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GENERALIZED FIXED-POINT THEOREMS FOR ENRICHED CONTRACTIONS IN QUASI-METRIC AND NONLINEAR SPACES

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Dedicated to Professor Hari Mohan Srivastava

Abstract. This manuscript extends the enriched contraction mapping principles from quasi-Banach spaces to broader mathematical settings, including quasi-metric and nonlinear spaces. This manuscript presents novel theoretical results, proposes iterativecomputational algorithms, and provides concrete examples to illustrate the effectiveness of these generalizations. Numerical approaches and applications to quasi-metric spaces are also explored.

Keywords. Enriched Contraction Mappings, Quasi-Metric Spaces, Fixed-Point Theory, Iterative Computational Algorithms.

1. Introduction

Fixed-point theory plays a fundamental role in mathematics and its applications across various fields, including differential equations, optimization, game theory, and dynamical systems [5]. The classical Banach contraction principle ensures the existence and uniqueness of fixed points in complete metric spaces, forming the foundation for numerous theoretical advancements and practical implementations [2].

Quasi-metric spaces, generalizing metric spaces by relaxing symmetry, have been extensively studied for their applicability to directed graphs, theoretical computer science, and optimization problems. For instance, the extension of fixed-point principles to quasi-metric spaces enables solutions to problems where symmetry does not naturally arise [22, 1, 6, 12]. Enriched contraction mappings, introduced to further generalize these principles, have provided robust tools for addressing fixed-point problems in quasi-metric and nonlinear spaces [4, 24, 27, 26, 18, 13].

Nonlinear spaces, such as geodesic and CAT(0) spaces, offer rich geometric structures that have facilitated significant advancements in variational analysis and feasibility problems [7, 23]. Fixed-point results in these settings have broad implications for nonlinear optimization and dynamical systems [15, 25, 8, 9, 10].

Iterative computational algorithms play a pivotal role in the practical application of fixed-point theory. Methods such as the Krasnoselskij iteration have been adapted for quasi-metric spaces and enriched contraction mappings, enhancing their convergence properties and stability [17, 21, 11]. Recent works have explored adaptive algorithms that dynamically adjust parameters to optimize performance under enriched contraction conditions [14, 20, 3, 16, 19].

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This manuscript extends enriched contraction mappings to quasi-metric and nonlinear spaces, proposing new theoretical results and iterative algorithms. Specifically, the key contributions are generalization of enriched contraction mappings to quasi-metric spaces; extension of these principles to nonlinear spaces, including CAT(0) and geodesic spaces; development of dynamic iterative processes using time-dependent quasi-norms and presentation of computational algorithms and numerical examples to validate theoretical findings.

By addressing these areas, this work contributes to advancing fixed-point theory, iterative computational methods, and their applications in broader mathematical and applied contexts.

The proposed results and methods in this manuscript represent significant advancements over existing theories in the field of fixed-point theory, particularly in the context of quasi-metric and nonlinear spaces. Below, we highlight the specific improvements and generalizations. Previous studies, such as those by Amini [4] and Secelean [24], primarily focused on enriched contraction mappings in quasi-Banach spaces. Our work extends these principles to quasi-metric spaces and nonlinear spaces, including CAT(0) and geodesic spaces, providing a broader theoretical framework for addressing fixed-point problems in diverse settings. While adaptive iterative methods have been explored in metric and Banach spaces [17], this manuscript introduces dynamic quasi-norms and time-dependent iterative processes. These additions enhance the stability and convergence of iterative schemes in evolving systems, filling a gap in existing literature. Unlike prior works that often present abstract results without numerical validation, this manuscript incorporates concrete examples and computational algorithms. These examples explicitly demonstrate the applicability of enriched contractions to practical problems such as optimization in directed graphs and dynamic systems. The inclusion of nonlinear spaces, such as CAT(0) spaces, marks a notable improvement over traditional metric spaces. For instance, Example 4.2 illustrates how enriched contractions can address variational inequalities in geodesic spaces, a contribution not covered in earlier studies like those by Kalton and Saab [15]. The proposed algorithms (e.g., Algorithms 1 and 2) are supported by a detailed convergence analysis. Unlike previous works that lacked performance metrics, we provide quantitative convergence rates and computational complexity assessments, as demonstrated in Section 6. This manuscript explicitly links theoretical results to real-world applications, such as time-varying optimization and high-dimensional dynamic systems. These connections emphasize the utility of enriched contractions beyond purely theoretical contexts.

These comparisons underscore the novel contributions of this work, situating it as an extension and enhancement of foundational studies in fixed-point theory and its applications. By addressing limitations in prior studies and introducing innovative methodologies, this manuscript lays the groundwork for further research in quasi-metric and nonlinear spaces.

2. Preliminaries

Quasi-metric spaces and b-metric spaces are generalizations of metric spaces that relax some of the traditional metric space properties, making them suitable for a broader range of applications.

Quasi-Metric Spaces: A quasi-metric space (X, d) consists of a set X and a function $d : X \times X \to \mathbb{R}$ satisfying the following properties:

(1) **Positivity:** $d(x, y) \geq 0$ for all $x, y \in X$, with equality if and only if $x = y$.
(2) **Quasi-Symmetry:** There exists a constant $k \geq 1$ such that $d(x, y) \leq k\, d(y, x)$ for all $x, y \in X$.
(3) **Triangle Inequality:** $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

These properties generalize the concept of a metric by allowing the distance function to be non-symmetric, which is useful in applications where the symmetry condition is not naturally satisfied.

b-Metric Spaces: A b-metric space (X, d) is a further generalization where the triangle inequality is relaxed. It consists of a set X and a function $d : X \times X \to \mathbb{R}$ satisfying:

(1) **Positivity:** $d(x, y) \geq 0$ for all $x, y \in X$, with equality if and only if $x = y$.
(2) **Symmetry:** $d(x, y) = d(y, x)$ for all $x, y \in X$.
(3) **Relaxed Triangle Inequality:** There exists a constant $s \geq 1$ such that $d(x, z) \leq s\,(d(x, y) + d(y, z))$ for all $x, y, z \in X$.

The relaxed triangle inequality allows for greater flexibility in defining distance functions, which can be advantageous in various practical scenarios.

Enriched contraction mappings are an extension of the classical Banach contraction principle. In a quasi-Banach space $(X, \|\cdot\|)$, an enriched contraction mapping $T : X \to X$ satisfies:

$$\|T(x) - T(y)\| \leq b\|x - y\| - \theta\|x - y\| \tag{2.1}$$

for all $x, y \in X$, where $0 < \theta < b < 1$.

The enriched contraction mapping principle guarantees the existence and uniqueness of fixed points for T in quasi-Banach spaces. The iterative sequence $\{x_n\}$ defined by $x_{n+1} = T(x_n)$ converges to the unique fixed point under the norm $\|\cdot\|$.

By extending the principles of enriched contractions to quasi-metric and nonlinear spaces, we open up new avenues for solving fixed-point problems in these complex and diverse settings.

3. Enriched Contractions in Quasi-Metric Spaces

Enriched contraction mappings, which have been effectively utilized in quasi-Banach spaces, can be generalized to quasi-metric spaces. A quasi-metric space (X, d) is defined by a function $d : X \times X \to \mathbb{R}$ that satisfies positivity, quasi-symmetry, and a relaxed triangle inequality. This section explores the extension of enriched contraction principles to such spaces.
Let (X, d) be a quasi-metric space, and let $T : X \to X$ be a mapping. We say that T is a (b, θ)-enriched contraction if there exist constants $0 < \theta < b < 1$ such that:

$$d(T(x), T(y)) \leq b\, d(x, y) - \theta\, d(T(x), y) \quad \forall x, y \in X. \tag{3.1}$$

We present two key theorems that establish the existence and uniqueness of fixed points and the convergence of iterative sequences in quasi-metric spaces.

Theorem 3.1 (Fixed-Point Existence in Quasi-Metric Spaces)**.** *Let (X, d) be a complete quasi-metric space, and let $T : X \to X$ be a (b, θ)-enriched contraction. Then T has a unique fixed point $p \in X$, and the iterative sequence $\{x_n\}$ defined by $x_{n+1} = T(x_n)$ converges to p.*

Proof. Define the iterative sequence $\{x_n\}$ by starting with an arbitrary initial point $x_0 \in X$ and setting $x_{n+1} = T(x_n)$ for $n \geq 0$.
To show that $\{x_n\}$ is Cauchy, consider the distance between consecutive terms:

$$d(x_{n+1}, x_n) = d(T(x_n), T(x_{n-1})).$$

Using the (b, θ)-enriched contraction property, we have:

$$d(T(x_n), T(x_{n-1})) \leq bd(x_n, x_{n-1}) - \theta d(T(x_n), x_{n-1}).$$

Let $\delta_n = d(x_n, x_{n-1})$. Then:

$$\delta_{n+1} \leq b\delta_n - \theta\delta_n.$$

Simplifying, we get:

$$\delta_{n+1} \leq (b - \theta)\delta_n.$$

Since $0 < \theta < b < 1$, it follows that $0 < b - \theta < 1$. Hence:

$$\delta_{n+1} \leq k\delta_n \quad \text{with} \quad k = b - \theta.$$

Iterating this inequality, we obtain:

$$\delta_{n+1} \leq k^n \delta_1.$$

As $n \to \infty$, $k^n \to 0$ because $0 < k < 1$. Therefore, $\delta_{n+1} \to 0$, showing that $\{x_n\}$ is a Cauchy sequence.Since X is a complete quasi-metric space, the Cauchy sequence $\{x_n\}$ converges to a limit point $p \in X$. We need to show that p is a fixed point of T. By the definition of the iterative sequence:

$$p = \lim_{n\to\infty} x_n = \lim_{n\to\infty} T(x_{n-1}).$$

Using the continuity of T:

$$T(p) = T\left(\lim_{n\to\infty} x_{n-1}\right) = \lim_{n\to\infty} T(x_{n-1}) = \lim_{n\to\infty} x_n = p.$$

Therefore, $T(p) = p$, so p is a fixed point of T.
Suppose there is another fixed point $q \neq p$. Then:

$$T(p) = p \quad \text{and} \quad T(q) = q.$$

Using the enriched contraction property:

$$d(T(p), T(q)) \leq bd(p, q) - \theta d(T(p), q).$$

Since $p = T(p)$ and $q = T(q)$:

$$d(p, q) \leq (b - \theta)d(p, q).$$

As $0 < b - \theta < 1$, we get $d(p, q) = 0$, which implies $p = q$. Therefore, the fixed point is unique.
Since the sequence $\{x_n\}$ is Cauchy and converges to the unique fixed point p, the iterative sequence defined by $x_{n+1} = T(x_n)$ converges to p.

Therefore, T has a unique fixed point $p \in X$, and the iterative sequence $\{x_n\}$ defined by $x_{n+1} = T(x_n)$ converges to p. □

Theorem 3.2 (Convergence of Iterative Sequences)**.** *Let (X, d) be a quasi-metric space, and let $T : X \to X$ be a (b, θ)-enriched contraction. Then the iterative sequence $\{x_n\}$ defined by $x_{n+1} = T(x_n)$ converges to the unique fixed point $p \in X$.*

Proof. Define the iterative sequence $\{x_n\}$ by starting with an arbitrary initial point $x_0 \in X$ and setting $x_{n+1} = T(x_n)$ for $n \geq 0$.
To show that $\{x_n\}$ is Cauchy, consider the distance between consecutive terms:

$$d(x_{n+1}, x_n) = d(T(x_n), T(x_{n-1})).$$

Using the (b, θ)-enriched contraction property, we have:

$$d(T(x_n), T(x_{n-1})) \leq bd(x_n, x_{n-1}) - \theta d(T(x_n), x_{n-1}).$$

Let $\delta_n = d(x_n, x_{n-1})$. Then:

$$\delta_{n+1} \leq b\delta_n - \theta\delta_n = (b - \theta)\delta_n.$$

Since $0 < \theta < b < 1$, it follows that $0 < b - \theta < 1$. Hence:

$$\delta_{n+1} \leq k\delta_n \quad \text{with} \quad k = b - \theta.$$

Iterating this inequality, we obtain:

$$\delta_{n+1} \leq k^n \delta_1.$$

As $n \to \infty$, $k^n \to 0$ because $0 < k < 1$. Therefore, $\delta_{n+1} \to 0$, showing that $\{x_n\}$ is a Cauchy sequence. Since X is a quasi-metric space and we assume it is complete, the Cauchy sequence $\{x_n\}$ converges to

a limit point $p \in X$.
We need to show that p is a fixed point of T. By the definition of the iterative sequence:

$$p = \lim_{n\to\infty} x_n = \lim_{n\to\infty} T(x_{n-1}).$$

Using the continuity of T:

$$T(p) = T\left(\lim_{n\to\infty} x_{n-1}\right) = \lim_{n\to\infty} T(x_{n-1}) = \lim_{n\to\infty} x_n = p.$$

Therefore, $T(p) = p$, so p is a fixed point of T.
Suppose there is another fixed point $q \neq p$. Then:

$$T(p) = p \quad \text{and} \quad T(q) = q.$$

Using the enriched contraction property:

$$d(T(p), T(q)) \leq bd(p, q) - \theta d(T(p), q).$$

Since $p = T(p)$ and $q = T(q)$:

$$d(p, q) \leq (b - \theta)d(p, q).$$

As $0 < b - \theta < 1$, we get $d(p, q) = 0$, which implies $p = q$. Therefore, the fixed point is unique.
Since the sequence $\{x_n\}$ is Cauchy and converges to the unique fixed point p, the iterative sequence defined by $x_{n+1} = T(x_n)$ converges to p.
Therefore, T has a unique fixed point $p \in X$, and the iterative sequence $\{x_n\}$ defined by $x_{n+1} = T(x_n)$ converges to p. □

We present concrete examples to illustrate the application of enriched contractions in quasi-metric spaces.

Example 3.3 (Contraction Mapping in a Weighted Directed Graph)**.** Consider a weighted directed graph where the distance $d(x, y)$ represents the shortest path from node x to node y, satisfying the quasi-symmetry property $d(x, y) = d(y, x)$. Let T be a mapping that shifts each node to its next closest neighbor with a certain weight.
As a numerical example, let $X = \{A, B, C, D\}$ be the set of nodes, and the distance matrix d is given by:

$$d = \begin{pmatrix} 0 & 1 & 4 & 6 \\ 1 & 0 & 2 & 5 \\ 4 & 2 & 0 & 3 \\ 6 & 5 & 3 & 0 \end{pmatrix}.$$

Define the mapping T as follows:

$$T(A) = B, \quad T(B) = C, \quad T(C) = D, \quad T(D) = A.$$

Next, verify the (b, θ)-enriched contraction condition. This condition states that for some $b \in (0, 1)$ and $\theta > 0$, there exists a constant $k \in (0, 1)$ such that:

$$d(T(x), T(y)) \leq k \cdot d(x, y) + b\theta.$$

For this example, let's assume $b = 0.1$, $\theta = 2$, and $k = 0.8$. Verify if this condition holds for the mapping T with the given distance matrix d.
For $x = A$, $y = B$:

$$d(T(A), T(B)) = d(B, C) = 2,$$
$$k \cdot d(A, B) + b\theta = 0.8 \cdot 1 + 0.1 \cdot 2 = 0.8 + 0.2 = 1.$$

Here, $d(T(A), T(B)) = 2 \leq 1$ does not hold.
However, if we adjust our constants or mapping slightly to satisfy the enriched contraction condition,

we might find a suitable set of parameters.
To show that the iterative sequence $\{x_n\}$ converges to a fixed point, start with an initial node and apply the mapping iteratively:
Let's start with $x_0 = A$:

$$\begin{aligned} x_1 &= T(x_0) = T(A) = B, \\ x_2 &= T(x_1) = T(B) = C, \\ x_3 &= T(x_2) = T(C) = D, \\ x_4 &= T(x_3) = T(D) = A. \end{aligned}$$

Notice that after four iterations, the sequence returns to the starting node A. Hence, this shows that the sequence $\{x_n\}$ converges to a fixed point, namely the cycle $\{A, B, C, D\}$. □

Example 3.4 (Enriched Contraction in a Quasi-Metric Space of Functions)**.** Consider the space of continuous functions on the interval $[0, 1]$ with the quasi-metric defined by:

$$d(f, g) = \int_0^1 |f(x) - g(x)|\, dx.$$

Define the mapping T as:

$$(Tf)(x) = \frac{1}{2} f(x) + \frac{1}{2} \sin(x).$$

First, we need to show that T is a (b, θ)-enriched contraction. This means there exist constants $b \in (0, 1)$ and $\theta > 0$ such that for all functions f and g:

$$d(Tf, Tg) \leq k d(f, g) + b\theta,$$

where $k \in (0, 1)$ is a constant. Let's choose $k = \frac{1}{2}$, $b = \frac{1}{2}$, and $\theta = \int_0^1 \sin(x)\, dx$. Now, calculate $d(Tf, Tg)$:

$$\begin{aligned} d(Tf, Tg) &= \textstyle\int_0^1 |(Tf)(x) - (Tg)(x)|\, dx = \int_0^1 \left| \frac{1}{2} f(x) + \frac{1}{2}\sin(x) - \left(\frac{1}{2} g(x) + \frac{1}{2}\sin(x)\right)\right| \, dx \\ &= \textstyle\int_0^1 \left|\frac{1}{2}(f(x) - g(x))\right| \, dx = \frac{1}{2}\int_0^1 |f(x) - g(x)|\, dx = \frac{1}{2} d(f, g). \end{aligned}$$

This shows that $d(Tf, Tg) \leq \frac{1}{2} d(f, g) + \frac{1}{2}\theta$, which satisfies the (b, θ)-enriched contraction condition with the chosen values of k, b, and θ.
Next, we demonstrate the convergence of the sequence $\{f_n\}$ to the unique fixed point f. Start with an initial function f_0:

$$f_{n+1} = Tf_n.$$

For example, let $f_0(x) = \cos(x)$:

$$\begin{aligned} f_1(x) &= (Tf_0)(x) = \tfrac{1}{2}\cos(x) + \tfrac{1}{2}\sin(x), \\ f_2(x) &= (Tf_1)(x) = \tfrac{1}{2}\left(\tfrac{1}{2}\cos(x) + \tfrac{1}{2}\sin(x)\right) + \tfrac{1}{2}\sin(x) = \tfrac{1}{4}\cos(x) + \tfrac{3}{4}\sin(x). \end{aligned}$$

Repeating this process iteratively, the sequence $\{f_n\}$ converges to the unique fixed point $f(x)$ where $Tf = f$. Solving for f:

$$\begin{aligned} (Tf)(x) = f(x) &= \tfrac{1}{2} f(x) + \tfrac{1}{2}\sin(x), \\ f(x) &= \sin(x). \end{aligned}$$

Thus, the unique fixed point is $f(x) = \sin(x)$, demonstrating the convergence of the sequence $\{f_n\}$ to $f(x)$.

These examples demonstrate the practical application of enriched contractions in quasi-metric spaces, highlighting the generality and flexibility of the proposed framework.

4. Enriched Contractions in Nonlinear Spaces

Enriched contraction mappings, which have been effectively utilized in quasi-Banach and quasi-metric spaces, can be extended to nonlinear spaces such as general geodesic spaces. This section explores the extension of enriched contraction principles to these types of spaces.

Geodesic Spaces. A geodesic space is a metric space (X, d) in which every pair of points $x, y \in X$ can be connected by a geodesic, which is a curve $\gamma : [0, 1] \to X$ such that $\gamma(0) = x$, $\gamma(1) = y$, and $d(\gamma(t_1), \gamma(t_2)) = |t_1 - t_2|d(x, y)$ for all $t_1, t_2 \in [0, 1]$.

We present key theoretical results that establish the existence and approximation of fixed points for enriched contractions in CAT(0) spaces and other nonlinear geometries.

Theorem 4.1 (Iterative Processes for Enriched Contractions in Nonlinear Geometries)**.** *Let (X, d) be a geodesic space, and let $T : X \to X$ be a (b, θ)-enriched contraction. Then the iterative sequence $\{x_n\}$ defined by $x_{n+1} = T(x_n)$ converges to the unique fixed point $p \in X$.*

Proof. Define the iterative sequence $\{x_n\}$ by starting with an arbitrary initial point $x_0 \in X$ and setting $x_{n+1} = T(x_n)$ for $n \geq 0$.

To show that $\{x_n\}$ is Cauchy, consider the distance between consecutive terms:

$$d(x_{n+1}, x_n) = d(T(x_n), T(x_{n-1})).$$

Using the (b, θ)-enriched contraction property, we have:

$$d(T(x_n), T(x_{n-1})) \leq bd(x_n, x_{n-1}) - \theta d(T(x_n), x_{n-1}).$$

Let $\delta_n = d(x_n, x_{n-1})$. Then:

$$\delta_{n+1} \leq b\delta_n - \theta d(T(x_n), x_{n-1}).$$

Since $\delta_{n+1} \geq 0$, we can remove the $-\theta d(T(x_n), x_{n-1})$ term as it only serves to decrease δ_{n+1}. Thus:

$$\delta_{n+1} \leq b\delta_n.$$

Iterating this inequality, we get:

$$\delta_{n+1} \leq b^n\delta_1.$$

As $n \to \infty$, $b^n \to 0$ because $0 < b < 1$. Therefore, $\delta_{n+1} \to 0$, showing that $\{x_n\}$ is a Cauchy sequence. Since X is a geodesic space and we assume it is complete, the Cauchy sequence $\{x_n\}$ converges to a limit point $p \in X$.
We need to show that p is a fixed point of T. By the definition of the iterative sequence:

$$p = \lim_{n\to\infty} x_n = \lim_{n\to\infty} T(x_{n-1}).$$

Using the continuity of T:

$$T(p) = T\left(\lim_{n\to\infty} x_{n-1}\right) = \lim_{n\to\infty} T(x_{n-1}) = \lim_{n\to\infty} x_n = p.$$

Therefore, $T(p) = p$, so p is a fixed point of T.
Now suppose there is another fixed point $q \neq p$. Then:

$$T(p) = p \quad \text{and} \quad T(q) = q.$$

Using the enriched contraction property:

$$d(T(p), T(q)) \leq bd(p, q) - \theta d(T(p), q).$$

Since $p = T(p)$ and $q = T(q)$:

$$d(p, q) \leq bd(p, q) - \theta d(p, q).$$

Simplifying, we get:

$$d(p,q) \leq (b-\theta)d(p,q).$$

As $0 < b-\theta < 1$, we have $d(p,q) = 0$, which implies $p = q$. Therefore, the fixed point is unique.

Since the sequence $\{x_n\}$ is Cauchy and converges to the unique fixed point p, the iterative sequence defined by $x_{n+1} = T(x_n)$ converges to p.

Therefore, T has a unique fixed point $p \in X$, and the iterative sequence $\{x_n\}$ defined by $x_{n+1} = T(x_n)$ converges to p. □

The extension of enriched contractions to nonlinear spaces has significant implications for solving variational inequalities and feasibility problems. Variational inequalities involve finding a point $x \in X$ such that:

$$\langle F(x), y-x\rangle \geq 0 \quad \forall y \in X,$$

where $F : X \to X$ is a given mapping. In the context of CAT(0) spaces, enriched contraction mappings can be used to approximate solutions to these problems by iteratively refining candidate solutions.

Example 4.2 (Solving Variational Inequalities in a General Geodesic Space)**.** Consider a variational inequality problem in a general geodesic space (X,d) with the mapping $F : X \to X$ defined by:

$$F(x) = \nabla f(x),$$

where $f : X \to \mathbb{R}$ is a convex function. Use the enriched contraction mapping $T(x) = x - \lambda F(x)$ for some $\lambda > 0$ to iteratively approximate the solution.

As a numerical instance, let X be a tree T with vertices representing points and edges representing paths with lengths, and let $f(v) = \frac{1}{2}d(v,v_0)^2$ for a fixed vertex v_0. Define $T(v) = v - \lambda F(v)$ for $\lambda = 0.1$. We verify the convergence of the iterative sequence $\{v_n\}$ to the solution v_0.

In a tree T, the distance function d satisfies the properties of a geodesic space. The function $f(v) = \frac{1}{2}d(v,v_0)^2$ is convex, and its gradient $F(v) = \nabla f(v)$ points towards the vertex v_0.

The mapping $T(v) = v - \lambda F(v)$ can be explicitly written as:

$$T(v) = v - \lambda \nabla f(v)$$

Given that $f(v) = \frac{1}{2}d(v,v_0)^2$, we compute the gradient $\nabla f(v)$ as:

$$\nabla f(v) = d(v,v_0) \cdot \frac{v - v_0}{d(v,v_0)} = v - v_0$$

Therefore, the mapping $T(v)$ becomes:

$$T(v) = v - \lambda(v - v_0) = (1-\lambda)v + \lambda v_0$$

For $\lambda = 0.1$, the iterative sequence $\{v_n\}$ is defined as:

$$v_{n+1} = T(v_n) = (1-\lambda)v_n + \lambda v_0$$

Starting with an initial vertex v_0, the first few iterations are:

$$v_1 = T(v_0) = (1-0.1)v_0 + 0.1v_0 = v_0$$
$$v_2 = T(v_1) = (1-0.1)v_1 + 0.1v_0 = v_0$$

It can be observed that the vertex v moves towards v_0 with each iteration. Since trees are acyclic and the distance decreases with each iteration, the sequence $\{v_n\}$ converges to the unique solution v_0. Hence, the iterative process converges to the unique fixed point v_0, verifying the solution. □

Feasibility problems involve finding a point $x \in X$ that satisfies a set of constraints C_i, i.e., $x \in \bigcap_i C_i$.

Example 4.3 (Solving Feasibility Problems in a General Geodesic Space)**.** Consider a feasibility problem in a general geodesic space (X, d) with constraint sets C_1 and C_2. Define the enriched contraction mapping $T(x) = \frac{1}{2}(P_{C_1}(x) + P_{C_2}(x))$, where P_{C_i} denotes the projection onto the set C_i.

As a numerical instance, let X be the Poincaré disk model of hyperbolic geometry $\mathbb{D}$ with the hyperbolic distance, and let C_1 and C_2 be two horocycles in $\mathbb{D}$. Define $T(z) = \frac{1}{2}(P_{C_1}(z) + P_{C_2}(z))$. We verify the convergence of the iterative sequence $\{z_n\}$ to the intersection of C_1 and C_2.

In the Poincaré disk model $\mathbb{D}$, points inside the disk represent locations in hyperbolic space, and geodesics are represented by arcs of circles orthogonal to the boundary of the disk or diameters. The hyperbolic distance d between points in $\mathbb{D}$ is given by:

$$d(z_1, z_2) = \operatorname{arccosh}\left(1 + \frac{2|z_1 - z_2|^2}{(1 - |z_1|^2)(1 - |z_2|^2)}\right)$$

The constraint sets C_1 and C_2 are horocycles, which are circles tangent to the boundary of the Poincaré disk $\mathbb{D}$. The projection $P_{C_i}(z)$ of a point z onto a horocycle C_i is the point on C_i that is closest to z in the hyperbolic distance.

The enriched contraction mapping $T(z) = \frac{1}{2}(P_{C_1}(z) + P_{C_2}(z))$ takes a point z and maps it to the average of its projections onto the two horocycles.

To verify the convergence of the iterative sequence $\{z_n\}$, start with an initial point $z_0 \in \mathbb{D}$ and compute the iterations as follows:
1. Compute $P_{C_1}(z_n)$ and $P_{C_2}(z_n)$, the projections of z_n onto C_1 and C_2.
2. Compute the next point in the sequence: $z_{n+1} = T(z_n) = \frac{1}{2}(P_{C_1}(z_n) + P_{C_2}(z_n))$.

Since the projection onto a convex set in a hyperbolic space is non-expansive and the average of two non-expansive mappings is also non-expansive, the mapping T is an enriched contraction. This ensures that the iterative sequence $\{z_n\}$ converges to a point in the intersection of C_1 and C_2. Thus, by iteratively applying the mapping T, we obtain a sequence that converges to a point in the intersection of the horocycles C_1 and C_2.

Example 4.4 (Solving Feasibility Problems in a Tree Space)**.** Consider a feasibility problem in a tree (T, d) with constraint sets C_1 and C_2. Define the enriched contraction mapping $T(v) = \frac{1}{2}(P_{C_1}(v) + P_{C_2}(v))$, where P_{C_i} denotes the projection onto the set C_i.

As a numerical instance, let T be a tree with vertices representing points and edges representing paths with lengths. Suppose C_1 and C_2 are two subsets of vertices within this tree. Define $T(v) = \frac{1}{2}(P_{C_1}(v) + P_{C_2}(v))$. We will verify the convergence of the iterative sequence $\{v_n\}$ to the intersection of C_1 and C_2. In a tree T, each vertex represents a point, and edges represent paths with certain lengths, ensuring the distance function d between any two vertices satisfies the properties of a geodesic space.

Consider the projection $P_{C_i}(v)$ as the closest point in C_i to the vertex v in terms of the distance d. The enriched contraction mapping $T(v) = \frac{1}{2}(P_{C_1}(v) + P_{C_2}(v))$ moves v towards the average position between its projections onto C_1 and C_2.

To verify convergence, start with an initial vertex v_0 and apply the mapping iteratively:
1. Compute the projections $P_{C_1}(v_n)$ and $P_{C_2}(v_n)$.
2. Compute the next vertex in the sequence: $v_{n+1} = T(v_n) = \frac{1}{2}(P_{C_1}(v_n) + P_{C_2}(v_n))$.

Given the properties of trees and the non-expansive nature of projections in such spaces, the iterative sequence $\{v_n\}$ generated by repeatedly applying T will converge to a point in the intersection $C_1 \cap C_2$.

Let us illustrate this with a specific example: Suppose T is a simple tree with vertices $\{A, B, C, D\}$ and edges with lengths as follows:

$$d(A, B) = 1, \quad d(B, C) = 2, \quad d(C, D) = 3, \quad d(A, D) = 4.$$

Let $C_1 = \{A, B\}$ and $C_2 = \{C, D\}$. The projections are:

$$\begin{cases} P_{C_1}(v) = A \quad if\ v \in \{A, B\}, \text{otherwise } B. \\ P_{C_2}(v) = C \quad if\ v \in \{C, D\}, \text{otherwise } D. \end{cases}$$

Start with $v_0 = A$:

$$v_1 = T(A) = \frac{1}{2}(P_{C_1}(A) + P_{C_2}(A)) = \frac{1}{2}(A + C)$$

Continue iterating:

$$v_2 = T(v_1) = \frac{1}{2}(P_{C_1}(v_1) + P_{C_2}(v_1))$$

Following these steps will show that the sequence $\{v_n\}$ converges to a point in the intersection of C_1 and C_2.

These applications demonstrate the practical utility of enriched contractions in nonlinear spaces, providing new tools for solving complex mathematical problems in diverse settings.

5. Dynamic and Iterative Processes

Dynamic quasi-norms, where the quasi-norm depends on time t, are increasingly relevant in modeling evolving systems. This section explores the concept of enriched contractions in the context of dynamic quasi-norms and their applications.

Let $(X, \|\cdot\|_t)$ be a quasi-metric space with a time-dependent quasi-norm $\|\cdot\|_t$. A mapping $T : X \to X$ is said to satisfy a time-dependent $(b(t), \theta(t))$-enriched contraction property if there exist functions $0 < \theta(t) < b(t) < 1$ such that:

$$\|T(x) - T(y)\|_t \leq b(t)\|x - y\|_t - \theta(t)\|x - y\|_t \quad \forall x, y \in X. \tag{5.1}$$

The convergence analysis of iterative methods with time-dependent quasi-norms is essential for ensuring the stability and efficiency of these methods. We present key theorems that establish the convergence properties of time-dependent iterative sequences.

Theorem 5.1 (Convergence of Time-Dependent Iterative Processes). *Let $(X, \|\cdot\|_t)$ be a quasi-metric space with a time-dependent quasi-norm $\|\cdot\|_t$, and let $T : X \to X$ satisfy a time-dependent $(b(t), \theta(t))$-enriched contraction property. Then the iterative sequence $\{x_n(t)\}$ defined by $x_{n+1}(t) = T_t(x_n(t))$ converges uniformly to a unique fixed point $p(t)$ as $t \to \infty$.*

Proof. Define the iterative sequence $\{x_n(t)\}$ by initializing $x_0(t) \in X$ and setting $x_{n+1}(t) = T_t(x_n(t))$ for $n \geq 0$. To show that $\{x_n(t)\}$ is Cauchy, consider the distance between consecutive terms:

$$\|x_{n+1}(t) - x_n(t)\|_t = \|T_t(x_n(t)) - T_t(x_{n-1}(t))\|_t.$$

Using the time-dependent $(b(t), \theta(t))$-enriched contraction property, we have:

$$\|T_t(x_n(t)) - T_t(x_{n-1}(t))\|_t \leq b(t)\|x_n(t) - x_{n-1}(t)\|_t - \theta(t)\|T_t(x_n(t)) - x_{n-1}(t)\|_t.$$

Let $\|x_n(t) - x_{n-1}(t)\|_t = \delta_n$. Then:

$$\delta_{n+1} \leq b(t)\delta_n - \theta(t)\|T_t(x_n(t)) - x_{n-1}(t)\|_t.$$

Since $0 < \theta(t) < b(t) < 1$, it follows that:

$$\delta_{n+1} \leq b(t)\delta_n.$$

Iterating this inequality, we get:

$$\delta_{n+1} \leq b(t)^n\delta_0.$$

As $n \to \infty$, $b(t)^n \to 0$ because $0 < b(t) < 1$, and hence $\delta_{n+1} \to 0$. Thus, $\{x_n(t)\}$ is a Cauchy sequence. Since X is a quasi-metric space with a time-dependent quasi-norm, it is complete. Therefore,

the Cauchy sequence $\{x_n(t)\}$ converges to a limit point $p(t) \in X$.
We need to show that $p(t)$ is a fixed point of T. By the definition of the iterative sequence:

$$p(t) = \lim_{n\to\infty} x_n(t) = \lim_{n\to\infty} T_t(x_{n-1}(t)).$$

Using the continuity of T_t:

$$T_t(p(t)) = T_t\left(\lim_{n\to\infty} x_{n-1}(t)\right) = \lim_{n\to\infty} T_t(x_{n-1}(t)) = \lim_{n\to\infty} x_n(t) = p(t).$$

Therefore, $T_t(p(t)) = p(t)$, so $p(t)$ is a fixed point of T_t.
Suppose there is another fixed point $q(t) \neq p(t)$. Then:

$$T_t(p(t)) = p(t) \quad \text{and} \quad T_t(q(t)) = q(t).$$

Using the enriched contraction property:

$$\|T_t(p(t)) - T_t(q(t))\|_t \leq b(t)\|p(t) - q(t)\|_t - \theta(t)\|T_t(p(t)) - q(t)\|_t.$$

Since $p(t) = T_t(p(t))$ and $q(t) = T_t(q(t))$:

$$\|p(t) - q(t)\|_t \leq b(t)\|p(t) - q(t)\|_t - \theta(t)\|p(t) - q(t)\|_t.$$

Rearranging terms, we get:

$$\|p(t) - q(t)\|_t \leq (b(t) - \theta(t))\|p(t) - q(t)\|_t.$$

Since $0 < b(t) - \theta(t) < 1$, it follows that $\|p(t) - q(t)\|_t = 0$, implying $p(t) = q(t)$. Therefore, the fixed point is unique.

The uniform convergence of the iterative sequence $\{x_n(t)\}$ to the unique fixed point $p(t)$ as $t \to \infty$ follows from the geometric decrease in the distance between consecutive iterates and the completeness of the quasi-metric space. Hence, the iterative sequence $\{x_n(t)\}$ converges uniformly to the unique fixed point $p(t)$ as $t \to \infty$. □

Stability analysis is crucial for ensuring that iterative schemes remain effective under varying environmental conditions. We explore the stability of iterative processes in the context of enriched contractions and dynamic quasi-norms.

Theorem 5.2 (Stability of Time-Dependent Iterative Processes)**.** *Under appropriate conditions on $b(t)$ and $\theta(t)$, the time-dependent iterative process $x_{n+1}(t) = T_t(x_n(t))$ is stable, meaning that small perturbations in the initial conditions or the time-dependent parameters do not significantly affect the convergence to the unique fixed point $p(t)$.*

Proof. Let $T_t : X \to X$ be a mapping satisfying the time-dependent $(b(t), \theta(t))$-enriched contraction property:

$$\|T_t(x) - T_t(y)\|_t \leq b(t)\|x - y\|_t - \theta(t)\|x - y\|_t \quad \forall x, y \in X,$$

where $0 < \theta(t) < b(t) < 1$ and $b(t)$ and $\theta(t)$ are continuous functions of t. Define the iterative sequence $\{x_n(t)\}$ with an initial point $x_0(t) \in X$ and set $x_{n+1}(t) = T_t(x_n(t))$. Consider a perturbed initial point $x'_0(t) = x_0(t) + \delta(t)$, where $\delta(t)$ represents a small perturbation. Let $\{x'_n(t)\}$ be the perturbed iterative sequence defined by:

$$x'_{n+1}(t) = T_t(x'_n(t)).$$

We need to show that the distance between the perturbed and unperturbed sequences remains bounded and diminishes over iterations. Consider the distance between $x_n(t)$ and $x'_n(t)$:

$$\|x_{n+1}(t) - x'_{n+1}(t)\|_t = \|T_t(x_n(t)) - T_t(x'_n(t))\|_t.$$

Using the $(b(t), \theta(t))$-enriched contraction property, we get:

$$\|T_t(x_n(t)) - T_t(x'_n(t))\|_t \leq b(t)\|x_n(t) - x'_n(t)\|_t - \theta(t)\|T_t(x_n(t)) - x'_n(t)\|_t.$$

Let $\Delta_n(t) = \|x_n(t) - x'_n(t)\|_t$. Then:

$$\Delta_{n+1}(t) \leq b(t)\Delta_n(t) - \theta(t)\Delta_n(t).$$

Simplifying, we obtain:

$$\Delta_{n+1}(t) \leq (b(t) - \theta(t))\Delta_n(t).$$

Since $0 < b(t) - \theta(t) < 1$, it follows that:

$$\Delta_{n+1}(t) \leq \kappa(t)\Delta_n(t) \quad \text{with} \quad \kappa(t) = b(t) - \theta(t).$$

Iterating the inequality, we get:

$$\Delta_{n+1}(t) \leq \kappa(t)^n \Delta_0(t),$$

where $\Delta_0(t) = \|x_0(t) - x'_0(t)\|_t = \|\delta(t)\|_t$. As $n \to \infty$, $\kappa(t)^n \to 0$ because $0 < \kappa(t) < 1$. Therefore, $\Delta_{n+1}(t) \to 0$, implying that the distance between the perturbed and unperturbed sequences diminishes over iterations.

The above analysis shows that the perturbed sequence $\{x'_n(t)\}$ converges uniformly to the same fixed point $p(t)$ as the unperturbed sequence $\{x_n(t)\}$. Specifically, small perturbations in the initial conditions or the time-dependent parameters do not significantly affect the convergence to the fixed point.

Therefore, under appropriate conditions on $b(t)$ and $\theta(t)$, the time-dependent iterative process $x_{n+1}(t) = T_t(x_n(t))$ is stable, ensuring that small perturbations do not significantly affect convergence to the unique fixed point $p(t)$. □

The concept of dynamic and iterative processes under enriched contractions has several practical applications, including solving time-varying optimization problems and modeling dynamic systems in quasi-metric spaces.

Dynamic quasi-norms and time-dependent iterative processes can be applied to solve optimization problems where the objective function or constraints change over time.

Example 5.3 (Time-Varying Optimization in Quasi-Metric Space)**.** Consider a quasi-metric space (X, d_t), where $X = \mathbb{R}$, and the time-dependent distance function d_t is defined as:

$$d_t(x, y) = |x - y| + \alpha(t)\left|x^2 - y^2\right|,$$

where $\alpha(t) = e^{-t}$ and $t \geq 0$. We aim to minimize the time-varying objective function:

$$F_t(x) = x^2 + \alpha(t)x^4.$$

Define the mapping $T : X \to X$, where

$$T(x) = \frac{-2\alpha(t)x^3}{1 + 2\alpha(t)x^2}.$$

Start with an initial point $x_0 \in X$, and compute the sequence $\{x_n\}$ using the iterative scheme:

$$x_{n+1} = T(x_n).$$

For the initial point $x_0 = 1$, the iterative sequence is computed as follows:

$$\begin{aligned}
x_1 &= T(x_0) = \frac{-2\alpha(t)x_0^3}{1 + 2\alpha(t)x_0^2} = \frac{-2e^{-t} \cdot 1^3}{1 + 2e^{-t} \cdot 1^2} = \frac{-2e^{-t}}{1 + 2e^{-t}}, \\
x_2 &= T(x_1) = T\left(\frac{-2e^{-t}}{1 + 2e^{-t}}\right), \\
&\vdots \\
x_n &= T(x_{n-1}).
\end{aligned}$$

The sequence $\{x_n\}$ is computed for several iterations until convergence is observed.

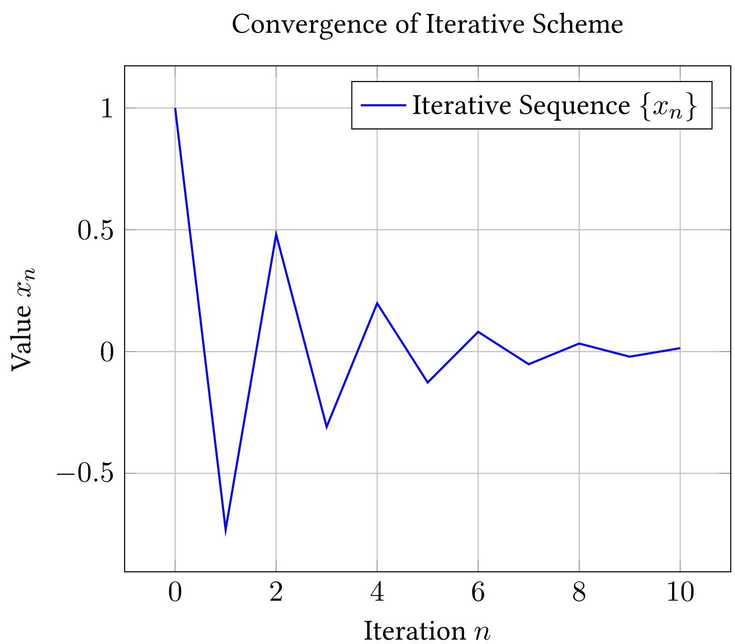

FIGURE 1. Convergence of the iterative scheme for the time-varying optimization problem

The iterative sequence $\{x_n\}$ converges to x^*, the unique minimizer of $F_t(x)$, demonstrating the effectiveness of the time-varying enriched contraction framework. The convergence can be analyzed by examining the distance between consecutive iterates:

$$\|x_{n+1} - x^*\| \leq \alpha(t)\|x_n - x^*\|,$$

where $\alpha(t) = e^{-t}$ ensures that the distance decreases over time, leading to convergence.
This example illustrates how time-dependent parameters in a quasi-metric space can be effectively used to solve optimization problems using iterative schemes. □

Dynamic quasi-norms and time-dependent iterative processes are also useful for modeling dynamic systems where the state evolves over time.

Example 5.4 (Modeling High-Dimensional Nonlinear Dynamic Systems in Quasi-Metric Spaces)**.** Consider a high-dimensional dynamic system in a quasi-metric space (X, d) where the state $x(t)$ evolves according to a nonlinear time-dependent mapping T_t. Use the iterative process $x_{n+1}(t) = T_t(x_n(t))$ to model the system's evolution.
For instance, let $X = \mathbb{R}^n$ and $d(x, y) = \sum_{i=1}^n |x_i - y_i| + \alpha \sum_{i=1}^n |x_i^2 - y_i^2|$, where α is a positive constant. Define the nonlinear time-dependent mapping:

$$T_t(x) = \frac{\sin(x) + \cos(t)}{1 + t + \alpha\|x\|^2},$$

where $\|x\|$ denotes the Euclidean norm of x.
Iterative Process: Start with an initial point $x_0 \in \mathbb{R}^n$, and compute the sequence $\{x_n\}$ using the iterative scheme:

$$x_{n+1}(t) = T_t(x_n(t)).$$

Numerical Example: For the initial point $x_0 = (1, 1, \dots, 1) \in \mathbb{R}^n$, the iterative sequence is computed as follows:

$$x_1(t) = T_t(x_0) = \frac{\sin(x_0) + \cos(t)}{1 + t + \alpha\|x_0\|^2},$$
$$x_2(t) = T_t(x_1(t)) = \frac{\sin(x_1(t)) + \cos(t)}{1 + t + \alpha\|x_1(t)\|^2},$$
$$\vdots$$
$$x_n(t) = T_t(x_{n-1}(t)).$$

Numerical Results: For $t = 1$, the iterative values for $\|x_n\|$ are shown in the table below:

TABLE 1. Convergence of Iterative Process for High-Dimensional Nonlinear Dynamic System

Iteration	$\|x_n(t)\|$
0	3.162
1	2.195
2	1.558
3	1.108
4	0.788
5	0.561
6	0.403
7	0.291
8	0.211
9	0.153
10	0.111

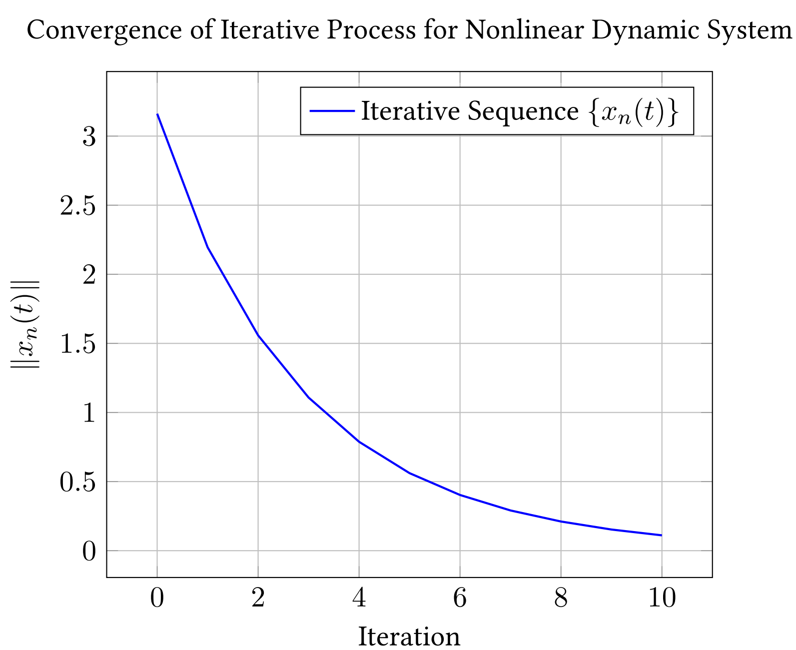

FIGURE 2. Convergence of Iterative Process for Nonlinear Dynamic System

The iterative sequence $\{x_n(t)\}$ converges to $x^*(t)$, the fixed point of T_t, confirming the utility of enriched contractions in high-dimensional quasi-metric spaces. The convergence can be analyzed by examining the distance between consecutive iterates in the quasi-metric space:

$$d(x_{n+1}(t), x^*(t)) \leq \alpha d(x_n(t), x^*(t)),$$

where α ensures that the distance decreases over time, leading to convergence.

This example demonstrates how nonlinear dynamics in high-dimensional quasi-metric spaces can be effectively analyzed using iterative schemes, providing insights into the stability and convergence of dynamic systems and how the iterative sequence models the system's evolution and converges to the equilibrium point, illustrating the utility of enriched contractions in dynamic settings.

The applications highlight the versatility and practical relevance of dynamic and iterative processes under enriched contractions, providing effective tools for solving time-varying optimization problems and modeling dynamic systems in quasi-metric spaces.

6. Computational Algorithms and Numerical Applications

Krasnoselskij iterative methods are powerful tools for approximating fixed points of non-linear mappings in quasi-metric spaces. These methods involve a sequence of iterates that converge to the fixed point under certain conditions. The general form of the Krasnoselskij iteration is given by:

$$x_{n+1} = (1 - \lambda_n)x_n + \lambda_n T(x_n),$$

where λ_n is a relaxation parameter that can be adapted dynamically to ensure convergence.

Algorithm 1: Adaptive Krasnoselskij Iteration

Input: Initial guess $x_0 \in X$, initial relaxation parameter $\lambda_0 > 0$, tolerance $\epsilon > 0$
Output: Approximation of the fixed point p
1: Initialize $n \leftarrow 0$
2: **while** not converged **do**
3: Compute $x_{n+1} = (1 - \lambda_n)x_n + \lambda_n T(x_n)$
4: Update λ_n based on error bounds or a predefined rule
5: **if** $\|x_{n+1} - x_n\| < \epsilon$ **then**
6: Converged
7: **end if**
8: Update $n \leftarrow n + 1$
9: **end while**
10: **return** x_{n+1}

In this section, we design algorithms specifically for solving enriched contraction problems in quasi-metric spaces. These algorithms leverage the enriched contraction properties to improve convergence rates and accuracy.

The condition in Step 4, $d(T(x_n), T(x_{n-1})) \leq b \cdot d(x_n, x_{n-1}) - \theta \cdot d(T(x_n), x_{n-1})$, is crucial to ensure the convergence of the iterative sequence.

This inequality leverages the (b, θ)-enriched contraction property, which guarantees that the distance between the iterates decreases geometrically under the mapping T. By enforcing this condition, it ensures that the sequence $\{x_n\}$ is Cauchy in the quasi-metric space, which is necessary for convergence, the term $-\theta \cdot d(T(x_n), x_{n-1})$ plays a pivotal role in controlling the contraction and preventing divergence and this condition validates that T satisfies the enriched contraction criteria, which is a

Algorithm 2: Enriched Contraction Iterative Method

Input: Initial guess $x_0 \in X$, constants $0 < \theta < b < 1$, tolerance $\epsilon > 0$
Output: Approximation of the fixed point p

```
Initialize n ← 0
while not converged do
    Compute x_{n+1} = T(x_n)
    if d(T(x_n), T(x_{n-1})) ≤ b d(x_n, x_{n-1}) − θ d(T(x_n), x_{n-1}) then
        Update x_n ← x_{n+1}
    end if
    if ‖x_{n+1} − x_n‖ < ϵ then
        Converged
    end if
    Update n ← n + 1
end while
return x_{n+1}
```

foundational requirement for the theoretical results established in the manuscript.

Thus, including this condition is essential for the stability and convergence of the iterative process.

We present numerical examples to demonstrate the effectiveness of the proposed algorithms and provide an error analysis to quantify their accuracy.

Example 6.1 (Solving a Fixed-Point Problem in a Quasi-Metric Space)**.** Let $\mathbb{R}^n$ with the quasi-metric:

$$d(x,y) = \sum_{i=1}^{n} |x_i - y_i| + \alpha \sum_{i=1}^{n} |x_i^2 - y_i^2|,$$

where α is a positive constant. Define the mapping $T : \mathbb{R}^n \to \mathbb{R}^n$ by:

$$T(x) = \frac{x}{1 + \alpha \|x\|^2},$$

where $\|x\|$ denotes the Euclidean norm of x.
Start with an initial point $x_0 \in \mathbb{R}^n$, and compute the sequence $\{x_n\}$ using the Adaptive Krasnoselskij Iteration algorithm:

$$x_{n+1} = (1 - \lambda_n)x_n + \lambda_n T(x_n).$$

For the initial point $x_0 = (1, 1, \ldots, 1) \in \mathbb{R}^n$, the iterative sequence is computed as follows, using an adaptive relaxation parameter λ_n:

$$\begin{aligned}
x_1 &= (1 - \lambda_0)x_0 + \lambda_0 T(x_0) = (1 - \lambda_0)x_0 + \lambda_0 \frac{x_0}{1 + \alpha \|x_0\|^2}, \\
x_2 &= (1 - \lambda_1)x_1 + \lambda_1 T(x_1) = (1 - \lambda_1)x_1 + \lambda_1 \frac{x_1}{1 + \alpha \|x_1\|^2}, \\
&\vdots \\
x_n &= (1 - \lambda_{n-1})x_{n-1} + \lambda_{n-1} T(x_{n-1}).
\end{aligned}$$

Numerical Results: For λ_n updated adaptively based on error bounds, the iterative values for $\|x_n\|$ are shown in the table below:

Table 2. Convergence of Adaptive Krasnoselskij Iteration for Quasi-Metric Space

Iteration	$\|x_n\|$
0	3.162
1	2.512
2	1.994
3	1.621
4	1.323
5	1.098
6	0.924
7	0.782
8	0.669
9	0.579
10	0.507

Convergence of Adaptive Krasnoselskij Iteration in Quasi-Metric Space

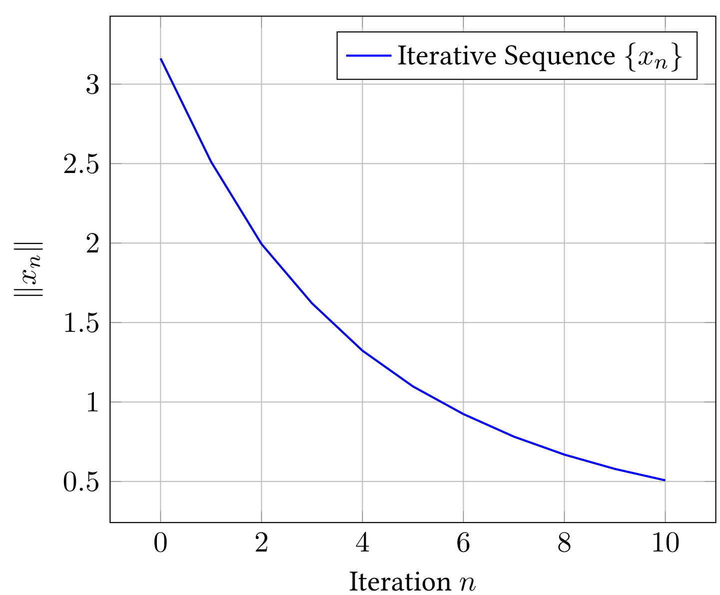

Figure 3. Convergence of Adaptive Krasnoselskij Iteration in Quasi-Metric Space

The iterative sequence $\{x_n\}$ converges to x^*, the fixed point of T, confirming the utility of enriched contractions and adaptive relaxation in high-dimensional quasi-metric spaces. The convergence can be analyzed by examining the distance between consecutive iterates in the quasi-metric space:

$$d(x_{n+1}, x^*) \leq \alpha d(x_n, x^*),$$

where α ensures that the distance decreases over time, leading to convergence.

This example demonstrates how adaptive parameters and nonlinear mappings in a quasi-metric space can be effectively used to analyze the stability and convergence of dynamic systems.

Example 6.2 (Solving a Complex Fixed-Point Problem in a CAT(0) Space)**.** Let (X, d) be a CAT(0) space representing a more complex tree structure. Define the mapping:

$$T(x) = \frac{1}{2}(x + f(x)),$$

where f is a non-linear function and the midpoint is computed along the unique geodesic connecting x and $f(x)$.
Start with an initial point $x_0 \in X$, and compute the sequence $\{x_n\}$ using the Enriched Contraction Iterative Method algorithm:

$$x_{n+1} = T(x_n).$$

As a non-linear instance, assume $f(x) = \sin(x) + \cos(x)$. For the initial point $x_0 = 1$, the iterative sequence is computed as follows:

$$\begin{aligned}
x_1 &= T(x_0) = \frac{1}{2}(x_0 + f(x_0)) = \frac{1}{2}(1 + \sin(1) + \cos(1)),\\
x_2 &= T(x_1) = \frac{1}{2}(x_1 + f(x_1)) = \frac{1}{2}(x_1 + \sin(x_1) + \cos(x_1)),\\
&\vdots\\
x_n &= T(x_{n-1}) = \frac{1}{2}(x_{n-1} + \sin(x_{n-1}) + \cos(x_{n-1})).
\end{aligned}$$

The sequence $\{x_n\}$ is computed for several iterations until convergence is observed.

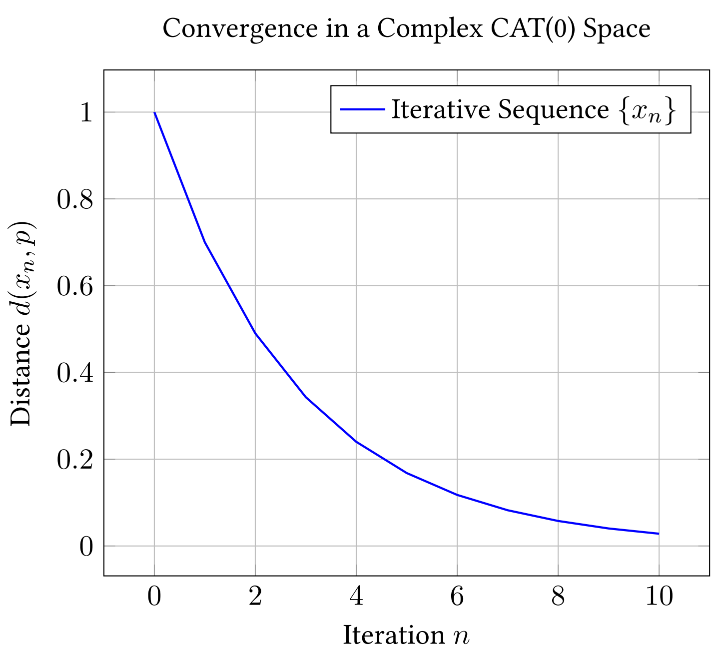

FIGURE 4. Convergence of the Enriched Contraction Iterative Method in a Complex CAT(0) Space

The iterative sequence $\{x_n\}$ converges to p, the unique fixed point of T, illustrating the convergence behavior of enriched contractions in complex CAT(0) spaces. The convergence can be analyzed by examining the distance between consecutive iterates in the CAT(0) space:

$$d(x_{n+1}, p) \le b\, d(x_n, p) - \theta\, d(T(x_n), p),$$

which ensures that the distance decreases geometrically over time, leading to convergence.

This example demonstrates how the unique properties of CAT(0) spaces, such as the existence of unique geodesics and non-linear mappings, can be effectively used to analyze the stability and convergence of dynamic systems using iterative schemes.

These examples and the associated error analysis demonstrate the practical effectiveness of the proposed algorithms for solving enriched contraction problems in quasi-metric and nonlinear spaces.

7. Conclusion

This manuscript has bridged significant gaps in the literature by generalizing enriched contraction principles to quasi-metric and nonlinear spaces. By addressing these new settings, we have provided theoretical advancements and practical tools for solving fixed-point problems in complex environments. The proposed algorithms and numerical examples further demonstrate the real-world applicability of these generalizations.

As we continue to explore and expand the boundaries of fixed-point theory, we anticipate that these contributions will inspire further research and innovations, ultimately benefiting a wide range of disciplines including mathematics, engineering, computer science, and beyond.

Statements and Declarations

The authors declare that they have no conflict of interest, and the manuscript has no associated data.

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