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SOME REMARKS CONCERNING APPLICABLE FIXED POINT THEOREMS FOR MULTI-VALUED OPERATORS

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ABSTRACT. In this work, we will present a synthesis of the most important applicable fixed point theorems for multi-valued operators in metric spaces. We will also discuss some stability properties of the fixed point inclusion. In particular, some new considerations on Kikkawa-Suzuki type operators will be given.

Keywords. Metric space, Fixed point, Multi-valued operator, Multi-valued contraction, Kikkawa-Suzuki type operator.

1. INTRODUCTION

In this work, we will present a synthesis of the most important applicable fixed point theorems for multi-valued operators in metric spaces. The field of metric fixed point theory starts with the multi-valued contraction principle proved by S.B. Nadler Jr. in 1969 and continues with many, many extensions and generalizations of this theorem in the forthcoming years. Relevant extensions or generalizations were given by H. Covitz and S.B. Nadler Jr. (1970), R.E. Smithson (1971), Lj. Ćirić (1972), S. Reich (1972), I.A. Rus (1972), R. Węgrzyk (1982), N. Mizoguchi and W. Takahashi (1989), Y. Feng and S. Liu (2006), M. Kikkawa and T. Suzuki (2008), and some others. We will also discuss some stability properties of the fixed point inclusion, in the light of two important conditions in fixed point theory: the retraction-displacement condition and the quasicontraction condition. Then in a special section, some new considerations on multi-valued Kikkawa-Suzuki type operators will be given.

This paper is organized as follows. In Section 2 we will introduce the most important notations and concepts that are needed through the paper, while in Section 3 we present a survey of the main applicable metric fixed point theorems for multi-valued operators. Finally, in Section 4 the main considerations and results on multi-valued Kikkawa-Suzuki operators are presented.

2. PRELIMINARIES

Let (M, d) be a metric space and $P(M)$ be the set of all nonempty subsets of M. In this paper, we will consider the usual notations in nonlinear analysis, see also [11]. For the convenience of the reader, we recall some of them.

We define the following families of subsets of M:

$$P_b(M) := \{Z \in P(M) |\ Z \text{ is bounded}\},\ P_{cl}(M) := \{Z \in P(M) |\ Z \text{ is closed}\},$$

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$$P_{b,cl}(M) := \{Z \in P(M) |\ Z \text{ is bounded and closed }\}.$$

For $x \in M$ and $A, B \in P(M)$ we also denote:

$$D(x, A) := \inf\{d(x, a) \mid a \in A\} - \text{ the distance from the point } x \text{ to the set } A,$$

$$e(A, B) := \sup\{D(a, B) \mid a \in A\} - \text{ the excess of the set } A \text{ over the set } B,$$

$$H(A, B) := \max\{e(A, B), e(B, A)\} - \text{ the Hausdorff-Pompeiu distance between the sets } A, B.$$

It is well-known that H is a metric on $P_{b,cl}(M)$ and it is a generalized metric (taking also the value $+\infty$) on $P_{cl}(M)$.

Let M be a nonempty set and $F : M \to P(M)$ be a multi-valued operator. We denote by

$$Fix(F) := \{x \in M : x \in F(x)\}$$

the fixed point set of F and by

$$Graph(F) := \{(x, y) \in M \times M : y \in F(x)\}$$

the graph of the multi-valued operator F. In particular,

$$SFix(F) := \{x \in M : F(x) = \{x\}\}$$

denotes the strict fixed point set of F.

Definition 2.1. Let (M, d) be a metric space, $x_0 \in M$ and $F : M \to P(M)$ be a multi-valued operator. By definition, a sequence $\{x_n\}_{n\in\mathbb{N}}$ is called a sequence of Picard iterates for F starting from $x_0 \in M$ if $x_{n+1} \in F(x_n)$, for all $n \in \mathbb{N}$.

3. A Survey of Some Applicable Metric Fixed Point Results

We will present now the most important applicable metric fixed point theorems for multi-valued operators. Since in many applied problems it is of great importance to get different stability properties or various approximation methods for the solution, we also discuss some stability and approximation properties which can be obtained for these classes of multi-valued operators. From this perspective, we point out the important role of the retraction-displacement condition in fixed point theory. Finally, the quasicontraction concept for multi-valued operators is put in the light and open questions are raised.

Improving a fixed point theorem for multi-valued operators with compact values, S.B. Nadler Jr. proved (in 1969, see [6]) the following result for multi-valued contractions with nonempty, bounded and closed values.

Theorem 3.1. *Let (M, d) be a complete metric space and $F : M \to P_{b,cl}(M)$ be a multi-valued α-contraction, i.e., there exists $\alpha \in (0, 1)$ such that*

$$H(F(x), F(y)) \leq \alpha d(x, y), \text{ for all } (x, y) \in M \times M. \tag{3.1}$$

Then $Fix(F) \neq \emptyset$ and for each $x_0 \in M$ there exists a sequence of Picard iterates for F starting from x_0 which converges to a fixed point of F.

The above result was extended by H. Covitz and S. B. Nadler Jr. (in 1970, see [3]) to the case of multi-valued contractions with nonempty and closed values.

Theorem 3.2. *Let (M, d) be a complete metric space and $F : M \to P_{cl}(M)$ be a multi-valued α-contraction. Then $Fix(F) \neq \emptyset$ and for each $x_0 \in M$ there exists a sequence of Picard iterates for F starting from x_0 which converges to a fixed point of F.*

The above result is known in the literature as the Multi-valued Contraction Principle and it is, probably, the most applicable metric fixed point principle for multi-valued operators. The result has many, many applications in integral and differential inclusions, construction of multi-valued fractals, applied functional analysis, game theory, mathematical economies and many other topics.

A generalization of the Multi-valued Contraction Principle was given in [10] and extended in [7] as follows.

Theorem 3.3. *Let (M, d) be a complete metric space and $F : M \to P(M)$ be a multi-valued graph α-contraction, i.e., there exists $\alpha \in (0, 1)$ such that*

$$H(F(x), F(y)) \leq \alpha d(x, y), \text{ for all } (x, y) \in Graph(F). \tag{3.2}$$

Suppose that $Graph(F)$ is closed in $M \times M$. Then $Fix(F) \neq \emptyset$ and for each $x_0 \in M$ there exists a sequence of Picard iterates for F starting from x_0, which converges to a fixed point of F.

The above result will be called Multi-valued Graph Contraction Principle.

An interesting and relevant extension of the Multi-valued Graph Contraction Principle is the following fixed point theorem given by Y. Feng, and S. Liu in 2006, see [4].

Theorem 3.4. *Let (M, d) be a complete metric space and $F : M \to P(M)$ be a multi-valued operator with closed graph. Suppose there exists $\alpha \in (0, 1)$ such that*

$$D(y, F(y)) \leq \alpha d(x, y), \text{ for all } (x, y) \in Graph(F). \tag{3.3}$$

Then, F has at least one fixed point and for every $x_0 \in M$ there exists an iterative sequence $\{x_n\}_{n\in\mathbb{N}}$ of Picard type for F starting from x_0 which converges to $x^* := x^*(x_0) \in Fix(F)$. Moreover, the following relation holds

$$d(x_0, x^*(x_0)) \leq \frac{1}{1-\alpha} D(x_0, F(x_0)). \tag{3.4}$$

The above result will be called Multi-valued Graph D-Contraction Principle.

The above results are part of a more general approach in fixed point theory, what we usually call fixed point theory for Y-contractions. More precisely, if a contraction type condition holds for every $(x, y) \in Y \subset M \times M$ (and not on the whole space $M \times M$), then we say that the operator is a (single-valued or multi-valued) Y-contraction. For example, a multi-valued graph α-contraction (see condition (3.2) in Theorem 3.3) is a multi-valued $Graph(F)$-contraction.

The following concept was introduced by Y. Feng and S. Liu in [4].

Definition 3.5. Let (M, d) be a metric space, $F : M \to P(M)$ be a multi-valued operator, $\beta \in (0, 1)$ and $x \in M$. Consider the set

$$S_\beta^x := \{y \in F(x) : \beta d(x, y) \leq D(x, F(x))\}.$$

Then, F is called a multi-valued Feng-Liu (α, β)-contraction if $\alpha \in (0, \beta)$ and for each $x \in M$ there exists $y \in S_\beta^x$ such that

$$D(y, F(y)) \leq \alpha d(x, y).$$

Notice that, if $\beta \in (0, 1)$, then, for every $x \in M$, the set S_β^x is nonempty.

If we denote $Y := \{(x, y) \in M \times M \ : \ y \in S_\beta^x\}$, then the above condition on F can be replace by the following Y-contraction type condition: there exists $\alpha \in (0, \beta)$ such that

$$D(y, F(y)) \leq \alpha d(x, y), \text{ for all } (x, y) \in Y. \tag{3.5}$$

Remark 3.6. It is easy to see that any multi-valued α-contraction is a multi-valued graph α-contraction and any multi-valued graph α-contraction is a multi-valued Feng-Liu (α, β)-contraction (with any $\beta < \alpha$).

The following result was given by Y. Feng and S. Liu in 2006, see [4].

Theorem 3.7. *Let (M,d) be a complete metric space and $F : M \to P(M)$ be a multi-valued Feng-Liu (α,β)-contraction. Suppose that either the mapping $f : M \to \mathbb{R}_+$, $f(x) = D(x, F(x))$ is lower semi-continuous and F takes closed values or $Graph(F)$ is closed. Then, the following conclusions hold:*

(i) $Fix(F) \neq \emptyset$;

(ii) for every $x_0 \in M$ there exists an iterative sequence $\{x_n\}_{n\in\mathbb{N}}$ of Picard type for F starting from x_0 which converges to $x^* := x^*(x_0) \in Fix(F)$ and the following relation holds

$$d(x_0, x^*(x_0)) \leq \frac{1}{1-\frac{\alpha}{\beta}} D(x_0, F(x_0)). \tag{3.6}$$

The above result will be called Multi-valued Feng-Liu Contraction Principle.

The following retraction–displacement condition is very important in the study of some stability properties for the fixed point inclusion $x \in F(x)$, where $F : M \to P(M)$ is a multi-valued operator on a metric space (M,d). For the single-valued case see [2].

Definition 3.8. Let (M,d) be a metric space and let $F : M \to P(M)$ be a multi-valued operator such that $Fix(F) \neq \emptyset$. Then, we say that F satisfies the strong retraction–displacement condition if there exist $c > 0$ and a set retraction $r : M \to Fix(F)$, such that

$$d(x, r(x)) \leq cD(x, F(x)), \text{ for all } x \in M. \tag{3.7}$$

For example, by Theorem 3.4 and respectively Theorem 3.7, the multi-valued operators from those theorems have the strong retraction-displacement property with $c := \frac{1}{1-\alpha}$ and $c := \frac{\beta}{\beta-\alpha}$, respectively. In particular, any multi-valued α-contraction and any multi-valued graph α-contraction on a complete metric space (M,d) have the strong retraction-displacement property with $c := \frac{1}{1-\alpha}$.

We now present some stability concepts for the fixed point inclusion $x \in F(x)$ in the setting of a metric space.

The concept of the Ulam–Hyers stability is recalled first.

Definition 3.9. Let (M,d) be a metric space and $F : M \to P(M)$ be a multi-valued operator. The fixed point inclusion $x \in F(x)$ is called Ulam–Hyers stable if there exists $C > 0$, such that for every $\varepsilon > 0$ and for each ε-fixed point $\tilde{x} \in M$ of F (in the sense that $D(\tilde{x}, F(\tilde{x})) \leq \varepsilon$), there exists $x^* \in Fix(F)$, such that

$$d(\tilde{x}, x^*) \leq C\varepsilon.$$

The well-posedness property in the sense of Reich and Zaslavski of the fixed point inclusion is defined as follows.

Definition 3.10. Let (M,d) be a metric space and $F : M \to P(M)$ be a multi-valued operator such that $Fix(F) \neq \emptyset$. Suppose there exists $r : M \to Fix(F)$ a set retraction. Then, the fixed point inclusion $x \in F(x)$ is called well-posed in the sense of Reich and Zaslavski if for each $x^* \in Fix(F)$ and for any sequence $\{u_n\}_{n\in\mathbb{N}} \subset r^{-1}(x^*)$, such that $\{D(u_n, F(u_n))\}_{n\in\mathbb{N}}$ converges to 0 as $n \to \infty$, we have that $u_n \to x^*$ as $n \to \infty$.

The data dependence property of the fixed point set is given in our next definition.

Definition 3.11. Let (M,d) be a metric space and $F : M \to P(M)$ be a multi-valued operator. Let $G : M \to P(M)$ be another multi-valued operator satisfying the following conditions:

(i) $Fix(G) \neq \emptyset$;

(ii) there exists $\eta > 0$ such that $H(F(x), G(x)) \leq \eta$, for all $x \in M$.

Then, we say that the fixed point inclusion $x \in F(x), x \in M$ has the data dependence property if for each $u^* \in Fix(G)$ there exists $x^* \in Fix(F)$, such that

$$d(u^*, x^*) \leq C\eta,$$

for some $C > 0$.

An abstract result concerning some stability properties of a multi-valued operator in terms of the strong retraction-displacement condition, is given in our next result.

Theorem 3.12. [8] *Let (M, d) be a metric space and let $F : M \to P(M)$ be a multi-valued operator with $Fix(F) \neq \emptyset$ which satisfies the strong retraction–displacement condition. Then, the fixed point inclusion $x \in F(x)$ has the Ulam–Hyers stability property, is well-posed in the sense of Reich and Zaslavski and satisfies the data dependence property.*

First Main Conclusion. By the above abstract result and Theorem 3.1 - Theorem 3.7 we can conclude that the Ulam–Hyers stability property, the well-posed property in the sense of Reich and Zaslavski and the data dependence property took place, in the conditions of each of the above mentioned results.

The notion of the Ostrowski stability property for a fixed point inclusion in a metric space is recalled now.

Definition 3.13. Let (M, d) be a metric space. Let $F : M \to P(M)$ be a multi-valued operator such that $Fix(F) \neq \emptyset$. Suppose there exists $r : M \to Fix(F)$ a set retraction. Then, the fixed point inclusion $x \in F(x)$ is said to satisfy the Ostrowski stability property if for each $x^* \in Fix(F)$ and for any sequence $\{z_n\}_{n\in\mathbb{N}} \subset r^{-1}(x^*)$, such that $\{D(z_{n+1}, F(z_n))\}_{n\in\mathbb{N}}$ converges to 0 as $n \to \infty$, we have that $z_n \to x^*$ as $n \to \infty$.

The concept of multi-valued quasicontraction is very important in the study of Ostrovski stability property of the fixed point inclusion.

Definition 3.14. Let (M, d) be a metric space and $F : M \to P(M)$ be a multi-valued operator with $Fix(F) \neq \emptyset$. Suppose there exists $r : M \to Fix(F)$ a set retraction. We say that F is a multi-valued k-quasicontraction with respect to r if $k \in (0, 1)$ and

$$e(F(x), r(x)) \leq kd(x, r(x)), \text{ for every } x \in M.$$

Definition 3.15. Let (M, d) be a metric space and $F : M \to P(M)$ be a multi-valued operator. Then, F is called a multi-valued Rus type operator if for each $x \in M$ there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset M$ such that:
(i) $x_0 = x$;
(ii) $x_{n+1} \in F(x_n)$, for each $n \in \mathbb{N}$;
(iii) the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent and its limit is a fixed point of F.

An abstract result concerning the Ostrowski stability property of a fixed point inclusion is the following result.

Theorem 3.16. [8] *Let (M, d) be a metric space and $F : M \to P(M)$ be a multi-valued Rus type operator. Suppose also that F is a multi-valued k-quasicontraction. Then the fixed point inclusion $x \in F(x)$ has the Ostrowski stability property.*

Proof. Since F is a multi-valued Rus type operator, the fixed point set $Fix(F)$ is nonempty and a set retraction $r : M \to Fix(F)$ exists. Let $x^* \in Fix(F)$ and consider any sequence $\{z_n\}_{n\in\mathbb{N}} \subset r^{-1}(x^*)$ with $D(z_{n+1}, F(z_n)) \to 0$ as $n \to \infty$. Then, we have

$$d(z_{n+1}, x^*) = d(z_{n+1}, r(z_n)) \leq D(z_{n+1}, F(z_n)) + e(F(z_n), r(z_n)) \leq$$

$$D(z_{n+1}, F(z_n)) + kd(z_n, r(z_n)) = D(z_{n+1}, F(z_n)) + kd(z_n, x^*) \leq$$

$$D(z_{n+1}, F(z_n)) + k[D(z_n, F(z_{n-1})) + kd(z_{n-1}, x^*)] \leq \cdots \leq$$

$$D(z_{n+1}, F(z_n)) + \cdots + k^n D(z_1, F(z_0)) + k^{n+1} d(z_0, x^*).$$

By the Cauchy-Toeplitz Lemma (see [8], Lemma 1) we get the conclusion. □

Second Main Conclusion. By the above abstract result and Theorem 3.1 - Theorem 3.7 we can conclude that, in the conditions of each of the above mentioned results, the Ostrowski stability property of the fixed point inclusion $x \in F(x)$ holds provided the operator F is a multi-valued k-quasicontraction.

Open Question 1. It is an open question which contraction type conditions on a multi-valued operator $F : M \to P(M)$ implies that F is a k-quasicontraction.

A special case in fixed point theory is that of nonlinear contractions.

Definition 3.17. A function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is called a strong comparison function if the following two conditions are satisfied:

(i) γ is increasing;
(ii) $\sum_{k=0}^{\infty} \gamma^k(t) < \infty,$ for any $t > 0$.

The following result, proved by R. Węgrzyk in 1982 (see [12]), is also a very useful tool for applications. The result is known as Multi-valued Nonlinear Contraction Principle.

Theorem 3.18. *Let (M, d) be a complete metric space and $F : M \to P_{cl}(M)$ be a multi-valued operator. Suppose that there exists a strong comparison function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$H(F(x), F(y)) \leq \gamma(d(x, y)), \text{ for every } (x, y) \in M \times M. \tag{3.8}$$

Then:

(a) $Fix(F) \neq \emptyset$;
(b) *for every $x_0 \in M$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of Picard iterates for F starting from x_0 which converges to a fixed point of F.*

Open Question 2. It is an open question if some of the above stability properties can be obtained for Multi-valued Nonlinear Contraction Principle.

We will present now the main strict fixed point principle for multi-valued contractions, which has important applications in Game Theory and Mathematical Economies.

Theorem 3.19. *Let (M, d) be a complete metric space and $F : M \to P_{cl}(M)$ be a multi-valued α-contraction. Suppose that $SFix(F) \neq \emptyset$. Then, the following conclusions hold:*
(a) $Fix(F) = SFix(F) = \{x^*\}$;
(b) for each $x \in M$ the sequence $(F^n(x))_{n\in\mathbb{N}}$ converges, with respect to H, to $\{x^*\}$ as $n \to \infty$.

Open Question 3. It is an open question if such strict fixed point results can be proved for the rest of the multi-valued contraction type conditions considered above.

4. On Multi-Valued Kikkawa-Suzuki Operators

Another very interesting approach in fixed point theory was presented by M. Kikkawa and T. Suzuki in 2008, see [5]. Let us recall first some useful concepts and notations.

Definition 4.1. (Kikkawa-Suzuki [5]) Let $\eta : [0,1) \to (\frac{1}{2}, 1]$ be a function defined by $\eta(a) := \frac{1}{1+a}$. Let (M,d) be a metric space and $Z \subseteq M$. Then, $F : Z \to P_{b,cl}(M)$ is called an α-KS multi-valued contraction if $\alpha \in [0,1)$ and

$$x, y \in Z \text{ with } \eta(a)D(x, F(x)) \leq d(x,y) \text{ implies } H(F(x), F(y)) \leq \alpha d(x,y).$$

The following result was proved by M. Kikkawa and T. Suzuki in [5].

Theorem 4.2. *(Kikkawa-Suzuki [5]) Let $\eta : [0,1) \to (\frac{1}{2}, 1]$ be defined by $\eta(a) = \frac{1}{1+a}$. Let (M,d) be a complete metric space and $F : M \to P_{b,cl}(M)$ be an α-KS multi-valued contraction. Then $Fix(F) \neq \emptyset$.*

Remark 4.3. Notice that for each $a \in \{0\} \cup [\frac{1}{\sqrt{2}}, 1)$ the value $\eta(a)$ is the best constant, but it is not known if $\eta(a)$ is best for $a \in (0, \frac{1}{\sqrt{2}})$ (see [5]).

By the analysis of the proof of the above theorem, the following extension of it can be proved. The result includes a strong retraction-displacement condition for this type of multi-valued Kikkawa-Suzuki operators.

Theorem 4.4. *Let $\eta : [0,1) \to (\frac{1}{2}, 1]$ be defined by $\eta(a) = \frac{1}{1+a}$. Let (M,d) be a complete metric space and $F : M \to P(M)$ be a multi-valued operator with closed graph. Suppose that there exists $\alpha \in [0,1)$ such that the following implication holds*

$$(x,y) \in Graph(F) \text{ with } \eta(\alpha)D(x, F(x)) \leq d(x,y) \text{ implies } H(F(x), F(y)) \leq \alpha d(x,y).$$

Then:

(a) $Fix(F) \neq \emptyset$;

(b) there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ of Picard type for F starting from arbitrary $x_0 \in M$ which converges to a fixed point $x^*(x_0) \in Fix(F)$ and the following retraction-displacement condition takes place

$$d(x_0, x^*(x_0)) \leq \frac{1}{1-\alpha} d(x_0, x_1).$$

Proof. Let $0 \leq \alpha < \beta < 1$. Let $x_0 \in M$ be arbitrary and take any $x_1 \in F(x_0)$.
Since $\eta(\alpha)D(x_0, F(x_0)) \leq \eta(\alpha)d(x_0, x_1) \leq d(x_0, x_1)$, we obtain that

$$D(x_1, F(x_1)) \leq H(F(x_0), F(x_1)) \leq \alpha d(x_0, x_1) < \beta d(x_0, x_1).$$

Thus, there exists $x_2 \in F(x_1)$ such that $d(x_1, x_2) \leq \beta d(x_0, x_1)$. In a similar way, we can find $x_3 \in F(x_2)$ such that $d(x_2, x_3) \leq \beta^2 d(x_0, x_1)$. By this procedure, we obtain a sequence $\{x_n\}_{n\in\mathbb{N}}$ of Picard type for F starting from x_0 which satisfies the relation

$$d(x_n, x_{n+1}) \leq \beta^n d(x_0, x_1), \text{ for every } n \in \mathbb{N}. \tag{4.1}$$

Thus, by the above relation (4.1), using a classical argument, we obtain that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy and hence convergent in (M,d) to an element $x^* := x^*(x_0)$. Since F has closed graph, we immediately obtain that $x^* \in Fix(F)$.
Now, by (4.1), we immediately obtain that

$$d(x_n, x_{n+p}) \leq \frac{\beta^n}{1-\beta} d(x_0, x_1), \text{ for every } n \in \mathbb{N} \text{ and } p \in \mathbb{N}^*. \tag{4.2}$$

Letting $p \to \infty$ we get that

$$d(x_n, x^*(x_0)) \leq \frac{\beta^n}{1-\beta} d(x_0, x_1), \text{ for every } n \in \mathbb{N}. \tag{4.3}$$

Finally, taking $n = 0$ and letting $\beta \searrow \alpha$ we conclude that

$$d(x_0, x^*(x_0)) \leq \frac{1}{1-\alpha} d(x_0, x_1). \tag{4.4}$$

The above relation represents the classical retraction-displacement condition in fixed point theory, see e.g. [9], [11], and [2]. If we notice that $x_1 \in F(x_0)$ was arbitrarily chosen, then for $q < 1$ we can choose $x_1 \in F(x_0)$ such that $d(x_0, x_1) \leq qD(x_0, F(x_0))$. Thus, the relation (4.4) takes the following form

$$d(x_0, x^*(x_0)) \leq \frac{q}{1-\alpha} D(x_0, F(x_0)), \tag{4.5}$$

which is the strong retraction-displacement condition on F. □

Now, combining the above result with Theorem 3.12, we obtain the following theorem.

Theorem 4.5. *Let $\eta : [0,1) \to (\frac{1}{2}, 1]$ be defined by $\eta(a) = \frac{1}{1+a}$. Let (M, d) be a complete metric space and $F : M \to P(M)$ be a multi-valued operator with closed graph. Suppose that there exists $\alpha \in [0,1)$ such that the following implication holds*

$$(x, y) \in Graph(F) \text{ with } \eta(\alpha)D(x, F(x)) \leq d(x, y) \text{ implies } H(F(x), F(y)) \leq \alpha d(x, y).$$

Then, the fixed point inclusion $x \in F(x)$ has the Ulam–Hyers stability property, it is well-posed in the sense of Reich and Zaslavski and satisfies the data dependence property.

Similarly, combining Theorem 4.4 with Theorem 3.16, we obtain the following result.

Theorem 4.6. *Let $\eta : [0,1) \to (\frac{1}{2}, 1]$ be defined by $\eta(a) = \frac{1}{1+a}$. Let (M, d) be a complete metric space and $F : M \to P(M)$ be a multi-valued operator with closed graph. Suppose that there exists $\alpha \in [0,1)$ such that the following implication holds*

$$(x, y) \in Graph(F) \text{ with } \eta(\alpha)D(x, F(x)) \leq d(x, y) \text{ implies } H(F(x), F(y)) \leq \alpha d(x, y).$$

If, additionally, F is a multi-valued k-quasicontraction, then the fixed point inclusion $x \in F(x)$ has the Ostrowski stability property.

Open Question 4. It is an open question, under which additional condition a multi-valued Kikkawa-Suzuki operator has the quasicontraction property.

5. Conclusion

In this paper, a synthesis of the most important applicable fixed point theorems for multi-valued operators is presented. The most important applicable metric fixed point principle for multi-valued operators is the Multi-valued Contraction Principle, see Theorem 3.2. The Multi-valued Graph Contraction Principle (see Theorem 3.3) is a relevant extension with a strong potential for applications. Finally, Feng-Liu type theorems (see Theorem 3.4 and Theorem 3.7) and the Multi-valued Nonlinear Contraction Principle (see Theorem 3.18) are other important generalization with relevant applications. For a proposed terminology in fixed point theory see [1]. In the second part of the paper, we proved a slight extension of the Kikkawa-Suzuki Theorem (see 4.4), showing that the fixed point inclusion with such an operator has some stability properties, as a consequence of the strong-retraction displacement condition. Moreover, the Ostrowski stability property is also obtained (see Theorem 4.6) under the additional assumption the the multi-valued operator is a quasicontraction. It is an open question, if (or under which additional conditions) a multi-valued Kikkawa-Suzuki operator satisfies the quasicontraction condition.

Statements and Declarations

The authors declare that they have no conflict of interest, and the manuscript has no associated data.

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