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ON PARALLEL MANN-TYPE EXTRAGRADIENT ALGORITHMS FOR SYSTEMS OF VARIATIONAL INEQUALITIES WITH CONSTRAINTS OF VARIATIONAL INCLUSION AND FIXED POINT PROBLEMS

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Abstract. In a uniformly convex and q-uniformly smooth Banach space with $q \in (1, 2]$, let the VI indicate a variational inclusion for two accretive operators and let the CFPP denote a common fixed point problem of a countable family of nonexpansive mappings. In this paper, we introduce a parallel Mann-type extragradient algorithm for solving a general system of variational inequalities (GSVI) with the VI and CFPP constraints. We then prove the strong convergence of the suggested algorithm to a solution of the GSVI with the VI and CFPP constraints under some suitable assumptions. As applications, we apply our main result to the variational inequality problem (VIP), split feasibility problem (SFP) and LASSO problem in Hilbert spaces.

Keywords. Parallel Mann-type extragradient algorithm; General system of variational inequalities; Variational inclusion; Common fixed point problem; Strong convergence; Banach space.

1. Introduction

In a real Hilbert space H, suppose that the inner product and induced norm are denoted by the notations $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Given a closed convex set $\emptyset \neq C \subset H$. Let P_C be the metric projection from H onto C. Given a mapping $A : C \to H$. Consider the classical variational inequality problem (VIP) of finding a point $u^* \in C$ s.t. $\langle Au^*, v - u^* \rangle \geq 0 \ \forall v \in C$. The solution set of the VIP is denoted by VI(C, A). In 1976, Korpelevich [23] first designed an extragradient method for solving the VIP. Whenever $\mathrm{VI}(C, A) \neq \emptyset$, this method has only weak convergence, and only requires that the mapping A is monotone and Lipschitz continuous. To the most of our knowledge, it has been one of the most popular approaches for solving the VIP up to now. Moreover, it has been improved and modified in various ways so that some new iterative methods happen to solve the VIP and related optimization problems; see e.g., [1, 2, 7, 8, 9, 10, 11, 13, 14, 15, 16, 17, 19, 20, 22, 25, 26, 29, 32, 33, 40, 41] and references therein, to name but a few.

Assume that $A : C \to H$ is an inverse-strongly monotone mapping, $B : D(B) \subset C \to 2^H$ is a maximal monotone operator, and $S : C \to C$ is a nonexpansive mapping. Consider the variational inclusion (VI) of finding a point $x^* \in C$ s.t. $0 \in (A + B)x^*$. In order to solve the FPP of S and the VI for A, B, Manaka and Takahashi [28] suggested an iterative process, i.e., for any given $x_0 \in C$, $\{x_j\}$ is the sequence generated by

$$x_{j+1} = \alpha_j x_j + (1 - \alpha_j) S J^B_{\lambda_j}(x_j - \lambda_j A x_j) \quad \forall j \geq 0, \tag{1.1}$$

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where $\{\alpha_j\} \subset (0,1)$ and $\{\lambda_j\} \subset (0,\infty)$. They proved weak convergence of $\{x_j\}$ to a point of $\mathrm{Fix}(S) \cap (A+B)^{-1}0$ under some suitable conditions.

Recently, Abdou et al. [2] suggested a parallel algorithm, i.e., for any given $x_0 \in C$, $\{x_j\}$ is the sequence generated by

$$x_{j+1} = (1-\zeta)Sx_j + \zeta J^B_{\lambda_j}(\alpha_j \gamma f(x_j) + (1-\alpha_j)x_j - \lambda_j A x_j) \quad \forall j \geq 0, \tag{1.2}$$

where S, A, B are the same as above, $\zeta \in (0,1)$, $\{\lambda_j\} \subset (0, 2\alpha)$ and $\{\alpha_j\} \subset (0,1)$. They proved strong convergence of $\{x_j\}$ to a point of $\mathrm{Fix}(S) \cap (A+B)^{-1}0$ under some appropriate conditions. In the practical life, many mathematical models have been formulated as the VI. Without question, many researchers have presented and developed a great number of iterative methods for solving the VI in various approaches; see e.g., [2, 8, 12, 16, 18, 24, 28, 33, 35] and the references therein. Due to the importance and interesting of the VI, many mathematicians are now interested in finding a common solution of the VI and FPP.

Furthermore, for $q \in (1,2]$, suppose that E is a uniformly convex and q-uniformly smooth Banach space with q-uniform smoothness coefficient κ_q. Assume that $f : E \to E$ is a ρ-contraction and $S : E \to E$ is a nonexpansive mapping. Let $A : E \to E$ be an α-inverse-strongly accretive mapping of order q and $B : E \to 2^E$ be an m-accretive operator. Very recently, Sunthrayuth and Cholamjiak [33] proposed a modified viscosity-type extragradient method for the FPP of S and the VI of finding $x^* \in E$ s.t. $0 \in (A+B)x^*$, i.e., for any given $x_0 \in E$, $\{x_j\}$ is the sequence generated by

$$\begin{cases} y_j = J^B_{\lambda_j}(x_j - \lambda_j A x_j), \\ z_j = J^B_{\lambda_j}(x_j - \lambda_j A y_j + r_j(y_j - x_j)), \\ x_{j+1} = \alpha_j f(x_j) + \beta_j x_j + \gamma_j S z_j \quad \forall j \geq 0, \end{cases} \tag{1.3}$$

where $J^B_{\lambda_j} = (I + \lambda_j B)^{-1}$, $\{r_j\}, \{\alpha_j\}, \{\beta_j\}, \{\gamma_j\} \subset (0,1)$ and $\{\lambda_j\} \subset (0,\infty)$ are such that: (i) $\alpha_j + \beta_j + \gamma_j = 1$; (ii) $\lim_{j\to\infty} \alpha_j = 0$, $\sum_{j=1}^{\infty} \alpha_j = \infty$; (iii) $\{\beta_j\} \subset [a,b] \subset (0,1)$; and (iv) $0 < \lambda \leq \lambda_j < \lambda_j / r_j \leq \mu < (\alpha q/\kappa_q)^{1/(q-1)}$, $0 < r \leq r_j < 1$. They proved the strong convergence of $\{x_j\}$ to a point of $\mathrm{Fix}(S) \cap (A+B)^{-1}0$, which solves a certain VIP.

On the other hand, let $J : E \to 2^{E^*}$ be the normalized duality mapping from E into 2^{E^*} defined by $J(x) = \{\phi \in E^* : \langle x, \phi \rangle = \|x\|^2 = \|\phi\|^2\}$ $\forall x \in E$, where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing between E and E^*. Recall that if E is smooth then J is single-valued. Let $B_1, B_2 : C \to E$ be two nonlinear mappings in a smooth Banach space E. Consider the general system of variational inequalities (GSVI) of finding $(x^*, y^*) \in C \times C$ s.t.

$$\begin{cases} \langle \mu_1 B_1 y^* + x^* - y^*, J(x - x^*) \rangle \geq 0 \quad \forall x \in C, \\ \langle \mu_2 B_2 x^* + y^* - x^*, J(x - y^*) \rangle \geq 0 \quad \forall x \in C, \end{cases} \tag{1.4}$$

where μ_i is a positive constant for $i = 1, 2$. In particular, if $E = H$ a real Hilbert space, it is easy to see that the GSVI (1.4) reduces to the GSVI considered in [19],

$$\begin{cases} \langle \mu_1 B_1 y^* + x^* - y^*, x - x^* \rangle \geq 0 \quad \forall x \in C, \\ \langle \mu_2 B_2 x^* + y^* - x^*, x - y^* \rangle \geq 0 \quad \forall x \in C. \end{cases} \tag{1.5}$$

In [19], problem (1.5) is transformed into a fixed point problem in the following way.

Lemma 1.1. *(see [19]). For given $x^*, y^* \in C$, (x^*, y^*) is a solution of problem (1.5) if and only if $x^* \in \mathrm{GSVI}(C, B_1, B_2)$, where $\mathrm{GSVI}(C, B_1, B_2)$ is the fixed point set of the mapping $G := P_C(I - \mu_1 B_1)P_C(I - \mu_2 B_2)$, and $y^* = P_C(I - \mu_2 B_2)x^*$.*

In addition, assume that $\{\mu_j\} \subset (0, \frac{1}{L})$, $\{\lambda_j\} \subset (0, 2\alpha]$ and $\{\alpha_j\}, \{\hat{\alpha}_j\} \subset (0,1]$ with $\alpha_j + \hat{\alpha}_j \leq 1$. Ceng et al. [8] introduced a Mann-type hybrid extragradient algorithm, i.e., for any initial $u_0 = u \in C$,

$\{u_j\}$ is the sequence generated by

$$\begin{cases} y_j = P_C(u_j - \mu_j \mathcal{A} u_j), \\ v_j = P_C(u_j - \mu_j \mathcal{A} y_j), \\ \hat{v}_j = J^B_{\lambda_j}(v_j - \lambda_j A v_j), \\ z_j = (1 - \alpha_j - \hat{\alpha}_j) u_j + \alpha_j \hat{v}_j + \hat{\alpha}_j S \hat{v}_j, \\ u_{j+1} = P_{C_j \cap Q_j} u \quad \forall j \geq 0, \end{cases}$$

where $C_j = \{x \in C : \|z_j - x\| \leq \|u_j - x\|\}$, $Q_j = \{x \in C : \langle u_j - x, u - u_j \rangle \geq 0\}$, $J^B_{\lambda_j} = (I + \lambda_j B)^{-1}$, $\mathcal{A} : C \to H$ is a monotone and L-Lipschitzian mapping, $A : C \to H$ is an α-inverse-strongly monotone mapping, B is a maximal monotone mapping with $D(B) = C$ and $S : C \to C$ is a nonexpansive mapping. They proved strong convergence of $\{u_j\}$ to the point $P_\Omega u$ in $\Omega = \text{Fix}(S) \cap (A + B)^{-1} 0 \cap \text{VI}(C, \mathcal{A})$ under some mild conditions.

In a uniformly convex and q-uniformly smooth Banach space with $q \in (1, 2]$, let the VI indicate a variational inclusion for two accretive operators and let the CFPP denote a common fixed point problem of a countable family of nonexpansive mappings. In this paper, we introduce a parallel Mann-type extragradient algorithm for solving the GSVI (1.4) with the VI and CFPP constraints. We then prove the strong convergence of the suggested algorithm to a solution of the GSVI (1.4) with the VI and CFPP constraints under some suitable assumptions. As applications, we apply our main result to the variational inequality problem (VIP), split feasibility problem (SFP) and LASSO problem in Hilbert spaces. Our results improve and extend the corresponding results in Manaka and Takahashi [28], Sunthrayuth and Cholamjiak [33], and Ceng et al. [8] to a certain extent.

2. Preliminaries

Let C be a nonempty closed convex subset of a real Banach space E with the dual E^*. For simplicity, we shall use the following notations: $x_n \to x$ indicates the strong convergence of the sequence $\{x_n\}$ to x and $x_n \rightharpoonup x$ denotes the weak convergence of the sequence $\{x_n\}$ to x. Given a self-mapping T on C. We use the notations $\mathbf{R}$ and $\text{Fix}(T)$ to stand for the set of all real numbers and the fixed point set of T, respectively. Recall that T is said to be nonexpansive if $\|Tu - Tv\| \leq \|u - v\| \; \forall u, v \in C$. A mapping $f : C \to C$ is called a contraction if $\exists \delta \in [0, 1)$ s.t. $\|f(u) - f(v)\| \leq \delta \|u - v\| \; \forall u, v \in C$. Also, recall that the normalized duality mapping J defined by

$$J(x) = \{\phi \in E^* : \langle x, \phi \rangle = \|x\|^2 = \|\phi\|^2\} \quad \forall x \in E \tag{2.1}$$

is the one from E into the family of nonempty (by Hahn-Banach's theorem) weak* compact subsets of E^*, satisfying $J(\tau u) = \tau J(u)$ and $J(-u) = -J(u)$ for all $\tau > 0$ and $u \in E$.

The modulus of convexity of E is the function $\delta_E : (0, 2] \to [0, 1]$ defined by

$$\delta_E(\epsilon) = \inf\{1 - \frac{\|u + v\|}{2} : u, v \in E, \; \|u\| = \|v\| = 1, \; \|u - v\| \geq \epsilon\}.$$

The modulus of smoothness of E is the function $\rho_E : \mathbf{R}_+ := [0, \infty) \to \mathbf{R}_+$ defined by

$$\rho_E(\tau) = \sup\{\frac{\|u + \tau v\| + \|u - \tau v\|}{2} - 1 : u, v \in E, \; \|u\| = \|v\| = 1\}.$$

A Banach space E is said to be uniformly convex if $\delta_E(\epsilon) > 0 \; \forall \epsilon \in (0, 2]$. It is said to be uniformly smooth if $\lim_{\tau \to 0^+} \rho_E(\tau)/\tau = 0$. Also, it is said to be q-uniformly smooth with $q > 1$ if $\exists c > 0$ s.t. $\rho_E(t) \leq ct^q \; \forall t > 0$. If E is q-uniformly smooth, then $q \leq 2$ and E is also uniformly smooth and if E is uniformly convex, then E is also reflexive and strictly convex. It is known that Hilbert space H is 2-uniformly smooth. Further, sequence space ℓ_p and Lebesgue space L_p are $\min\{p, 2\}$-uniformly smooth for every $p > 1$ [38].

Let $q > 1$. The generalized duality mapping $J_q : E \to 2^{E^*}$ is defined by

$$J_q(x) = \{\phi \in E^* : \langle x, \phi\rangle = \|x\|^q,\ \|\phi\| = \|x\|^{q-1}\}, \tag{2.2}$$

where $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing between E and E^*. In particular, if $q = 2$, then $J_2 = J$ is the normalized duality mapping of E. It is known that $J_q(x) = \|x\|^{q-2}J(x)\ \forall x \neq 0$ and that J_q is the subdifferential of the functional $\frac{1}{q}\|\cdot\|^q$. If E is uniformly smooth, the generalized duality mapping J_q is one-to-one and single-valued. Furthermore, J_q satisfies $J_q = J_p^{-1}$, where J_p is the generalized duality mapping of E^* with $\frac{1}{p} + \frac{1}{q} = 1$. Note that no Banach space is q-uniformly smooth for $q > 2$; see [34] for more details.

The following lemma is an immediate consequence of the subdifferential inequality of the functional $\frac{1}{q}\|\cdot\|^q$.

Lemma 2.1. *Let $q > 1$ and E be a real normed space with the generalized duality mapping J_q. Then*

$$\|x + y\|^q \le \|x\|^q + q\langle y, j_q(x + y)\rangle \quad \forall x, y \in E,\ j_q(x + y) \in J_q(x + y). \tag{2.3}$$

The following lemma can be obtained from the result in [38].

Lemma 2.2. *Let $q > 1$ and $r > 0$ be two fixed real numbers and let E be uniformly convex. Then there exist strictly increasing, continuous and convex functions $g, h : \mathbf{R}_+ \to \mathbf{R}_+$ with $g(0) = 0$ and $h(0) = 0$ such that*

(a) $\|\mu u + (1-\mu)v\|^q \le \mu\|u\|^q + (1-\mu)\|v\|^q - \mu(1-\mu)g(\|u - v\|)$ with $\mu \in [0, 1]$;
(b) $h(\|u - v\|) \le \|u\|^q - q\langle u, j_q(v)\rangle + (q-1)\|v\|^q$
for all $u, v \in B_r$ and $j_q(v) \in J_q(v)$, where $B_r := \{y \in E : \|y\| \le r\}$.

The following lemma is an analogue of Lemma 2.2 (a).

Lemma 2.3. *Let $q > 1$ and $r > 0$ be two fixed real numbers and let E be uniformly convex. Then there exists a strictly increasing, continuous and convex function $g : \mathbf{R}_+ \to \mathbf{R}_+$ with $g(0) = 0$ such that*

$$\|\lambda u + \mu v + \nu w\|^q \le \lambda\|u\|^q + \mu\|v\|^q + \nu\|w\|^q - \lambda\mu g(\|u - v\|)$$

for all $u, v, w \in B_r$ and $\lambda, \mu, \nu \in [0, 1]$ with $\lambda + \mu + \nu = 1$.

Proposition 2.4. [4] *Let $\{S_n\}_{n=0}^{\infty}$ be a sequence of self-mappings on C such that $\sum_{n=1}^{\infty}\sup_{x\in C}\|S_n x - S_{n-1}x\| < \infty$. Then for each $y \in C$, $\{S_n y\}$ converges strongly to some point of C. Moreover, let S be a self-mapping on C defined by $Sy = \lim_{n\to\infty} S_n y$ for all $y \in C$. Then $\lim_{n\to\infty}\sup_{x\in C}\|S_n x - Sx\| = 0$.*

Proposition 2.5. [38] *Let $q \in (1, 2]$ a fixed real number and let E be q-uniformly smooth. Then $\|x + y\|^q \le \|x\|^q + q\langle y, J_q(x)\rangle + \kappa_q\|y\|^q\ \forall x, y \in E$, where κ_q is the q-uniform smoothness coefficient of E.*

Let D be a subset of C and let Π be a mapping of C into D. Then Π is said to be sunny if $\Pi[\Pi(x) + t(x - \Pi(x))] = \Pi(x)$, whenever $\Pi(x) + t(x - \Pi(x)) \in C$ for $x \in C$ and $t \ge 0$. A mapping Π of C into itself is called a retraction if $\Pi^2 = \Pi$. If a mapping Π of C into itself is a retraction, then $\Pi(z) = z$ for each $z \in R(\Pi)$, where $R(\Pi)$ is the range of Π. A subset D of C is called a sunny nonexpansive retract of C if there exists a sunny nonexpansive retraction from C onto D. In terms of [30], we know that if E is smooth and Π is a retraction of C onto D, then the following statements are equivalent:

(i) Π is sunny and nonexpansive;
(ii) $\|\Pi(x) - \Pi(y)\|^2 \le \langle x - y, J(\Pi(x) - \Pi(y))\rangle\ \forall x, y \in C$;
(iii) $\langle x - \Pi(x), J(y - \Pi(x))\rangle \le 0\ \forall x \in C, y \in D$.

Let $B : C \to 2^E$ be a set-valued operator with $Bx \neq \emptyset \, \forall x \in C$. Let $q > 1$. An operator B is said to be accretive if for each $x, y \in C$, $\exists j_q(x-y) \in J_q(x-y)$ s.t. $\langle u - v, j_q(x-y)\rangle \geq 0 \; \forall u \in Bx, v \in By$. An accretive operator B is said to be α-inverse-strongly accretive of order q if for each $x, y \in C$, $\exists j_q(x-y) \in J_q(x-y)$ s.t. $\langle u - v, j_q(x-y)\rangle \geq \alpha \|u - v\|^q \; \forall u \in Bx, v \in By$ for some $\alpha > 0$. If $E = H$ a Hilbert space, then B is called α-inverse-strongly monotone. An accretive operator B is said to be m-accretive if $(I + \lambda B)C = E$ for all $\lambda > 0$. For an accretive operator B, we define the mapping $J_\lambda^B : (I + \lambda B)C \to C$ by $J_\lambda^B = (I + \lambda B)^{-1}$ for each $\lambda > 0$. Such J_λ^B is called the resolvent of B for $\lambda > 0$.

Lemma 2.6. [24, 16] *Let $B : C \to 2^E$ be an m-accretive operator. Then the following statements hold:*

(i) *the resolvent identity: $J_\lambda^B x = J_\mu^B(\frac{\mu}{\lambda}x + (1 - \frac{\mu}{\lambda})J_\lambda^B x) \; \forall \lambda, \mu > 0, \; x \in E$;*

(ii) *if J_λ^B is a resolvent of B for $\lambda > 0$, then J_λ^B is a firmly nonexpansive mapping with $\mathrm{Fix}(J_\lambda^B) = B^{-1}0$, where $B^{-1}0 = \{x \in C : 0 \in Bx\}$;*

(iii) *if $E = H$ a Hilbert space, B is maximal monotone.*

Let $A : C \to E$ be an α-inverse-strongly accretive mapping of order q and $B : C \to 2^E$ be an m-accretive operator. In the sequel, we will use the notation $T_\lambda := J_\lambda^B(I - \lambda A) = (I + \lambda B)^{-1}(I - \lambda A) \; \forall \lambda > 0$.

Proposition 2.7. [24] *The following statements hold:*

(i) $\mathrm{Fix}(T_\lambda) = (A + B)^{-1}0 \; \forall \lambda > 0$*;*

(ii) $\|y - T_\lambda y\| \leq 2\|y - T_r y\|$ *for $0 < \lambda \leq r$ and $y \in C$.*

Proposition 2.8. [37] *Let E be uniformly smooth, $T : C \to C$ be a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$ and $f : C \to C$ be a fixed contraction. For each $t \in (0,1)$, let $z_t \in C$ be the unique fixed point of the contraction $C \ni z \mapsto tf(z) + (1-t)Tz$ on C, i.e., $z_t = tf(z_t) + (1-t)Tz_t$. Then $\{z_t\}$ converges strongly to a fixed point $x^* \in \mathrm{Fix}(T)$, which solves the VIP: $\langle (I - f)x^*, J(x^* - x)\rangle \leq 0 \; \forall x \in \mathrm{Fix}(T)$.*

Proposition 2.9. [24] *Let E be q-uniformly smooth with $q \in (1, 2]$. Suppose that $A : C \to E$ is an α-inverse-strongly accretive mapping of order q. Then, for any given $\lambda \geq 0$,*

$$\|(I - \lambda A)u - (I - \lambda A)v\|^q \leq \|u - v\|^q - \lambda(\alpha q - \kappa_q \lambda^{q-1})\|Au - Av\|^q \quad \forall u, v \in C,$$

where $\kappa_q > 0$ is the q-uniform smoothness coefficient of E. In particular, if $0 \leq \lambda \leq (\frac{q\alpha}{\kappa_q})^{\frac{1}{q-1}}$, then $I - \lambda A$ is nonexpansive.

Proposition 2.10. [32] *Let E be q-uniformly smooth with $q \in (1, 2]$. Let Π_C be a sunny nonexpansive retraction from E onto C. Suppose that $B_1, B_2 : C \to E$ are α-inverse-strongly accretive mapping of order q and β-inverse-strongly accretive mapping of order q, respectively. Let $G : C \to C$ be a mapping defined by $G := \Pi_C(I - \mu_1 B_1)\Pi_C(I - \mu_2 B_2)$, and GSVI($C, B_1, B_2$) denote the fixed point set of G. If $0 \leq \mu_1 \leq (\frac{q\alpha}{\kappa_q})^{\frac{1}{q-1}}$ and $0 \leq \mu_2 \leq (\frac{q\beta}{\kappa_q})^{\frac{1}{q-1}}$, then G is nonexpansive.*

Lemma 2.11. [32] *Let E be q-uniformly smooth with $q \in (1, 2]$. Let Π_C be a sunny nonexpansive retraction from E onto C. Suppose that $B_1, B_2 : C \to E$ are two nonlinear mappings. For given $x^*, y^* \in C$, (x^*, y^*) is a solution of problem (1.4) if and only if $x^* \in \mathrm{GSVI}(C, B_1, B_2)$, where $\mathrm{GSVI}(C, B_1, B_2)$ is the fixed point set of the mapping $G := \Pi_C(I - \mu_1 B_1)\Pi_C(I - \mu_2 B_2)$, and $y^* = \Pi_C(I - \mu_2 B_2)x^*$.*

Lemma 2.12. [3] *Let E be smooth, $A : C \to E$ be accretive and Π_C be a sunny nonexpansive retraction from E onto C. Then $\mathrm{VI}(C, A) = \mathrm{Fix}(\Pi_C(I - \lambda A)) \; \forall \lambda > 0$, where $\mathrm{VI}(C, A)$ is the solution set of the VIP of finding $z \in C$ s.t. $\langle Az, J(z - y)\rangle \leq 0 \; \forall y \in C$.*

Recall that if $E = H$ a Hilbert space, then the sunny nonexpansive retraction Π_C from E onto C coincides with the metric projection P_C from H onto C. Moreover, if E is uniformly smooth and T

is a nonexpansive self-mapping on C with $\mathrm{Fix}(T) \neq \emptyset$, then $\mathrm{Fix}(T)$ is a sunny nonexpansive retract from E onto C [31]. By Lemma 2.12 we know that, $x^* \in \mathrm{Fix}(T)$ solves the VIP in Proposition 2.8 if and only if x^* solves the fixed point equation $x^* = \Pi_{\mathrm{Fix}(T)} f(x^*)$.

Lemma 2.13. [27] *Let $\{\Gamma_n\}$ be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence $\{\Gamma_{n_i}\}$ of $\{\Gamma_n\}$ which satisfies $\Gamma_{n_i} < \Gamma_{n_i+1}$ for each integer $i \geq 1$. Define the sequence $\{\tau(n)\}_{n\geq n_0}$ of integers as follows:*

$$\tau(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\},$$

where integer $n_0 \geq 1$ such that $\{k \leq n_0 : \Gamma_k < \Gamma_{k+1}\} \neq \emptyset$. Then, the following hold:

(i) $\tau(n_0) \leq \tau(n_0 + 1) \leq \cdots$ *and* $\tau(n) \to \infty$*;*
(ii) $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ *and* $\Gamma_n \leq \Gamma_{\tau(n)+1}$ $\forall n \geq n_0$.

Lemma 2.14. [5] *Let E be strictly convex, and $\{T_n\}_{n=0}^{\infty}$ be a sequence of nonexpansive mappings on C. Suppose that $\bigcap_{n=0}^{\infty} \mathrm{Fix}(T_n)$ is nonempty. Let $\{\lambda_n\}$ be a sequence of positive numbers with $\sum_{n=0}^{\infty} \lambda_n = 1$. Then a mapping S on C defined by $Sx = \sum_{n=0}^{\infty} \lambda_n T_n x$ $\forall x \in C$ is defined well, nonexpansive and $\mathrm{Fix}(S) = \bigcap_{n=0}^{\infty} \mathrm{Fix}(T_n)$ holds.*

Lemma 2.15. [37] *Let $\{a_n\}$ be a sequence in $[0, \infty)$ such that $a_{n+1} \leq (1 - s_n)a_n + s_n\nu_n$ $\forall n \geq 0$, where $\{s_n\}$ and $\{\nu_n\}$ satisfy the conditions: (i) $\{s_n\} \subset [0, 1]$, $\sum_{n=0}^{\infty} s_n = \infty$; (ii) $\limsup_{n\to\infty} \nu_n \leq 0$ or $\sum_{n=0}^{\infty} |s_n\nu_n| < \infty$. Then $\lim_{n\to\infty} a_n = 0$.*

3. Main Results

Throughout this paper, suppose that E is a q-uniformly smooth and uniformly convex Banach space with $q \in (1, 2]$. Let C be a nonempty closed convex subset of E and Π_C be a sunny nonexpansive retraction from E onto C. Let $f : C \to C$ be a ϱ-contraction with constant $\varrho \in [0, \frac{1}{q})$ and $\{S_n\}_{n=0}^{\infty}$ be a countable family of nonexpansive self-mappings on C. Let $A : C \to E$ and $B : C \to 2^E$ be a σ-inverse-strongly accretive mapping of order q and an m-accretive operator, respectively. Suppose that $B_1, B_2 : C \to E$ are α-inverse-strongly accretive mapping of order q and β-inverse-strongly accretive mapping of order q, respectively. Assume that $\Omega := \bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n) \cap \mathrm{GSVI}(C, B_1, B_2) \cap (A+B)^{-1}0 \neq \emptyset$ where $\mathrm{GSVI}(C, B_1, B_2)$ is the same as defined in Lemma 2.11.

Algorithm 3.1. *Parallel Mann-type extragradient algorithm for the GSVI (1.4) with the VI and CFPP constraints.*

Initial Step: *Given $\zeta \in (0, 1)$ and $x_0 \in C$ arbitrarily.*
Iteration Steps: *Given the current iterate x_n, compute x_{n+1} as follows:*
Step 1 *Calculate* $w_n = s_n x_n + (1 - s_n)Gx_n$*;*
Step 2 *Calculate* $v_n = \Pi_C(w_n - \mu_2 B_2 w_n)$*;*
Step 3 *Calculate* $u_n = \Pi_C(v_n - \mu_1 B_1 v_n)$*;*
Step 4 *Calculate* $x_{n+1} = (1-\zeta)S_n u_n + \zeta J_{\lambda_n}^B(\alpha_n f(u_n) + (1-\alpha_n)u_n - \lambda_n A u_n)$, *where* $\{s_n\}, \{\alpha_n\} \subset (0, 1)$ *and* $\{\lambda_n\} \subset (0, \infty)$.
Set $n := n + 1$ and go to Step 1.

Lemma 3.2. *Let $\{x_n\}$ be the sequence generated by Algorithm 3.1. Then $\{x_n\}$ is bounded.*

Proof. Let $p \in \Omega := \bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n) \cap \mathrm{GSVI}(C, B_1, B_2) \cap (A + B)^{-1}0$. Then we observe that

$$p = Gp = S_n p = J_{\lambda_n}^B(p - \lambda_n Ap) = J_{\lambda_n}^B(\alpha_n p + (1 - \alpha_n)(p - \frac{\lambda_n}{1 - \alpha_n} Ap)).$$

By Propositions 2.9 and 2.10, we know that $I-\mu_1B_1,\ I-\mu_2B_2$ and $G:=\Pi_C(I-\mu_1B_1)\Pi_C(I-\mu_2B_2)$ are nonexpansive mappings. Since $G:C\to C$ is a nonexpansive mapping, by Lemma 2.2 (a) we get

$$\begin{aligned}\|w_n-p\|^q &\le s_n\|x_n-p\|^q+(1-s_n)\|Gx_n-p\|^q-s_n(1-s_n)\tilde{g}(\|x_n-Gx_n\|)\\ &\le \|x_n-p\|^q-s_n(1-s_n)\tilde{g}(\|x_n-Gx_n\|).\end{aligned} \tag{3.1}$$

Using the nonexpansivity of G again, we obtain from $u_n=Gw_n$ that

$$\|u_n-p\|\le\|w_n-p\|\le\|x_n-p\|\quad\forall n\ge 0. \tag{3.2}$$

Put $y_n:=J^B_{\lambda_n}z_n$ and $z_n:=\alpha_nf(u_n)+(1-\alpha_n)u_n-\lambda_nAu_n$ for all $n\ge 0$. Since $J^B_{\lambda_n}$, S_n and $I-\frac{\lambda_n}{1-\alpha_n}A$ are nonexpansive for all $n\ge 0$, we obtain from (3.2) that

$$\begin{aligned}&\|y_n-p\| \qquad\qquad\qquad\qquad (3.3)\\ &=\|J^B_{\lambda_n}(\alpha_nf(u_n)+(1-\alpha_n)u_n-\lambda_nAu_n)-p\|\\ &=\|J^B_{\lambda_n}(\alpha_nf(u_n)+(1-\alpha_n)(u_n-\frac{\lambda_n}{1-\alpha_n}Au_n))-J^B_{\lambda_n}(\alpha_np+(1-\alpha_n)(p-\frac{\lambda_n}{1-\alpha_n}Ap))\|\\ &\le\|(\alpha_nf(u_n)+(1-\alpha_n)(u_n-\frac{\lambda_n}{1-\alpha_n}Au_n))-(\alpha_np+(1-\alpha_n)(p-\frac{\lambda_n}{1-\alpha_n}Ap))\|\\ &=\|(1-\alpha_n)((u_n-\frac{\lambda_n}{1-\alpha_n}Au_n)-(p-\frac{\lambda_n}{1-\alpha_n}Ap))+\alpha_n(f(u_n)-p)\|\\ &\le(1-\alpha_n)\|u_n-p\|+\alpha_n\|f(u_n)-f(p)\|+\alpha_n\|f(p)-p\|\\ &\le(1-\alpha_n(1-\varrho))\|u_n-p\|+\alpha_n\|f(p)-p\|\\ &\le(1-\alpha_n(1-\varrho))\|x_n-p\|+\alpha_n\|f(p)-p\|\\ &=(1-\alpha_n(1-\varrho))\|x_n-p\|+\alpha_n(1-\varrho)\frac{\|f(p)-p\|}{1-\varrho}\\ &\le\max\{\|x_n-p\|,\frac{\|f(p)-p\|}{1-\varrho}\}.\end{aligned}$$

Hence, from (3.2) and (3.3) we get

$$\begin{aligned}\|x_{n+1}-p\| &\le (1-\zeta)\|S_nu_n-p\|+\zeta\|y_n-p\|\\ &\le (1-\zeta)\|u_n-p\|+\zeta\max\{\|x_n-p\|,\frac{\|f(p)-p\|}{1-\varrho}\}\\ &\le (1-\zeta)\|x_n-p\|+\zeta\max\{\|x_n-p\|,\frac{\|f(p)-p\|}{1-\varrho}\}\\ &\le \max\{\|x_n-p\|,\frac{\|f(p)-p\|}{1-\varrho}\}.\end{aligned}$$

By induction, we have

$$\|x_n-p\|\le\max\{\|x_0-p\|,\frac{\|f(p)-p\|}{1-\varrho}\}\quad\forall n\ge 0.$$

Consequently, $\{x_n\}$ is bounded, and so are $\{u_n\},\{w_n\},\{y_n\},\{z_n\},\{S_nu_n\}$, and $\{Au_n\}$. This completes the proof. □

Theorem 3.3. *Let $\{x_n\}$ be the sequence generated by Algorithm 3.1. Suppose that the following conditions hold:*

(C1) $\lim_{n\to\infty}\alpha_n=0$ *and* $\sum_{n=0}^\infty\alpha_n=\infty$;

(C2) $0<a\le\frac{\lambda_n}{1-\alpha_n}\le b<(\frac{\sigma q}{\kappa_q})^{\frac{1}{q-1}}$ *and* $0<c\le s_n\le d<1$;

(C3) $0<\mu_1<(\frac{\alpha q}{\kappa_q})^{\frac{1}{q-1}}$ *and* $0<\mu_2<(\frac{\beta q}{\kappa_q})^{\frac{1}{q-1}}$.

Assume that $\sum_{n=0}^{\infty}\sup_{x\in D}\|S_{n+1}x - S_n x\| < \infty$ *for any bounded subset* D *of* C*. Let* $S : C \to C$ *be a mapping defined by* $Sx = \lim_{n\to\infty} S_n x\ \forall x \in C$*, and suppose that* $\mathrm{Fix}(S) = \bigcap_{n=0}^{\infty}\mathrm{Fix}(S_n)$*. Then* $x_n \to x^* \in \Omega$*, which is the unique solution to the VIP:* $\langle (I-f)x^*, J(x^*-p)\rangle \le 0\ \forall p \in \Omega$*, i.e., the fixed point equation* $x^* = \Pi_\Omega f(x^*)$*.*

Proof. First of all, let $x^* \in \Omega$ and $y^* = \Pi_C(x^* - \mu_2 B_2 x^*)$. Using Proposition 2.9 we get

$$\begin{aligned}\|v_n - y^*\|^q &= \|\Pi_C(w_n - \mu_2 B_2 w_n) - \Pi_C(x^* - \mu_2 B_2 x^*)\|^q \\ &\le \|w_n - x^*\|^q - \mu_2(\beta q - \kappa_q \mu_2^{q-1})\|B_2 w_n - B_2 x^*\|^q,\end{aligned}$$

and

$$\begin{aligned}\|u_n - x^*\|^q &= \|\Pi_C(v_n - \mu_1 B_1 v_n) - \Pi_C(y^* - \mu_1 B_1 y^*)\|^q \\ &\le \|v_n - y^*\|^q - \mu_1(\alpha q - \kappa_q \mu_1^{q-1})\|B_1 v_n - B_1 y^*\|^q.\end{aligned}$$

Combining the last two inequalities, we have

$$\begin{aligned}\|u_n - x^*\|^q \le\ & \|w_n - x^*\|^q - \mu_2(\beta q - \kappa_q \mu_2^{q-1})\|B_2 w_n - B_2 x^*\|^q \\ & -\mu_1(\alpha q - \kappa_q \mu_1^{q-1})\|B_1 v_n - B_1 y^*\|^q.\end{aligned} \tag{3.4}$$

Also, using Propositions 2.5 and 2.9 and the convexity of $\|\cdot\|^q$, from (3.3) and (3.4) we get

$$\begin{aligned}&\|y_n - x^*\|^q \\ \le\ & \|(1-\alpha_n)((u_n - \frac{\lambda_n}{1-\alpha_n}Au_n) - (x^* - \frac{\lambda_n}{1-\alpha_n}Ax^*)) + \alpha_n(f(u_n) - x^*)\|^q \\ \le\ & (1-\alpha_n)^q\|(u_n - \frac{\lambda_n}{1-\alpha_n}Au_n) - (x^* - \frac{\lambda_n}{1-\alpha_n}Ax^*)\|^q \\ & +q\alpha_n(1-\alpha_n)^{q-1}\langle f(u_n) - x^*, J_q(u_n - x^* - \frac{\lambda_n}{1-\alpha_n}(Au_n - Ax^*))\rangle + \kappa_q\alpha_n^q\|f(u_n) - x^*\|^q \\ \le\ & (1-\alpha_n)[\|u_n - x^*\|^q - \frac{\lambda_n}{1-\alpha_n}(\sigma q - \kappa_q(\frac{\lambda_n}{1-\alpha_n})^{q-1})\|Au_n - Ax^*\|^q] \\ & +q\alpha_n(1-\alpha_n)^{q-1}\langle f(u_n) - f(x^*), J_q(u_n - x^* - \frac{\lambda_n}{1-\alpha_n}(Au_n - Ax^*))\rangle \\ & +q\alpha_n(1-\alpha_n)^{q-1}\langle f(x^*) - x^*, J_q(u_n - x^* - \frac{\lambda_n}{1-\alpha_n}(Au_n - Ax^*))\rangle + \kappa_q\alpha_n^q\|f(u_n) - x^*\|^q \\ \le\ & (1-\alpha_n(1-q\varrho))\|u_n - x^*\|^q - \lambda_n(\sigma q - \kappa_q(\frac{\lambda_n}{1-\alpha_n})^{q-1})\|Au_n - Ax^*\|^q \\ & +q\alpha_n(1-\alpha_n)^{q-1}\langle f(x^*) - x^*, J_q(u_n - x^* - \frac{\lambda_n}{1-\alpha_n}(Au_n - Ax^*))\rangle + \kappa_q\alpha_n^q\|f(u_n) - x^*\|^q \\ \le\ & (1-\alpha_n(1-q\varrho))[\|w_n - x^*\|^q - \mu_2(\beta q - \kappa_q\mu_2^{q-1})\|B_2 w_n - B_2 x^*\|^q - \mu_1(\alpha q - \kappa_q\mu_1^{q-1}) \\ & \times\|B_1 v_n - B_1 y^*\|^q] - \lambda_n(\sigma q - \kappa_q(\frac{\lambda_n}{1-\alpha_n})^{q-1})\|Au_n - Ax^*\|^q \\ & +q\alpha_n(1-\alpha_n)^{q-1}\langle f(x^*) - x^*, J_q(u_n - x^* - \frac{\lambda_n}{1-\alpha_n}(Au_n - Ax^*))\rangle + \kappa_q\alpha_n^q\|f(u_n) - x^*\|^q.\end{aligned} \tag{3.5}$$

Using Lemma 2.2 (a) again, from (3.1), (3.2) and (3.5) we get

$$
\begin{aligned}
&\|x_{n+1}-x^*\|^q \qquad (3.6)\\
\le\;& (1-\zeta)\|S_nu_n-x^*\|^q+\zeta\|y_n-x^*\|^q-\zeta(1-\zeta)g(\|S_nu_n-y_n\|)\\
\le\;& (1-\zeta)\|w_n-x^*\|^q+\zeta\{(1-\alpha_n(1-q\varrho))[\|w_n-x^*\|^q-\mu_2(\beta q-\kappa_q\mu_2^{q-1})\|B_2w_n-B_2x^*\|^q\\
&-\mu_1(\alpha q-\kappa_q\mu_1^{q-1})\|B_1v_n-B_1y^*\|^q]-\lambda_n(\sigma q-\kappa_q(\frac{\lambda_n}{1-\alpha_n})^{q-1})\|Au_n-Ax^*\|^q\\
&+q\alpha_n(1-\alpha_n)^{q-1}\langle f(x^*)-x^*,J_q(u_n-x^*-\frac{\lambda_n}{1-\alpha_n}(Au_n-Ax^*))\rangle\\
&+\kappa_q\alpha_n^q\|f(u_n)-x^*\|^q\}-\zeta(1-\zeta)g(\|S_nu_n-y_n\|)\\
\le\;& (1-\alpha_n\zeta(1-q\varrho))[\|x_n-x^*\|^q-s_n(1-s_n)\tilde g(\|x_n-Gx_n\|)]\\
&-\zeta(1-\alpha_n(1-q\varrho))[\mu_2(\beta q-\kappa_q\mu_2^{q-1})\|B_2w_n-B_2x^*\|^q\\
&+\mu_1(\alpha q-\kappa_q\mu_1^{q-1})\|B_1v_n-B_1y^*\|^q]-\zeta\lambda_n(\sigma q-\kappa_q(\frac{\lambda_n}{1-\alpha_n})^{q-1})\|Au_n-Ax^*\|^q\\
&+\zeta q\alpha_n(1-\alpha_n)^{q-1}\langle f(x^*)-x^*,J_q(u_n-x^*-\frac{\lambda_n}{1-\alpha_n}(Au_n-Ax^*))\rangle\\
&+\zeta\kappa_q\alpha_n^q\|f(u_n)-x^*\|^q-\zeta(1-\zeta)g(\|S_nu_n-y_n\|).
\end{aligned}
$$

For each $n\ge 0$, we set

$$
\begin{aligned}
\varGamma_n &= \|x_n-x^*\|^q,\\
\epsilon_n &= \alpha_n\zeta(1-q\varrho),\\
\eta_n &= \zeta(1-\alpha_n(1-q\varrho))[\mu_2(\beta q-\kappa_q\mu_2^{q-1})\|B_2w_n-B_2x^*\|^q\\
&\quad+\mu_1(\alpha q-\kappa_q\mu_1^{q-1})\|B_1v_n-B_1y^*\|^q]+\zeta\lambda_n(\sigma q-\kappa_q(\frac{\lambda_n}{1-\alpha_n})^{q-1})\|Au_n-Ax^*\|^q\\
&\quad+\zeta(1-\zeta)g(\|S_nu_n-y_n\|)+(1-\alpha_n\zeta(1-q\varrho))s_n(1-s_n)\tilde g(\|x_n-Gx_n\|),\\
\delta_n &= \zeta q\alpha_n(1-\alpha_n)^{q-1}\langle f(x^*)-x^*,J_q(u_n-x^*-\frac{\lambda_n}{1-\alpha_n}(Au_n-Ax^*))\rangle\\
&\quad+\zeta\kappa_q\alpha_n^q\|f(u_n)-x^*\|^q.
\end{aligned}
$$

Then (3.6) can be rewritten as the following formula:

$$\varGamma_{n+1}\le(1-\epsilon_n)\varGamma_n-\eta_n+\delta_n\quad\forall n\ge 0, \tag{3.7}$$

and hence

$$\varGamma_{n+1}\le(1-\epsilon_n)\varGamma_n+\delta_n\quad\forall n\ge 0. \tag{3.8}$$

We next show the strong convergence of $\{\varGamma_n\}$ by the following two cases:
Case 1. Suppose that there exists an integer $n_0\ge 1$ such that $\{\varGamma_n\}$ is non-increasing. Then

$$\varGamma_n-\varGamma_{n+1}\to 0.$$

From (3.7), we get

$$0\le\eta_n\le\varGamma_n-\varGamma_{n+1}+\delta_n-\epsilon_n\varGamma_n.$$

Since $\alpha_n\to 0$, $\epsilon_n\to 0$ and $\delta_n\to 0$, we have $\eta_n\to 0$. This ensures that $\lim_{n\to\infty}g(\|S_nu_n-y_n\|)=\lim_{n\to\infty}\tilde g(\|x_n-Gx_n\|)=0$,

$$\lim_{n\to\infty}\|B_2w_n-B_2x^*\|=\lim_{n\to\infty}\|B_1v_n-B_1y^*\|=0, \tag{3.9}$$

and

$$\lim_{n\to\infty}\|Au_n-Ax^*\|=0. \tag{3.10}$$

Note that g and $\tilde{g}$ are a strictly increasing, continuous and convex functions with $g(0) = \tilde{g}(0) = 0$. So it follows that

$$\lim_{n\to\infty} \|S_n u_n - y_n\| = \lim_{n\to\infty} \|x_n - Gx_n\| = 0. \tag{3.11}$$

On the other hand, using Lemma 2.2 (b) and the firm nonexpansivity of Π_C, we have

$$\begin{aligned}
\|v_n - y^*\|^q &= \|\Pi_C(w_n - \mu_2 B_2 w_n) - \Pi_C(x^* - \mu_2 B_2 x^*)\|^q \\
&\le \langle w_n - \mu_2 B_2 w_n - (x^* - \mu_2 B_2 x^*), J_q(v_n - y^*)\rangle \\
&= \langle w_n - x^*, J_q(v_n - y^*)\rangle + \mu_2\langle B_2 x^* - B_2 w_n, J_q(v_n - y^*)\rangle \\
&\le \frac{1}{q}[\|w_n - x^*\|^q + (q-1)\|v_n - y^*\|^q - \tilde{h}_1(\|w_n - x^* - v_n + y^*\|)] \\
&\quad + \mu_2\langle B_2 x^* - B_2 w_n, J_q(v_n - y^*)\rangle,
\end{aligned}$$

which hence attains

$$\|v_n - y^*\|^q \le \|w_n - x^*\|^q - \tilde{h}_1(\|w_n - v_n - x^* + y^*\|) + q\mu_2\|B_2 x^* - B_2 w_n\|\|v_n - y^*\|^{q-1}.$$

In a similar way, we get

$$\begin{aligned}
\|u_n - x^*\|^q &= \|\Pi_C(v_n - \mu_1 B_1 v_n) - \Pi_C(y^* - \mu_1 B_1 y^*)\|^q \\
&\le \langle v_n - \mu_1 B_1 v_n - (y^* - \mu_1 B_1 y^*), J_q(u_n - x^*)\rangle \\
&= \langle v_n - y^*, J_q(u_n - x^*)\rangle + \mu_1\langle B_1 y^* - B_1 v_n, J_q(u_n - x^*)\rangle \\
&\le \frac{1}{q}[\|v_n - y^*\|^q + (q-1)\|u_n - x^*\|^q - \tilde{h}_2(\|v_n - y^* - u_n + x^*\|)] \\
&\quad + \mu_1\langle B_1 y^* - B_1 v_n, J_q(u_n - x^*)\rangle,
\end{aligned}$$

which hence attains

$$\begin{aligned}
\|u_n - x^*\|^q &\le \|v_n - y^*\|^q - \tilde{h}_2(\|v_n - y^* - u_n + x^*\|) + q\mu_1\|B_1 y^* - B_1 v_n\|\|u_n - x^*\|^{q-1} \\
&\le \|x_n - x^*\|^q - \tilde{h}_1(\|w_n - v_n - x^* + y^*\|) + q\mu_2\|B_2 x^* - B_2 w_n\|\|v_n - y^*\|^{q-1} \\
&\quad - \tilde{h}_2(\|v_n - u_n + x^* - y^*\|) + q\mu_1\|B_1 y^* - B_1 v_n\|\|u_n - x^*\|^{q-1}.
\end{aligned} \tag{3.12}$$

Since $J^B_{\lambda_n}$ is firmly nonexpansive (due to Lemma 2.6 (ii)), by Lemma 2.2 (b) we get

$$\begin{aligned}
ll\|y_n - x^*\|^q &= \|J^B_{\lambda_n}(\alpha_n f(u_n) + (1-\alpha_n)u_n - \lambda_n A u_n) - J^B_{\lambda_n}(x^* - \lambda_n A x^*)\|^q \\
&\le \langle(\alpha_n f(u_n) + (1-\alpha_n)u_n - \lambda_n A u_n) - (x^* - \lambda_n A x^*), J_q(y_n - x^*)\rangle \\
&\le \frac{1}{q}[\|(\alpha_n f(u_n) + (1-\alpha_n)u_n - \lambda_n A u_n) - (x^* - \lambda_n A x^*)\|^q + (q-1)\|y_n - x^*\|^q \\
&\quad - h_1(\|\alpha_n f(u_n) + (1-\alpha_n)u_n - \lambda_n(Au_n - Ax^*) - y_n\|)],
\end{aligned}$$

which together with the convexity of $\|\cdot\|^q$ and the nonexpansivity of $I - \frac{\lambda_n}{1-\alpha_n}A$, implies that

$$\begin{aligned}
\|y_n - x^*\|^q &\le \|(\alpha_n f(u_n) + (1-\alpha_n)u_n - \lambda_n A u_n) - (x^* - \lambda_n A x^*)\|^q \\
&\quad - h_1(\|\alpha_n f(u_n) + (1-\alpha_n)u_n - \lambda_n(Au_n - Ax^*) - y_n\|) \\
&= \|(1-\alpha_n)((u_n - \frac{\lambda_n}{1-\alpha_n}Au_n) - (x^* - \frac{\lambda_n}{1-\alpha_n}Ax^*)) + \alpha_n(f(u_n) - x^*)\|^q \\
&\quad - h_1(\|\alpha_n f(u_n) + (1-\alpha_n)u_n - \lambda_n(Au_n - Ax^*) - y_n\|) \\
&\le (1-\alpha_n)\|(u_n - \frac{\lambda_n}{1-\alpha_n}Au_n) - (x^* - \frac{\lambda_n}{1-\alpha_n}Ax^*)\|^q + \alpha_n\|f(u_n) - x^*\|^q \\
&\quad - h_1(\|\alpha_n f(u_n) + (1-\alpha_n)u_n - \lambda_n(Au_n - Ax^*) - y_n\|) \\
&\le (1-\alpha_n)\|u_n - x^*\|^q + \alpha_n\|f(u_n) - x^*\|^q \\
&\quad - h_1(\|\alpha_n f(u_n) + (1-\alpha_n)u_n - \lambda_n(Au_n - Ax^*) - y_n\|).
\end{aligned}$$

This together with (3.2) and (3.12), implies that

$$\begin{aligned}
&\|x_{n+1}-x^*\|^q \\
&\le (1-\zeta)\|S_nu_n-x^*\|^q+\zeta\|y_n-x^*\|^q \\
&\le (1-\zeta)\|u_n-x^*\|^q+\zeta[(1-\alpha_n)\|u_n-x^*\|^q+\alpha_n\|f(u_n)-x^*\|^q \\
&\quad -h_1(\|\alpha_nf(u_n)+(1-\alpha_n)u_n-\lambda_n(Au_n-Ax^*)-y_n\|)] \\
&= (1-\zeta\alpha_n)\|u_n-x^*\|^q+\zeta\alpha_n\|f(u_n)-x^*\|^q \\
&\quad -\zeta h_1(\|\alpha_nf(u_n)+(1-\alpha_n)u_n-\lambda_n(Au_n-Ax^*)-y_n\|) \\
&\le (1-\zeta\alpha_n)[\|x_n-x^*\|^q-\tilde{h}_1(\|w_n-v_n-x^*+y^*\|)+q\mu_2\|B_2x^*-B_2w_n\|\|v_n-y^*\|^{q-1} \\
&\quad -\tilde{h}_2(\|v_n-u_n+x^*-y^*\|)+q\mu_1\|B_1y^*-B_1v_n\|\|u_n-x^*\|^{q-1}]+\zeta\alpha_n\|f(u_n)-x^*\|^q \\
&\quad -\zeta h_1(\|\alpha_nf(u_n)+(1-\alpha_n)u_n-\lambda_n(Au_n-Ax^*)-y_n\|) \\
&\le \zeta\alpha_n\|f(u_n)-x^*\|^q+\|x_n-x^*\|^q-\{(1-\zeta\alpha_n)[\tilde{h}_1(\|w_n-v_n-x^*+y^*\|) \\
&\quad +\tilde{h}_2(\|v_n-u_n+x^*-y^*\|)]+\zeta h_1(\|\alpha_nf(u_n)+(1-\alpha_n)u_n-\lambda_n(Au_n-Ax^*)-y_n\|)\} \\
&\quad +q\mu_1\|B_1y^*-B_1v_n\|\|u_n-x^*\|^{q-1}+q\mu_2\|B_2x^*-B_2w_n\|\|v_n-y^*\|^{q-1},
\end{aligned}$$

which immediately yields

$$\begin{aligned}
&(1-\zeta\alpha_n)[\tilde{h}_1(\|w_n-v_n-x^*+y^*\|)+\tilde{h}_2(\|v_n-u_n+x^*-y^*\|)] \\
&\quad +\zeta h_1(\|\alpha_nf(u_n)+(1-\alpha_n)u_n-\lambda_n(Au_n-Ax^*)-y_n\|) \\
&\le \zeta\alpha_n\|f(u_n)-x^*\|^q+\Gamma_n-\Gamma_{n+1}+q\mu_1\|B_1y^*-B_1v_n\|\|u_n-x^*\|^{q-1} \\
&\quad +q\mu_2\|B_2x^*-B_2w_n\|\|v_n-y^*\|^{q-1}.
\end{aligned}$$

Since $\tilde{h}_1, \tilde{h}_2$ and h_1 are strictly increasing, continuous and convex functions with $\tilde{h}_1(0)=\tilde{h}_2(0)=h_1(0)=0$, from (3.9) we conclude that $\|w_n-v_n-x^*+y^*\|\to 0$, $\|v_n-u_n+x^*-y^*\|\to 0$ and $\|\alpha_nf(u_n)+(1-\alpha_n)u_n-\lambda_n(Au_n-Ax^*)-y_n\|\to 0$ as $n\to\infty$. Note that

$$\|w_n-u_n\|\le\|w_n-v_n-x^*+y^*\|+\|v_n-u_n+x^*-y^*\|,$$

and

$$\begin{aligned}
&\|u_n-y_n\| \\
&= \|\alpha_nf(u_n)+(1-\alpha_n)u_n-\lambda_n(Au_n-Ax^*)-y_n+\alpha_n(u_n-f(u_n))+\lambda_n(Au_n-Ax^*)\| \\
&\le \|\alpha_nf(u_n)+(1-\alpha_n)u_n-\lambda_n(Au_n-Ax^*)-y_n\|+\alpha_n\|u_n-f(u_n)\|+\lambda_n\|Au_n-Ax^*\|.
\end{aligned}$$

So it follows from (3.10) that

$$\lim_{n\to\infty}\|w_n-u_n\|=\lim_{n\to\infty}\|u_n-y_n\|=0. \tag{3.13}$$

Also, since $w_n=s_nx_n+(1-s_n)Gx_n$, from (3.11) and (3.12) we infer that

$$\|w_n-x_n\|=(1-s_n)\|Gx_n-x_n\|\le\|Gx_n-x_n\|\to 0\quad(n\to\infty),$$

$$\|x_n-u_n\|\le\|x_n-w_n\|+\|w_n-u_n\|\to 0\quad(n\to\infty), \tag{3.14}$$

and hence

$$\begin{aligned}
\|S_nx_n-x_n\| &\le \|S_nx_n-S_nu_n\|+\|S_nu_n-y_n\|+\|y_n-u_n\|+\|u_n-x_n\| \\
&\le 2\|x_n-u_n\|+\|S_nu_n-y_n\|+\|y_n-u_n\|\to 0\quad(n\to\infty).
\end{aligned}$$

Moreover, using Proposition 2.4 we get

$$\lim_{n\to\infty}\|S_nx_n-Sx_n\|=0.$$

So, it follows that

$$\|Sx_n - x_n\| \leq \|Sx_n - S_n x_n\| + \|S_n x_n - x_n\| \to 0 \quad (n \to \infty). \tag{3.15}$$

For each $n \geq 0$, we put $T_{\lambda_n} := J^B_{\lambda_n}(I - \lambda_n A)$. Then from (3.13) and $\alpha_n \to 0$, we get

$$\begin{aligned}\|u_n - T_{\lambda_n} u_n\| &\leq \|u_n - J^B_{\lambda_n}(\alpha_n f(u_n) + (1-\alpha_n)u_n - \lambda_n A u_n)\| \\ &\quad + \|J^B_{\lambda_n}(\alpha_n f(u_n) + (1-\alpha_n)u_n - \lambda_n A u_n) - J^B_{\lambda_n}(u_n - \lambda_n A u_n)\| \\ &\leq \|u_n - y_n\| + \|(\alpha_n f(u_n) + (1-\alpha_n)u_n - \lambda_n A u_n) - (u_n - \lambda_n A u_n)\| \\ &= \|u_n - y_n\| + \alpha_n \|f(u_n) - u_n\| \to 0 \quad (n \to \infty).\end{aligned}$$

Since $\lim_{n\to\infty} a(1-\alpha_n) = a > 0$, Without loss of generality, we may assume that $\exists \lambda > 0$ s.t. $\lambda \leq a(1-\alpha_n) \leq \lambda_n \ \forall n \geq 0$. Using Proposition 2.7 (ii), we obtain from (3.14) that

$$\begin{aligned}\|T_\lambda x_n - x_n\| &\leq \|T_\lambda x_n - T_\lambda u_n\| + \|T_\lambda u_n - u_n\| + \|u_n - x_n\| \\ &\leq 2\|x_n - u_n\| + \|T_\lambda u_n - u_n\| \\ &\leq 2\|x_n - u_n\| + 2\|T_{\lambda_n} u_n - u_n\| \to 0 \quad (n \to \infty).\end{aligned} \tag{3.16}$$

We define the mapping $\Phi : C \to C$ by $\Phi x := \theta_1 Sx + \theta_2 Gx + (1 - \theta_1 - \theta_2)T_\lambda x \ \forall x \in C$ with $\theta_1 + \theta_2 < 1$ for constants $\theta_1, \theta_2 \in (0,1)$. Then by Lemma 2.14 and Proposition 2.7 (i), we know that Φ is nonexpansive and

$$\mathrm{Fix}(\Phi) = \mathrm{Fix}(S) \cap \mathrm{Fix}(G) \cap \mathrm{Fix}(T_\lambda) = \bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n) \cap \mathrm{GSVI}(C, B_1, B_2) \cap (A+B)^{-1}0 \ (=: \Omega).$$

Taking into account that

$$\|\Phi x_n - x_n\| \leq \theta_1 \|Sx_n - x_n\| + \theta_2 \|Gx_n - x_n\| + (1 - \theta_1 - \theta_2)\|T_\lambda x_n - x_n\|,$$

we deduce from (3.11), (3.15) and (3.16) that

$$\lim_{n\to\infty} \|\Phi x_n - x_n\| = 0. \tag{3.17}$$

Let $z_s = sf(z_s) + (1-s)\Phi z_s \ \forall s \in (0,1)$. Then it follows from Proposition 2.8 that $\{z_s\}$ converges strongly to a point $x^* \in \mathrm{Fix}(\Phi) = \Omega$, which solves the VIP:

$$\langle (I - f)x^*, J(x^* - p)\rangle \leq 0 \quad \forall p \in \Omega.$$

Also, from Lemma 2.1 we get

$$\begin{aligned}&\|z_s - x_n\|^q \\ &= \|s(f(z_s) - x_n) + (1-s)(\Phi z_s - x_n)\|^q \\ &\leq (1-s)^q \|\Phi z_s - x_n\|^q + qs\langle f(z_s) - x_n, J_q(z_s - x_n)\rangle \\ &= (1-s)^q \|\Phi z_s - x_n\|^q + qs\langle f(z_s) - z_s, J_q(z_s - x_n)\rangle + qs\langle z_s - x_n, J_q(z_s - x_n)\rangle \\ &\leq (1-s)^q (\|\Phi z_s - \Phi x_n\| + \|\Phi x_n - x_n\|)^q + qs\langle f(z_s) - z_s, J_q(z_s - x_n)\rangle + qs\|z_s - x_n\|^q \\ &\leq (1-s)^q (\|z_s - x_n\| + \|\Phi x_n - x_n\|)^q + qs\langle f(z_s) - z_s, J_q(z_s - x_n)\rangle + qs\|z_s - x_n\|^q,\end{aligned}$$

which immediately attains

$$\langle f(z_s) - z_s, J_q(x_n - z_s)\rangle \leq \frac{(1-s)^q}{qs}(\|z_s - x_n\| + \|\Phi x_n - x_n\|)^q + \frac{qs-1}{qs}\|z_s - x_n\|^q.$$

From (3.17), we have

$$\limsup_{n\to\infty} \langle f(z_s) - z_s, J_q(x_n - z_s)\rangle \leq \frac{(1-s)^q}{qs} M + \frac{qs-1}{qs} M = \frac{(1-s)^q + qs - 1}{qs} M, \tag{3.18}$$

where M is a constant such that $\|z_s - x_n\|^q \le M$ for all $n \ge 0$ and $s \in (0,1)$. It is easy to see that $((1-s)^q + qs - 1)/qs \to 0$ as $s \to 0$. Since J_q is norm-to-norm uniformly continuous on bounded subsets of E and $z_s \to x^*$, we get

$$\|J_q(x_n - z_s) - J_q(x_n - x^*)\| \to 0 \quad (s \to 0).$$

So we obtain

$$\begin{aligned}
&|\langle f(z_s) - z_s, J_q(x_n - z_s)\rangle - \langle f(x^*) - x^*, J_q(x_n - x^*)\rangle| \\
&\quad = |\langle f(z_s) - f(x^*), J_q(x_n - z_s)\rangle + \langle f(x^*) - x^*, J_q(x_n - z_s)\rangle + \langle x^* - z_s, J_q(x_n - z_s)\rangle \\
&\qquad -\langle f(x^*) - x^*, J_q(x_n - x^*)\rangle| \\
&\quad \le |\langle f(x^*) - x^*, J_q(x_n - z_s) - J_q(x_n - x^*)\rangle| + |\langle f(z_s) - f(x^*), J_q(x_n - z_s)\rangle| \\
&\qquad +|\langle x^* - z_s, J_q(x_n - z_s)\rangle| \\
&\quad \le \|f(x^*) - x^*\|\|J_q(x_n - z_s) - J_q(x_n - x^*)\| + (1+\delta)\|z_s - x^*\|\|x_n - z_s\|^{q-1}.
\end{aligned}$$

Hence, for each $n \ge 0$, we get

$$\lim_{s\to 0}\langle f(z_s) - z_s, J_q(x_n - z_s)\rangle = \langle f(x^*) - x^*, J_q(x_n - x^*)\rangle.$$

From (3.18), as $s \to 0$, it follows that

$$\limsup_{n\to\infty}\langle f(x^*) - x^*, J_q(x_n - x^*)\rangle \le 0. \tag{3.19}$$

By (C2), (3.10) and (3.14), we get

$$\begin{aligned}
&\|u_n - x^* - \frac{\lambda_n}{1-\alpha_n}(Au_n - Ax^*) - (x_n - x^*)\| \\
&\quad \le \|u_n - x_n\| + \frac{\lambda_n}{1-\alpha_n}\|Au_n - Ax^*\| \\
&\quad \le \|u_n - x_n\| + b\|Au_n - Ax^*\| \to 0 \quad (n \to \infty).
\end{aligned} \tag{3.20}$$

In addition, from (3.11), (3.13) and (3.14) it is easy to see that as $n \to \infty$,

$$\|x_{n+1} - x_n\| \le (1-s_n)\|S_n u_n - x_n\| + s_n\|y_n - x_n\| \le \|S_n u_n - y_n\| + \|y_n - u_n\| + \|u_n - x_n\| \to 0.$$

Using (3.19) and (3.20), we have

$$\limsup_{n\to\infty}\langle f(x^*) - x^*, J_q(u_n - x^* - \frac{\lambda_n}{1-\alpha_n}(Au_n - Ax^*))\rangle \le 0. \tag{3.21}$$

Now, from (3.6) it is easy to see that

$$\begin{aligned}
&\|x_{n+1} - x^*\|^q \\
&\quad \le (1-\alpha_n\zeta(1-q\varrho))\|x_n - x^*\|^q \\
&\qquad +\zeta q\alpha_n(1-\alpha_n)^{q-1}\langle f(x^*) - x^*, J_q(u_n - x^* - \frac{\lambda_n}{1-\alpha_n}(Au_n - Ax^*))\rangle + \zeta\kappa_q\alpha_n^q\|f(u_n) - x^*\|^q \\
&\quad = (1-\alpha_n\zeta(1-q\varrho))\|x_n - x^*\|^q \\
&\qquad +\alpha_n\zeta(1-q\varrho)[\frac{q(1-\alpha_n)^{q-1}\langle f(x^*) - x^*, J_q(u_n - x^* - \frac{\lambda_n}{1-\alpha_n}(Au_n - Ax^*))\rangle}{1-q\varrho} \\
&\qquad +\frac{\kappa_q\alpha_n^{q-1}\|f(u_n) - x^*\|^q}{1-q\varrho}].
\end{aligned} \tag{3.22}$$

Note that $\{\alpha_n\zeta(1-q\varrho)\} \subset [0,1]$, $\sum_{n=0}^\infty \alpha_n\zeta(1-q\varrho) = \infty$ and

$$\limsup_{n\to\infty}[\frac{q(1-\alpha_n)^{q-1}\langle f(x^*) - x^*, J_q(u_n - x^* - \frac{\lambda_n}{1-\alpha_n}(Au_n - Ax^*))\rangle}{1-q\varrho} + \frac{\kappa_q\alpha_n^{q-1}\|f(u_n) - x^*\|^q}{1-q\varrho}] \le 0.$$

Applying Lemma 2.15 to (3.22), we infer that $\Gamma_n \to 0$ as $n \to \infty$. Thus, $x_n \to x^*$ as $n \to \infty$.

Case 2. Suppose that there exists a subsequence $\{\Gamma_{n_k}\}$ of $\{\Gamma_n\}$ s.t. $\Gamma_{n_k} < \Gamma_{n_k+1}$ $\forall k \in \mathbf{N}$, where $\mathbf{N}$ is the set of all positive integers. Define the mapping $\tau : \mathbf{N} \to \mathbf{N}$ by

$$\tau(n) := \max\{k \le n : \Gamma_k < \Gamma_{k+1}\}.$$

Using Lemma 2.13, we have

$$\Gamma_{\tau(n)} \le \Gamma_{\tau(n)+1} \quad \text{and} \quad \Gamma_n \le \Gamma_{\tau(n)+1}.$$

Putting $\Gamma_n = \|x_n - x^*\|^q$ $\forall n \in \mathbf{N}$ and using the same inference as in Case 1, we can obtain

$$\lim_{n\to\infty} \|x_{\tau(n)+1} - x_{\tau(n)}\| = 0 \tag{3.23}$$

and

$$\limsup_{n\to\infty} \langle f(x^*) - x^*, J_q(u_{\tau(n)} - x^* - \frac{\lambda_{\tau(n)}}{1-\alpha_{\tau(n)}}(Au_{\tau(n)} - Ax^*))\rangle \le 0. \tag{3.24}$$

Because of $\Gamma_{\tau(n)} \le \Gamma_{\tau(n)+1}$ and $\alpha_{\tau(n)} > 0$, we conclude from (3.6) that

$$\begin{aligned} \|x_{\tau(n)} - x^*\|^q \quad &\le \quad \frac{q(1-\alpha_{\tau(n)})^{q-1}}{1-q\varrho}\langle f(x^*) - x^*, J_q(u_{\tau(n)} - x^* - \frac{\lambda_{\tau(n)}}{1-\alpha_{\tau(n)}}(Au_{\tau(n)} - Ax^*))\rangle \\ &\quad + \frac{\kappa_q \alpha_{\tau(n)}^{q-1}}{1-q\varrho}\|f(u_{\tau(n)}) - x^*\|^q, \end{aligned}$$

and hence

$$\limsup_{n\to\infty} \|x_{\tau(n)} - x^*\|^q \le 0.$$

Thus, we have

$$\lim_{n\to\infty} \|x_{\tau(n)} - x^*\|^q = 0.$$

Using Proposition 2.5 and (3.23), we obtain

$$\begin{aligned} &\|x_{\tau(n)+1} - x^*\|^q - \|x_{\tau(n)} - x^*\|^q \\ &\le \quad q\langle x_{\tau(n)+1} - x_{\tau(n)}, J_q(x_{\tau(n)} - x^*)\rangle + \kappa_q\|x_{\tau(n)+1} - x_{\tau(n)}\|^q \\ &\le \quad q\|x_{\tau(n)+1} - x_{\tau(n)}\|\|x_{\tau(n)} - x^*\|^{q-1} + \kappa_q\|x_{\tau(n)+1} - x_{\tau(n)}\|^q \to 0 \quad (n \to \infty). \end{aligned}$$

Taking into account $\Gamma_n \le \Gamma_{\tau(n)+1}$, we have

$$\begin{aligned} \|x_n - x^*\|^q \quad &\le \quad \|x_{\tau(n)+1} - x^*\|^q \\ &\le \quad \|x_{\tau(n)} - x^*\|^q + q\|x_{\tau(n)+1} - x_{\tau(n)}\|\|x_{\tau(n)} - x^*\|^{q-1} + \kappa_q\|x_{\tau(n)+1} - x_{\tau(n)}\|^q. \end{aligned}$$

It is easy to see from (3.23) that $x_n \to x^*$ as $n \to \infty$. This completes the proof. □

We also obtain the strong convergence result for the parallel Mann-type extragradient algorithm in a real Hilbert space H. It is well known that $\kappa_2 = 1$ [38]. Thus, by Theorem 3.3 we derive the following conclusion.

Corollary 3.4. *Let $\emptyset \ne C \subset H$ be a closed convex set. Let $f : C \to C$ be a ϱ-contraction with constant $\varrho \in [0, \frac{1}{2})$ and $\{S_n\}_{n=0}^{\infty}$ be a countable family of nonexpansive self-mappings on C. Let $A : C \to H$ and $B : C \to 2^H$ be a σ-inverse-strongly monotone mapping and a maximal monotone operator,* *respectively. Suppose that* $B_1, B_2 : C \to H$ *are* α*-inverse-strongly monotone mapping and* β*-inverse-strongly monotone mapping, respectively. Assume that* $\Omega := \bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n) \cap \mathrm{GSVI}(C, B_1, B_2) \cap (A + B)^{-1}0 \neq \emptyset$*. For any given* $x_0 \in C$ *and* $\zeta \in (0,1)$*, let* $\{x_n\}_{n=0}^{\infty}$ *be the sequence generated by*

$$\begin{cases} w_n = s_n x_n + (1 - s_n) G x_n, \\ v_n = P_C(w_n - \mu_2 B_2 w_n), \\ u_n = P_C(v_n - \mu_1 B_1 v_n), \\ x_{n+1} = (1 - \zeta) S_n u_n + \zeta J_{\lambda_n}^B (\alpha_n f(u_n) + (1 - \alpha_n) u_n - \lambda_n A u_n) \quad \forall n \geq 0, \end{cases} \tag{3.25}$$

where the sequences $\{s_n\}, \{\alpha_n\} \subset (0,1)$ *and* $\{\lambda_n\} \subset (0, \infty)$ *are such that*

- (C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$*;*
- (C2) $0 < a \leq \frac{\lambda_n}{1-\alpha_n} \leq b < 2\sigma$ *and* $0 < c \leq s_n \leq d < 1$*;*
- (C3) $0 < \mu_1 < 2\alpha$ *and* $0 < \mu_2 < 2\beta$*.*

Assume that $\sum_{n=0}^{\infty} \sup_{x \in D} \|S_{n+1}x - S_n x\| < \infty$ *for any bounded subset* D *of* C*. Let* $S : C \to C$ *be a mapping defined by* $Sx = \lim_{n\to\infty} S_n x \ \forall x \in C$*, and suppose that* $\mathrm{Fix}(S) = \bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n)$*. Then* $x_n \to x^* \in \Omega$*, which is the unique solution to the VIP:* $\langle (I - f)x^*, p - x^* \rangle \geq 0 \ \forall p \in \Omega$*, i.e., the fixed point equation* $x^* = P_\Omega f(x^*)$*.*

Remark 3.5. Compared with the corresponding results in Manaka and Takahashi [28], Sunthrayuth and Cholamjiak [33], and Ceng et al. [8], our results improve and extend them in the following aspects.

- (i) The problem of solving the VI for two monotone operators A, B with the FPP constraint of a nonexpansive mapping S in [28, Theorem 3.1] is extended to develop our problem of solving the GSVI (1.4) with the constraints of the VI for two accretive operators A, B and the CFPP of $\{S_n\}_{n=0}^{\infty}$ a countable family of nonexpansive mappings. The Mann-type iterative scheme with weak convergence in [28, Theorem 3.1] is extended to develop our parallel Mann-type extragradient algorithm with strong convergence.
- (ii) The problem of solving the VI for two monotone operators with the constraints of the FPP of a nonexpansive mapping S and the VIP for a monotone and Lipschitzian mapping in [8, Theorem 3.1], is extended to develop our problem of solving solving the GSVI (1.4) with the constraints of the VI for two accretive operators A, B and the CFPP of $\{S_n\}_{n=0}^{\infty}$ a countable family of nonexpansive mappings. The Mann-type hybrid extragradient method in [8, Theorem 3.1] is extended to develop our parallel Mann-type extragradient algorithm.
- (iii) The problem of solving the VI for two accretive operators A, B with the FPP constraint of a nonexpansive mapping S in [33, Theorem 3.3] is extended to develop our problem of solving the GSVI (1.4) with the constraints of the VI for two accretive operators A, B and the CFPP of $\{S_n\}_{n=0}^{\infty}$ a countable family of nonexpansive mappings. The modified viscosity-type extragradient method in [33, Theorem 3.3] is extended to develop our parallel Mann-type extragradient algorithm.

4. Some Applications

In this section, we give some applications of Corollary 3.4 to important mathematical problems in the setting of Hilbert spaces.

4.1. **Application to variational inequality problem.** Given a nonempty closed convex subset $C \subset H$ and a nonlinear monotone operator $A : C \to H$. Consider the classical VIP of finding $u^* \in C$ s.t.

$$\langle Au^*, v - u^* \rangle \geq 0 \quad \forall v \in C. \tag{4.1}$$

The solution set of problem (4.1) is denoted by $\mathrm{VI}(C, A)$. It is clear that $u^* \in C$ solves VIP (4.1) if and only if it solves the fixed point equation $u^* = P_C(u^* - \lambda Au^*)$ with $\lambda > 0$. Let i_C be the indicator function of C defined by

$$i_C(u) = \begin{cases} 0 & \text{if } u \in C, \\ \infty & \text{if } u \notin C. \end{cases}$$

We use $N_C(u)$ to indicate the normal cone of C at $u \in H$, i.e., $N_C(u) = \{w \in H : \langle w, v-u \rangle \le 0 \, \forall v \in C\}$. It is known that i_C is a proper, convex and lower semicontinuous function and its subdifferential ∂i_C is a maximal monotone mapping [10]. We define the resolvent operator $J_\lambda^{\partial i_C}$ of ∂i_C for $\lambda > 0$ by

$$J_\lambda^{\partial i_C}(x) = (I + \lambda \partial i_C)^{-1}(x) \quad \forall x \in H,$$

where

$$\begin{aligned} \partial i_C(u) &= \{w \in H : i_C(u) + \langle w, v-u \rangle \le i_C(v) \, \forall v \in H\} \\ &= \{w \in H : \langle w, v-u \rangle \le 0 \, \forall v \in C\} = N_C(u) \quad \forall u \in C. \end{aligned}$$

Hence, we get

$$\begin{aligned} u = J_\lambda^{\partial i_C}(x) &\Leftrightarrow x - u \in \lambda N_C(u) \\ &\Leftrightarrow \langle x-u, v-u \rangle \le 0 \quad \forall v \in C \\ &\Leftrightarrow u = P_C(x), \end{aligned}$$

where P_C is the metric projection of H onto C. Moreover, we also have $(A + \partial i_C)^{-1}0 = \mathrm{VI}(C, A)$ [35].

Thus, putting $B = \partial i_C$ in Corollary 3.4, we obtain the following result:

Theorem 4.1. *Let f, A, B_1, B_2 and $\{S_n\}_{n=0}^\infty$ be the same as in Corollary 3.4. Suppose that $\Omega := \bigcap_{n=0}^\infty \mathrm{Fix}(S_n) \cap \mathrm{GSVI}(C, B_1, B_2) \cap \mathrm{VI}(C, A) \neq \emptyset$. For any given $x_0 \in C$ and $\zeta \in (0,1)$, let $\{x_n\}_{n=0}^\infty$ be the sequence generated by*

$$\begin{cases} w_n = s_n x_n + (1-s_n) G x_n, \\ v_n = P_C(w_n - \mu_2 B_2 w_n), \\ u_n = P_C(v_n - \mu_1 B_1 v_n), \\ x_{n+1} = (1-\zeta) S_n u_n + \zeta P_C(\alpha_n f(u_n) + (1-\alpha_n) u_n - \lambda_n A u_n) \quad \forall n \ge 0, \end{cases} \tag{4.2}$$

where the sequences $\{s_n\}, \{\alpha_n\} \subset (0,1)$ and $\{\lambda_n\} \subset (0, \infty)$ are such that the conditions (C1)-(C3) in Corollary 3.4 hold. Then $x_n \to x^* \in \Omega$, which is the unique solution to the VIP: $\langle (I-f)x^*, p - x^* \rangle \ge 0 \, \forall p \in \Omega$, i.e., the fixed point equation $x^* = P_\Omega f(x^*)$.

4.2. **Application to split feasibility problem.** Let H_1 and H_2 be two real Hilbert spaces. Consider the following split feasibility problem (SFP) of finding

$$u \in C \text{ s.t. } Tu \in Q, \tag{4.3}$$

where C and Q are closed convex subsets of H_1 and H_2, respectively, and $T : H_1 \to H_2$ is a bounded linear operator with its adjoint T^*. The solution set of SFP is denoted by $\mho := C \cap T^{-1}Q = \{u \in C : Tu \in Q\}$. In 1994, Censor and Elfving [21] first introduced the SFP for modelling inverse problems of radiation therapy treatment planning in a finite dimensional Hilbert space, which arise from phase retrieval and in medical image reconstruction.

It is known that $z \in C$ solves the SFP (4.3) if and only if z is a solution of the minimization problem: $\min_{y \in C} g(y) := \frac{1}{2}\|Ty - P_Q Ty\|^2$. Note that the function g is differentiable convex and has the Lipschitzian gradient defined by $\nabla g = T^*(I - P_Q)T$. Moreover, ∇g is $\frac{1}{\|T\|^2}$-inverse-strongly monotone, where $\|T\|^2$ is the spectral radius of T^*T [6]. So, $z \in C$ solves the SFP if and only if it solves the variational inclusion problem of finding $z \in H_1$ s.t.

$$\begin{aligned} 0 \in \nabla g(z) + \partial i_C(z) \quad &\Leftrightarrow \quad 0 \in z + \lambda \partial i_C(z) - (z - \lambda \nabla g(z)) \\ &\Leftrightarrow \quad z - \lambda \nabla g(z) \in z + \lambda \partial i_C(z) \\ &\Leftrightarrow \quad z = (I + \lambda \partial i_C)^{-1}(z - \lambda \nabla g(z)) \\ &\Leftrightarrow \quad z = P_C(z - \lambda \nabla g(z)). \end{aligned}$$

Now, setting $A = \nabla g,\ B = \partial i_C$ and $\sigma = \frac{1}{\|T\|^2}$ in Corollary 3.4, we obtain the following result:

Theorem 4.2. *Let f, B_1, B_2 and $\{S_n\}_{n=0}^\infty$ be the same as in Corollary 3.4. Assume that $\Omega := \bigcap_{n=0}^\infty \mathrm{Fix}(S_n) \cap \mathrm{GSVI}(C, B_1, B_2) \cap \mho \neq \emptyset$. For any given $x_0 \in C$ and $\zeta \in (0,1)$, let $\{x_n\}_{n=0}^\infty$ be the sequence generated by*

$$\begin{cases} w_n = s_n x_n + (1 - s_n) G x_n, \\ v_n = P_C(w_n - \mu_2 B_2 w_n), \\ u_n = P_C(v_n - \mu_1 B_1 v_n), \\ x_{n+1} = (1 - \zeta) S_n u_n + \zeta P_C(\alpha_n f(u_n) + (1 - \alpha_n) u_n - \lambda_n T^*(I - P_Q) T u_n) \quad \forall n \geq 0, \end{cases} \tag{4.4}$$

where the sequences $\{s_n\}, \{\alpha_n\} \subset (0,1)$ and $\{\lambda_n\} \subset (0,\infty)$ are such that the conditions (C1)-(C3) in Corollary 3.1 hold where $\sigma = \frac{1}{\|T\|^2}$. Then $x_n \to x^* \in \Omega$, which is the unique solution to the VIP: $\langle (I - f)x^*, p - x^* \rangle \geq 0\ \forall p \in \Omega$, i.e., the fixed point equation $x^* = P_\Omega f(x^*)$.

4.3. **Application to LASSO problem.** In this subsection, we first recall the least absolute shrinkage and selection operator (LASSO) [36], which can be formulated as a convex constrained optimization problem:

$$\min_{y \in H} \frac{1}{2} \|Ty - b\|_2^2 \quad \text{subject to } \|y\|_1 \leq s, \tag{4.5}$$

where $T : H \to H$ is a bounded operator on H, b is a fixed vector in H and $s > 0$. Let $\mho$ be the solution set of LASSO (4.5). The LASSO has received much attention because of the involvement of the ℓ_1 norm which promotes sparsity, phenomenon of many practical problems arising in statics model, image compression, compressed sensing and signal processing theory.

In terms of the optimization theory, ones know that the solution to the LASSO problem (4.5) is a minimizer of the following convex unconstrained minimization problem so-called Basis Pursuit denoising problem:

$$\min_{y \in H} g(y) + h(y), \tag{4.6}$$

where $g(y) := \frac{1}{2}\|Ty - b\|_2^2,\ h(y) := \lambda \|y\|_1$ and $\lambda \geq 0$ is a regularization parameter. It is known that $\nabla g(y) = T^*(Ty - b)$ is $\frac{1}{\|T^*T\|}$-inverse-strongly monotone. Hence, we have that z solves the LASSO if and only if z solves the variational inclusion problem of finding $z \in H$ s.t.

$$\begin{aligned} 0 \in \nabla g(z) + \partial h(z) \quad &\Leftrightarrow \quad 0 \in z + \lambda \partial h(z) - (z - \lambda \nabla g(z)) \\ &\Leftrightarrow \quad z - \lambda \nabla g(z) \in z + \lambda \partial h(z) \\ &\Leftrightarrow \quad z = (I + \lambda \partial h)^{-1}(z - \lambda \nabla g(z)) \\ &\Leftrightarrow \quad z = \mathrm{prox}_h(z - \lambda \nabla g(z)), \end{aligned}$$

where $\mathrm{prox}_h(y)$ is the proximal of $h(y) := \lambda \|y\|_1$ given by

$$\mathrm{prox}_h(y) = \mathrm{argmin}_{u \in H} \{\lambda \|u\|_1 + \frac{1}{2}\|u - y\|_2^2\} \quad \forall y \in H,$$

which is separable in indices. Then, for $y \in H$,

$$\begin{aligned}\operatorname{prox}_h(y) &= \operatorname{prox}_{\lambda\|\cdot\|_1}(y)\\ &= (\operatorname{prox}_{\lambda|\cdot|}(y_1), \operatorname{prox}_{\lambda|\cdot|}(y_2), ..., \operatorname{prox}_{\lambda|\cdot|}(y_n)),\end{aligned}$$

where $\operatorname{prox}_{\lambda|\cdot|}(y_i) = \operatorname{sgn}(y_i)\max\{|y_i| - \lambda, 0\}$ for $i = 1, 2, ..., n$.

In 2014, Xu [39] suggested the following proximal-gradient algorithm (PGA):

$$x_{k+1} = \operatorname{prox}_h(x_k - \lambda_k \Gamma^*(\Gamma x_k - b)).$$

He proved the weak convergence of the PGA to a solution of the LASSO problem (4.5).

Next, putting $C = H,\ A = \nabla g,\ B = \partial h$ and $\sigma = \frac{1}{\|T^*T\|}$ in Corollary 3.4, we obtain the following result:

Theorem 4.3. *Let f, B_1, B_2 and $\{S_n\}_{n=0}^\infty$ be the same as in Corollary 3.4 with $C = H$. Assume that $\Omega := \bigcap_{n=0}^\infty \operatorname{Fix}(S_n) \cap \operatorname{GSVI}(H, B_1, B_2) \cap \mho \neq \emptyset$. For any given $x_0 \in H$ and $\zeta \in (0,1)$, let $\{x_n\}_{n=0}^\infty$ be the sequence generated by*

$$\begin{cases} w_n = s_n x_n + (1 - s_n)Gx_n,\\ v_n = w_n - \mu_2 B_2 w_n,\\ u_n = v_n - \mu_1 B_1 v_n,\\ x_{n+1} = (1-\zeta)S_n u_n + \zeta \operatorname{prox}_h(\alpha_n f(u_n) + (1-\alpha_n)u_n - \lambda_n T^*(Tu_n - b)) \quad \forall n \geq 0, \end{cases} \tag{4.7}$$

where the sequences $\{s_n\}, \{\alpha_n\} \subset (0,1)$ and $\{\lambda_n\} \subset (0,\infty)$ are such that the conditions (C1)-(C3) in Corollary 3.4 hold where $\sigma = \frac{1}{\|T^*T\|}$. Then $x_n \to x^* \in \Omega$, which is the unique solution to the VIP: $\langle (I - f)x^*, p - x^*\rangle \geq 0\ \forall p \in \Omega$, i.e., the fixed point equation $x^* = P_\Omega f(x^*)$.

Statements and Declarations

The authors declare that they have no conflict of interest, and the manuscript has no associated data.

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