t);\mathcal{M}w(t)-\mathcal{M}w_n(t)\right)dt\\ &\le \int_0^T \jmath_t^0\left(\varsigma(t),\mathcal{M}w(t);0\right)dt\\ &=0.\end{aligned}$$

Hence (2.5)(a) is valid, and $\mathcal{L}$ is quasimonotone. Using the upper semicontinuity of $J^0(\cdot,z)$ for all $z\in\mathscr{V}$, we determine that the condition (2.5)(b). Since $J^0(w,\cdot)$ is positively homogeneous and subadditive for all $w\in\mathscr{V}$, therefore, it is convex and implies the condition (2.5)(c). Hence, (2.5)(d) is held, and we conclude that Claim 3 is supported.

Claim 4. *The condition* (2.6) *is valid.*

The condition (2.6) holds if

$$\frac{\Upsilon_\lambda(w,v_0)}{\|w-v_0\|_{\mathscr{V}}}\longrightarrow -\infty \text{ uniformly in } \lambda, \text{ as } \|w-v_0\|\longrightarrow+\infty, \text{ for some } v_0\in\mho_1, \tag{3.25}$$

where

$$\Upsilon_\lambda(w,v_0)=\Upsilon(w,v_0)+\mathcal{L}(w,v_0)+\lambda\langle \jmath w, v_0-w\rangle_{\mathscr{V}^*\times\mathscr{V}}, \forall\, w\in\mho_1.$$

From (3.3)(c),(d) and (3.4)(d),(e), we have

$$\begin{aligned}\Upsilon_\lambda(w,0) &= \langle \mathscr{A}w-f,-w\rangle_{\mathscr{V}^*\times\mathscr{V}}+J^0(w;-w)+\lambda\langle \jmath w,-w\rangle_{\mathscr{V}^*\times\mathscr{V}}\\ &= \langle \mathscr{A}w-\mathscr{A}0,0-w\rangle_{\mathscr{V}^*\times\mathscr{V}}+\langle \mathscr{A}0,0-w\rangle_{\mathscr{V}^*\times\mathscr{V}}+\langle f,w\rangle_{\mathscr{V}^*\times\mathscr{V}}\\ &\quad+J^0(w;0-w)+J^0(0;w-0)-J^0(0;w-0)+\lambda\langle \jmath w,-w\rangle_{\mathscr{V}^*\times\mathscr{V}}\\ &\le -\alpha_{\mathcal{A}}\zeta_{\mathcal{A}}^2\|w\|_{\mathscr{V}}^2+\beta_{\mathcal{A}}\|w\|_{\mathscr{V}}^2+\|\varrho_0\|_{L^2}\|w\|_{\mathscr{V}}+\alpha_2^j\left(\|\mathcal{A}_{\mathcal{M}}\|\|w\|_{\mathscr{V}}+\|\mathcal{M}_0\|_{\mathbb{X}}\right)^2\\ &\quad+\left(\vartheta_0^j+\vartheta_1^j\|\varsigma\|_{L^2(0,T;\mathbb{Z})}\right)\left(\|\mathcal{A}_{\mathcal{M}}\|\|w\|_{\mathscr{V}}+\|\mathcal{M}_0\|_{\mathbb{X}}\right)+\|f\|_{\mathscr{V}^*}\|w\|_{\mathscr{V}}-\lambda\|w\|_{\mathscr{V}}^2\\ &\le -\left(\alpha_{\mathcal{A}}\zeta_{\mathcal{A}}^2-\beta_{\mathcal{A}}-\alpha_2^j\|\mathcal{A}_{\mathcal{M}}\|^2\right)\|w\|_{\mathscr{V}}^2+\left(\|\varrho_0\|_{L^2}+2\alpha_2^j\|\mathcal{A}_{\mathcal{M}}\|\|\mathcal{M}_0\|_{\mathbb{X}}+\|f\|_{\mathscr{V}^*}\right.\\ &\quad\left.+\left(\vartheta_0^j+\vartheta_1^j\|\varsigma\|_{L^2(0,T;\mathbb{Z})}\right)\|\mathcal{A}_{\mathcal{M}}\|\right)\|w\|_{\mathscr{V}}+\left(\vartheta_0^j+\vartheta_1^j\|\varsigma\|_{L^2(0,T;\mathbb{Z})}\right)\|\mathcal{M}_0\|_{\mathbb{X}}+\alpha_2^j\|\mathcal{M}_0\|_{\mathbb{X}}^2.\end{aligned}$$

Hence,

$$\begin{aligned}\frac{\Upsilon_\lambda(w,0)}{\|w\|_{\mathscr{V}}} &\le -(\alpha_{\mathcal{A}}\zeta_{\mathcal{A}}^2-\beta_{\mathcal{A}}+\alpha_2^j\|\mathcal{A}_{\mathcal{M}}\|^2)\|w\|_{\mathscr{V}}+\|\varrho_0\|_{L^2}+2\alpha_2^j\|\mathcal{A}_{\mathcal{M}}\|\|\mathcal{M}_0\|_{\mathbb{X}}+\|f\|_{\mathscr{V}^*}\\ &\quad+\left(\vartheta_0^j+\vartheta_1^j\|\varsigma\|_{L^2(0,T;\mathbb{Z})}\right)\|\mathcal{A}_{\mathcal{M}}\|+\frac{\left(\vartheta_0^j+\vartheta_1^j\|\varsigma\|_{L^2(0,T;\mathbb{Z})}\right)\|\mathcal{M}_0\|_{\mathbb{X}}+\alpha_2^j\|\mathcal{M}_0\|_{\mathbb{X}}^2}{\|w\|_{\mathscr{V}}}.\end{aligned}$$

As a result, (3.25) is held with $v_0=0$.

According to Theorem 2.5, the equation (3.23) has a solution $w\in\mho_1$. Hence, there is a solution to (3.15). From Step 1, we may say that (3.15) has a unique solution $w\in\mathscr{W}$ and the proof is therefore complete.

Step 3. Let $(\xi_i,\eta_i,\varsigma_i)\in L^2(0,T;\mathbb{V}^*\times\mathbb{Y}\times\mathbb{Z}), i=1,2$ and $w_1=w_{\xi_1\eta_1\varsigma_1}\in\mathscr{W}$, $w_2=w_{\xi_2\eta_2\varsigma_2}\in\mathscr{W}$ with $w_1(t),w_2(t)\in\Omega$ for a.e. $t\in(0,T)$, be the unique solutions to (3.15) corresponding to $(\xi_1,\eta_1,\varsigma_1)$ and $(\xi_2,\eta_2,\varsigma_2)$, *respectively.* We will display the estimate below:

$$\|w_1-w_2\|_{L^2(0,t;\mathbb{V})}\le\vartheta\left(\|\varsigma_1-\varsigma_2\|_{L^2(0,t;\mathbb{Z})}+\|\eta_1-\eta_2\|_{L^2(0,t;\mathbb{Y})}+\|\xi_1-\xi_2\|_{L^2(0,t;\mathbb{V}^*)}\right), \forall t\in[0,T], \tag{3.26}$$

where ϑ is a positive constant.

From (3.15), we have

$$\langle w_1'(t) + \mathcal{A}_t(w_1(t)) - f(t) + \xi_1(t), w_2(t) - w_1(t)\rangle_{\mathbb{V}^*\times\mathbb{V}} + \jmath_t^0(\varsigma_1(t), \mathcal{M}w_1(t); \mathcal{M}w_2(t) - \mathcal{M}w_1(t)) \\ + \varphi_t(\eta_1(t), \mathcal{M}w_2(t)) - \varphi_t(\eta_1(t), \mathcal{M}w_1(t)) \geq 0, \text{ for a.e. } t \in (0,T) \tag{3.27}$$

and

$$\langle w_2'(t) + \mathcal{A}_t(w_2(t)) - f(t) + \xi_2(t), w_1(t) - w_2(t)\rangle_{\mathbb{V}^*\times\mathbb{V}} + \jmath_t^0(\varsigma_2(t), \mathcal{M}w_2(t); \mathcal{M}w_1(t) - \mathcal{M}w_2(t)) \\ + \varphi_t(\eta_2(t), \mathcal{M}w_1(t)) - \varphi_t(\eta_2(t), \mathcal{M}w_2(t)) \geq 0, \text{ for a.e. } t \in (0,T). \tag{3.28}$$

and, assume that

$$w_1(0) = w_2(0) = 0.$$

From (3.27) and (3.28), we have

$$\langle w_1'(t) - w_2'(t), w_2(t) - w_1(t)\rangle_{\mathbb{V}^*\times\mathbb{V}} + \langle \mathcal{A}_t(w_1(t)) - \mathcal{A}_t(w_2(t)), w_2(t) - w_1(t)\rangle_{\mathbb{V}^*\times\mathbb{V}} \\ + \jmath_t^0(\varsigma_1(t), \mathcal{M}w_1(t); \mathcal{M}w_2(t) - \mathcal{M}w_1(t)) + \jmath_t^0(\varsigma_2(t), \mathcal{M}w_2(t); \mathcal{M}w_1(t) - \mathcal{M}w_2(t)) \\ + \varphi_t(\eta_1(t), \mathcal{M}w_2(t)) - \varphi_t(\eta_1(t), \mathcal{M}w_1(t)) + \varphi_t(\eta_2(t), \mathcal{M}w_1(t)) - \varphi_t(\eta_2(t), \mathcal{M}w_2(t)) \\ \geq \langle \xi_1(t), w_1(t) - w_2(t)\rangle_{\mathbb{V}^*\times\mathbb{V}} - \langle \xi_2(t), w_2(t) - w_1(t)\rangle_{\mathbb{V}^*\times\mathbb{V}}, \text{ for a.e. } t \in (0,T). \tag{3.29}$$

Now, we integrate the aforementioned inequality on $(0,t)$, and use the assumptions (3.3)(d), (3.4)(e) and (3.5)(d) to obtain

$$\frac{1}{2}\|w_1(t) - w_2(t)\|_{\mathbb{H}}^2 - \frac{1}{2}\|w_1(0) - w_2(0)\|_{\mathbb{H}}^2 + (-\alpha_{\mathcal{A}}\zeta_{\mathcal{A}}^2 + \beta_{\mathcal{A}}) \int_0^t \|w_1(s) - w_2(s)\|_{\mathbb{V}}^2 ds \\ \leq \bar{\alpha}_2^j \|\mathcal{A}_{\mathcal{M}}\| \int_0^t \|\varsigma_1(s) - \varsigma_2(s)\|_{\mathbb{Z}} \|w_1(s) - w_2(s)\|_{\mathbb{V}} ds \\ + \alpha_2^j \|\mathcal{A}_{\mathcal{M}}\|^2 \int_0^t \|w_1(s) - w_2(s)\|_{\mathbb{V}} ds + \beta_\varphi \|\mathcal{A}_{\mathcal{M}}\| \int_0^t \|\eta_1(s) - \eta_2(s)\|_{\mathbb{Y}} \|w_1(s) - w_2(s)\|_{\mathbb{V}} ds \\ + \int_0^t \|\xi_1(s) - \xi_2(s)\|_{\mathbb{V}^*} \|w_1(s) - w_2(s)\|_{\mathbb{V}} ds, \ \forall t \in [0,T].$$

Next from (3.10) and the Hölder inequality, we have

$$\left(-\alpha_{\mathcal{A}}\zeta_{\mathcal{A}}^2 + \beta_{\mathcal{A}} - \alpha_2^j \|\mathcal{A}_{\mathcal{M}}\|^2\right) \|w_1 - w_2\|_{L^2(0,t;\mathbb{V})}^2 \leq \bar{\alpha}_2^j \|\mathcal{A}_{\mathcal{M}}\| \|\varsigma_1 - \varsigma_2\|_{L^2(0,t;\mathbb{Z})} \|w_1 - w_2\|_{L^2(0,t;\mathbb{V})} \\ + \beta_\varphi \|\mathcal{A}_{\mathcal{M}}\| \|\eta_1 - \eta_2\|_{L^2(0,t;\mathbb{Y})} \|w_1 - w_2\|_{L^2(0,t;\mathbb{V})} + \|\xi_1 - \xi_2\|_{L^2(0,t;\mathbb{V}^*)} \|w_1 - w_2\|_{L^2(0,t;\mathbb{V})}, \forall t \in [0,T].$$

Therefore, the inequality (3.26) follows from (3.10).

Step 4. In this stage, we use the argument of fixed point to define the operator $\Im : L^2(0,T;\mathbb{V}^* \times \mathbb{Y} \times \mathbb{Z}) \longrightarrow L^2(0,T;\mathbb{V}^* \times \mathbb{Y} \times \mathbb{Z})$ by

$$\Im(\xi,\eta,\varsigma) = (\mathcal{R}_1 w_{\xi\eta\varsigma}, \mathcal{R}_2 w_{\xi\eta\varsigma}, \mathcal{S} w_{\xi\eta\varsigma}), \forall (\xi,\eta,\varsigma) \in L^2(0,T;\mathbb{V}^* \times \mathbb{Y} \times \mathbb{Z}),$$

where $w_{\xi\eta\varsigma} \in \mathscr{W}$ is the unique solution to (3.11) corresponding to (ξ, η, ς). From (3.6), (3.25) and the Hölder inequality, we discover a constant $\vartheta > 0$ such that

$$\begin{aligned}
\|\Im(\xi_1, \eta_1, \varsigma_1)(t) - \Im(\xi_2, \eta_2, \varsigma_2)(t)\|^2_{\mathbb{V}^* \times \mathbb{Y} \times \mathbb{Z}} &= \|(\mathcal{R}_1 w_1)(t) - (\mathcal{R}_1 w_2)(t)\|^2_{\mathbb{V}^*} \\
&+ \|(\mathcal{R}_2 w_1)(t) - (\mathcal{R}_2 w_2)(t)\|^2_{\mathbb{Y}} + \|(\mathcal{S} w_1)(t) - (\mathcal{S} w_2)(t)\|^2_{\mathbb{Z}} \\
&\leq \left(\vartheta^{\mathcal{R}_1} \int_0^t \|w_1(s) - w_2(s)\|_{\mathbb{V}} ds\right)^2 + \left(\vartheta^{\mathcal{R}_2} \int_0^t \|w_1(s) - w_2(s)\|_{\mathbb{V}} ds\right)^2 \\
&+ \left(\vartheta^{\mathcal{S}} \int_0^t \|w_1(s) - w_2(s)\|_{\mathbb{V}} ds\right)^2 \\
&\leq \vartheta \|w_1 - w_2\|^2_{L^2(0,t,V^*)} \\
&\leq \vartheta \left(\|\varsigma_1 - \varsigma_2\|^2_{L^2(0,t;\mathbb{Z})} + \|\eta_1 - \eta_2\|^2_{L^2(0,t;\mathbb{Y})} + \|\xi_1 - \xi_2\|^2_{L^2(0,t;\mathbb{V}^*)}\right)
\end{aligned}$$

implies that

$$\begin{aligned}
&\|\Im(\xi_1, \eta_1, \varsigma_1)(t) - \Im(\xi_2, \eta_2, \varsigma_2)(t)\|^2_{\mathbb{V}^* \times \mathbb{Y} \times \mathbb{Z}} \\
&\leq \vartheta \int_0^t \|(\xi_1, \eta_1, \varsigma_1)(s) - (\xi_2, \eta_2, \varsigma_2)(s)\|^2_{\mathbb{V}^* \times \mathbb{Y} \times \mathbb{Z}}, ds \text{ for a.e. } t \in (0, T).
\end{aligned} \tag{3.30}$$

From Lemma 2.7, we have a unique fixed point $(\xi^*, \eta^*, \varsigma^*)$ of $\Im$, *i.e.,*

$$(\xi^*, \eta^*, \varsigma^*) \in L^2(0, T; \mathbb{V}^* \times \mathbb{Y} \times \mathbb{Z}) \text{ and } \Im(\xi^*, \eta^*, \varsigma^*) = (\xi^*, \eta^*, \varsigma^*).$$

Step 5. Let $(\xi^*, \eta^*, \varsigma^*) \in L^2(0, T; \mathbb{V}^* \times \mathbb{Y} \times \mathbb{Z})$ be the unique fixed point of $\Im$. We define $w_{\xi^*\eta^*\varsigma^*}$ to be the unique solution to (3.11) corresponding to $(\xi^*, \eta^*, \varsigma^*)$. From the definition of $\Im$, we have

$$\xi^* = \mathcal{R}_1(w_{\xi^*\eta^*\varsigma^*}),\ \eta^* = \mathcal{R}_2(w_{\xi^*\eta^*\varsigma^*}) \text{ and } \varsigma^* = \mathcal{S}(w_{\xi^*\eta^*\varsigma^*}).$$

Finally, we use these relations in (3.11), and come to the conclusion that $w_{\xi^*\eta^*\varsigma^*}$ is the unique solution to (3.2) and completes the proof.

Next, we define the constrained mixed variational-hemivariational inequality problem with the non-homogeneous initial condition $0 \neq w_0 \in \mathbb{V}$ for finding $w \in \mathscr{W}$ such that $w(t) \in \Omega$ for a.e. $t \in (0, T)$ and

$$\begin{cases}
\langle w'(t) + \mathcal{A}_t(w(t)) + (\mathcal{R}_1 w)(t) - f(t), v - w(t)\rangle_{\mathbb{V}^* \times \mathbb{V}} + \jmath_t^0((\mathcal{S}w)(t), \mathcal{M}w(t); \mathcal{M}v - \mathcal{M}w(t)) \\
+\varphi_t((\mathcal{R}_2 w)(t), \mathcal{M}v) - \varphi_t((\mathcal{R}_2 w)(t), \mathcal{M}w(t)) \geq 0, \forall v \in \Omega, \\
w(0) = w_0.
\end{cases} \tag{3.31}$$

Theorem 3.2. *Assume that* (3.2)-(3.6) *hold, then the equation* (3.31) *has a unique solution.*

Proof. Let $\bar{w}(t) = w(t) - w_0$ and $\bar{\Omega} = \{v - w_0 | v \in \Omega\} \subset \mathbb{V}$. We define the operator $\bar{\mathcal{A}} : (0, T) \times \mathbb{V} \longrightarrow \mathbb{V}^*$ and $\bar{\mathcal{M}} : \mathbb{V} \longrightarrow \mathbb{X}$ by

$$\bar{\mathcal{A}}_t(v) = \mathcal{A}_t(v + w_0),\ \text{for } v \in \mathbb{V},\ \text{a.e. } t \in (0, T), \tag{3.32}$$

$$\bar{\mathcal{M}}v = \mathcal{M}(v + w_0),\ \text{for } v \in \mathbb{V}. \tag{3.33}$$

Now, we reformulate the problem (3.31) as follows.

Find $\bar{w} \in \mathscr{W}$ such that $\bar{w}(t) \in \bar{\Omega}$ for a.e. $t \in (0, T)$ and

$$\begin{cases}
\langle \bar{w}'(t) + \bar{\mathcal{A}}_t(\bar{w}(t)) + (\mathcal{R}_1 \bar{w})(t) - f(t), v - \bar{w}(t)\rangle_{\mathbb{V}^* \times \mathbb{V}} \\
+\jmath_t^0((\mathcal{S}(\bar{w} + w_0))(t), \bar{\mathcal{M}}\bar{w}(t); \bar{\mathcal{M}}v - \bar{\mathcal{M}}\bar{w}(t)) \\
+\varphi_t((R_2(\bar{w} + w_0))(t), \bar{\mathcal{M}}v) - \varphi_t((\mathcal{R}_2(\bar{w} + w_0))(t), \bar{\mathcal{M}}\bar{w}(t)) \geq 0, \forall v \in \bar{\Omega}, \\
\bar{w}(0) = 0.
\end{cases} \tag{3.34}$$

From Theorem 3.1 to deduce that (3.34) has a unique solution $\bar{w} \in \mathscr{W}$. Therefore it is sufficient to show that $\bar{\mathcal{A}}_t$ and $\bar{\mathcal{M}}$ satisfy the condition (3.3) and (3.8).

Now we cinfirm that $\bar{\mathcal{A}}_t$ meets (3.3). Since (3.3)(a),(b) are clear. For $v \in \mathbb{V}$, a.e. $t \in (0,T)$, we have

$$\|\bar{\mathcal{A}}_t(v)\|_{\mathbb{V}^*} = \|\mathcal{A}_t(v+w_0)\|_{\mathbb{V}^*} \le \varrho_0(t) + \varrho_1\|v+w_0\|_{\mathbb{V}} \le \varrho_0(t) + \varrho_1\|w_0\|_{\mathbb{V}} + \varrho_1\|v\|_{\mathbb{V}}.$$

Therefore, with $\bar{\varrho}_0(t) = \varrho_0(t) + \varrho_1\|w_0\|_{\mathbb{V}}$ and $\bar{\varrho}_1 = \varrho_1$, the assumption (3.3)(c) holds. Additionally, for $v_1, v_2 \in \mathbb{V}$, a.e. $t \in (0,T)$, we obtain

$$\begin{aligned}
\langle \bar{\mathcal{A}}_t(v_1) - \bar{\mathcal{A}}_t(v_2), v_1 - v_2\rangle_{\mathbb{V}^*\times\mathbb{V}} &= \langle \mathcal{A}_t(v_1+w_0) - \mathcal{A}_t(v_2+w_0), v_1 - v_2\rangle_{\mathbb{V}^*\times\mathbb{V}} \\
&= \langle \mathcal{A}_t(v_1+w_0) - \mathcal{A}_t(v_2+w_0), (v_1+w_0) - (v_2+w_0)\rangle_{\mathbb{V}^*\times\mathbb{V}} \\
&\ge -\alpha_{\mathcal{A}}\|\mathcal{A}_t(v_1+w_0) - \mathcal{A}_t(v_2+w_0)\|^2_{\mathbb{V}^*} + \beta_{\mathcal{A}}\|(v_1+w_0) - (v_2+w_0)\|^2_{\mathbb{V}} \\
&\ge -\alpha_{\mathcal{A}}\zeta^2_{\mathcal{A}}\|(v_1+w_0) - (v_2+w_0)\|^2_{\mathbb{V}} + \beta_{\mathcal{A}}\|(v_1+w_0) - (v_2+w_0)\|^2_{\mathbb{V}} \\
&= (-\alpha_{\mathcal{A}}\zeta^2_{\mathcal{A}} + \beta_{\mathcal{A}})\|v_1 - v_2\|^2_{\mathbb{V}}.
\end{aligned} \tag{3.35}$$

Hence, (3.3)(d) holds with $\alpha_{\bar{\mathcal{A}}}\zeta^2_{\bar{\mathcal{A}}} - \beta_{\bar{\mathcal{A}}} = \alpha_{\mathcal{A}}\zeta^2_{\mathcal{A}} - \beta_{\mathcal{A}}$.

Next, using $\mathcal{A}_{\bar{\mathcal{M}}} = \mathcal{A}_{\mathcal{M}}$ and $\bar{\mathcal{M}}_0 = \mathcal{A}_{\mathcal{M}}w_0 + \mathcal{M}_0$ to confirm that $\bar{\mathcal{M}}$ satisfies (3.8). Given that $\mathcal{A}_{\mathcal{M}} : \mathbb{V} \longrightarrow \mathbb{X}$ is linear and $\mathcal{A}_{\mathcal{M}}v = \mathcal{M}v - \mathcal{M}_0$ for $v \in \mathbb{V}$ and from (3.33), we have

$$\mathcal{A}_{\bar{\mathcal{M}}}v = \bar{\mathcal{M}}v - \bar{\mathcal{M}}_0 = \mathcal{M}(v+w_0) - \mathcal{M}w_0 = \mathcal{A}_{\mathcal{M}}(v+w_0) + \mathcal{M}_0 - (\mathcal{A}_{\mathcal{M}}w_0 + \mathcal{M}_0) = \mathcal{A}_{\mathcal{M}}v, \text{ for } v \in \mathbb{V},$$

and

$$\bar{\mathcal{M}}_0 = \mathcal{M}w_0 = \mathcal{A}_{\mathcal{M}}w_0 + \mathcal{M}_0,$$

imply that $\bar{\mathcal{M}}$ is an affine operator. Furthermore, for $v \in \mathbb{V}$, a.e. $t \in (0,T)$, we get

$$(\bar{\mathscr{M}}v)(t) = \bar{\mathcal{M}}(v(t)) = \mathcal{M}(v(t) + w_0) = \mathcal{M}(v+w_0)(t). \tag{3.36}$$

According to the compactness of $\mathscr{M}$, $\bar{\mathscr{M}}$ and (3.36) is compact. Hence, $\bar{\mathcal{M}}$ satisfies (3.8). Therefore, from Theorem 3.1, (3.31) has a unique solution $\bar{w} \in \mathscr{W}$. Therefore $w \in \mathscr{W}$ given by $w(t) = \bar{w}(t) + w_0$ is a unique solution to (3.31) and the proof is completed. □

4. Applications

In this section, we present a classical contact problem in a variational formulation and prove its existence of unique weak solution. Consider a viscoelastic body occupies a bounded domain $\Omega \subset \mathbb{R}^\iota, \iota = 1,2,3$. The boundary of Ω, denoted by $\daleth$, is assumed to be Lipschitz continuous and $\boldsymbol{\nu}$ is a outward unit normal at $\daleth$. Suppose that $\daleth$ consists of three mutually disjoint and measurable parts $\bar{\daleth}_D$, $\bar{\daleth}_N$ and $\bar{\daleth}_C$ such that meas$(\daleth_D) > 0$. The symbol $\mathbb{S}^\iota$ denotes the space of $\iota \times \iota$ symmetric matrices. The standard inner products and norms on $\mathbb{R}^\iota$ and $\mathbb{S}^\iota$ are given by

$$\mathbf{u}\cdot\mathbf{v} = u_iv_i, \ \|\mathbf{u}\| = \sqrt{(\mathbf{u}\cdot\mathbf{u})} \text{ for } \mathbf{u} = (u_i), \mathbf{v} = (v_i) \in \mathbb{R}^\iota,$$
$$\boldsymbol{\sigma}\cdot\boldsymbol{\tau} = \sigma_i\tau_i, \ \|\boldsymbol{\sigma}\| = \sqrt{(\boldsymbol{\sigma}\cdot\boldsymbol{\sigma})} \text{ for } \boldsymbol{\sigma} = (\sigma_{ij}), \tau = (\tau_{ij}) \in \mathbb{S}^\iota.$$

For a vector field $\mathbf{v}$, v_ν and $\mathbf{v}_\tau$ denote its normal and tangential components on the boundary defined by

$$v_\nu = \mathbf{v}\cdot\boldsymbol{\nu} \text{ and } \mathbf{v}_\tau = \mathbf{v} - v_\nu\boldsymbol{\nu}.$$

Given a tensor $\boldsymbol{\sigma}$, the symbols σ_ν and $\boldsymbol{\sigma}_\tau$ denote its normal and tangential components on the boundary, *i.e.*,

$$\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu})\cdot\boldsymbol{\nu} \text{ and } \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu\boldsymbol{\nu}.$$

Consider a classical model for the contact process on the finite time interval for finding a displacement field $\mathbf{u} : \Omega \times (0,T) \longrightarrow \mathbb{R}^\iota$ and a stress field $\boldsymbol{\sigma} : \Omega \times (0,T) \longrightarrow \mathbb{S}^\iota$ such that for all $t \in (0,T)$,

$$\boldsymbol{\sigma}(t) = \mathfrak{A}\boldsymbol{\varepsilon}(\mathbf{u}'(t)) + \mathfrak{B}\boldsymbol{\varepsilon}(\mathbf{u}(t)) + \int_0^t \mathfrak{C}(t-s)\boldsymbol{\varepsilon}(\mathbf{u}'(s))ds \text{ in } \Omega, \tag{4.1}$$

represents the constitutive law for viscoelastic materials with long memory in which $\mathfrak{A}$ is the viscosity operators, $\mathfrak{B}$ represents the elasticity operator and $\mathfrak{C}$ is the relaxation tensor, and $\boldsymbol{\varepsilon}(\mathbf{u})$ denotes the linearized strain tensor defined by

$$\boldsymbol{\varepsilon}(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u})),\ \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}) \text{ in } \Omega.$$

The motion of the equation

$$\mathbf{u}''(t) = Div\boldsymbol{\sigma}(t) + \mathbf{f}_0(t) \text{ in } \Omega, \tag{4.2}$$

where $Div\boldsymbol{\sigma} = (\sigma_{ij,j})$ and $\mathbf{f}_0$ is a density of the body forces.

$$\mathbf{u}(t) = \mathbf{0} \text{ on } \daleth_D, \tag{4.3}$$

is a displacement homogeneous boundary where the body is fixed on $\daleth_D$.

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = f_N(t) \text{ on } \daleth_N \tag{4.4}$$

is a traction boundary condition with surface tractions of density $\mathbf{f}_N$ acting on $\daleth_N$.

$$\begin{aligned} &u'_\nu(t) \le g, \sigma_\nu(t) + \eta(t) \le 0, (u'_\nu(t) - g)(\sigma_\nu(t) + \eta(t)) = 0, \\ &\eta(t) \le k(u_\nu(t))\partial j_\nu(u'_\nu(t)) \text{ on} \daleth_C, \end{aligned} \tag{4.5}$$

is a Signorini unilateral contact boundary condition for the normal velocity in which $g > 0$ and ∂j_ν denotes the Clarke subgradient of a prescribed function j_ν. Condition $\eta(t) \in \kappa(u_\nu(t))\partial j_\nu(u'_\nu(t))$ on $\daleth_C$ is a normal damped response condition where κ is a given damper coefficient depending on the normal displacement.

$$\begin{aligned} \|\boldsymbol{\sigma}_\tau(t)\| &\le \mathcal{F}_b\left(\int_0^t \|\mathbf{u}_\tau(s)\|ds\right), \\ -\boldsymbol{\sigma}_\tau &= \mathcal{F}_b\left(\int_0^t \|\mathbf{u}_\tau(s)\|ds\right)\frac{\mathbf{u}'_\tau(t)}{\|\mathbf{u}'_\tau(t)\|},\ \text{if } \mathbf{u}'_\tau(t) \ne \mathbf{0} \text{ on } \daleth_C, \end{aligned} \tag{4.6}$$

is a Coulomb law of dry friction in which $\mathcal{F}_b$ denotes the friction bound and

$$\mathbf{u}(0) = \mathbf{u}_0,\ \mathbf{u}'(0) = \mathbf{w}_0 \text{ in } \Omega, \tag{4.7}$$

are the initial conditions, where $\mathbf{u}_0$ is a initial displacement and $\mathbf{w}_0$ a initial velocity.

The total accumulated slip is represented by

$$\int_0^t \|\mathbf{u}_\tau(\mathbf{x},s)\|ds \text{ at } x \in \daleth_C,\ t \in [0,t].$$

To provide the weak formulation of (4.1)-(4.7), we use the following spaces

$$\mathscr{H} = L^2(\Omega;\mathbb{S}^\iota),\ \mathbb{V} = \{\mathbf{v} \in \mathbb{H}^1(\Omega;\mathbb{R}^\iota) | \mathbf{v} = \mathbf{0} \text{ on } \daleth_D\}$$

where $\mathscr{H}$ is a Hilbert space with the inner product

$$\langle \boldsymbol{\sigma}, \boldsymbol{\varepsilon}\rangle_{\mathscr{H}} = \int_\Omega \sigma_{ij}(\mathbf{x})\varepsilon_{ij}(\mathbf{x})dx,\ \forall \boldsymbol{\sigma}, \boldsymbol{\varepsilon} \in \mathscr{H},$$

and the norm $\|\cdot\|_{\mathscr{H}}$. The inner product and the corresponding norm on $\mathbb{V}$ are given by

$$(\mathbf{u}, \mathbf{v})_{\mathbb{V}} = (\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathscr{H}},\ \|\mathbf{v}\|_{\mathbb{V}} = \|\boldsymbol{\varepsilon}(\mathbf{v})\|_{\mathscr{H}},\ \forall \mathbf{u}, \mathbf{v} \in \mathbb{V}.$$

However, the continuity of the trace operator $\rho : \mathbb{V} \longrightarrow L^2(\daleth_C;\mathbb{R}^\iota)$ implies

$$\|\mathbf{v}\|_{L^2(\daleth;\mathbb{R}^\iota)} \leq \|\rho\|\|\mathbf{v}\|_{\mathbb{V}},\ \forall \mathbf{v} \in \mathbb{V},$$

where ρ is a norm of the trace operator in $\mathscr{L}(\mathbb{V}, L^2(\daleth_C;\mathbb{R}^\iota))$. We define a space of fourth order tensor fields

$$\leftthreetimes_\infty = \{\boldsymbol{\sigma} = (\sigma_{ijkl})|\sigma_{ijkl} = \sigma_{jikl} = \sigma_{klij} \in L^\infty(\Omega),\ 1 \leq i,j,k,l \leq \iota\}$$

is a real Banach space with the norm

$$\|\boldsymbol{\sigma}\|_{\leftthreetimes_\infty} = \sum_{1\leq i,j,k,l\leq\iota} \|\sigma_{ijkl}\|_{L^\infty(\Omega)},\ \forall \boldsymbol{\sigma} \in \leftthreetimes_\infty.$$

Now we suggest the following hypotheses for (4.1)-(4.7) as:

$$\mathfrak{A} : \Omega \times \mathbb{S}^\iota \longrightarrow \mathbb{S}^\iota \text{ is such that}$$
$$\begin{cases} (a)\ \mathfrak{A}(\cdot,\varepsilon) \text{ is measurable on } \Omega \text{ for all } \varepsilon \in \mathbb{S}^\iota. \\ (b) \text{ there exists } \mathscr{L}_{\mathfrak{A}} > 0 \text{ such that} \\ \|\mathfrak{A}(\mathbf{x},\boldsymbol{\varepsilon}_1) - \mathfrak{A}(\mathbf{x},\boldsymbol{\varepsilon}_2)\| \leq \mathscr{L}_{\mathfrak{A}}\|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|,\ \forall\, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^\iota \text{ a.e. } \mathbf{x} \in \Omega. \\ (c) \text{ there exist } \alpha_{\mathfrak{A}} > 0, \beta_{\mathfrak{A}} > 0 \text{ such that} \\ (\mathfrak{A}(\mathbf{x},\boldsymbol{\varepsilon}_1) - \mathfrak{A}(\mathbf{x},\boldsymbol{\varepsilon}_2)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \geq -\alpha_{\mathfrak{A}}\mathscr{L}^2_{\mathfrak{A}}\|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2 + \beta_{\mathfrak{A}}\|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2, \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^\iota. \\ (d)\ \mathfrak{A}(\mathbf{x},\mathbf{0}) = \mathbf{0},\ \text{for a.e. } \mathbf{x} \in \Omega. \end{cases} \tag{4.8}$$

$$\mathfrak{B} : \Omega \times \mathbb{S}^\iota \longrightarrow \mathbb{S}^\iota \text{ is such that}$$
$$\begin{cases} (a)\ \mathfrak{B}(\cdot,\boldsymbol{\varepsilon}) \text{ is measurable on } \Omega \text{ for all } \boldsymbol{\varepsilon} \in \mathbb{S}\iota. \\ (b) \text{ there exists } \mathscr{L}_{\mathfrak{B}} > 0 \text{ such that} \\ \|\mathfrak{B}(\mathbf{x},\boldsymbol{\varepsilon}_1) - \mathfrak{B}(\mathbf{x},\boldsymbol{\varepsilon}_2)\| \leq \mathscr{L}_{\mathfrak{B}}\|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|, \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^\iota \text{ a.e. } \mathbf{x} \in \Omega. \\ c) \text{ there exist } \alpha_{\mathfrak{B}} > 0, \beta_{\mathfrak{B}} > 0 \text{ such that} \\ (\mathfrak{B}(\mathbf{x},\boldsymbol{\varepsilon}_1) - \mathfrak{B}(\mathbf{x},\boldsymbol{\varepsilon}_2)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \geq -\alpha_{\mathfrak{B}}\mathscr{L}^2_{\mathfrak{B}}\|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2 + \beta_{\mathfrak{B}}\|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2, \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^\iota. \\ (d)\ \mathfrak{B}(\mathbf{x},\mathbf{0}) = \mathbf{0} \text{ for a.e. } \mathbf{x} \in \Omega. \end{cases} \tag{4.9}$$

$$\mathfrak{C} \in C([0,T];\leftthreetimes_\infty). \tag{4.10}$$

$$\kappa : \daleth_C \times \mathbb{R} \longrightarrow \mathbb{R} \text{ is such that}$$
$$\begin{cases} (a)\ \kappa(\cdot,r) \text{ is measurable on } \daleth_C \text{ for all } r \in \mathbb{R}. \\ (b) \text{ there exist } \kappa_1, \kappa_2 \text{ such that} \\ 0 < \kappa_1 \leq \kappa(\mathbf{x},t) \leq \kappa_2\ \forall\, r \in \mathbb{R},\ \text{a.e. } \mathbf{x} \in \daleth_C. \\ (c) \text{ there exists } \mathscr{L}_\kappa > 0 \text{ such that} \\ |\kappa(\mathbf{x},r_1) - \kappa(\mathbf{x},r_2)| \leq \mathscr{L}_\kappa|r_1 - r_2|, \forall r_1, r_2 \in \mathbb{R},\ \text{a.e. } \mathbf{x} \in \daleth_C. \end{cases} \tag{4.11}$$

$\jmath_\nu : \daleth_C \times \mathbb{R} \longrightarrow \mathbb{R}$ is such that

$$\begin{cases} (a)\ \jmath_\nu(\cdot, r) \text{ is measurable on } \daleth_C, \forall\, r \in \mathbb{R} \text{ and there exists } e \in L^2(\daleth_C) \text{ such that} \\ \jmath_\nu(\cdot, e(\cdot)) \in L^1(\daleth_C). \\ (b)\ \jmath_\nu(\mathbf{x}, \cdot) \text{ is locally Lipschitz on } \mathbb{R} \text{ for a.e. } \mathbf{x} \in \daleth_C. \\ (c) \text{ there are } \bar{\vartheta}_0, \bar{\vartheta}_1 \geq 0 \text{ such that} \\ |\partial \jmath_\nu(\mathbf{x}, r)| \leq \bar{\vartheta}_0 + \bar{\vartheta}_1 |r|, \forall r \in \mathbb{R}, \text{ a.e. } \mathbf{x} \in \daleth_C. \\ (d) \text{ there exists } \alpha^{\jmath_\nu} \geq 0 \text{ such that} \\ \jmath_\nu^0(\mathbf{x}, r_1; r_2 - r_1) + \jmath_\nu^0(\mathbf{x}, r_2; r_1 - r_2) \leq \alpha^{\jmath_\nu} |r_1 - r_2|^2, \forall r_1, r_2 \in \mathbb{R}, \text{ a.e. } \mathbf{x} \in \daleth_C. \end{cases} \tag{4.12}$$

$$\mathcal{F}_b : \daleth_C \times \mathbb{R} \longrightarrow \mathbb{R} \text{ is such that}$$

$$\begin{cases} (a)\ \mathcal{F}_b(\cdot, r) \text{ is measurable on } \daleth_C, \forall r \in \mathbb{R}. \\ (b) \text{ there exists } \mathscr{L}_{\mathcal{F}_b} > 0 \text{ such that} \\ |\mathcal{F}_b(\mathbf{x}, r_1) - \mathcal{F}_b(\mathbf{x}, r_2)| \leq \mathscr{L}_{\mathcal{F}_b} |r_1 - r_2|, \forall r_1, r_2 \in \mathbb{R}, \text{ a.e. } \mathbf{x} \in \daleth_C. \\ (c)\ \mathcal{F}_b(\mathbf{x}, r) = 0, \forall r \leq 0, \text{ and } \mathcal{F}_b(\mathbf{x}, r) \geq 0, \forall r \geq 0 \text{ for a.e. } \mathbf{x} \in \daleth_C. \end{cases} \tag{4.13}$$

$$\begin{aligned} &\mathbf{f}_0 \in L^2(0, T; L^2(\Omega; \mathbb{R}^\iota)), \\ &\mathbf{f}_N \in L^2(0, T; L^2(\daleth_N; \mathbb{R}^\iota)),\ \mathbf{u}_0, \mathbf{w}_0 \in \mathbb{V}. \end{aligned} \tag{4.14}$$

Additionally, we introduce the set of admissible velocity fields $\mathbb{U}$ described by

$$\mathbb{U} = \{\mathbf{v} \in \mathbb{V} | v_\nu \leq g \text{ on } \daleth_C\},$$

and an element $\mathbf{f} \in \mathbb{V}^*$ by

$$\langle \mathbf{f}, \mathbf{v} \rangle_{\mathbb{V}^* \times \mathbb{V}} = \langle \mathbf{f}_0, \mathbf{v} \rangle_{L^2(\Omega; \mathbb{R}^\iota)} + \langle \mathbf{f}_N, \mathbf{v} \rangle_{L^2(\daleth_N; \mathbb{R}^\iota)} \forall \mathbf{v} \in \mathbb{V}. \tag{4.15}$$

Now from the weak formulation of (4.1)-(4.7). We assume that $\mathbf{v} \in \mathbb{U}$ and $t \in (0, T)$, and multiply (4.2) by $\mathbf{v} - \mathbf{u}'(t)$ and integrate by parts together with (4.3) and (4.4) to obtain

$$\begin{aligned} \int_\Omega \mathbf{u}''(t) \cdot (\mathbf{v} - \mathbf{u}'(t)) dx + \int_\Omega \boldsymbol{\sigma}(t) \cdot (\boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\mathbf{u}'(t))) dx &= \int_\Omega \mathbf{f}_0(t) \cdot (\mathbf{v} - \mathbf{u}'(t)) dx \\ &+ \int_{\daleth_N} \mathbf{f}_N(t) \cdot (\mathbf{v} - \mathbf{u}'(t)) d\daleth + \int_{\daleth_C} \boldsymbol{\sigma}(t) \boldsymbol{\nu} \cdot (\mathbf{v} - \mathbf{u}'(t)) d\daleth. \end{aligned}$$

Using (4.5) and the concept of the Clarke's subgradient, we have

$$\begin{aligned} \sigma_\nu(t)(v_\nu - u'_\nu(t)) &= (\sigma_\nu(t) + \eta(t))(v_\nu - g) - (\sigma_\nu(t) + \eta(t))(u'_\nu(t) - g) - \eta(t)(v_\nu - u'_\nu(t)) \\ &\geq -\kappa(u_\nu(t)) \jmath_\nu^0(u'_\nu(t); v_\nu - u'_\nu(t)) \text{ on } \daleth_C. \end{aligned} \tag{4.16}$$

From the friction law, (4.6) can be expressed as

$$\boldsymbol{\sigma}_\tau(t) \cdot (\mathbf{v}_\tau - \mathbf{u}'_\nu(t)) \geq -\mathcal{F}_b \left(\int_0^t \|\mathbf{u}_\tau(s)\| ds \right) \left(\|\mathbf{v}_\tau\| - \|\mathbf{u}'_\tau(t)\| \right) \text{ on } \daleth_C. \tag{4.17}$$

Combining (4.16), (4.17) and the decomposition formula [5], we obtain

$$\mathcal{F}_b \left(\int_0^t \|\mathbf{u}_\tau(s)\| ds \right) \left(\|\mathbf{v}_\tau\| - \|\mathbf{u}'_\tau(t)\| \right) + \kappa(u_\nu(t)) \jmath_\nu^0(u'_\nu(t); v_\nu - u'_\nu(t)) + \boldsymbol{\sigma}(t) \boldsymbol{\nu} \cdot (\mathbf{v} - \mathbf{u}'(t)) \geq 0 \text{ on } \daleth_C. \tag{4.18}$$

Again, concluding from (4.15)

$$\int_{\Omega}\mathbf{u}''(t)\cdot(\mathbf{v}-\mathbf{u}'(t))dx+\int_{\urcorner_C}\mathcal{F}_b\left(\int_0^t\|\mathbf{u}_\tau(s)\|ds\right)\left(\|\mathbf{v}_\tau\|-\|\mathbf{u}'_\tau(t)\|\right)d\urcorner$$
$$+\langle\boldsymbol{\sigma}(t),\boldsymbol{\varepsilon}(\mathbf{v})-\boldsymbol{\varepsilon}(\mathbf{u}'(t))\rangle_{\mathscr{H}}+\int_{\urcorner_C}k(u_\nu(t))j^0_\nu(u'_\nu(t);v_\nu-u'_\nu(t))d\urcorner\geq\langle\mathbf{f},\mathbf{v}-\mathbf{u}\rangle_{\mathbb{V}^*\times\mathbb{V}}. \quad (4.19)$$

Finally, from (4.1)-(4.7), we have following problem for finding $\mathbf{u}:(0,T)\longrightarrow\mathbb{V}$ such that $\mathbf{u}(0)=\mathbf{u}_0,\mathbf{u}'(0)=\mathbf{w}_0$ and

$$\int_{\Omega}\mathbf{u}''(t)\cdot(\mathbf{v}-\mathbf{u}'(t))dx+\langle\mathfrak{A}(\boldsymbol{\varepsilon}(\mathbf{u}'(t))),\boldsymbol{\varepsilon}(\mathbf{v})-\boldsymbol{\varepsilon}(\mathbf{u}'(t))\rangle_{\mathscr{H}}+\langle\mathfrak{B}(\boldsymbol{\varepsilon}(\mathbf{u}(t)))$$
$$+\int_0^t\mathfrak{C}(t-s)\boldsymbol{\varepsilon}(\mathbf{u}'(s))ds,\boldsymbol{\varepsilon}(\mathbf{v}-\boldsymbol{\varepsilon}(\mathbf{u}'(t)))\rangle_{\mathscr{H}}+\int_{\urcorner_C}\mathcal{F}_b\left(\int_0^t\|\mathbf{u}_\tau(s)\|ds\right)\left(\|\mathbf{v}_\tau\|-\|\mathbf{u}'_\tau(t)\|\right)d\urcorner$$
$$+\int_{\urcorner_C}k(u_\nu(t))j^0_\nu(u'_\nu(t);v_\nu-u\prime_\nu(t))d\urcorner\geq\langle\mathbf{f},\mathbf{v}-\mathbf{u}\rangle_{\mathbb{V}^*\times\mathbb{V}}\forall\mathbf{v}\in\mathbb{U},\ \text{a.e. } t\in(0,T). \quad (4.20)$$

Now, we prove the unique solvability of (4.20).

Theorem 4.1. *Assume that* (4.8)-(4.14) *and*

$$\beta_{\mathfrak{A}}>\alpha_{\mathfrak{A}}\mathscr{L}^2_{\mathfrak{A}}+\alpha^{\jmath_\nu}\kappa_2\|\rho\|^2$$

holds. Then (4.20) *has a unique solution* $\mathbf{u}\in C([0,T];\mathbb{V})$, $\mathbf{u}'\in\mathscr{W}$ *with* $\mathbf{u}'(t)\in\mathbb{U}$ *for a.e.* $t\in(0,T)$.

Proof. Using the Theorem 3.2 with $\mathbb{X}=\mathbb{Y}=\mathbb{Z}=L^2(\urcorner_C),\Omega=\mathbb{U}$, and $\mathcal{M}=\rho$. Let the operator $\mathcal{A}:\mathbb{V}\times\mathbb{V}\longrightarrow\mathbb{V}^*$, and functions $\varphi:\mathbb{Y}\times\mathbb{X}\longrightarrow\mathbb{R}$ and $\jmath:\mathbb{Z}\times\mathbb{X}\longrightarrow\mathbb{R}$ be defined by

$$\langle\mathcal{A}(\mathbf{w}),\mathbf{v}\rangle_{\mathbb{V}^*\times\mathbb{V}}=\langle\mathfrak{A}(\boldsymbol{\varepsilon}(\mathbf{w})),\boldsymbol{\varepsilon}(\mathbf{v})\rangle_{\mathscr{H}},\forall\mathbf{w},\mathbf{v}\in\mathbb{V},$$
$$\varphi(y,x)=\int_{\urcorner_C}\mathcal{F}_b(y)\|x\|d\urcorner,\forall y\in\mathbb{Y},x\in\mathbb{X},$$
$$\jmath(z,x)=\int_{\urcorner_C}\kappa(z)\jmath_\nu(x)\,d\urcorner,\forall z\in\mathbb{Z},x\in\mathbb{X}.$$

Now, the operators $\mathcal{R}_1:\mathscr{V}\longrightarrow\mathscr{V}^*$, $\mathcal{R}_2:\mathscr{V}\longrightarrow L^2(0,T;\mathbb{Y})$, and $\mathcal{S}:\mathbb{V}\longrightarrow L^2(0,T;\mathbb{Z})$ specified by

$$\langle(\mathcal{R}_1\mathbf{w})(t),\mathbf{v}\rangle_{\mathbb{V}^*\times\mathbb{V}}=\langle\mathfrak{B}(\boldsymbol{\varepsilon}(\mathbf{u}_0)+\int_0^t\boldsymbol{\varepsilon}(\mathbf{w}(s))ds),\boldsymbol{\varepsilon}(\mathbf{v})\rangle_{\mathscr{H}}$$
$$+\langle\int_0^t\mathfrak{C}(t-s)\boldsymbol{\varepsilon}(\mathbf{w}(s))ds,\boldsymbol{\varepsilon}(\mathbf{v})\rangle_{\mathscr{H}},\forall\mathbf{w}\in\mathscr{V},\mathbf{v}\in\mathbb{V},t\in(0,T),$$
$$(\mathcal{R}_2\mathbf{w})(t)=\int_0^t\|\int_0^s\mathbf{w}_\tau(r)dr+\mathbf{u}_{0\tau}\|ds,\forall\mathbf{w}\in\mathscr{V},t\in(0,T),$$
$$(\mathcal{S}\mathbf{w})(t)=\int_0^t w_\nu(s)ds+u_{0\nu},\forall\mathbf{w}\in\mathscr{V},t\in(0,T).$$

Let $\mathbf{w}(t)=\mathbf{u}'(t)\forall t\in(0,T)$. Then, we have the following problems.
Find $\mathbf{w}\in\mathscr{W}$ such that $\mathbf{w}(t)\in\mathbb{U}$ for a.e. $t\in(0,T)$, $\mathbf{w}(0)=\mathbf{w}_0$ and

$$\langle\mathbf{w}'(t)+\mathcal{A}(\mathbf{w}(t))+(\mathcal{R}_1\mathbf{w})(t)-\mathbf{f}(t),\mathbf{v}-\mathbf{w}(t)\rangle_{\mathbb{V}^*\times\mathbb{V}}+\jmath^0((\mathcal{S}\mathbf{w})(t),\mathcal{M}\mathbf{w}(t);\mathcal{M}\mathbf{v}-\mathcal{M}\mathbf{w}(t))$$
$$+\varphi((\mathcal{R}_2\mathbf{w})(t),\mathcal{M}\mathbf{v})-\varphi((\mathcal{R}_2\mathbf{w})(t),\mathcal{M}\mathbf{w}(t))\geq0,\forall\mathbf{v}\in\mathbb{U},\ \text{a.e. } t\in(0,T). \quad (4.21)$$

Due to the fact that the set $\Omega=\mathbb{U}$ is a closed and convex subset of $\mathbb{V}$ with $\mathbf{0}\in\Omega$, and (3.7) holds. Then from [5], (4.21) together with Theorem 4.1 has a unique solution $\mathbf{w}\in\mathscr{W}$ such that $\mathbf{w}(t)\in\mathbb{U}$ for a.e. $t\in(0,T)$, and completing the proof. □

Statements and Declarations

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References

[1] P. D. Panagiotopoulos. *Inequality Problems in Mechanics and Applications.* Convex and Nonconvex Energy Functions. Birkhäuser, Basel, 1985.

[2] C. Eck, J. Jarusek, and M. Krbec. Unilateral Contact Problems - Variational Methods and Existence Theorems. *Pure and Applied Mathematics*, Vol. 270, Chapman/CRC Press, New York, 2005.

[3] J. Haslinger, M. Miettinen, and P. D. Panagiotopoulos. *Finite Element Method for Hemivariational Inequalities.* Theory, Methods and Applications, Springer New York, NY, 1999.

[4] W. Han and M. Sofonea. Quasistatic Contact Problems in Viscoelasticity and Viscoplasticity. *Studies in Advanced Mathematics, vol. 30. Americal Mathematical Society*, Providence, RI–International Press, Somerville, MA 2002.

[5] S. Migórski, A. Ochal, and M. Sofonea. Nonlinear Inclusions and Hemivariational Inequalities. *Models and Analysis of Contact Problems, Advances in Mechanics and Mathematics*, vol. 26, Springer, New York, 2013.

[6] S. S. Chang, S. Salahuddin, A. A. H. Ahmadini, and L Wang. The penalty method for generalized mixed variational-hemivariational inequality problems. *Carpathian Journal of Mathematics*, 38(2): 357-381, 2022.

[7] S. Salahuddin. Solution of variational inclusions over the sets of common fixed points in Banach spaces. *Journal of Applied Nonlinear Dynamics*, 11(1): 75-85, 2021.

[8] S. Migórski and S. D. Zeng. Rothe method and numerical analysis for history-dependent hemivariational inequalities with applications to contact mechanics. *Numerical Algorithms*, 82(2): 423-450, 2019.

[9] S. Migórski, A. Ochal, and M. Sofonea. History-dependent subdifferential inclusions and hemivariational inequalities in contact mechanics. *Nonlinear Analysis: Real World Applications*, 12: 3384-3396, 2011.

[10] M. Shillor, M. Sofonea, and J. J. Telega. *Models and Analysis of Quasistatic Contact. Lecture Notes in Physics*, vol. 655, Springer, New York, 2004.

[11] M. Sofonea and A. Matei. *Mathematical Models in Contact Mechanics, London Mathematical Society Lecture Notes Series*, vol. 398. Cambridge University Press, Cambridge, 2012.

[12] S. S. Chang, Salahuddin, L Wang, J. Tang, and M. Zhaoli. The Convergence Results of Differential Variational Inequality Problems. *Symmetry*, 14: 760, 2022.

[13] W. Han, S. Migórski, and M. Sofonea. Analysis of a general dynamic history-dependent variational-hemivariational inequality. *Nonlinear Analysis: Real World Applications*, 36: 69-88, 2017.

[14] D. Goeleven, D. Motreanu, Y. Dumont, and M. Rochdi. *Variational and Hemivariational Inequalities, Theory, Methods and Applications, Vol. I: Unilateral Analysis and Unilateral Mechanics.* Kluwer Academic Publishers, Boston, 2003.

[15] M. Cocou. Existence of solutions of a dynamic Signorini's problem with nonlocal friction in viscoelasticity. *Zeitschrift für angewandte Mathematik und Physik ZAMP*, 53: 1099-1109, 2002.

[16] El-H Essoufi and M. Kabbaj. Existence of solutions of a dynamic Signorini's problem with nonlocal friction for viscoelastic piezoelectric materials. *Bulletin mathématique de la Société des Sciences Mathématiques de Roumanie*, 48: 181-195, 2005.

[17] K. Kuttler and M. Shillor. Dynamic contact with Signorini's condition and slip rate dependent friction. *Electronic Journal of Differential Equations*, 83: 1-21, 2004.

[18] Z. Denkowski, S. Migórski, and N. S. Papageorgiou. *An Introduction to Nonlinear Analysis: Applications.* Springer New York, NY, 2003.

[19] S. Migórski and B. Zeng. A new class of history-dependent evolutionary variational-hemivariational inequalities with unilateral constraints. *Applied Mathematics and Optimization*, 84: 2671–2697, 2021.

[20] O. Chadli, Q. H. Ansari, and S. Al-Homidan. Existence of solutions for nonlinear implicit differential equations. An equilibrium problem approach. *Numerical Functional Analysis and Optimization*, 37: 1385-1419, 2016.

[21] F. H. Clarke. *Optimization and Nonsmooth Analysis.* Wiley, New York, 1983.

[22] Lions, J.-L.:Quelques methodes de resolution des problemes aux limites non lineaires. Dunod, Gauthier-Villars, Paris, 1969.

[23] I. Ekeland and R. Temam. *Convex Analysis and Variational Problems.* North-Holland, Amsterdam, 1976.

[24] C. Carstensen and J. Gwinner. A theory of discretization for nonlinear evolution inequalities applied to parabolic signorini problems. *Annali di Matematica Pura ed Applicata*, 177: 363-394, 1999.

[25] E. Zeidler. *Nonlinear Functional Analysis and Applications II A/B.* Springer, New York, 1990.

[26] J. K. Kim, A. H. Dar, and Salahuddin. Existence solution for the generalized relaxed pseudo monotone variational inequalities. *Nonlinear Functional Analysis and Application*, 25(1): 25-34, 2020.