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FRACTIONAL SWEEPING PROCESS WITH CAPUTO TYPE VELOCITY CONSTRAINT

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Abstract. This paper deals with the existence of solutions for a novel variant of fractional sweeping processes, where the Caputo type derivative belongs to the set of constraints which is assumed to be closed convex and varies in a Hölderian way. By using a modified catching-up algorithm, we construct a family of approximate solutions that converges to a Hölderain solution of the evolution inclusion under the semicoercivity condition of the considered operator. The Cauchy criterion of the approximate solutions in an infinite dimensional Hilbert space is obtained under some additional condition.

Keywords. Normal cone, Fractional evolution inclusion, Fractional sweeping processes, Riemann-Liouville fractional order integral, Caputo fractional order derivative.

1. Introduction

The sweeping process, first introduced and extensively studied by J.J. Moreau in a series of papers [16, 17, 18, 19], was originally designed to model quasi-static evolution of elastoplastic systems in unilateral mechanics. His work not only established a precise mathematical framework but also broadly used as a foundation for numerous extensions and practical applications in the decades that followed such as: switched electrical circuits [1, 3], nonsmooth mechanics [13], crowd motion [14] among others.

Let $\mathbb{H}$ be a Hilbert space and let $T > 0$ be a nonnegative real number. Moreau's approach provides a way to describe the evolution of a point that is swept by a moving set $\Omega : [0, T] \rightrightarrows \mathbb{H}$. Formally, such model can be presented by the following generalized Cauchy problem

$$\begin{cases} -\dot{u}(t) \in \mathrm{N}_{\Omega(t)}(u(t)) & \text{a.e.} \quad t \in [0, T] \\ u(0) = u_0 \in \Omega(0), \end{cases} \tag{SP}$$

where $\dot{u}(t)$ stands for the time derivative of $u(t)$ and a.e. (almost everywhere) means that the inclusion holds on a set in $[0, T]$ of full Lebesgue measure. Whereas, $u(t)$ represents the state of the system at time t, $\Omega(t)$ is a time-dependent (moving) set, usually convex, representing the set of constraints, minus sign reflects resistance to leaving the set $\Omega(t)$ and $\mathrm{N}_{\Omega(t)}(u(t))$ denotes the *(outward) normal cone* to the set $\Omega(t)$ at the point $u(t)$ in the sense of *convex analysis*.

The Fractional calculus and its associated differential equations and inclusions has emerged as a powerful mathematical tool due to its numerous applications in applied mathematics, unilateral mechanics

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and various fields of sciences and engineering. Recently in [25, 26] Zeng et al. introduced and studied for the first time the so-called *fractional sweeping process* defined as follows

$$\begin{cases} \mathcal{A}^C\mathbb{D}_0^\alpha u(t) \in -\mathrm{N}_{\Omega(t)}(u(t)) + f(t,u(t)) \quad \text{a.e.} \quad t \in [0,T] \\ u(0) = u_0 \in \Omega(0), \end{cases}$$

where ${}^C\mathbb{D}_0^\alpha u(t)$ is *the (left-sided) Caputo fractional derivative of order* $0 < \alpha < 1$ of $u(\cdot)$ and the sets $\Omega(t)$ are assumed convex or prox-regular and varies in α-Hölder continuous way. Later in [7, 5] the authors proposed a new variant with Caputo fractional derivative in the constraint and provided an application in contact problems. This novel variant has the following form

$$\begin{cases} \mathcal{A}^C\mathbb{D}_0^\alpha u(t) \in -\mathrm{N}_{\Omega(t)}\left(\mathcal{A}^C\mathbb{D}_0^\alpha u(t) + \mathcal{B}u(t)\right) \quad \text{a.e.} \quad t \in [0,T] \\ u(0) = u_0 \in \Omega(0), \end{cases}$$

where $\mathcal{A}, \mathcal{B} : \mathbb{H} \to \mathbb{H}$ are two linear bounded operators such that $\mathcal{A}$ is coercive and B is semi-definite.

Let $\varphi : \mathbb{H} \to \mathbb{H}$ be a mapping. In this paper, we are interested in a novel variant of the fractional sweeping sweeping process, say the *implicit fractional sweeping process with Caputo velocity constraint*

$$\begin{cases} \mathcal{A}^C\mathbb{D}_0^\alpha u(t) + \mathcal{B}u(t) \in -\mathrm{N}_{\Omega(t)}\left({}^C\mathbb{D}_0^\alpha u(t)\right) + \varphi(t) \text{ a.e } t \in [0,T], \\ \\ u(0) = u_0. \end{cases} \tag{$\mathcal{IFSP}$}$$

As we see, the fractional derivative ${}^C\mathbb{D}_0^\alpha u(t)$ appears in both sides of the first relationship of ($\mathcal{IFSP}$) which translates the implicit aspect of the problem. This study represents an extension of what exists in the usual implicit sweeping processes (see [3, 2]) to the fractional setting. We use a modified catching-up algorithm to construct a sequence of approximate solutions $(u_n(\cdot))_n$ and then, prove its uniform convergence to the desired solution $u(\cdot)$. This will be done in a finite dimensional Hilbert space under the assumptions that the set of constraints $\Omega(t)$ is supposed to vary in α-Hölder continuous way with respect to time and $\mathcal{A}$ is semi-coercive operator. The finite dimensional condition limits the applications for this new variant in some Hilbert space, for this reason, we assume in the last section that $\mathcal{A}$ is coercive wich makes the differential inclusion ($\mathcal{IFSP}$) having Hölderian solution even in infinite dimensional space by showing the Cauchy criterion of the approximate solutions $(u_n(\cdot))_n$.

The paper is organized as follows. In the next section, we recall some standard tools from convex analysis and fractional calculus which are involved throughout the paper. In section 3, we gather the notations and the hypotheses used along the paper. We prove in section 4 some auxiliary results which will be needed in the rest of the paper as well as the existence result of the evolution inclusion ($\mathcal{IFSP}$). The last section is devoted to establish the existence of solution in a general Hilbert space under the coercivity condition on the operator $\mathcal{A}$.

2. Notation and Preliminaries

In all the paper, unless otherwise stated, $J := [0,T], T > 0$ is an interval of $\mathbb{R}$ and $\mathbb{H}$ is a (real) infinite dimensional *Hilbert space* whose scalar product will be denoted by $\langle \cdot, \cdot \rangle$ and the associated norm by $\|\cdot\|$. For any $x \in \mathbb{H}$ and $\eta \geq 0$, the closed (respectively open) ball centered at x with radius η will be denoted by $B[x, \eta]$ (respectively $B(x, \eta)$). For $x = 0$ and $\eta = 1$, we will put $\mathbb{B}_{\mathbb{H}}$ or $\mathbb{B}$ in place of $B[0, 1]$. Further, if Ω is a subset of $\mathbb{H}$, we denote by $\delta_\Omega(\cdot)$ or $\delta(\cdot, \Omega)$ the indicator function of Ω, that is, $\delta(x, \Omega) = 0$ if $x \in \Omega$ and $+\infty$ otherwise.

We will denote by $\mathcal{C}(J; \mathbb{H})$ the space of continuous maps from J to $\mathbb{H}$. It is well known that $\mathcal{C}(J; \mathbb{H})$ is a Banach space equipped with the norm of the uniform convergence denoted by $\| \cdot \|_{\mathcal{C}(J;\mathbb{H})}$ or $\| \cdot \|_\infty$

and defined as follow

$$\|\Phi\|_{\infty} := \max_{t\in J} \|\Phi(t)\|, \text{ for all } \Phi \in \mathcal{C}(J;\mathbb{H}).$$

For $p \in [1,+\infty]$, we denote by $\mathsf{L}^p(J;\mathbb{H})$ the quotient of all λ-Bochner measurable maps $\Phi : J \to \mathbb{H}$ such that $\|\Phi(\cdot)\|$ belongs to $\mathsf{L}^p(J;\mathbb{R})$. The space $\mathsf{L}^p(J;\mathbb{H})$ will be endowed with the norm $\|\cdot\|_p$ given by

$$\|\Phi\|_p := \left(\int_0^T \|\Phi(t)\|^p dt\right)^{\frac{1}{p}},$$

whereas, the one on the space $\mathsf{L}^{\infty}(J;\mathbb{H})$ of essentially bounded measurable maps will be denoted by $\|\cdot\|_{\mathsf{L}^\infty}$ and given by

$$\|\Phi\|_{\mathsf{L}^\infty} := \inf\{c > 0 : \|\Phi(t)\| \le c \text{ for a.e } t \in J\}.$$

Whenever there is no ambiguity concerning either the norm $\|\cdot\|_{\infty}$ or $\|\cdot\|_{\mathsf{L}^\infty}$, we will merely denote $\|\cdot\|_{\infty}$ in place of $\|\cdot\|_{\mathsf{L}^\infty}$.

Given an extended real-valued function $\Phi : H \to \mathbb{R} \cup \{+\infty\}$, *the subdifferential of* Φ *at a point* $x \in \operatorname{dom}\Phi$ (in the sense of convex analysis) is the set (may be empty) defined by

$$\partial\Phi(x) := \{v \in H : \langle v, y - x\rangle \le \Phi(y) - \Phi(x), \text{ for all } y \in \mathbb{H}\}, \tag{2.1}$$

where $\operatorname{dom}\Phi := \{y \in \mathbb{H} : \Phi(y) < +\infty\}$ is the effective domain of Φ. When $\Phi(x) = +\infty$, by convention $\partial\varphi(x) = \emptyset$, that is $x \notin \operatorname{Dom}\partial\Phi$, where $\operatorname{Dom} F := \{x \in \mathbb{H} : F(x) \neq \emptyset\}$ is the domain of a set-valued map $F : \mathbb{H} \rightrightarrows \mathbb{H}$ and

$$\operatorname{gph} F := \{(x, y) \in \mathbb{H} \times \mathbb{H} : y \in F(x)\},$$

is the graph of F.

Let Ω be a nonempty closed convex subset of $\mathbb{H}$. We define three functions that are of great interest in modern convex analysis (see [22, 23]). Those particular functions correspond to the support functions $\sigma(\Omega,\cdot)$ of Ω and to the distance function $d_\Omega(\cdot)$ from the set Ω respectively, given by

$$\sigma(\Omega,\cdot) : \mathbb{H} \to \mathbb{R} \cup \{+\infty\} \text{ with } \sigma(\Omega,\zeta) := \sup_{x\in\Omega} \langle x, \zeta\rangle,$$

$$d_\Omega(\cdot) : \mathbb{H} \to \mathbb{R}_+ \text{ with } d_\Omega(x) := \inf_{y\in\Omega} \|x - y\|.$$

The notion of support function $\sigma(\Omega,\cdot)$ can be used to characterizes the closed convex set Ω through the following equivalence property

$$x \in \Omega \text{ if and only if } \langle \zeta, x\rangle \le \sigma(\Omega,\zeta) \text{ for all } \zeta \in \mathbb{H}. \tag{2.2}$$

According to (2.1) and for $x \in \Omega$, it is straightforward to see that an element $\xi \in \partial\psi_\Omega(x)$ if and only if $\langle \xi, v - x\rangle \le 0$ for all $v \in \Omega$, so $\partial\psi_\Omega(x)$ is the set of outward normals of Ω at the point $x \in \Omega$ denoted by $N_\Omega(x)$ and defined by

$$\mathrm{N}_\Omega(x) = \{\vartheta \in \mathbb{H} : \langle \vartheta, z - x\rangle \le 0 \text{ for all } z \in \Omega\}.$$

We derive from the last relationship that

$$\vartheta \in \mathrm{N}_\Omega(x) \Leftrightarrow \sigma(\Omega,\vartheta) = \langle \vartheta, x\rangle \text{ and } x \in \Omega. \tag{2.3}$$

Moreover, for any nonempty subsets $\Omega_1, \Omega_2 \subset H$ we have the representation

$$\sigma(\Omega_1 + \Omega_2,\cdot) = \sigma(\Omega_1,\cdot) + \sigma(\Omega_2,\cdot). \tag{2.4}$$

Now, we recall some basic definitions and properties related to fractional calculus; we refer the reader to [21, 10, 8] for more details and discussions. Let $x(\cdot) \in \mathsf{L}^1([0,T];\mathbb{H})$, the (left-sided) Riemann-Liouville fractional (Bochner) integral of order $0 < \alpha < 1$ is defined by

$$(I_0^\alpha x)(t) = I_0^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s)ds,$$

where $\Gamma(\cdot)$ is the Euler's Gamma function defined by $\Gamma(\alpha) := \int_0^{+\infty} t^{\alpha-1}e^{-t}dt$. The following proposition collects some fundamental characterizations and properties of the Riemann-Liouville fractional integral.

Proposition 2.1 (See [8]). *Let $\alpha \in]0,1]$ and $p \in]\frac{1}{\alpha}, +\infty]$. Then*

(a) *For any $\vartheta(\cdot) \in \mathsf{L}^p([0,T];\mathbb{H})$, the value $(I_0^\alpha \vartheta)(t)$ is well defined for any $t \in [0,T]$, and $(I_0^\alpha \vartheta)(0) = 0$.*

(b) *There exists $L_\alpha > 0$ such that, for any $\vartheta(\cdot) \in \mathsf{L}^p([0,T];\mathbb{H})$ and any $t, s \in [0,T]$, the inequality below is valid*

$$\|(I_0^\alpha \vartheta)(t) - (I_0^\alpha \vartheta)(s)\| \le L_\alpha \|\vartheta\|_p |t-s|^{\alpha - \frac{1}{p}},$$

where $\frac{1}{p} = 0$ if $p = \infty$. In particular, $(I^\alpha \vartheta)(\cdot) \in \mathcal{C}([0,T];\mathbb{H})$.

For a mapping $x(\cdot) : [0,T] \to \mathbb{H}$, the (left-sided) Riemann-Liouville fractional derivative of order $0 < \alpha < 1$ is defined by

$$({}^L\mathbb{D}_0^\alpha x)(t) = {}^L\mathbb{D}_0^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} x(s)ds.$$

One also defines the (left-sided) Caputo fractional derivative of order $0 < \alpha < 1$ by

$$({}^C\mathbb{D}_0^\alpha x)(t) = {}^C\mathbb{D}_0^\alpha x(t) = {}^L\mathbb{D}_0^\alpha (x(\cdot) - x(0))(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} (x(s) - x(0))ds,$$

provided that the right hand side is well-defined.

Let us describe some properties of the Riemann-Liouville fractional derivative. Prior this, let us define the set $\mathcal{I}^\alpha(\mathsf{L}^\infty([0,T];\mathbb{H}))$ by

$$\mathcal{I}^\alpha(\mathsf{L}^\infty([0,T];\mathbb{H})) := \{x(\cdot) : [0,T] \to \mathbb{H} : \exists v(\cdot) \in \mathsf{L}^\infty([0,T];\mathbb{H}) : x(t) = I_0^\alpha v(t), \forall t \in [0,T]\}.$$

Proposition 2.2 (See [8]). *Let $0 < \alpha < 1$ then*

(a) *For any mapping $x(\cdot) \in \mathcal{I}^\alpha(\mathsf{L}^\infty([0,T];\mathbb{H}))$ one has*

- (a_1) *The mapping $t \mapsto ({}^L\mathbb{D}_0^\alpha x)(t)$ is well defined for almost every $t \in [0,T]$, and $({}^L\mathbb{D}_0^\alpha x)(\cdot) \in \mathsf{L}^\infty([0,T];\mathbb{H})$.*
- (a_2) *The equality $(I_0^\alpha {}^L\mathbb{D}_0^\alpha x)(t) = x(t)$ is valid for any $t \in [0,T]$.*

(b) *For any mapping $w(\cdot) \in \mathsf{L}^\infty([0,T];\mathbb{H}) : ({}^L\mathbb{D}_0^\alpha I_0^\alpha w)(t) = w(t)$ for almost every $t \in [0,T]$.*

The next lemma provides an extension of this famous property in the fractional framework.

Lemma 2.3. *Let $0 < \alpha < 1$ and let $x(\cdot) \in \mathcal{I}^\alpha(\mathsf{L}^\infty([0,T];\mathbb{H}))$ then $[0,T] \ni t \mapsto \|x(t)\|^2 \in \mathcal{I}^\alpha(\mathsf{L}^\infty([0,T];\mathbb{H}))$ and*

$$^L\mathbb{D}_0^\alpha \|x(t)\|^2 \le 2\langle {}^L\mathbb{D}_0^\alpha x(t), x(t)\rangle, \qquad a.e\ t \in [0,T]. \tag{2.5}$$

Now, let us recall a fractional version of Gronwall inequality proved for the first time in [9, Lemma 7.1.1]. We also refer the reader to [24, 12] and [4, Theorem 4.2] for other related results.

Lemma 2.4. *Given a nondecreasing function* $a(\cdot) \in \mathsf{L}^1([0,T];\mathbb{R}_+)$. *Let* $\varrho > 0, \mu \geq 0$ *and* $u(\cdot) \in \mathsf{L}^1([0,T];\mathbb{R}_+)$. *Assume that for any* $t \in [0,T]$

$$u(t) \leq \mu I_0^\varrho u(t) + a(t),$$

then

$$u(t) \leq a(t) E_\varrho(\mu t^\varrho) \quad \text{on } [0,T],$$

where $E_\varrho(\cdot)$ *is the Mittag-Leffler function defined by*

$$E_\varrho(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\varrho k + 1)}.$$

We also need the following discrete version of Gronwall's inequality.

Lemma 2.5. *Let* $\eta > 0$ *and let* $(\theta_k), (a_k)$ *be sequences of nonnegative real numbers such that*

$$\theta_k \leq \eta + \sum_{p=1}^{k-1} a_p \theta_p, \text{ for } k \in \mathbb{N},$$

then

$$\theta_k \leq \eta \exp\left(\sum_{p=1}^{k-1} a_p\right).$$

To conclude this section, we establish an auxiliary result which will be used to give some boundedness property of the constructed sequence.

Lemma 2.6. *Let* $(\Omega(t))_{t\in[0,T]}$ *be a family of nonempty sets such that* $\Omega(s) \subset \Omega(t) + \omega(|t-s|)\mathbb{B}, \forall t, s \in [0,T]$ *where* $\omega(\cdot) : [0,T] \to \mathbb{R}_+$ *is a continuous function. Then, for any* $t \in [0,T]$ *we can find* $z \in \Omega(t)$ *such that*

$$\|z\| \leq \|z_0\| + \|\omega\|_\infty,$$

for any fixed element $z_0 \in \Omega(0)$. *In particular, under the assumption* $(\mathcal{H}_\Omega)$ *stated below, we obtain*

$$\|z\| \leq \|z_0\| + \mathcal{K}T^\alpha. \tag{2.6}$$

Proof. Fix any $z_0 \in \Omega(0)$ and take arbitrary time $t \in [0,T]$ then $z_0 \in \Omega(t) + w(t)\mathbb{B}$, this means that there exists $z \in \Omega(t)$ such that $z_0 \in \{z\} + \omega(t)\mathbb{B}$, which in turn implies that $z \in \{z_0\} + \omega(t)\mathbb{B}$. Then

$$\|z\| \leq \|z_0\| + \omega(t) \leq \|z_0\| + \|\omega\|_\infty.$$

The inequality (2.6) is a direct consequence of the latter one with the choice $w(t) := \mathcal{K}t^\alpha$. □

3. Mild Assumptions

For the sake of readability, in this section we collect the hypotheses used throughout the development of the paper.

($\mathcal{H}_\Omega$) $\Omega(\cdot) : [T_0, T] \rightrightarrows H$ is a multimapping such that for each $t \in [T_0, T]$, $\Omega(t)$ is a nonempty closed convex subset of H which moves in a α-Hölderian way for some $\alpha \in]0,1[$, that is

$$\Omega(t) \subset \Omega(s) + \mathcal{K}|t-s|^\alpha \mathbb{B}, \quad \forall\, t, s \in [0,T].$$

($\mathcal{H}_\varphi$) $\varphi : [0,T] \to \mathbb{H}$ is a continuous mapping.

($\mathcal{H}_\mathcal{B}$) $\mathcal{B} : \mathbb{H} \to \mathbb{H}$ is a bounded, symmetric, linear, and semi-definite operator, that is $\langle \mathcal{B}x, x\rangle \geq 0$ for any $x \in \mathbb{H}$.

($\mathcal{H}_\mathcal{A}$) $\mathcal{A} : \mathbb{H} \to \mathbb{H}$ is a bounded, symmetric, linear, and semi-definite operator such that

$$\langle \mathcal{A}x, x\rangle \geq \rho\|x\|^2 - \beta,\ \forall x \in \Omega(0) \quad \text{for some } \rho > 0 \text{ and } \beta > 0. \tag{3.1}$$

($\mathcal{H}'_{\mathcal{A}}$) $\mathcal{A}:\mathbb{H}\to\mathbb{H}$ is a bounded, symmetric, linear operator such that

$$\langle \mathcal{A}x, x\rangle \geq \rho\|x\|^2,\ \forall x\in\Omega(0) \quad \text{for some } \rho>0. \tag{3.2}$$

This last property means that the operator $\mathcal{A}$ is coercive on $\Omega(0)$.

Lemma 3.1. *Under assumptions* $(\mathcal{H}_\Omega)$, $(\mathcal{H}_\mathcal{A})$ *and* $(\mathcal{H}_\mathcal{B})$, *the set-valued mapping* $\mathbb{H}\ni x\mapsto \left(\mathrm{N}_{\Omega(t)}+\mathcal{A}+\mu\mathcal{B}\right)(x)$ *is surjective for every* $t\in[0,T]$ *and any* $\mu>0$.

Proof. Firstly, let us prove that the operator $\mathcal{A}$ satisfies the inequality (3.1) over all the sets $\Omega(t), t\in[0,T]$. Pick any $t\in[0,T]$ then $\Omega(t)\subset\Omega(0)+t^\alpha\mathbb{B}$. Let $x\in\Omega(t)$, then $x-t^\alpha v\in\Omega(0)$ for some $v\in\mathbb{B}$, this entails trough (3.1)

$$\begin{aligned}&\langle\mathcal{A}(x-t^\alpha v), x-t^\alpha v\rangle \geq \rho\|x-t^\alpha v\|^2-\beta\\ &\Rightarrow \langle\mathcal{A}x,x\rangle\geq\rho\|x\|^2-\beta+t^{2\alpha}(\rho\|v\|^2-\|A\|)+2t^\alpha(\langle Ax,v\rangle-\rho\langle x,v\rangle),\ \text{for any } t\in[0,T],\end{aligned}$$

letting $t\downarrow 0$, we get the desired property. Now, observe that the operator $Q:=\mathcal{A}x+\mu\mathcal{B}, \mu>0$ is bounded, symmetric, linear, semi-definite and satisfies the following inequality

$$\langle Qx,x\rangle=\langle\mathcal{A}x,x\rangle+\mu\langle\mathcal{B}x,x\rangle\geq\rho\|x\|^2-\beta \ \text{ for any } x\in\Omega(t) \text{ and any } t\in[0,T].$$

Since the latter property is valid over all the sets $\Omega(t)$, we are in a position to apply [2, Lemma 1] to get the surjectivity of $\mathrm{N}_{\Omega(t)}+\mathcal{A}+\mu\mathcal{B}$. □

4. Main Results

In this section, we use an implicit scheme to approximate the problem ($\mathcal{IFSP}$). In details, let be given some positive integer n, we consider the partition of interval $[0,T]$ with the points $t_k^n=k\delta_n$ with $\delta_n=\frac{T}{n}$ and let us set

$$\varphi_k^n=\varphi(t_k^n),\quad \gamma_{n,\alpha}=\frac{\delta_n^\alpha}{\Gamma(\alpha+1)},$$

it results that $\|\varphi_k^n\|\leq\|\varphi\|_\infty$. Starting by the initial guess $u_0^n=u_0$, we construct a sequence $(u_k^n), k=1,\dots,n$ by using the following iterate procedure

$$\mathcal{A}z_k^n+\mathcal{B}u_k^n\in-\mathrm{N}_{\Omega(t_k^n)}(z_k^n)+\varphi_k^n \tag{4.1}$$

$$u_k^n=u_0+\gamma_{n,\alpha}\sum_{p=1}^{k}z_p^n\left\{(k-p+1)^\alpha-(k-p)^\alpha\right\},\quad u_0^n=u_0. \tag{4.2}$$

Let us justify the well-posedness of the suggested numerical method. Prior this, we introduce a sequence $(\omega_k)_{1\leq k\leq n}$ as follows

$$\omega_{k-1}^n=u_0+\gamma_{n,\alpha}\sum_{p=1}^{k-1}z_p^n\left((k-p+1)^\alpha-(k-p)^\alpha\right),\quad \omega_0^n=u_0, \tag{4.3}$$

then

$$u_k^n=\omega_{k-1}^n+\gamma_{n,\alpha}z_k^n. \tag{4.4}$$

The equality (4.4) allows us to rewrite the inclusion (4.1) as follows

$$(\mathcal{A}+\gamma_{n,\alpha}\mathcal{B})(z_k^n)+\mathcal{B}\omega_{k-1}^n\in-\mathrm{N}_{\Omega(t_k^n)}(z_k^n)+\varphi_k^n,$$

which is equivalent to

$$\varphi_k^n-\mathcal{B}\omega_{k-1}^n\in\left(\mathrm{N}_{\Omega(t_k^n)}+\mathcal{A}+\gamma_{n,\alpha}\mathcal{B}\right)(z_k^n).$$

The assumptions $(\mathcal{H}_{\mathcal{A}})$ and $(\mathcal{H}_{\mathcal{B}})$ with the help of the Lemma 3.1 guarantee the surjectivity of the set-valued operator $\left(\mathrm{N}_{\Omega(t_k^n)} + \mathcal{A} + \gamma_{n,\alpha}\mathcal{B}\right)$, in other words, the element z_k^n exists and

$$z_k^n \in \left(\mathcal{A} + \gamma_{n,\alpha}\mathcal{B} + \mathrm{N}_{\Omega(t_k^n)}\right)^{-1} (\varphi_k^n - \mathcal{B}\omega_{k-1}^n), \tag{4.5}$$

which entails the existence of the claimed sequence $(u_k^n), k = 1, \dots, n$.

Lemma 4.1. *Assume that hypotheses $(\mathcal{H}_\Omega), (\mathcal{H}_{\mathcal{A}}), (\mathcal{H}_{\mathcal{B}})$ and $(\mathcal{H}_\varphi)$ are made. There exists two real numbers $c_1 > 0$ and $c_2 > 0$ such that, for any integer $n \geq 1$ the following uniform estimates hold*

$$\|z_k^n\| \leq c_1 \quad \text{and} \quad \|u_k^n\| \leq c_2, \quad k = 1, \dots, n, \tag{4.6}$$

where

$$c_1 := \left\{ \left[\left(\frac{4}{\rho^2} a_{n,\alpha}^2 + \frac{2}{\rho}\|\mathcal{B}\|^2 \right) (\|y_0\| + \mathcal{K}T^\alpha)^2 + \frac{4}{\rho}\left(\frac{1}{\rho}\|\varphi\|_\infty^2 + \beta + \|\varphi_\infty\|(\|y_0\| + \mathcal{K}T^\alpha) \right) \right]^{\frac{1}{2}} \right.$$
$$\left. + \|u_0\|\sqrt{\frac{2}{\rho} + \frac{4}{\rho^2}\|\mathcal{B}\|^2} \right\} \exp\left(\frac{T^\alpha}{\Gamma(\alpha+1)}\sqrt{\frac{2}{\rho} + \frac{4}{\rho^2}\|\mathcal{B}\|^2} \right),$$

and

$$c_2 = \|u_0\| + \frac{c_1 T^\alpha}{\Gamma(\alpha+1)},$$

for any fixed point $y_0 \in \Omega(0)$.

Proof. The inclusion (4.1) implies that

$$\langle \varphi_k^n - \mathcal{A}z_k^n - \mathcal{B}u_k^n, y - z_k^n \rangle \leq 0, \quad \text{for any } y \in \Omega(t_k^n),$$

which gives by (4.4)

$$\langle \mathcal{A}z_k^n + \mathcal{B}\omega_{k-1}^n + \gamma_{n,\alpha}\mathcal{B}z_k^n - \varphi_k^n, z_k^n - y \rangle \leq 0, \quad \text{for any } y \in \Omega(t_k^n),$$

hence

$$\langle \mathcal{A}z_k^n + \gamma_{n,\alpha}\mathcal{B}z_k^n, z_k^n \rangle \leq \langle \mathcal{A}z_k^n + \gamma_{n,\alpha}\mathcal{B}z_k^n, y \rangle + \langle \varphi_k^n - \mathcal{B}\omega_{k-1}^n, z_k^n - y \rangle, \quad \text{for any } y \in \Omega(t_k^n),$$

the imposed assumptions $(\mathcal{H}_{\mathcal{A}})$ and $(\mathcal{H}_{\mathcal{B}})$ ensure that for any $y \in \Omega(t_k^n)$

$$\begin{aligned}
\rho\|z_k^n\|^2 - \beta &\leq \langle \mathcal{A}z_k^n, z_k^n \rangle + \gamma_{n,\alpha}\underbrace{\langle \mathcal{B}z_k^n, z_k^n \rangle}_{\geq 0} \leq \langle \mathcal{A}z_k^n + \gamma_{n,\alpha}\mathcal{B}z_k^n, y \rangle + \langle \varphi_k^n - \mathcal{B}\omega_{k-1}^n, z_k^n - y \rangle \\
&\leq a_{n,\alpha}\|y\|\|z_k^n\| + \left(\|\mathcal{B}\|\|\omega_{k-1}^n\| + \|\varphi_k^n\| \right)(\|y\| + \|z_k^n\|) \\
&= a_{n,\alpha}\|y\|\|z_k^n\| + \|\mathcal{B}\|\|y\|\|\omega_{k-1}^n\| + \|\mathcal{B}\|\|\omega_{k-1}^n\|\|z_k^n\| + \|\varphi_k^n\|\|z_k^n\| + \|\varphi_k^n\|\|y\|, \quad (4.7)
\end{aligned}$$

where $a_{n,\alpha} := \|\mathcal{A}\| + \gamma_{n,\alpha}\|\mathcal{B}\|$. On the other side, due to the classical inequality $pq \leq \frac{1}{\rho}p^2 + \frac{\rho}{4}q^2, p, q \in \mathbb{R}$, it follows that

$$\begin{aligned}
&a_{n,\alpha}\|y\|\|z_k^n\| \leq \frac{a_{n,\alpha}^2}{\rho}\|y\|^2 + \frac{\rho}{4}\|z_k^n\|^2 \\
&\|\mathcal{B}\|\|y\|\|\omega_{k-1}^n\| \leq \frac{1}{2}\|\mathcal{B}\|^2\|y\|^2 + \frac{1}{2}\|\omega_{k-1}^n\|^2 \\
&\|\mathcal{B}\|\|\omega_{k-1}^n\|\|z_k^n\| \leq \frac{1}{\rho}\|\mathcal{B}\|^2\|\omega_{k-1}^n\|^2 + \frac{\rho}{4}\|z_k^n\|^2, \quad \|\varphi_k^n\|\|z_k^n\| \leq \frac{1}{\rho}\|\varphi_k^n\|^2 + \frac{\rho}{4}\|z_k^n\|^2.
\end{aligned}$$

Accordingly, the inequality (4.7) brings us to

$$\begin{aligned}\|z_k^n\|^2 &\leq \left(\frac{2}{\rho}+\frac{4}{\rho^2}\|\mathcal{B}\|^2\right)\|\omega_{k-1}^n\|^2+\left(\frac{4}{\rho^2}a_{n,\alpha}^2+\frac{2}{\rho}\|\mathcal{B}\|^2\right)\|y\|^2+\frac{4}{\rho}\left(\frac{1}{\rho}\|\varphi_k^n\|^2+\beta+\|\varphi_k^n\|\|y\|\right)\\ &\leq \left(\sqrt{\frac{2}{\rho}+\frac{4}{\rho^2}\|\mathcal{B}\|^2}\|\omega_{k-1}^n\|+\left\{\left(\frac{4}{\rho^2}a_{n,\alpha}^2+\frac{2}{\rho}\|\mathcal{B}\|^2\right)\|y\|^2+\frac{4}{\rho}\left(\frac{1}{\rho}\|\varphi_k^n\|^2+\beta+\|\varphi_k^n\|\|y\|\right)\right\}^{\frac{1}{2}}\right)^2,\end{aligned}$$

consequently, for any $y\in\Omega(t_k^n)$,

$$\|z_k^n\|\leq \sqrt{\frac{2}{\rho}+\frac{4}{\rho^2}\|\mathcal{B}\|^2}\|\omega_{k-1}^n\|+\left\{\left(\frac{4}{\rho^2}a_{n,\alpha}^2+\frac{2}{\rho}\|\mathcal{B}\|^2\right)\|y\|^2+\frac{4}{\rho}\left(\frac{1}{\rho}\|\varphi_k^n\|^2+\beta+\|\varphi_k^n\|\|y\|\right)\right\}^{\frac{1}{2}}.$$

Since the latter inequality holds true for any $y\in\Omega(t_k^n)$ then, the assumption $(\mathcal{H}_C)$ along with the Lemma 2.6 guarantees the existence of some $y\in\Omega(t_k^n)$ such that

$$\|y\|\leq\|y_0\|+\mathcal{K}T^\alpha,\qquad \text{for any fixed point } y_0\in\Omega(0),$$

this leads us to the following estimate

$$\|z_k^n\|\leq\sqrt{\frac{2}{\rho}+\frac{4}{\rho^2}\|\mathcal{B}\|^2}\|\omega_{k-1}^n\|+r_{n,\alpha,y_0},$$

where

$$r_{n,\alpha,y_0}:=\left\{\left(\frac{4}{\rho^2}a_{n,\alpha}^2+\frac{2}{\rho}\|\mathcal{B}\|^2\right)(\|y_0\|+\mathcal{K}T^\alpha)^2+\frac{4}{\rho}\left(\frac{1}{\rho}\|\varphi\|_\infty^2+\beta+\|\varphi\|_\infty(\|y_0\|+\mathcal{K}T^\alpha)\right)\right\}^{\frac{1}{2}},$$

using the latter and (4.3), we obtain

$$\|z_k^n\|\leq r_{n,\alpha,y_0}+\|u_0\|\sqrt{\frac{2}{\rho}+\frac{4}{\rho^2}\|\mathcal{B}\|^2}+\gamma_{n,\alpha}\sqrt{\frac{2}{\rho}+\frac{4}{\rho^2}\|\mathcal{B}\|^2}\sum_{p=1}^{k-1}\|z_p^n\|\left((k-p+1)^\alpha-(k-p)^\alpha\right),$$

applying discrete gronwall's inequality stated in Lemma 2.5 yields

$$\begin{aligned}\|z_k^n\|&\leq\left(r_{n,\alpha,y_0}+\|u_0\|\sqrt{\frac{2}{\rho}+\frac{4}{\rho^2}\|\mathcal{B}\|^2}\right)\exp\left(\gamma_{n,\alpha}\sqrt{\frac{2}{\rho}+\frac{4}{\rho^2}\|\mathcal{B}\|^2}\sum_{p=1}^{k-1}\left((k-p+1)^\alpha-(k-p)^\alpha\right)\right)\\ &\leq\left(r_{n,\alpha,y_0}+\|u_0\|\sqrt{\frac{2}{\rho}+\frac{4}{\rho^2}\|\mathcal{B}\|^2}\right)\exp\left(\frac{T^\alpha}{\Gamma(\alpha+1)}\sqrt{\frac{2}{\rho}+\frac{4}{\rho^2}\|\mathcal{B}\|^2}\right),\end{aligned}$$

finally, the sequence $(z_k^n)_{1\leq k\leq n}$ is uniformly bounded and

$$\begin{aligned}\|z_k^n\|\leq&\left(\left\{\left(\frac{4}{\rho^2}a_{n,\alpha}^2+\frac{2}{\rho}\|\mathcal{B}\|^2\right)(\|y_0\|+\mathcal{K}T^\alpha)^2+\frac{4}{\rho}\left(\frac{1}{\rho}\|\varphi\|_\infty^2+\beta+\|\varphi_\infty\|(\|y_0\|+\mathcal{K}T^\alpha)\right)\right\}^{\frac{1}{2}}\right.\\ &\left.+\|u_0\|\sqrt{\frac{2}{\rho}+\frac{4}{\rho^2}\|\mathcal{B}\|^2}\right)\exp\left(\frac{T^\alpha}{\Gamma(\alpha+1)}\sqrt{\frac{2}{\rho}+\frac{4}{\rho^2}\|\mathcal{B}\|^2}\right)=:c_1.\end{aligned}$$

On the other side, the right boundedness property in (4.6) is a direct consequence of the last estimate and (4.2), and therefore we are done with the proof of this lemma. □

After sequences (u_k^n) and (z_k^n) are introduced via (4.2) and (4.5) respectively, we are now in a position to define the sequences of functions $u_n(\cdot)$ and $z_n(\cdot)$ as follows: $u_n(0)=0$ and

$$u_n(t)=u_k^n,\quad z_n(t)=z_k^n,\qquad \text{for every } t\in]t_{k-1}^n,t_k^n],\quad 1\leq k\leq n. \tag{4.8}$$

It is obvious that $z_n(\cdot)\in\mathsf{L}^\infty([0,T];\mathbb{H})$ and then, according to Proposition 2.1, the mapping $I_0^\alpha z_n(\cdot)$ is well-defined and $I_0^\alpha z_n(0)=0$.

We consider the function $\theta_n(\cdot)$ defined by $\theta_n(t):=t_k^n$ for any $t\in]t_{k-1}^n,t_k^n]$ with $\theta_n(0)=0$. It is not difficult to show that

$$\sup_{t\in[0,T]}|\theta_n(t)-t|\leq\delta_n\to 0 \text{ as } n\to 0.$$

Finally, we introduce the function $\widehat{u}_n : [0,T] \to \mathbb{H}$ that will play the role of the approximate solutions of the problem ($\mathcal{IFSP}$):

$$\widehat{u}_n(t) = u_0 + I_0^\alpha z_n(t), \qquad \text{for all } t \in [0,T]. \tag{4.9}$$

Lemma 4.2. *For any $t \in [0,T]$(and then $t \in]t_{k-1}^n, t_k^n]$ for some $1 \le k \le n$), one has*

$$\widehat{u}_n(t) = u_0 + \frac{1}{\Gamma(\alpha+1)} \left(z_k^n (t - t_{k-1}^n)^\alpha - \sum_{p=1}^{k-1} z_p^n [(t-t_p^n)^\alpha - (t - t_{p-1}^n)^\alpha] \right), \tag{4.10}$$

in particular

$$\widehat{u}_n(t_k^n) = u_k^n. \tag{4.11}$$

Proof. Let $t \in [0,T]$ and let $k \in \{1, \ldots, n\}$ be such that $t \in]t_{k-1}^n, t_k^n]$ then

$$\begin{aligned}
\widehat{u}_n(t) &= u_0 + I_0^\alpha z_n(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} z_n(s) ds \\
&= u_0 + \frac{1}{\Gamma(\alpha)} \left(\sum_{p=1}^{k-1} \int_{t_{p-1}^n}^{t_p^n} (t-s)^{\alpha-1} z_n(s) ds + \int_{t_{k-1}^n}^{t} (t-s)^{\alpha-1} z_n(s) ds \right) \\
&= u_0 + \frac{1}{\Gamma(\alpha)} \left(\sum_{p=1}^{k-1} z_p^n \int_{t_{p-1}^n}^{t_p^n} (t-s)^{\alpha-1} ds + z_k^n \int_{t_{k-1}^n}^{t} (t-s)^{\alpha-1} ds \right),
\end{aligned}$$

thus

$$\widehat{u}_n(t) = u_0 + \frac{1}{\Gamma(\alpha+1)} \left(z_k^n (t - t_{k-1}^n)^\alpha - \sum_{p=1}^{k-1} z_p^n [(t-t_p^n)^\alpha - (t - t_{p-1}^n)^\alpha] \right),$$

which gives the desired formula of $\widehat{u}_n(t)$, as well as the relationship (4.11). ☐

Lemma 4.3. *Assume that hypotheses $(\mathcal{H}_\Omega), (\mathcal{H}_\mathcal{A}), (\mathcal{H}_\mathcal{B})$ and $(\mathcal{H}_\varphi)$ are made. The following assertions hold*

$$(z_n)_n \text{ converges weakly to some } z(\cdot) \text{ in } \mathsf{L}^\infty([0,T];\mathbb{H}) \tag{4.12}$$

$$(u_n(t)) \text{ converges weakly in } \mathbb{H} \text{ to } u_0 + I_0^\alpha z(t), \text{ for all } t \in [0,T] \tag{4.13}$$

$$\lim_{n\to\infty} (u_n(t) - \widehat{u}_n(t)) = 0. \tag{4.14}$$

Here, the convergence of $(z_n)_n$ to $z(\cdot)$ is considered in the weak star topology of $\mathsf{L}^\infty([0,T];\mathbb{H})$, that is

$$\lim_{n\to\infty} \int_0^T \langle z_n(t), \zeta(t) \rangle dt = \int_0^T \langle z(t), \zeta(t) \rangle dt, \qquad \textit{for any } \zeta \in \mathsf{L}^1([0,T];\mathbb{H}).$$

Proof. The claimed properties follow directly from the proof of Theorem 3.5 in [25]. However, the property (4.14) can be also proved as follows, let $t \in [0,T]$ and let $k \in \{1, \ldots, n\}$ such that $t \in]t_{k-1}^n, t_k^n]$ then

$$\|\widehat{u}_n(t) - u_n(t)\| = \|\widehat{u}_n(t) - \widehat{u}_n(t_k^n)\| = \|I_0^\alpha z_n(t) - I_0^\alpha z_n(t_k^n)\| = \|I_0^\alpha z_n(t) - I_0^\alpha z_n(\theta_n(t))\|,$$

by what precedes, $z_n(\cdot) \in \mathsf{L}^\infty([0,T];\mathbb{H})$ and $\|z_n\|_\infty \le c_1$, this combined with the Proposition 2.1 ensures the existence of some $L_\alpha > 0$ such that

$$\|\widehat{u}_n(t) - u_n(t)\| \le c_1 L_\alpha |\theta_n(t) - t|^\alpha \to 0 \text{ as } n \to \infty.$$

☐

After establishing all the above auxiliary properties, we come to our main result in this work which provides the well-posedness (in the sense of existence) of the inclusion ($\mathcal{IFSP}$) in a finite dimensional Hilbert space. That will be done under the assumption that the operator $\mathcal{A}$ is semicoercive.

Theorem 4.4. *Assume that hypotheses* $(\mathcal{H}_\Omega), (\mathcal{H}'_\mathcal{A}), (\mathcal{H}_\mathcal{B})$ *and* $(\mathcal{H}_\varphi)$ *are satisfied. Then, for any initial condition* $u_0 \in H$*, the implicit fractional sweeping process* ($\mathcal{IFSP}$) *has at least one* α*-Hölderian solution* $u : [0, T] \to H$ *satisfying*

$$u(\cdot) \in \{u_0\} + \mathcal{I}^\alpha(\mathsf{L}^\infty([0, T]); \mathbb{H})), \tag{4.15}$$

$$^{C}\mathbb{D}_0^\alpha u(t) \in \Omega(t) \text{ a.e } t \in [0, T], \tag{4.16}$$

and

$$\|^{C}\mathbb{D}_0^\alpha u(t)\| \le c_1 \quad \text{a.e } t \in [0, T]. \tag{4.17}$$

Proof. The proof will be divided into several steps.
Step 1: We prove that

$$(\widehat{u}_n(t)) \text{ converges in } \mathbb{H} \text{ to } u(t) := u_0 + I_0^\alpha z(t), \text{ for all } t \in [0, T]. \tag{4.18}$$

Due to the inequality (4.6), it results that for any $t \in [0, T]$, $\|\widehat{u}_n(t)\| \le \|u_0\| + \frac{Tc_1}{\Gamma(\alpha)}$. This means that (since $\dim(\mathbb{H}) < +\infty$) the set $\{u_n(t), n \in \mathbb{N}\}$ is relatively compact in $\mathbb{H}$ for every $t \in [0, T]$. On the other hand, by (a) of Proposition 2.1, one can find some real $L_\alpha > 0$ such that for all $t, s \in [0, T]$

$$\|\widehat{u}_n(t) - \widehat{u}_n(s)\| = \|I_0^\alpha z_n(t) - I_0^\alpha z_n(t)\| \le c_1 L_\alpha |t - s|^\alpha,$$

then, for any $\varepsilon > 0$ there exists some real $\delta(\varepsilon) := \big(\varepsilon/(c_1 L_\alpha)\big)^{1/\alpha} > 0$ such that all $t, s \in [0, T]$ and for any $n \in \mathbb{N}$

$$|t - s| < \delta(\varepsilon) \Rightarrow \|\widehat{u}_n(t) - \widehat{u}_n(s)\| < \varepsilon,$$

which translates the equicontinuity of the family $\{x_n(\cdot), n \in \mathbb{N}\}$. Getting all the above together and using Arzelà-Ascoli's theorem, we deduce that $(\widehat{u}_n(\cdot))_n$ has a subsequence (not relabeled) converging uniformly to a mapping $u(\cdot) \in \mathcal{C}([0, T]; \mathbb{H})$ with $u(0) := u_0$. Pick an arbitrary time $t \in [0, T]$ then

$$\|u(t) - u_0 - I_0^\alpha z(t)\| \le \|u(t) - \widehat{u}_n(t)\| + \|\widehat{u}_n(t) - u_n(t)\| + \|u_n(t) - u_0 - I_0^\alpha z(t)\|,$$

since $\dim(\mathbb{H}) < +\infty$, it results form (4.13) and (4.14), after passing to the limit as $n \to +\infty$, that $u(t) = u_0 + I_0^\alpha z(t)$, which gives the inclusion (4.15). Furthermore, the assertion (b) of Proposition 2.2 allows us to get

$$^{C}\mathbb{D}_0^\alpha u(t) = z(t), \quad \text{a.e } t \in [0, T].$$

Bearing in mind that $^{C}\mathbb{D}_0^\alpha \widehat{u}_n(t) = z_n(t)$ a.e $t \in [0, T]$ so, the weak convergence of $^{C}\mathbb{D}_0^\alpha \widehat{u}_n(\cdot)$ to $^{C}\mathbb{D}_0^\alpha u(\cdot)$ in $\mathsf{L}^\infty([0, T]; \mathbb{H})$ is a direct consequence of (4.12). On the other hand, for all $t, s \in [0, T]$

$$\|u(t) - u(s)\| = \lim_{n \to +\infty} \|\widehat{u}_n(t) - \widehat{u}_n(s)\| \le c_1 L_\alpha |t - s|^\alpha, \tag{4.19}$$

which translates the α-Hölder property of $u(\cdot)$. By what precedes, for almost every $t \in [0, T]$

$$\|^{C}\mathbb{D}_0^\alpha u(t)\| \le \|^{C}\mathbb{D}_0^\alpha u\|_{\mathsf{L}^\infty([0,T];\mathbb{H})} \le \liminf_{n \to +\infty} \|^{C}\mathbb{D}_0^\alpha \widehat{u}_n\|_{\mathsf{L}^\infty([0,T];\mathbb{H})} \le c_1,$$

this ensures the claimed inequality (4.17).

Step 2: We show that $^{C}\mathbb{D}_0^\alpha u(t) \in \Omega(t)$ **for almost every** $t \in [0, T]$**.**
According to the above analysis and by virtue of Mazur's lemma, for each $n \in \mathbb{N}$ there exists some sequence of convex combinations of the form

$$\left(\sum_{k=n}^{K(n)} \sigma_{k,n} \, ^{C}\mathbb{D}_0^\alpha \widehat{u}_k \right)_n \quad \text{with} \quad \sigma_{k,n} \ge 0 \quad \text{and} \quad \sum_{k=n}^{K(n)} \sigma_{k,n} = 1,$$

converging strongly to ${}^{C}\mathbb{D}_0^\alpha u$ in $\mathsf{L}^\infty([0,T];\mathbb{H})$. Extracting a subsequence, we may suppose that there exists some negligible set $\mathcal{N} \subset [0,T]$ such that $\left(\sum_{k=n}^{K(n)} \sigma_{k,n} {}^{C}\mathbb{D}_0^\alpha \widehat{u}_k(t)\right)_n$ converges in $\mathbb{H}$ to ${}^{C}\mathbb{D}_0^\alpha u(t)$ as $n \to +\infty$ for all $t \in [0,T]\backslash\mathcal{N}$. The inclusion ${}^{C}\mathbb{D}_0^\alpha \widehat{u}_n(t) \in \Omega(\theta_n(t))$ a.e $t \in [0,T]$ with the help of [15, Proposition 2.6] and the convexity of $\Omega(\theta_n(t))$ entails that for all $t \in [0,T]\backslash\mathcal{N}$ (without loss of generality, we keep the same negligible set $\mathcal{N}$)

$$\sum_{k=n}^{K(n)} \sigma_{k,n} {}^{C}\mathbb{D}_0^\alpha \widehat{u}_k(t) \in \Omega(\theta_n(t)),$$

this yields, for all $t \in [0,T]\backslash\mathcal{N}$

$$\begin{aligned} d_{\Omega(t)}({}^{C}\mathbb{D}_0^\alpha u(t)) &= d_{\Omega(t)}({}^{C}\mathbb{D}_0^\alpha u(t)) - d_{\Omega(t)}\left(\sum_{k=n}^{K(n)} \sigma_{k,n} {}^{C}\mathbb{D}_0^\alpha \widehat{u}_k(t)\right) + d_{\Omega(t)}\left(\sum_{k=n}^{K(n)} \sigma_{k,n} {}^{C}\mathbb{D}_0^\alpha \widehat{u}_k(t)\right) \\ &\quad - d_{\Omega(\theta_n(t))}\left(\sum_{k=n}^{K(n)} \sigma_{k,n} {}^{C}\mathbb{D}_0^\alpha \widehat{u}_k(t)\right) \\ &\le \left\| {}^{C}\mathbb{D}_0^\alpha u(t) - \sum_{k=n}^{K(n)} \sigma_{k,n} {}^{C}\mathbb{D}_0^\alpha \widehat{u}_k(t) \right\| + \mathcal{K}|t-\theta_n(t)|^\alpha, \end{aligned}$$

letting $n \to \infty$ and taking the closedness of $\Omega(t)$ into account, we obtain that ${}^{C}\mathbb{D}_0^\alpha u(t) \in \Omega(t)$ for almost every $t \in [T_0,T]$.

Step 3 We prove that

$$\mathcal{A}{}^{C}\mathbb{D}_0^\alpha u(t) + \mathcal{B}u(t) \in -N_{\Omega(t)}\left({}^{C}\mathbb{D}_0^\alpha u(t)\right) + \varphi(t) \text{ a.e } t \in [0,T].$$

Fix any integer $n \in \mathbb{N}$, the inclusion (4.1) along with the the above considerations bring us to the inclusion

$$\varphi(\theta_n(t)) - \mathcal{A}z_n(t) - \mathcal{B}\widehat{u}_n(\theta_n(t)) \in \mathrm{N}_{\Omega(\theta_n(t))}(z_n(t)),$$

the application of (2.3) gives

$$\sigma(\Omega(\theta_n(t)), \xi_n(t)) = \langle \xi_n(t), z_n(t)\rangle, \ \text{ a.e } t \in [0,T], \tag{4.20}$$

where $\xi_n(t) := \varphi(\theta_n(t)) - \mathcal{A}z_n(t) - \mathcal{B}\widehat{u}_n(\theta_n(t))$. Further, according to the Hölderian behavior of the sets $\Omega(t), t \in [0,T]$ with the help of the equality (2.4) one has, for any $a \in \mathbb{H}$

$$\begin{aligned} \sigma(\Omega(t), a) &= \sup_{y\in\Omega(t)} \langle y, a\rangle \le \sup_{y\in\Omega(\theta_n(t))+|t-\theta_n(t)|^\alpha\mathbb{B}} \langle y, a\rangle = \sigma(\Omega(\theta_n(t)) + |t-\theta_n(t)|^\alpha\mathbb{B}, a) \\ &= \sigma(\Omega(\theta_n(t)), a) + \sigma(|t-\theta_n(t)|^\alpha\mathbb{B}, a) \\ &\le \sigma(\Omega(\theta_n(t)), a) + |t-\theta_n(t)|^\alpha\|a\|, \end{aligned}$$

which yields by (4.11) and (4.20)

$$\sigma(\Omega(t), \xi_n(t)) \le \sigma(\Omega(\theta_n(t)), \xi_n(t)) + \nu_n(t) = \langle \xi_n(t), z_n(t)\rangle + \nu_n(t), \tag{4.21}$$

where

$$\nu_n(t) := (\|\varphi\|_\infty + c_1\|\mathcal{A}\| + c_2\|\mathcal{B}\|)|t-\theta_n(t)|^\alpha \to 0 \text{ as } n \to +\infty.$$

Integrating the inequality (4.21) brings us to

$$\int_0^T (\sigma(\Omega(t), \xi_n(t)) - \langle \xi_n(t), z_n(t)\rangle)\, dt \le \int_0^T \nu_n(t) dt,$$

passing to the inferior limit when $n \to +\infty$ in the previous inequality and using the Lebesgue Dominated Theorem, one gets

$$\liminf_{n\to+\infty}\left(\int_0^T \sigma(\Omega(t),\xi_n(t))dt+\int_0^T\langle -\xi_n(t),z_n(t)\rangle dt\right)\le 0. \tag{4.22}$$

Recalling that the mapping

$$\zeta\in \mathsf{L}^2([0,T];\mathbb{H})\mapsto \int_0^T \sigma(\Omega(t),\zeta(t))dt,$$

is lower semicontinuous with respect to the weak topology of $\mathsf{L}^2([0,T];\mathbb{H})$ thanks to [20, Corollary p227] (one can also consult [6] for various extensions in more general settings). Further, it is not difficult to check that $\xi_n(\cdot)$ converges to $\xi(\cdot):=\varphi(\cdot)-\mathcal{A}z(\cdot)-\mathcal{B}u(\cdot)$ in the weak star topology of $\mathsf{L}^\infty([0,T];\mathbb{H})$, which in turn implies its weak convergence in $\mathsf{L}^2([0,T];\mathbb{H})$. Consequently

$$\int_0^T \sigma(\Omega(t),\xi(t))dt\le \liminf_{n\to+\infty}\int_0^T \sigma(\Omega(t),\xi_n(t))dt. \tag{4.23}$$

On the other side, since the operator $\mathcal{A}$ is semi-definite and symmetric, one gets

$$2\liminf_{n\to+\infty}\int_0^T\langle Az(t),z_n(t)\rangle dt-\int_0^T\langle Az(t),z(t)\rangle dt\le \liminf_{n\to+\infty}\int_0^T\langle Az_n(t),z_n(t)\rangle dt,$$

this yields

$$\int_0^T\langle Az(t),z(t)\rangle dt\le \liminf_{n\to+\infty}\int_0^T\langle Az_n(t),z_n(t)\rangle dt, \tag{4.24}$$

the last inequality being due to the weak convergence of $z_n(\cdot)$ to $z(\cdot)$ in $\mathsf{L}^\infty([0,T];\mathbb{H})$. By the previous development, we have

$$\begin{aligned}
&\left|\int_0^T\big(\langle\mathcal{B}\widehat{u}_n(\theta_n(t)),z_n(t)\rangle-\langle\mathcal{B}u(t),z(t)\rangle\big)dt\right|\\
&=\left|\int_0^T\left(\langle\mathcal{B}\widehat{u}_n(\theta_n(t))-\mathcal{B}u(t),z_n(t)\rangle+\langle\mathcal{B}u(t),z_n(t)-z(t)\rangle\right)dt\right|\\
&\le c_1\|\mathcal{B}\|\int_0^T\|\widehat{u}_n(\theta_n(t))-u(t)\|dt+\left|\int_0^T\langle\mathcal{B}u(t),z_n(t)-z(t)\rangle dt\right|,
\end{aligned} \tag{4.25}$$

by passage to the limit as $n\to\infty$ with the use of the weak convergence of $z_n(\cdot)$ to $z(\cdot)$ and the uniform convergence of $\widehat{u}_n(\cdot)$ to $u(\cdot)$, we deduce that

$$\lim_{n\to+\infty}\int_0^T\langle\mathcal{B}\widehat{u}_n(\theta_n(t)),z_n(t)\rangle dt=\int_0^T\langle\mathcal{B}u(t),z(t)\rangle dt. \tag{4.26}$$

By what precedes, $z_n(\cdot)$ converges weakly to $z(\cdot)$ in $\mathsf{L}^\infty([0,T];\mathbb{H})$ and $\varphi(\theta_n(\cdot))$ converges strongly to $\varphi(\cdot)$ in $\mathsf{L}^1([0,T];\mathbb{H})$, it follows directly that

$$\lim_{n\to+\infty}\int_0^T\langle\varphi(\theta_n(t)),z_n(t)\rangle dt=\int_0^T\langle\varphi(t),z(t)\rangle.$$

Combining the latter equality with (4.22), (4.23), (4.24) and (4.26) to obtain

$$\int_0^T\big(\sigma(\Omega(t),\xi(t))-\langle\xi(t),{}^{C}\mathbb{D}_0^\alpha u(t)\rangle\big)dt\le 0, \tag{4.27}$$

bearing in mind that ${}^{C}\mathbb{D}_0^\alpha u(t) \in \Omega(t)$ for a.e $t \in [0,T]$, so $\sigma(\Omega(t),\xi(t)) - \langle \xi(t), {}^{C}\mathbb{D}_0^\alpha u(t)\rangle \geq 0$ for a.e $t \in [0,T]$ and then

$$\int_0^T \left(\sigma(\Omega(t),\xi(t)) - \langle \xi(t), {}^{C}\mathbb{D}_0^\alpha u(t)\rangle\right) dt \geq \int_0^\tau \left(\sigma(\Omega(t),\xi(t)) - \langle \xi(t), {}^{C}\mathbb{D}_0^\alpha u(t)\rangle\right) dt \geq 0, \text{ for all } \tau \in [0,T],$$

taking (4.27) into account, we get

$$\int_0^\tau \left(\sigma(\Omega(t),\xi(t)) - \langle \xi(t), {}^{C}\mathbb{D}_0^\alpha u(t)\rangle\right) dt = 0, \text{ for all } \tau \in [0,T]$$
$$\Rightarrow \sigma(\Omega(t),\xi(t)) = \langle \xi(t), {}^{C}\mathbb{D}_0^\alpha u(t)\rangle, \text{ a.e } t \in [0,T],$$

with the above notations and through the equality (2.3), we conclude that

$$\mathcal{A}{}^{C}\mathbb{D}_0^\alpha u(t) + \mathcal{B}u(t) \in -\mathrm{N}_{\Omega(t)}\left({}^{C}\mathbb{D}_0^\alpha u(t)\right) + \varphi(t) \text{ a.e } t \in [0,T].$$

This completes the proof of the theorem. □

Remark 4.5. As we see in (4.25), the strong convergence of $\widehat{u}_n(t)$ to $u(t)$ is required to prove that

$$\int_0^T \|\widehat{u}_n(\theta_n(t)) - u(t)\| dt \longrightarrow 0. \tag{4.28}$$

So, contrary to what is known in classical sweeping process with the usual derivative $\dot{u}(t)$, the weak convergence of $\widehat{u}_n(t)$ to $u(t)$, which can be proved in the setting of an infinite dimensional Hilbert space with fractional derivative, is not sufficient to get the convergence in (4.28). This comes down to the fact that the operator $\mathcal{A}$ is semi-coercive, which prevents us from using the Cauchy approach (see the next section). To get around this difficulty, we have assumed that $\dim H < \infty$ which allows us to benefit from the Arzelà-Ascoli's theorem. Aiming to highlight this situation in the usual setting, i.e., where we deal with the classical implicit sweeping process, see for example [2] or [3], the weak convergence of $u_n(t)$ to $u(t)$ is sufficient to obtain the inequality (4.26). Indeed, in such a case, one has

$$\begin{aligned}\int_0^T \langle \mathcal{B}u(t), \dot{u}(t)\rangle dt &= \frac{1}{2}\langle Bu(T), u(T)\rangle - \frac{1}{2}\langle \mathcal{B}u(0), u(0)\rangle \\ &\leq \liminf_{n\to+\infty}\left(\frac{1}{2}\langle \mathcal{B}u_n(T), u_n(T)\rangle - \frac{1}{2}\langle \mathcal{B}u_n(0), u_n(0)\rangle\right) \\ &= \liminf_{n\to+\infty}\int_0^T \langle \mathcal{B}u_n(t), \dot{u}_n(t)\rangle dt,\end{aligned}$$

where these estimates are due to the following famous rule (given by an equality, see [11])

$$\frac{d}{dt}\|v(t)\|^2 = \frac{d}{dt}\langle v(t), v(t)\rangle = 2\langle v(t), \dot{v}(t)\rangle, \tag{4.29}$$

for some absolutely continuous mapping $v(\cdot)$. Whereas, in the fractional framework, as mentioned above in (2.5), the equality (4.29) becomes an inequality and then we cannot benefit from the inferior limit.

The next proposition presents a uniqueness result related to Theorem 4.4 in the case where the linear operator $\mathcal{B}$ is coercive.

Proposition 4.6. *In addition to the assumptions imposed in Theorem 4.4, we assume that the operator $\mathcal{B}$ is coercive, that is, for all $x \in \mathbb{H} : \langle \mathcal{B}x, x\rangle \geq \mu\|x\|^2$ for some $\mu > 0$. Then, the sweeping process ($\mathcal{IFSP}$) admits one and only one α-Hölderian solution.*

Proof. Since the existence result is established in Theorem 4.4, we only prove the uniqueness of solution. To this end, let $u_i(\cdot) : [0,T] \to \mathbb{H}, i = 1,2$ be two solutions of ($\mathcal{IFSP}$) with initial conditions $u_1(0) = u_2(0) = u_0$. Then, for a.e $t \in [0,T]$

$$\left\langle \varphi(t) - \mathcal{A}{}^{C}\mathbb{D}_0^\alpha u_i(t) - \mathcal{B}u_i(t), \zeta - {}^{C}\mathbb{D}_0^\alpha u_i(t)\right\rangle \leq 0, \text{ for any } \zeta \in \Omega(t).$$

Taking into account the inclusions ${}^{C}\mathbb{D}_0^\alpha u_i(t) \in \Omega(t), i = 1, 2$ a.e $t \in [0, T]$ and applying the last inequality one time with $\zeta = {}^{C}\mathbb{D}_0^\alpha u_2(t)$ and $i = 1$ and another time with $\zeta = {}^{C}\mathbb{D}_0^\alpha u_1(t)$ and $i = 2$ and adding the resulting inequalities, we get for a.e $t \in [0, T]$

$$\left\langle {}^{C}\mathbb{D}_0^\alpha u_1(t) - {}^{C}\mathbb{D}_0^\alpha u_2(t), \mathcal{A}\left({}^{C}\mathbb{D}_0^\alpha u_1(t) - {}^{C}\mathbb{D}_0^\alpha u_2(t)\right)\right\rangle + \left\langle {}^{C}\mathbb{D}_0^\alpha u_1(t) - {}^{C}\mathbb{D}_0^\alpha u_2(t), \mathcal{B}\left(u_1(t) - u_2(t)\right)\right\rangle \leq 0,$$

using the fact $\mathcal{A}$ is semi-definite yields

$$\left\langle {}^{C}\mathbb{D}_0^\alpha u_1(t) - {}^{C}\mathbb{D}_0^\alpha u_2(t), \mathcal{B}\left(u_1(t) - u_2(t)\right)\right\rangle \leq 0. \tag{4.30}$$

Setting $[0, T] \ni t \mapsto \Psi(t) := \langle \mathcal{B}(u_1(t) - u_2(t)), u_1(t) - u_2(t)\rangle$, that is

$$\Psi(t) = V(u_1(t) - u_2(t)) \quad \text{where} \quad V(x) := \langle \mathcal{B}x, x\rangle, \ \forall x \in \mathbb{H}.$$

Having in hand these notations and the inclusion $u_1(\cdot) - u_2(\cdot) \in \mathcal{I}^\alpha(\mathsf{L}^\infty([0, T]); \mathbb{H}))$, we are in a position to apply [8, Lemma 4.1] to conclude that $\Psi(\cdot) \in \mathcal{I}^\alpha(\mathsf{L}^\infty([0, T]); \mathbb{H}))$ and

$${}^{C}\mathbb{D}_0^\alpha \Psi(t) \leq \langle \nabla V(u_1(t) - u_2(t)), {}^{C}\mathbb{D}_0^\alpha u_1(t) - {}^{C}\mathbb{D}_0^\alpha u_2(t)\rangle,$$

this combined with (4.30) entails

$${}^{C}\mathbb{D}_0^\alpha \langle \mathcal{B}(u_1(t) - u_2(t)), u_1(t) - u_2(t)\rangle \leq 2\left\langle {}^{C}\mathbb{D}_0^\alpha u_1(t) - {}^{C}\mathbb{D}_0^\alpha u_2(t), \mathcal{B}(u_1(t) - u_2(t))\right\rangle \leq 0,$$

by virtue of (a_2) of Proposition 2.2 and the coercivity of $\mathcal{B}$, it follows that

$$\mu\|u_1(t) - u_2(t)\|^2 \leq \langle \mathcal{B}(u_1(t) - u_2(t)), u_1(t) - u_2(t)\rangle \leq 0, \ \text{for all } t \in [0, T],$$

which ensures the equality $u_1(\cdot) = u_2(\cdot)$ and translates the claimed uniqueness. □

5. Coercivity Settings

In this section, we are going to strengthen the hypothesis (3.1) imposed on $\mathcal{A}$ by assuming that $\mathcal{A}$ is ρ-coercive. More precisely, in place of $(\mathcal{H}_\mathcal{A})$, we will consider the assumption $(\mathcal{H}'_\mathcal{A})$, that is

$$\langle \mathcal{A}x, x\rangle \geq \rho\|x\|^2, \ \forall x \in C(0) \quad \text{for some } \rho > 0. \tag{5.1}$$

In such a case, some nice properties can be obtained concerning the approximate solutions $(u_n(\cdot))_n$ and the approach used along the paper as well.

Proposition 5.1. *Let $\mathbb{H}$ be an* dimensional Hilbert space. *Assume that the conditions $(\mathcal{H}_\Omega)$, $(\mathcal{H}'_\mathcal{A})$, $(\mathcal{H}_\mathcal{B})$ and $(\mathcal{H}_\varphi)$ are made. Then $(\widehat{u}_n)_n$ is a Cauchy sequence in $\mathcal{C}([0, T]; \mathbb{H})$ converging uniformly in $\mathcal{C}([0, T]; \mathbb{H})$ to a solution $u : [0, T] \to \mathbb{H}$ and satisfying the properties (4.15), (4.15) and (4.17).*

Proof. Let us fix $n, m \in \mathbb{N}$, the inclusion (4.1) along with the formulas of $\widehat{u}_n$, z_n and θ_n bring us to the inclusions

$$\begin{cases} \varphi(\theta_n(t)) - \mathcal{A}z_n(t) - \mathcal{B}\widehat{u}_n(\theta_n(t)) \in \mathrm{N}_{\Omega(\theta_n(t))}(z_n(t)) \\ \\ \varphi(\theta_m(t)) - \mathcal{A}z_m(t) - \mathcal{B}\widehat{u}_m(\theta_m(t)) \in \mathrm{N}_{\Omega(\theta_m(t))}(z_m(t)) \end{cases}, \ \text{a.e } t \in [0, T], \tag{5.2}$$

making use of the definition of the normal cone, we get

$$\langle \varphi(\theta_n(t)) - \mathcal{A}z_n(t) - \mathcal{B}\widehat{u}_n(\theta_n(t)), y_1 - z_n(t)\rangle \leq 0, \forall y_1 \in \Omega(\theta_n(t)), \tag{5.3}$$

and

$$\langle \varphi(\theta_m(t)) - \mathcal{A}z_m(t) - \mathcal{B}\widehat{u}_m(\theta_m(t)), y_2 - z_m(t)\rangle \leq 0, \forall y_2 \in \Omega(\theta_m(t)). \tag{5.4}$$

The implicit constraints in (5.2) combined with the assumption $(\mathcal{H}_\Omega)$ give

$$z_n(t) \in \Omega(\theta_n(t)) \subset \Omega(\theta_m(t)) + \mathcal{K}|\theta_n(t) - \theta_m(t)|^\alpha \mathbb{B}$$
$$z_m(t) \in \Omega(\theta_m(t)) \subset \Omega(\theta_n(t)) + \mathcal{K}|\theta_n(t) - \theta_m(t)|^\alpha \mathbb{B},$$

coming back to (5.3) and (5.4) we obtain that for any $v_1, v_2 \in \mathbb{B}$

$$\begin{aligned}\langle Az_n(t), z_n(t) - z_m(t)\rangle \leq\ & \langle \mathcal{B}\widehat{u}_n(\theta_n(t)) - \varphi(\theta_n(t)), z_m(t) - z_n(t)\rangle \\ & + \mathcal{K}|\theta_n(t) - \theta_m(t)|^\alpha \langle \varphi(\theta_n(t)) - \mathcal{A}z_n(t) - \mathcal{B}\widehat{u}_n(\theta_n(t)), v_1\rangle\end{aligned}$$

and

$$\begin{aligned}\langle -Az_m(t), z_n(t) - z_m(t)\rangle \leq\ & \langle \mathcal{B}\widehat{u}_m(\theta_m(t)) - \varphi(\theta_m(t)), z_n(t) - z_m(t)\rangle \\ & + \mathcal{K}|\theta_n(t) - \theta_m(t)|^\alpha \langle \varphi(\theta_m(t)) - \mathcal{A}z_m(t) - \mathcal{B}\widehat{u}_m(\theta_m(t)), v_2\rangle,\end{aligned}$$

putting these last two inequalities together and using the coercivity of $\mathcal{A}$ and the inclusions $z_n(t) \in \Omega(\theta_n(t))$, $z_m(t) \in \Omega(\theta_m(t))$, we observe after simplifying

$$\begin{aligned}\rho\|z_n(t) - z_m(t)\|^2 \leq\ & \langle \mathcal{B}\widehat{u}_m(\theta_m(t)) - \mathcal{B}\widehat{u}_n(\theta_n(t)), z_n(t) - z_m(t)\rangle \\ & + \langle \varphi(\theta_n(t)) - \varphi(\theta_m(t)), z_n(t) - z_m(t)\rangle + Q_{n,m}(t),\end{aligned}$$

where

$$\begin{aligned}Q_{n,m}(t) :=\ & \mathcal{K}|\theta_n(t) - \theta_m(t)|^\alpha \langle \varphi(\theta_n(t)) - \mathcal{A}z_n(t) - \mathcal{B}\widehat{u}_n(\theta_n(t)), v_1\rangle \\ & + \mathcal{K}|\theta_n(t) - \theta_m(t)|^\alpha \langle \varphi(\theta_m(t)) - \mathcal{A}z_m(t) - \mathcal{B}\widehat{u}_m(\theta_m(t)), v_2\rangle.\end{aligned}$$

Let us establish an upper bound of the mapping $Q_{n,m}$. This will be done by applying Proposition 2.1 as follows

$$\|\widehat{u}_n(t)\| \leq \|u_0\| + \|I_0^\alpha z_n(t) - I_0^\alpha z_n(0)\| \leq \|u_0\| + L_\alpha c_1 T^\alpha =: \widehat{c},$$

for some real number $L_\alpha > 0$. Consequently

$$|Q_{n,m}(t)| \leq 2(\|\varphi\|_\infty + c_1\|\mathcal{A}\| + \widehat{c}\|\mathcal{B}\|)\mathcal{K}|\theta_n(t) - \theta_m(t)|^\alpha,$$

this leads to

$$\|z_n(t) - z_m(t)\|^2 - \frac{\|\mathcal{B}\|}{\rho}\|\widehat{u}_n(\theta_n(t)) - \widehat{u}_m(\theta_m(t))\|\|z_n(t) - z_m(t)\| - E_{n,m}(t) \leq 0, \tag{5.5}$$

where

$$E_{n,m}(t) := \frac{2}{\rho}(\|\varphi\|_\infty + c_1\|\mathcal{A}\| + \widehat{c}\|\mathcal{B}\|)\mathcal{K}|\theta_n(t) - \theta_m(t)|^\alpha + \frac{2c_1}{\rho}\|\varphi(\theta_n(t)) - \varphi(\theta_m(t))\|.$$

The expression (5.5) represents a quadratic inequality of the form $r^2 + a_1 r + a_2 \leq 0$, its discriminant $\Delta = a_1^2 - 4a_2$ is nonnegative where

$$r := \|z_n(t) - z_m(t)\|,\ a_1 := -\frac{\|\mathcal{B}\|}{\rho}\|\widehat{u}_n(\theta_n(t)) - \widehat{u}_m(\theta_m(t))\|,\ a_2 := -E_{n,m}(t).$$

Then, the estimate $0 \leq r \leq \frac{1}{2}(-a_1 + \sqrt{\Delta})$ is satisfied, which entails that

$$r^2 \leq \frac{1}{4}(-a_1 + \sqrt{\Delta})^2 \leq \frac{1}{2}(a_1^2 + \Delta) = a_1^2 - 2a_2,$$

thus

$$\begin{aligned}\|z_n(t) - z_m(t)\|^2 &\leq \frac{\|\mathcal{B}\|^2}{\rho^2}\|\widehat{u}_n(\theta_n(t)) - \widehat{u}_m(\theta_m(t))\|^2 + 2E_{n,m}(t) \\ &\leq \frac{\|\mathcal{B}\|^2}{\rho^2}\big(\|\widehat{u}_n(t) - \widehat{u}_m(t)\| + \|\widehat{u}_n(\theta_n(t)) - \widehat{u}_n(t)\| + \|\widehat{u}_m(t) - \widehat{u}_m(\theta_m(t))\|\big)^2 + 2E_{n,m}(t) \\ &\leq 2\frac{\|\mathcal{B}\|^2}{\rho^2}\|\widehat{u}_n(t) - \widehat{u}_m(t)\|^2 + F_{n,m}(t),\end{aligned} \tag{5.6}$$

where

$$F_{n,m}(t) := 2\frac{\|\mathcal{B}\|^2}{\rho^2}\Big(c_1 L_{1,\alpha}\left(|\theta_n(t) - t|^\alpha + |\theta_m(t) - t|^\alpha\right)\Big)^2 + 2E_{n,m}(t),$$

for some $L_{1,\alpha} > 0$. Because of the boundedness $z_n(\cdot)$, the Proposition 2.2 allows us to write ${}^C\mathbb{D}_0^\alpha \widehat{u}_n(t) = z_n(t)$ for almost every $t \in [0,T]$, which gives thanks to (5.6)

$$\begin{aligned}\langle {}^C\mathbb{D}_0^\alpha \widehat{u}_n(t) - {}^C\mathbb{D}_0^\alpha \widehat{u}_m(t), \widehat{u}_n(t) - \widehat{u}_m(t)\rangle &\leq \frac{1}{2}\|z_n(t) - z_m(t)\|^2 + \frac{1}{2}\|\widehat{u}_n(t) - \widehat{u}_m(t)\|^2 \\ &\leq \left(\frac{\|\mathcal{B}\|^2}{\rho^2} + \frac{1}{2}\right)\|\widehat{u}_n(t) - \widehat{u}_m(t)\|^2 + \frac{1}{2}F_{n,m}(t). \qquad (5.7)\end{aligned}$$

Further, putting $g_{n,m}(t) := \widehat{u}_n(t) - \widehat{u}_m(t) = I_0^\alpha(z_n - z_m)(t)$ where $(z_n - z_m)$ is a (uniformly) bounded mapping. Then $g_{n,m}(\cdot) \in \mathcal{I}^\alpha(\mathsf{L}^\infty([0,T];\mathbb{H}))$, which in turn ensures the inclusion $\|g_{n,m}(\cdot)\|^2 \in \mathcal{I}^\alpha(\mathsf{L}^\infty([0,T];\mathbb{H}))$ according to Lemma 2.3. Keeping in mind that $g_{n,m}(0) = 0$ and applying Lemma 2.3 again, one obtains

$${}^C\mathbb{D}_0^\alpha\|g_{n,m}(t)\|^2 \leq 2\langle {}^C\mathbb{D}_0^\alpha g_{n,m}(t), g_{n,m}(t)\rangle, \text{ a.e } t \in [0,T],$$

the latter together with (5.7) gives us

$${}^C\mathbb{D}_0^\alpha\|g_{n,m}(t)\|^2 \leq \left(\frac{2\|\mathcal{B}\|^2}{\rho^2} + 1\right)\|g_{n,m}(t)\|^2 + F_{n,m}(t), \qquad (5.8)$$

since $g_{n,m}(\cdot) \in \mathcal{I}^\alpha(\mathsf{L}^\infty([0,T];\mathbb{H}))$, the Proposition 2.2 guarantees that

$$\|g_{n,m}(t)\|^2 \leq \left(\frac{2\|\mathcal{B}\|^2}{\rho^2} + 1\right) I_0^\alpha\|g_{n,m}(t)\|^2 + I_0^\alpha F_{n,m}(t).$$

It is not difficult to prove that the mapping

$$[0,T] \ni t \mapsto h_{n,m}(t) := I_0^\alpha F_{n,m}(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\underbrace{F_{n,m}(s)}_{\geq 0}ds,$$

is nondecreasing and belongs to $\mathsf{L}^1([0,T];\mathbb{R})$. So, by applying Gronwall inequality stated in Lemma 2.4, we get the following estimate

$$\|g_{n,m}(t)\|^2 \leq h_{n,m}(t)E_\alpha\left(\left(\frac{2\|\mathcal{B}\|^2}{\rho^2} + 1\right)t^\alpha\right) \leq E_\alpha\left(\left(\frac{2\|\mathcal{B}\|^2}{\rho^2} + 1\right)T^\alpha\right)h_{n,m}(t).$$

On the one hand, for any $t \in [0,T]$ and any $s \in [0,t]$, elementary computations allow us to find some real number $R > 0$ such that

$$\frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}F_{n,m}(s) \leq R,$$

on the other hand, through the continuity of $\varphi(\cdot)$ and the convergence of $\theta_n(t)$ to t, we see that $\lim_{n,m\to\infty} F_{n,m}(t) = 0$. Making use of the Lebesgue dominated Theorem, we obtain

$$\lim_{n,m\to\infty} h_{n,m}(t) = 0,$$

which entails that $\lim_{n,m\to\infty}\sup_{t\in[0,T]}\|\widehat{u}_n(t) - \widehat{u}_m(t)\| = 0$ and translates the Cauchy criterion of $(\widehat{u}_n)_n$ in $\mathcal{C}([0,T];\mathbb{H})$. Consequently, $(\widehat{u}_n)_n$ converges uniformly to some mapping $u \in \mathcal{C}([0,T];\mathbb{H})$ which represents a solution of the main problem ($\mathcal{IFSP}$). $\square$

Statements and Declarations

The authors declare that they have no conflict of interest, and the manuscript has no associated data.

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