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VECTOR SOLUTIONS OF IMPLICIT FUNCTIONAL-INTEGRAL EQUATIONS WITH HIGHLY DISCONTINUOUS OPERATORS

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Abstract. Let $I \subseteq \mathbf{R}$ be a compact interval. In this paper we prove an existence result for solutions $u \in L^p(I, \mathbf{R}^n)$, with $p \in]1, +\infty]$, of the implicit functional-integral equation

$$f\big(t, u(t), \int_I \xi(t,s)\, u(\varphi(s))\, ds\big) = 0 \quad \text{for almost every} \quad t \in I,$$

where $f : I \times S \times \mathbf{R}^n \to \mathbf{R}$, $\varphi : I \to I$ and $\xi : I \times I \to [0, +\infty[$ are given functions, and $S \subseteq \mathbf{R}^n$ is a suitable closed connected and locally connected set. The main peculiarity of our result is the regularity assumption on f with respect to the third variable, considerably weaker than the usual continuity required in the literature. A function f satisfying the assumptions of our result can be discontinuous, with respect to the third variable, even at each point $x \in \mathbf{R}^n$. Our result extends a very recent result proved in the scalar case $n = 1$. Such an extension is not trivial and requires more articulated assumptions, together with a more articulated and delicate technical construction.

Keywords. Vector functional-integral equations, Integrable solutions, Discontinuous selections, Lower semicontinuous multifunctions, Operator inclusions.

1. Introduction

Let $I := [a, b]$ be a compact interval, and let us consider the implicit functional-integral equation

$$f\big(t, u(t), \int_I \xi(t,s)\, u(\varphi(s))\, ds\big) = 0 \quad \text{for a.e.} \quad t \in I, \tag{1.1}$$

where $f : I \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$, $\xi : I \times I \to [0, +\infty[$ and $\varphi : I \to I$ are given functions. Such an equation, which is motivated by applications in itself (see, for instance, [27, 28]) contains as special cases several integral equations widely studied in the literature (see, for instance, [1, 2, 3, 7, 12, 13, 18, 19, 20] and the references therein). Beyond theoretical interest, such an interest is motivated by the applications to several field of research, including physics, economics, engineering, and also nonlinear boundary value problems for ordinary differential equations (see [3, 20] and references therein).

For what concerns equation (1.1) and its special cases, a very common assumption is the continuity of f with respect to third variable (see, for instance, [3, 18, 19, 20, 28, 27] and the references therein). It is worth noticing that this is an usual requirement even in more general equations, which contain special cases of equation (1.1) (see, for instance [16, 23] and the references therein).

In the last years, much effort has been spent in order to weaken such a continuity assumption, by imposing conditions that have been gradually refined and sharpened. In this direction, various special

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cases of equation (1.1) have been considered ([1, 2, 7, 14, 12, 13]). Actually, as regards such a field of research, the most general result appears to be Theorem 3.1 of [14], where equation (1.1) has been considered in its full generality. In this latter result, which extends and improves the previous ones, the following basic assumption was made for what concern the regularity of f with respect to third variable:

- (a_1) There exist be a closed real interval Y, with $0 \notin Y$, two dense subsets D_1 and D_2 of Y, and a null-measure set $E \subseteq \mathbf{R}$, such that for a.e. $t \in I$, and for all $y \in D_1$, the function $f(t, y, \cdot)|_{\mathbf{R} \setminus E}$ is lower semicontinuous, and for a.e. $t \in I$, and for all $y \in D_2$, the function $f(t, y, \cdot)|_{\mathbf{R} \setminus E}$ is upper semicontinuous.

As showed in [14] by examples, such a condition does not imply any kind of continuity for f. That is, a function f satisfying the assumptions of Theorem 3.1 of [14] can be discontinuous, with respect to third variable, even at all points $x \in \mathbf{R}$.

At this point, we observe that a further effort in the recent research has been to consider integral equations associated to vector-valued functions. In this connection, some of the above quoted results have been extended to the case where the function u takes its values in $\mathbf{R}^n$ (see [8, 9, 15]). It is worth noticing that such extensions are not trivial, and require much more delicate techniques. Thus, from a theoretical point of view, it is natural to ask if Theorem 3.1 of [14], which concern the general equation (1.1), can be extended to the vector valued case. The aim of this paper is exactly to provide such an extension (Theorem 3.1 below). We point out that such an extension is not trivial, and requires a much more delicate proof. In particular, as regards the kind of discontinuity allowed for f with respect to the third variable, assumption (a_1) cannot be extended in an obvious way. That is, the set E cannot be replaced by an arbitrary null-measure set $E \subseteq \mathbf{R}^n$. On the contrary, the null-measure set E must belong to a particular family $\mathcal{A}_n$ defined below, whose members have a suitable geometry. As a matter of fact, our result fully extends Theorem 3.1 of [14] to the vector case. In particular, our assumptions do not imply any kind of continuity for f. That is, a function f satisfying the assumptions of our result can be discontinuous, with respect to third variable, even at all points $x \in \mathbf{R}^n$. As regards the function $f(t, \cdot, x)$, which will be defined over a suitable closed connected set $S \subseteq \mathbf{R}^n$, we only require that it is continuous, that changes its sign over S, and that is is not identically zero over any open subset of S.

Our main result will be stated and proved in Section 3, while in Section 2 we shall fix some preliminar notatiions and definitions, and we shall recall some results which will be fundamental in the sequel. Finally, in Section 4, we shall discuss briefly about some open problems concerning possible further improvements of our main result.

2. Preliminaries

From now on, the word "measurable" means "Lebesgue measurable." For any fixed $k \in \mathbf{N}$, we shall denote by $|\cdot|_k$ and m_k, respectively, the Euclidean norm in $\mathbf{R}^k$ and the Lebesgue measure in $\mathbf{R}^k$. If $A \subseteq \mathbf{R}^k$, we denote by $\overline{\operatorname{conv}}\,(A)$ the closed convex hull of the set A. If $A \subseteq B \subseteq \mathbf{R}^k$, we shall denote by $\operatorname{int}_B(A)$ the interior of A in B. Also, if $C \subseteq \mathbf{R}^k$ is a Lebesgue measurable set, we denote by $\mathcal{L}(C)$ the family of all Lebesgue measurable subsets of C.

In what follows, $n \in \mathbf{N}$ is a fixed positive integer. If $x_0 \in \mathbf{R}^n$ and $r > 0$, we denote by $\overline{B}(x_0, r)$ the closed ball in $\mathbf{R}^n$ centered at x_0 with radius r, with respect to the Euclidean norm $|\cdot|_n$. For each $i \in \{1, \ldots, n\}$, we denote by $\Pi_{n,i} : \mathbf{R}^n \to \mathbf{R}$ the projection over the i-th axis. Finally, the symbol $0_{\mathbf{R}^n}$ denotes the origin of $\mathbf{R}^n$.

We now introduce two families of subsets of $\mathbf{R}^n$ which play a crucial role in the sequel. First of all, we denote by $\mathcal{A}_n$ the family of all subsets $S \subseteq \mathbf{R}^n$ such that, for every $i = 1, \ldots, n$, the supremum and the infimum of the projection of $\overline{\operatorname{conv}}\,(S)$ on the i-th axis are both positive or both negative. Moreover, we denote by $\mathcal{H}_n$ the family of all subsets $V \subseteq \mathbf{R}^n$ such that there exist sets $V_1, V_2, \ldots, V_n \subseteq \mathbf{R}^n$, with

$m_1(\Pi_{n,i}(V_i)) = 0$ for all $i = 1 \dots, n$, such that $V = \bigcup_{i=1}^n V_i$. Of course, if $V \in \mathcal{H}_n$, then $V \in \mathcal{L}(\mathbf{R}^n)$ and $m_n(V) = 0$.

Henceforth, if $J \subseteq \mathbf{R}$ is any compact interval, we denote by $AC(J, \mathbf{R}^n)$ the set of all absolutely continuous functions from J into $\mathbf{R}^n$. If $p \in [1, +\infty]$ and $[a, b]$ is a compact interval, the space $L^p([a, b], \mathbf{R}^n)$ will be considered with the usual norm

$$\|u\|_{L^p([a,b],\mathbf{R}^n)} := \begin{cases} \left(\int_a^b |u(t)|_n^p \, dt \right)^{\frac{1}{p}} & \text{if } p < +\infty, \\ \operatorname{ess\,sup}_{t \in [a,b]} |u(t)|_n & \text{if } p = +\infty. \end{cases}$$

As usual, we put $L^p([a, b]) := L^p([a, b], \mathbf{R})$.

For the basic definitions and properties concerning continuity of multifunctions, we refer to [17, 25]. As regards measurable multifunctions, we refer to [22, 25]. Here, we only recall that if Y is a topological space and $(S, \mathcal{G})$ is a measurable space, we say that a multifunction $F : S \to 2^Y$ is $\mathcal{G}$*-measurable* (resp., $\mathcal{G}$*-weakly measurable*) in S if for any closed (resp., open) set $C \subseteq Y$ one has

$$F^-(C) := \{x \in S : \ F(x) \cap C \neq \emptyset\} \in \mathcal{G}.$$

If $S \in \mathcal{L}(\mathbf{R}^k)$, we say that a $F : S \to 2^Y$ is *measurable* (resp., *weakly measurable*) if it is $\mathcal{L}(S)$-*measurable* (resp., $\mathcal{L}(S)$-*weakly measurable*). Finally, we denote by $\mathcal{B}(Y)$ the Borel family of the topological space Y. As regards Souslin sets and their properties, we refer to [4].

For the reader's convenience, we now state explicitly some results that will be crucial keys in the proof of our result. Firstly, we recall the following selection result (here, $\mathcal{T}_\mu$ denotes the completion of the σ-algebra $\mathcal{B}(T)$ with respect to the measure μ).

Theorem 2.1 (Theorem 2.1 of [10]). *Let T and $X_1, X_2, \dots X_k$ be complete separable metric spaces, with $k \in \mathbf{N}$, and let $X := \prod_{j=1}^k X_j$ (endowed with the product topology). Let $\mu, \psi_1, \dots, \psi_k$ be positive regular Borel measures over $T, X_1, X_2, \dots X_k$, respectively, with μ finite and $\psi_1, \dots, \psi_k$ σ-finite.*

Let S be a separable metric space, $W \subseteq X$ a Souslin set, and let $F : T \times W \to 2^S$ be a multifunction with nonempty complete values. Let $E \subseteq W$ be a given set. Finally, for all $i \in \{1, \dots, k\}$, let $P_{*,i} : X \to X_i$ be the projection over X_i. Assume that:

(i) *the multifunction F is $\mathcal{T}_\mu \otimes \mathcal{B}(W)$-weakly measurable;*
(ii) *for a.e. $t \in T$, one has*

$$\big\{x = (x_1, \dots, x_k) \in W : F(t, \cdot\,) \text{ is not lower semicontinuous at } x\big\} \subseteq E.$$

Then, there exist sets $Q_1, \dots, Q_k$, with $Q_i \in \mathcal{B}(X_i)$ and $\psi_i(Q_i) = 0$ for all $i = 1, \dots, k$, and a function $\phi : T \times W \to S$ such that:

(a) *$\phi(t, x) \in F(t, x)$ for all $(t, x) \in T \times W$;*
(b) *for all $x := (x_1, x_2, \dots, x_k) \in W \setminus \Big[\big(\bigcup_{i=1}^k P_{*,i}^{-1}(Q_i)\big) \cup E\Big]$, the function $\phi(\,\cdot\,, x)$ is $\mathcal{T}_\mu$-measurable over T;*
(c) *for a.e. $t \in T$, one has*

$$\big\{x = (x_1, x_2, \dots, x_k) \in W : \phi(\,t, \cdot\,) \text{ is discontinuous at } x\big\} \subseteq E \cup \Big[W \cap \Big(\bigcup_{i=1}^k P_{*,i}^{-1}(Q_i)\Big)\Big].$$

The following is a deep existence result for operator inclusions, due to O. Naselli Ricceri and B. Ricceri.

Theorem 2.2 (Theorem 1 of [29]). *Let $(T, \mathcal{F}, \mu)$ be a finite non-atomic complete measure space; V a nonempty set; $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ two separable real Banach spaces, with Y finite-dimensional; $p, q, s \in [1, +\infty]$, with $q < +\infty$ and $q \leq p \leq s$; $\Psi : V \to L^s(T, Y)$ a surjective and one-to-one operator;*

$\Phi : V \to L^1(T, X)$ *an operator such that, for every* $v \in L^s(T, Y)$ *and every sequence* $\{v_n\}$ *in* $L^s(T, Y)$ *weakly converging to* v *in* $L^q(T, Y)$*, the sequence* $\{\Phi(\Psi^{-1}(v_n))\}$ *converges strongly to* $\Phi(\Psi^{-1}(v))$ *in* $L^1(T, X)$*;* $\varphi : [0, +\infty[\to [0, +\infty]$ *a non-decreasing function such that*

$$\operatorname*{ess\,sup}_{t \in T} \|\Phi(u)(t)\|_X \leq \varphi(\|\Psi(u)\|_{L^p(T,Y)})$$

for all $u \in V$*.*

Further, let $F : T \times X \to 2^Y$ *be a multifunction, with nonempty closed convex values, satisfying the following conditions:*

(i) *for* μ*-almost every* $t \in T$*, the multifunction* $F(t, \cdot)$ *has closed graph;*
(ii) *the set*

$$\{x \in X : \text{ the multifunction } F(\,\cdot, x) \text{ is } \mathcal{F}\text{-weakly measurable}\}$$

is dense in X*;*
(iii) *there exists a number* $r > 0$ *such that the function*

$$t \to \sup_{\|x\|_X \leq \varphi(r)} d(0_Y, F(t, x))$$

belongs to $L^s(T)$ *and its norm in* $L^p(T)$ *is less or equal to* r*.*

Under such hypotheses, there exists $\tilde{u} \in V$ *such that*

$$\Psi(\tilde{u})(t) \in F(t, \Phi(\tilde{u})(t)) \quad \mu\text{-a.e.},$$

$$\|\Psi(\tilde{u})(t)\|_Y \leq \sup_{\|x\|_X \leq \varphi(r)} d(0_Y, F(t, x)) \quad \mu\text{-a.e. in } T.$$

Finally, we recall the following proposition, concernig the convex-valued regularization of a single-valued function.

Proposition 2.3 (Proposition 2.2 of [11]). *Let* $\psi : [a, b] \times \mathbf{R}^n \to \mathbf{R}^k$ *be a given function,* $E \subseteq \mathbf{R}^n$ *a Lebesgue measurable set, with* $m_n(E) = 0$*, and let* D *be a countable dense subset of* $\mathbf{R}^n$*, with* $D \cap E = \emptyset$*. Assume that:*

(i) *for all* $t \in [a, b]$*, the function* $\psi(t, \cdot\,)$ *is bounded;*
(ii) *for all* $x \in D$*, the function* $\psi(\,\cdot\,, x)$ *is* $\mathcal{L}([a, b])$*-measurable.*

Let $G : [a, b] \times \mathbf{R}^n \to 2^{\mathbf{R}^k}$ *be the multifunction defined by setting, for each* $(t, x) \in [a, b] \times \mathbf{R}^n$*,*

$$G(t, x) := \bigcap_{m \in \mathbf{N}} \overline{\operatorname{conv}} \overline{\Bigg(\bigcup_{\substack{y \in D \\ \|y - x\|_n \leq \frac{1}{m}}} \{\psi(t, y)\} \Bigg)}.$$

Then, one has:

(a) G *has nonempty closed convex values;*
(b) *for all* $x \in \mathbf{R}^n$*, the multifunction* $G(\,\cdot\,, x)$ *is* $\mathcal{L}([a, b])$*-measurable;*
(c) *for all* $t \in [a, b]$*, the multifunction* $G(\,t,\,\cdot\,)$ *has closed graph;*
(d) *if* $t \in [a, b]$*, and* $\psi(t, \cdot)|_{\mathbf{R}^n \setminus E}$ *is continuous at* $x \in \mathbf{R}^n \setminus E$*, then one has*

$$G(t, x) = \{\psi(t, x)\}.$$

3. The Main Result

The following is our main result.

Theorem 3.1. *Let $I := [a,b]$ be a compact interval, and let $S \in \mathcal{A}_n$ be a closed, connected and locally connected subset of $\mathbf{R}^n$. Let $f : I \times S \times \mathbf{R}^n \to \mathbf{R}$, $\xi : I \times I \to [0,+\infty[$ and $\varphi : I \to I$ be given functions, and let $K := \varphi(I)$.*

Let $\Delta \subseteq S \times S$ be a countable set, dense in $S \times S$, and let S' and S'' be two dense subsets of S. Let $p, j \in]1,+\infty]$, with $j \geq p'$. Let $\gamma \in L^p(I)$ be a positive function, and let $\eta_0, \eta_1 : I \to \mathbf{R}$ be two functions. Finally, let $V \in \mathcal{H}_n$. Assume that:

(i) *φ is absolutely continuous and strictly increasing, and there exists $C > 0$ such that $\varphi' \geq C$ a.e. in I; moreover, assume that $\eta_0(\varphi^{-1}) \in L^j(K)$ and $\eta_1(\varphi^{-1}) \in L^{p'}(K)$;*

(ii) *for all $(y', y'') \in \Delta$, one has*
$$\big\{(t,x) \in I \times (\mathbf{R}^n \setminus V) : f(t,y',x) < 0 < f(t,y'',x)\big\} \in \mathcal{L}(I) \otimes \mathcal{B}(\mathbf{R}^n \setminus V);$$

(iii) *for a.e. $t \in I$, and for all $y \in S'$, the function $f(t,y,\cdot)|_{\mathbf{R}^n \setminus E}$ is lower semicontinuous (in the sense of single-valued real functions);*

(iv) *for a.e. $t \in I$, and for all $y \in S''$, the function $f(t,y,\cdot)|_{\mathbf{R}^n \setminus E}$ is upper semicontinuous (in the sense of single-valued real functions);*

(v) *for a.e. $t \in I$, and for all $x \in \mathbf{R}^n \setminus V$, the function $f(t,\cdot,x)$ is continuous in S, and one has*
$$0 \in \mathrm{int}_{\mathbf{R}}(f(t,S,x)) \qquad \text{and} \qquad \mathrm{int}_S(\{y \in S : \ f(t,y,x) = 0\}) = \emptyset;$$

(vi) *for a.e. $t \in I$, and for all $x \in \mathbf{R}^n \setminus V$, one has*
$$\sup\,\{|y|_n : \ y \in S \text{ and } \ f(t,y,x) = 0\ \} \ \leq \ \gamma(t).$$

(vii) *for every $t \in I$, the function $\xi(t,\cdot)$ is a Borel function;*

(viii) *for a.e. $s \in I$, the function $\xi(\cdot,s)$ is continuous in I, differentiable in $]a,b[$ and*
$$\xi(t,s) \leq \eta_0(s), \quad 0 < \frac{\partial \xi}{\partial t}(t,s) \leq \eta_1(s) \quad \text{for all} \quad t \in]a,b[\,.$$

Then, there exists $u \in L^p(I], \mathbf{R}^n)$ satisfying
$$f\Big(t, u(t), \int_I \xi(t,s)\, u(\varphi(s))\, ds\Big) = 0 \quad \text{for a.e.} \quad t \in I,$$
and also
$$|u(t)|_n \leq \gamma(t) \quad \text{and} \quad \int_I \xi(t,s)\, u(\varphi(s))\, ds \in \mathbf{R}^n \setminus V \quad \text{for a.e.} \quad t \in I.$$

Proof. It is not restrictive to assume that assumptions (iii)–(vi) are satisfied for every $t \in I$, and $j < +\infty$.

Since $V \in \mathcal{H}_n$, there exist sets $V_1, \ldots, V_n \subseteq \mathbf{R}^n$ such that $m_1(\Pi_{n,i}(V_i)) = 0$ for all $i = 1, \ldots, n$, and $V = \bigcup_{i=1}^n V_i$. Choose sets $H_1, \ldots, H_n \in \mathcal{B}(\mathbf{R})$ in such a way that $m_1(H_i) = 0$ and $\Pi_{n,i}(V_i)) \subseteq H_i$ for all $i = 1, \ldots, n$.

Let $H^* := \bigcup_{i=1}^n \Pi_{n,i}^{-1}(H_i)$. Of course, one has $H^* \in \mathcal{B}(\mathbf{R}^n) \cap \mathcal{H}_n$, hence $m_n(H^*) = 0$. Of course, one has $\mathbf{R}^n \setminus H^* = \prod_{i=1}^n (\mathbf{R} \setminus H_i)$. Moreover, $V \subseteq H^*$, hence $\mathbf{R}^n \setminus H^* \subseteq \mathbf{R}^n \setminus V$. Now, let the multifunctions
$$Q_0 : I \times (\mathbf{R}^n \setminus H^*) \to 2^S, \quad Q_1 : I \times (\mathbf{R}^n \setminus H^*) \to 2^S, \quad Q_2 : I \times (\mathbf{R}^n \setminus H^*) \to 2^S$$
by defined by setting, for each $(t,x) \in I \times (\mathbf{R}^n \setminus H^*)$,
$$\begin{aligned} &Q_0(t,x) := \{y \in S : \ f(t,y,x) = 0\}, \\ &Q_1(t,x) := \{y \in S : \ y \text{ is a local extremum for } \ f(t,\cdot,x)\}, \\ &Q_2(t,x) := Q_0(t,x) \setminus Q_1(t,x). \end{aligned}$$

By assumption (vi), we have

$$Q_2(t,x) \subseteq Q_0(t,x) \subseteq \overline{B}(0_{\mathbf{R}^n}, \gamma(t)) \quad \text{for all} \quad (t,x) \in I \times (\mathbf{R}^n \setminus H^*). \tag{3.1}$$

By Theorem 2.2 of [30], taking into account assumptions (iii), (iv) and (v), we have that, for every $t \in I$, the multifunction $Q_2(t,\cdot\,)$ is lower semicontinuous in $\mathbf{R}^n \setminus H^*$, with nonempty closed (in Y, hence in $\mathbf{R}^n$) values.

We now prove that the multifunction Q_2 is $\mathcal{L}(I) \otimes \mathcal{B}(\mathbf{R}^n \setminus H^*)$- weakly measurable. Since our assumptions on S imply that S has a countable base of connected open (in S) sets, it suffices to prove that $Q_2^-(U) \in \mathcal{L}(I) \otimes \mathcal{B}(\mathbf{R}^n \setminus H^*)$ for every open (in S) connected set $U \subseteq S$. To this aim, let $U \subseteq S$ be a fixed nonempty connected open set (in S), such that $Q_2^-(U) \neq \emptyset$. We claim that

$$Q_2^-(U) = \bigcup_{(y',y'') \in \Delta \cap (U \times U)} \Big\{(t,x) \in I \times (\mathbf{R}^n \setminus H^*) : f(t,y',x) < 0 < f(t,y'',x)\Big\}. \tag{3.2}$$

To see this, fix a point $(y',y'') \in (U \times U) \cap \Delta$, and let $(t,x) \in I \times (\mathbf{R}^n \setminus H^*)$ be such that

$$f(t,y',x) < 0 < f(t,y'',x). \tag{3.3}$$

By assumption (v) and by the connectedness of U, it follows that there exists $y_0 \in U$ such that $f(t,y_0,x) = 0$. Now, we consider two cases. First, assume that y_0 is not a local extremum for the function $f(t,\cdot\,,x)$. Then we get at once that $y_0 \in U \cap Q_2(t,x)$, hence $(t,x) \in Q_2^-(U)$. Conversely, assume that y_0 is a local extremum for the function $f(t,\cdot\,,x)$ (not absolute by assumption (v)). Then, y_0 is a local extremum for the function $f(t,\cdot\,,x)|_U$ (not absolute by (3.3)). Since U is open in S, by assumption (v) we have

$$\operatorname{int}_U(\{y \in U : \ f(t,y,x) = 0\}) = \emptyset.$$

Hence, by Lemma 2.1 of [30], we get that there exists a point $y_1 \in U$ such that $f(t,y_1,x) = 0$ and y_1 is not a local extremum for the function $f(t,\cdot\,,x)|_U$. This easily implies that y_1 is not a local extremum for the function $f(t,\cdot\,,x)$ in S. Consequently, we have $y_1 \in Q_2(t,x) \cap U$, and thus we get again $(t,x) \in Q_2^-(U)$. Thus, we have proved that

$$\bigcup_{(y',y'') \in \Delta \cap (U \times U)} \Big\{(t,x) \in I \times (\mathbf{R}^n \setminus H^*) : f(t,y',x) < 0 < f(t,y'',x)\Big\} \subseteq Q_2^-(U).$$

In order to prove the converse inclusion, choose a point $(t^*,x^*) \in Q_2^-(U)$. Hence, $(t^*,x^*) \in I \times (\mathbf{R}^n \setminus H^*)$ and there exists $y^* \in U \cap Q_2(t^*,x^*)$. Hence, $f(t^*,y^*,x^*) = 0$ and y^* is not a local extremum for the function $f(t^*,\cdot\,,x^*)$. Thus, there exist $y_0, y_1 \in U$ such that $f(t^*,y_0,x^*) < 0 < f(t^*,y_1,x^*)$. Since by assumption (v) the function $f(t^*,\cdot\,,x^*)$ is continuous in S, there exist two open (in S) sets $\Omega_0, \Omega_1 \subseteq S$ such that $y_0 \in \Omega_0$, $y_1 \in \Omega_1$, and also

$$f(t^*,w,x^*) < 0 < f(t^*,z,x^*) \quad \text{for all} \quad (w,z) \in \Omega_0 \times \Omega_1. \tag{3.4}$$

Thus, the set $(\Omega_0 \cap U) \times (\Omega_1 \cap U)$ is an open neighborhood of (y_0,y_1) in $S \times S$. Choose a point $(y',y'') \in ((\Omega_0 \cap U) \times (\Omega_1 \cap U)) \cap \Delta$ (this last set is nonempty by the density of Δ in S). By (3.4) we get $f(t^*,y',x^*) < 0 < f(t^*,y'',x^*)$, hence the point (t^*,x^*) belongs to the right-hand side of (3.2). Thus, the equality (3.2) is proved.

By assumption (ii), since Δ is countable, we immediately get that $Q_2^-(U) \in \mathcal{L}(I) \otimes \mathcal{B}(\mathbf{R}^n \setminus H^*)$, as desired. Hence, the the multifunction Q_2 is $\mathcal{L}(I) \otimes \mathcal{B}(\mathbf{R}^n \setminus H^*)$-weakly measurable, as claimed. We also observe that, by Theorem 3.5 of [22], the multifunction Q_2 is also $\mathcal{L}(I) \otimes \mathcal{B}(\mathbf{R}^n \setminus H^*)$-measurable.

The set $\mathbf{R}^n \setminus H^*$ is a Souslin set by Corollary 6.6.7 of [4]). By Theorem 2.1, there exist a set $U_0 \in \mathcal{L}(I)$, with $m_1(U_0) = 0$, n sets $J_1, \ldots, J_n \in \mathcal{B}(\mathbf{R})$, with $m_1(\bigcup_{i=1}^n J_i) = 0$ and a function $\omega : I \times (\mathbf{R}^n \setminus H^*) \to S$ such that:

(b_1) $\omega(t,x) \in Q_2(t,x)$ for all $(t,x) \in I \times (\mathbf{R}^n \setminus H^*)$;

(b_2) for every $x \in (\mathbf{R}^n \setminus H^*) \setminus [\bigcup_{i=1}^n \Pi_{n,i}^{-1}(J_i)]$, the function $\omega(\,\cdot\,, x)$ is measurable;
(b_3) for each $t \in I \setminus U_0$, one has

$$\{x \in \mathbf{R}^n \setminus H^* : \omega(t,\cdot\,) \text{ is discontinuous at } x\} \subseteq (\mathbf{R}^n \setminus H^*) \cap \big[\bigcup_{i=1}^n \Pi_{n,i}^{-1}(J_i)\big].$$

By (3.1) and (b_1) we immediately get

$$\omega(t,x) \in S \cap \overline{B}(0_{\mathbf{R}^n}, \gamma(t)) \quad \text{for all} \quad (t,x) \in I \times (\mathbf{R}^n \setminus H^*). \tag{3.5}$$

Put $J^* := \bigcup_{i=1}^n \Pi_{n,i}^{-1}(J_i)$. Of course, $J^* \in \mathcal{H}_n$ and $m_n(J^*) = 0$. Let $\omega^* : I \times \mathbf{R}^n \to \mathbf{R}^n$ be defined by

$$\omega^*(t,x) = \begin{cases} \omega(t,x) & \text{if } t \in I \text{ and } x \in \mathbf{R}^n \setminus H^* \\ 0_{\mathbf{R}^n} & \text{if } t \in I \text{ and } x \in H^*. \end{cases}$$

Fix a countable set $D_0 \subseteq \mathbf{R}^n \setminus (H^* \cup J^*)$, such that D_0 is dense in $\mathbf{R}^n$. Of course, such a set D_0 exists since $m_n(H^* \cup J^*) = 0$. Let $F : I \times \mathbf{R}^n \to 2^{\mathbf{R}^n}$ be the multifunction defined by

$$F(t,x) := \bigcap_{m \in \mathbf{N}} \overline{\operatorname{conv}}\,\overline{\Big(\bigcup_{\substack{z \in D_0 \\ |z-x|_n \le \frac{1}{m}}} \{\,\omega^*(t,z)\,\} \Big)} = \bigcap_{m \in \mathbf{N}} \overline{\operatorname{conv}}\,\overline{\Big(\bigcup_{\substack{z \in D_0 \\ |z-x|_n \le \frac{1}{m}}} \{\,\omega(t,z)\,\} \Big)}$$

for each $(t,x) \in I \times \mathbf{R}^n$. By (3.5), we get

$$F(t,x) \subseteq \overline{\operatorname{conv}}\,(S) \cap \overline{B}(0_{\mathbf{R}^n}, \gamma(t)) \quad \text{for every} \quad (t,x) \in I \times \mathbf{R}^n. \tag{3.6}$$

By (b_2) and Proposition 2.3, taking into account property (b_3), we get:
(c_1) F has nonempty closed convex values;
(c_2) for every $x \in \mathbf{R}^n$, the multifunction $F(\,\cdot\,, x)$ is $\mathcal{L}(I)$-measurable;
(c_3) for every $t \in I$, the multifunction $F(t,\,\cdot\,)$ has closed graph;
(c_4) $F(t,x) = \{\omega^*(t,x)\} = \{\omega(t,x)\}$ for every $(t,x) \in (I \setminus U_0) \times (\mathbf{R}^n \setminus (H^* \cup J^*))$.

By assumption (i) and by Theorem 2 of [32], the function $\varphi^{-1} : K \to I$ is absolutely continuous on the compact interval K. Let $T_0 \subseteq I$ be such that $m_1(T_0) = 0$ and $\varphi'(t) \ge C$ for all $t \in I \setminus T_0$. Put $T_1 := \varphi(T_0)$. By the absolute continuity of ϕ, we get $m_1(T_1) = 0$ (see Theorem 18.25 of [21]).

Let us define a function $\xi_1 : I \times K \to \mathbf{R}$ by putting

$$\xi_1(t,\tau) = \begin{cases} \xi(t,\varphi^{-1}(\tau)) \cdot \frac{1}{\varphi'(\varphi^{-1}(\tau))} & \text{if } t \in I \text{ and } \tau \in K \setminus T_1\,, \\ \xi(t,\varphi^{-1}(\tau)) \cdot \frac{1}{C} & \text{if } t \in I \text{ and } \tau \in T_1\,. \end{cases}$$

We observe that the function ξ_1 satisfies the two following properties.

(d_1) For every $t \in I$, the function $\xi_1(t,\cdot)$ is measurable in K. To see this, fix $t \in I$. By the continuity of φ^{-1} and by assumption (vii), the function $\xi(t,\varphi^{-1}(\cdot))$ is measurable in K. On the other side, for every $\tau \in K \setminus T_1$ we have

$$\frac{1}{\varphi'(\varphi^{-1}(\tau))} = (\varphi^{-1})'(\tau).$$

The absolute continuity of φ^{-1} implies that $(\varphi^{-1})'|_{K \setminus T_1}$ is measurable, hence our claim follows.

(d_2) For almost every $\tau \in K$, one has that the function $\xi_1(\,\cdot\,,\tau)$ is continuous in I, differentiable in $]a,b[$, and

$$\xi_1(t,\tau) \le \frac{1}{C}\,\eta_0(\varphi^{-1}(\tau)), \quad 0 < \frac{\partial \xi_1}{\partial t}(t,\tau) \le \frac{1}{C}\,\eta_1(\varphi^{-1}(\tau)) \quad \text{for all} \quad t \in]a,b[\,. \tag{3.7}$$

Indeed, let $T_0' \subseteq I$ be such that $m_1(T_0') = 0$ and, for every $s \in I \setminus T_0'$, assumption (viii) holds. Put $T_1' := T_1 \cup \varphi(T_0')$. By Theorem 18.25 of [21] we get $m_1(T_1') = 0$. At this point, taking into account assumption (viii) and the definition of ξ_1, it is immediate to check that (3.7) holds for every point $\tau \in K \setminus T_1'$.

Now we intend to apply Theorem 2.2 to the multifunction F, by choosing $T = K$ (endowed with the usual Lebesgue structure), $X = Y = \mathbf{R}^n$, $s = p$, $q = j'$, $V = L^p(K, \mathbf{R}^n)$, $\Psi(u) = u$, $r = \|\gamma\|_{L^p(K)}$, $\varphi \equiv +\infty$, and

$$\Phi(u)(t) = \int_K \xi_1(t, \tau)\, u(\tau)\, d\tau$$

for each $u \in L^p(K)$ and each $t \in K$. In order to achieve our goal, we observe the following facts.

- (e_1) One has $\Phi(L^p(K, \mathbf{R}^n)) \subseteq AC(K, \mathbf{R}^n)$. This follows at once by Proposition 2.6 of [33], taking into account assumption (i) and properties (d_1) and (d_2).
- (e_2) If $v \in L^p(K, \mathbf{R}^n)$ and $\{v^k\}$ is a sequence in $L^p(K, \mathbf{R}^n)$, weakly convergent to v in $L^{j'}(K, \mathbf{R}^n)$, then the sequence $\{\Phi(v^k)\}$ converges to $\Phi(v)$ strongly in $L^1(K, \mathbf{R}^n)$. To see this, observe that by (d_1), (d_2) and by the classical Scorza-Dragoni's theorem [31], the function ξ_1 is measurable on $K \times K$ (see also the Lemma at p. 198 of [26]). Since ξ_1 is j-th power summable in $K \times K$, our claim follows by Theorem 2 at p. 326 of [24] (see also [6], p. 171).
- (e_3) Let $g : K \to [0, +\infty]$ be defined by $g(t) = \sup_{x \in \mathbf{R}^n} \inf_{z \in F(t,x)} |z|_n$. Then, we get $g \in L^p(K)$ and $\|g\|_{L^p(K)} \leq \|\gamma\|_{L^p(K)}$. Indeed, our claim follows immediately by (3.5) (for what concerns the measurability of g, we refer to [29], p. 262).

Thus, all the assumptions of Theorem 2.2 are fulfilled. Consequently, there exist a function $\psi \in L^p(K, \mathbf{R}^n)$ and a set $K_0 \subseteq K$ such that $m_1(U_1) = 0$ and

$$\psi(t) \in F(\,t\,, \Phi(\psi)(t)) = F(\,t\,, \int_K \xi_1(t, \tau)\, \psi(\tau)\, d\tau) \quad \text{for all} \quad t \in K \setminus K_0. \tag{3.8}$$

By putting together (3.5) and (3.8) we get

$$\psi(t) \in \overline{B}(0_{\mathbf{R}^n}, \gamma(t)) \cap \overline{\operatorname{conv}}\,(S) \quad \text{for all} \quad t \in K \setminus K_0. \tag{3.9}$$

Now, fix $i \in \{1, \ldots, n\}$. Let us denote by ψ_i the i-th component of the function ψ, and let $\mu_i : K \to \mathbf{R}$ be defined by putting, for each $t \in K$,

$$\mu_i(t) := \int_K \xi_1(t, \tau)\, \psi_i(\tau)\, d\tau$$

(that is $\mu_i(t)$ is the i-th component of $\Phi(\psi)(t)$).

By (3.9) and by the definition of the family $\mathcal{G}_n$, it follows that ψ_i has constant sign in $K \setminus K_0$. Let us assume that $\psi_i(\tau) > 0$ for all $\tau \in K \setminus K_0$ (if $\psi_i(\tau) < 0$ for all $\tau \in K \setminus K_0$, then one can use an analogous argument). By (d_1), (d_2), and by Proposition 2.6 of [33], we then have that μ_i is absolutely continuous in K, and

$$\mu_i'(t) = \int_K \frac{\partial \xi_1}{\partial t}(t, s)\, \psi_i(\tau)\, d\tau > 0 \quad \text{for almost every} \quad t \in K\,.$$

Consequently, the absolutely continuous function μ_i is strictly increasing in K. By Theorem 2 of [32], we get that $\mu_i^{-1} : \mu_i(K) \to K$ is also absolutely continuous. Let

Now, let $K_1 \subseteq K$ be the set defined by

$$K_1 := K_0 \cup (U_0 \cap K) \cup \Big(\bigcup_{i=1}^{n} \mu_i^{-1}((H_i \cup J_i) \cap \mu_i(K)) \Big).$$

Again by Theorem 18.25 of [21], we get $m_1(K_1) = 0$. Now we claim that

$$f\big(t, \psi(t), \int_K \xi_1(t, \tau)\, \psi(\tau)\, d\tau\big) = 0 \quad \text{for all} \quad t \in K \setminus K_1. \tag{3.10}$$

Indeed, pick $t \in K \setminus K_1$. Firstly, observe that for every $i = 1, \ldots, n$, we have $\mu_i(t) \notin H_i \cup J_i$, hence $\Phi(\psi)(t) \in \mathbf{R}^n \setminus (H^* \cup J^*)$. By the property (c_4), we get $F(t, \Phi(\psi)(t)) = \{\omega(t, \Phi(\psi)(t))\}$. Consequently, by (3.8) and by the property (b_1), we get

$$\psi(t) = \omega(t, \Phi(\psi)(t)) \in Q_2(t, \Phi(\widetilde{v})(t)) \subseteq Q_0(t, \Phi(\widetilde{v})(t)),$$

hence $f\big(t, \psi(t), \Phi(\psi)(t)\big) = 0$, that is our claim. Thus, (3.10) holds.

By the definition of ξ_1 and by the change of variables formula for absolutely continuous transformations (see Corollary 5.4.4 of [5]), for every $t \in I$, and for all $i \in \{1, \ldots, n\}$ we have

$$\begin{aligned} \int_K \xi_1(t,\tau)\, \psi_i(\tau)\, d\tau &= \int_I \xi_1(t, \varphi(s))\, \psi_i(\varphi(s))\, \varphi'(s)\, ds \\ &= \int_{I \setminus T_0} \xi(t,s)\, \frac{1}{\varphi'(s)}\, \psi_i(\varphi(s))\, \varphi'(s)\, ds \\ &= \int_I \xi(t,s)\, \psi_i(\varphi(s))\, ds\,. \end{aligned} \tag{3.11}$$

Consequently, we have

$$\int_J \xi_1(t,\tau)\, \psi(\tau)\, d\tau = \int_I \xi(t,s)\, \psi(\varphi(s))\, ds \quad \text{for all} \quad t \in I. \tag{3.12}$$

and thus, by (3.10), we infer

$$f\Big(t, \psi(t), \int_I \xi(t,s)\, \psi(\varphi(s))\, ds\Big) = 0 \quad \text{for all} \quad t \in K \setminus K_1. \tag{3.13}$$

At this point, we need to extend appropriately the function ψ outside K. First of all, for each $i \in \{1, \ldots, n\}$ we extend the function μ_i to the whole I by defining , for each $t \in I$,

$$\mu_i^*(t) := \int_K \xi_1(t,\tau)\, \psi_i(\tau)\, d\tau = \int_I \xi(t,s)\, \psi_i(\varphi(s))\, ds \tag{3.14}$$

(where we have used (3.11)). Hence, we have $\mu_i^* : I \to \mathbf{R}$ and $\mu_i^*|_K = \mu_i$.

Fix $i \in \{1, \ldots, n\}$. We have already observed that the function ψ_i has constant sign in $K \setminus K_0$. As before, assume that $\psi_i(\tau) > 0$ for all $\tau \in K \setminus K_0$ (if $\psi_i(\tau) < 0$ for all $\tau \in K \setminus K_0$, then one can use an analogous argument). Again by (d_1), (d_2) and by Proposition 2.6 of [33], we have that μ_i^* is absolutely continuous in I and

$$(\mu_i^*)'(t) = \int_K \frac{\partial \xi_1}{\partial t}(t,\tau)\, \psi_i(\tau)\, d\tau > 0 \quad \text{for almost every} \quad t \in I.$$

Hence, μ_i^* is strictly increasing in I, and by Theorem 2 of [32], the function $(\mu_i^*)^{-1}$ is absolutely continuous in $\mu_i^*(I)$.

Now, let $Y := \bigcup_{i=1}^n (\mu_i^*)^{-1}((H_i \cup J_i) \cap \mu_i^*(I))$. A further application of Theorem 18.25 of [21] gives $m_1(Y) = 0$. For each $t \in I$, put $\mu^*(t) = (\mu_1^*(t), \ldots, \mu_n^*(t))$. Hence, we get $\mu^* : I \to \mathbf{R}^n$, and $\mu^*(t) = \Phi(\psi)(t)$ for all $t \in K$. By the properties (b_2) and (b_3), and by Theorem 6.5 of [22], the function

$$t \in I \setminus (Y \cup U_0) \to \omega(t, \mu^*(t))$$

is measurable in $I \setminus (Y \cup U_0)$. Now, observe that, by the definition of Y, we have that $\mu^*(t) \in \mathbf{R}^n \setminus (H^* \cup J^*)$ for all $t \in I \setminus (Y \cup U_0)$. Consequently, by (3.5), we have

$$|\omega(t, \mu^*(t))|_n \le \gamma(t) \quad \text{for all} \quad t \in I \setminus (Y \cup U_0). \tag{3.15}$$

Now, let us define a function $\overline{u} : I \to \mathbf{R}^n$ by

$$\overline{u}(t) = \begin{cases} \psi(t) & \text{if } t \in K \\ \omega(t, \mu^*(t)) & \text{if } t \in I \setminus (K \cup Y \cup U_0) \\ 0_{\mathbf{R}^n} & \text{if } t \in (I \setminus K) \cap (Y \cup U_0). \end{cases}$$

We claim that the function $\overline{u}$ satisfies the conclusion. In order to prove our claim, we firstly observe that, by putting together (3.9) and (3.15), we immediately obtain that $\overline{u} \in L^p(I, \mathbf{R}^n)$ and

$$|\overline{u}(t)|_n \leq \gamma(t) \quad \text{for almost every} \quad t \in I.$$

To prove the remaining part of the conclusion, choose a point $t \in I \setminus (K_1 \cup Y \cup U_0)$. If $t \in K$, then, in particular, we have $t \in K \setminus K_1$. hence, as we have already observed, we have $\Phi(\psi)(t) \in \mathbf{R}^n \setminus (H^* \cup J^*)$ By (3.12) we then get

$$\begin{aligned} \int_I \xi(t,s)\, \overline{u}(\varphi(s))\, ds &= \int_I \xi(t,s)\, \psi(\varphi(s))\, ds \\ &= \int_K \xi_1(t,\tau)\, \psi(\tau)\, d\tau \\ &= \Phi(\psi)(t) \\ &\in \mathbf{R}^n \setminus (H^* \cup J^*) \\ &\subseteq \mathbf{R}^n \setminus V. \end{aligned}$$

Moreover, by (3.13) and by the definition of $\overline{u}$ we immediately get

$$f\big(t, \overline{u}(t), \int_I \xi(t,s)\, \overline{u}(\varphi(s))\, ds\big) = f\big(t, \psi(t), \int_I \xi(t,s)\, \psi(\varphi(s))\, ds\big) = 0$$

Conversely, assume that $t \in I \setminus K$. Since $t \notin Y \cup U_0$, we have already observed that $\mu^*(t) \in \mathbf{R}^n \setminus (H^* \cup J^*)$. By (3.14) we then get

$$\int_I \xi(t,s)\, \overline{u}(\varphi(s))\, ds = \int_I \xi(t,s)\, \psi(\varphi(s))\, ds = \mu^*(t) \in \mathbf{R}^n \setminus (H^* \cup J^*) \subseteq \mathbf{R}^n \setminus V.$$

Moreover, by the definition of $\overline{u}$ and by the property (b_1), we get

$$\overline{u}(t) = \omega(t, \mu^*(t)) \in Q_2(t, \omega(t)) \subseteq Q_0(t, \omega(t)).$$

Hence, by (3.14) we get

$$0 = f(t, \overline{u}(t), \mu^*(t)) = f\big(t, \overline{u}(t), \int_I \xi(t,s)\, \psi(\varphi(s))\, ds\big) = f\big(t, \overline{u}(t), \int_I \xi(t,s)\, \overline{u}(\varphi(s))\, ds\big)\,.$$

Since $m_1(K_1 \cup Y \cup U_0) = 0$, this prove our claim. This completes the proof. □

Before concluding the paper, we make the following remark.

Remark 3.2. We have already observed in Section 1 that a function $f : I \times S \times \mathbf{R}^n \to \mathbf{R}$ satisfying the assumption of Theorem 3.1 can be discontinuous, with respect to the third variable, even at all points $x \in \mathbf{R}^n$. The example given in Remark 3.2 of [14] for the case $n = 1$ illustates such a circumstance.

Moreover, the example in Remark 3.3 of [14] shows that assumption (viii) of Theorem 3.1 cannot be weakened by assuming that $0 \leq \frac{\partial \xi}{\partial t}(t,s) \leq \eta_1(s)$. That is, Theorem 3.1 is no longer true if we allow the derivative $\frac{\partial \xi}{\partial t}$ to be zero.

Finally, as showed in the Remark 3.2 of [12], in order to make the proof of Theorem 3.1 work, it is not enough to assume that $\eta_0 \in L^j(I)$ and $\eta_1 \in L^{p'}(I)$. Indeed, if $j < +\infty$, taking into account assumption (i) and the change of variables formula for absolutely continuous transformations (Corollary 5.4.4 of [5]), it can be checked that the assumption $\eta_0(\varphi^{-1}) \in L^j(J)$ implies that $\eta_0 \in L^j(I)$, while the converse implication is not necessarily true in general. Analogously, the assumption $\eta_1(\varphi^{-1}) \in L^{p'}(J)$ implies that $\eta_1 \in L^{p'}(I)$, but the converse implication is not necessarily true in general.

4. Conclusion

In conclusion of the paper, we want to point out some open problems concerning further possible improvements of Theorem 3.1.

Firstly, on the basis of Remark 3.2, it is natural to ask if the conditions $\eta_0(\varphi^{-1}) \in L^j(K)$ and $\eta_1(\varphi^{-1}) \in L^{p'}(K)$, required in assumption (i) of Theorem 3.1, can be replaced, respectively, by the weaker conditions $\eta_0 \in L^j(I)$ and $\eta_1 \in L^{p'}(I)$. Moreover, it is natural to ask if the boundedness assumption (vi) can be replaced by a weaker condition of the type

$$\sup \{|y|_n : \ y \in S \text{ and } \ f(t,y,x) = 0 \} \ \leq \ \gamma(t) + C|x|_n^\sigma,$$

for almost every $t \in I$ and for all $x \in \mathbf{R}^n \setminus V$, with $\gamma \in L^p(I)$, $C \geq 0$ and $\sigma > 0$.

Finally, it would be interesting to extend Theorem 3.1, if possible, to the case of nonlinear Hammerstein integral operators of the type

$$\Phi(u)(t) = \int_I \xi(t,s)\, g(s, u(\varphi(s)))\, ds.$$

Statements and Declarations

The authors declare that they have no conflict of interest, and the manuscript has no associated data.

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