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EXISTENCE AND UNIQUENESS OF FIXED POINTS VIA SIMULATION FUNCTIONS IN METRIC SPACES

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Abstract. In this paper, we investigate the existence and uniqueness of fixed points for almost contraction mappings via simulation functions in metric spaces. Moreover, some examples and applications in order to illustrate the reality of our generalizations and usability of the results are given. The main results of the article extend the published corresponding results in this area, especially the paper published in Carpathian Mathematical Publications **11** (2) (2019), 475-492.

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1. Introduction and Preliminaries

The F-contraction is introduced by Wardowski [13] in order to generalize the Banach contraction principle.

The family of all functions $F : (0, +\infty) \to \mathbb{R}$ that satisfy the following conditions:

(F1) F is strictly increasing,
(F2) for every sequence $\{\alpha_n\}$ in $(0, +\infty)$ we have $\lim_{n\to\infty} F(\alpha_n) = -\infty$ iff $\lim_{n\to\infty} \alpha_n = 0$,
(F3) there exists a number $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = -\infty$,

is denoted by $\mathcal{F}$ (see, [13]) and the collection of all functions $F : (0, +\infty) \to \mathbb{R}$ satisfying the following conditions:

(G1) F is strictly increasing,
(G2) there exists a sequence $\{\alpha_n\}$ in $(0, +\infty)$ such that $\lim_{n\to\infty} F(\alpha_n) = -\infty$, or $\inf F = -\infty$,
(G3) F is a continuous map,

is denoted by $\mathcal{G}$ ([11]).

We need the following definitions in the sequel.

Definition 1.1. [13] Let (X, d) be a metric space. The mapping $T : X \to X$ is called an F-contraction, if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that, for all $x, y \in X$ with $d(Tx, Ty) > 0$ we have

$$\tau + F(d(Tx, Ty)) \leq F(d(x, y)).$$

Definition 1.2. [10, 4] Let $\alpha : X \times X \to (0, +\infty)$ be a given mapping. The mapping $T : X \to X$ is said to be an α-admissible, whenever $\alpha(Tx, Ty) \geq 1$ provided $\alpha(x, y) \geq 1$ and $x, y \in X$.

Definition 1.3. [1] An α-admissible map T is said to have the K-property, while for each sequence $\{x_n\} \subseteq X$ with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}_0$, the nonnegative integer numbers, there exists a positive integer number k such that $\alpha(Tx_n, Tx_m) \geq 1$, for all $m > n \geq k$.

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The following lemmas play crucial role in proving main results.

Lemma 1.4. [9] *Let $F : (0, +\infty) \to \mathbb{R}$ be an increasing function and $\{\alpha_n\}$ be a sequence of positive real numbers. Then the following holds:*

(a) *if $\lim_{n\to\infty} F(\alpha_n) = -\infty$, then $\lim_{n\to\infty} \alpha_n = 0$,*
(b) *if $\inf F = -\infty$, and $\lim_{n\to\infty} \alpha_n = 0$, then $\lim_{n\to\infty} F(\alpha_n) = -\infty$.*

Lemma 1.5. [3] *Let (X, d) be a metric space, and $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. If $\{x_n\}$ is not a Cauchy sequence then there exists $\epsilon > 0$ and two sequences of positive integers $\{n_k\}$ and $\{m_k\}$ with $n_k > m_k > k$ such that $d(x_{m_k}, x_{n_k}) > \epsilon$, $d(x_{m_k}, x_{n_k-1}) < \epsilon$ and*

(1) $\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \epsilon$.
(2) $\lim_{k\to\infty} d(x_{m_k-1}, x_{n_k}) = \epsilon$.
(3) $\lim_{k\to\infty} d(x_{m_k}, x_{n_k+1}) = \epsilon$.
(4) $\lim_{k\to\infty} d(x_{m_k-1}, x_{n_k+1}) = \epsilon$.

In [7, 6, 8] the simulation function introduced as follows

Definition 1.6. Let $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ be a mapping, then ζ is called a simulation function if satisfies the following conditions:

(ζ1) $\zeta(0, 0) = 0$;
(ζ2) $\zeta(t, s) < s - t$ for all $t, s > 0$;
(ζ3) if $\{t_n\}$, $\{s_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$ and $t_n < s_n$ for all $n \in \mathbb{N}$, then

$$\limsup_{n\to\infty} \zeta(t_n, s_n) < 0.$$

We denote the set of all simulation functions by $\mathcal{Z}$.

2. Main Results

The next result provides sufficient conditions for existing a fixed point.

Theorem 2.1. *Let (X, d) be a complete metric space, $\alpha : X \times X \to (0, +\infty)$ be a symmetric function and $T : X \to X$ be a mapping which there exist $F \in \mathcal{F}$, $\tau > 0$, $L \geq 0$ and simulation function ζ such that for all $x, y \in X$ and $d(Tx, Ty) > 0$,*

$$\zeta(\tau + \alpha(x, y)F(d(Tx, Ty)), F(m(x, y) + LN_1(x, y))) \geq 0, \tag{2.1}$$

where

$$m(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\right\}$$

and

$$N_1(x, y) = \min\{d(x, Ty), d(y, Tx)\},$$

and satisfying the following conditions:

(i): *T is α-admissible,*
(ii): *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$.*

Then, T has a fixed point if at least one of the following cases holds:
(a) T is continuous.
(b) F is continuous and

(iii): *if $\{x_n\}$ is a sequence in X such that $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}_0$.*

Proof. Let $x_0 \in X$ be such that $\alpha(x_0, Tx_0) \geq 1$. For any $n \in \mathbb{N}_0$, define:

$$x_{n+1} = T(x_n).$$

If $x_{n_0+1} = x_{n_0}$ for some $n_0 \in \mathbb{N}_0$ then x_{n_0} is a fixed point of T. So, we can assume that $x_{n+1} \neq x_n$ for each $n \in \mathbb{N}_0$. Since T is α-admissible, then

$$\alpha(x_n, x_{n+1}) \geq 1, \ \forall n \in \mathbb{N}_0. \tag{2.2}$$

Now, if $d(Tx, Ty) > 0$, from (2.1) and $(\zeta 2)$, then

$$\begin{aligned} 0 &\leq \zeta(\tau + \alpha(x,y)F(d(Tx,Ty)), F(m(x,y) + LN_1(x,y))) \\ &\leq F(m(x,y) + LN_1(x,y)) - (\tau + \alpha(x,y)F(d(Tx,Ty))). \end{aligned}$$

Hence

$$\tau + \alpha(x,y)F(d(Tx,Ty)) \leq F(m(x,y) + LN_1(x,y)). \tag{2.3}$$

Therefore, by (2.2) and (2.3)

$$\begin{aligned} \tau + F(d(Tx_n, Tx_{n+1})) &\leq \tau + \alpha(x_n, x_{n+1})F(d(Tx_n, Tx_{n+1})) \\ &\leq F(m(x_n, x_{n+1}) + LN_1(x_n, x_{n+1})) \\ &\leq F(m(x_n, x_{n+1}) + Ld(x_{n+1}, Tx_n)) \\ &= F(m(x_n, x_{n+1}) + 0) \\ &= F(m(x_n, x_{n+1})), \end{aligned}$$

Hence we have

$$\tau + F(d(x_{n+1}, x_{n+2}) \leq F(m(x_n, x_{n+1})). \tag{2.4}$$

But

$$\begin{aligned} m(x_n, x_{n+1}) &= \max\Big\{d(x_n, x_{n+1}), d(x_n, Tx_n), d(x_{n+1}, Tx_{n+1}), \\ &\qquad \tfrac{d(x_n, Tx_{n+1}) + d(x_{n+1}, Tx_n)}{2}\Big\} \\ &= \max\Big\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \tfrac{d(x_n, x_{n+2})}{2}\Big\} \\ &\leq \max\Big\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \tfrac{d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})}{2}\Big\} \\ &\leq \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}. \end{aligned}$$

If $d(x_{n_0+1}, x_{n_0+2}) \geq d(x_{n_0}, x_{n_0+1})$ for some $n_0 \in \mathbb{N}_0$, then

$$m(x_{n_0}, x_{n_0+1}) \leq d(x_{n_0+1}, x_{n_0+2}),$$

and since F is strictly increasing,

$$F(m(x_{n_0}, x_{n_0+1})) \leq F(d(x_{n_0+1}, x_{n_0+2})).$$

So, it follow from (2.4) that

$$\tau + F(d(x_{n_0+1}, x_{n_0+2}) \leq F(d(x_{n_0+1}, x_{n_0+2})).$$

So, $\tau \leq 0$, which is a contradiction. Consequently

$$d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1}), \ \forall n \in \mathbb{N}_0. \tag{2.5}$$

Hence, from (2.4) and (2.5) we have

$$\tau + F(d(x_{n+1}, x_{n+2}) \leq F(d(x_n, x_{n+1})),$$

or,

$$F(d(x_{n+1}, x_{n+2}) \leq F(d(x_n, x_{n+1})) - \tau.$$

In general, one can get

$$F(d(x_{n+1},x_{n+2}) \leq F(d(x_0,x_1)) - n\tau. \tag{2.6}$$

Hence,

$$\lim_{n\to\infty} F(d(x_n,x_{n+1})) = -\infty.$$

So, from (F_2) we have

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = 0.$$

Therefore, with notice to (F_3), there exists $k \in (0,1)$ such that

$$\lim_{n\to\infty} (d(x_n,x_{n+1}))^k F(d(x_n,x_{n+1})) = 0.$$

Now, (2.6) implies that

$$(d(x_n,x_{n+1}))^k F(d(x_n,x_{n+1})) \leq (d(x_n,x_{n+1}))^k (F(d(x_0,x_1)) - n\tau).$$

Then, it can be easily seen that

$$\lim_{n\to\infty} n(d(x_n,x_{n+1}))^k = 0.$$

So, there exists $n_0 \in \mathbb{N}_0$ such that

$$d(x_n,x_{n+1}) \leq \frac{1}{n^{\frac{1}{k}}}\ ,\ \forall n \geq n_0.$$

Consequently, if $m > n > n_0$, then

$$d(x_n,x_m) \quad \leq \quad \sum_{i=n}^{m} d(x_i,x_{i+1}) \leq \sum_{i=n}^{m} \frac{1}{i^{\frac{1}{k}}} \leq \sum_{i=n_0}^{\infty} \frac{1}{i^{\frac{1}{k}}}.$$

Since $k \in (0,1)$, the series $\sum_{i=n_0}^{\infty} \frac{1}{i^{\frac{1}{k}}}$ is convergent. Therefore $\{x_n\}$ is a cauchy sequence, and since X is complete, there exists $u \in X$ such that $x_n \to u$ as $n \to \infty$. We claim that u is a fixed point of T. Now, we show that u is a fixed point of T under any of the cases (a) and (b).

First, we suppose that T is continuous case (a), then we have

$$u = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = T(u),$$

and so u is a fixed point of T. This completes the proof of Theorem by using case (a).

Now, suppose that case (b) is true. If $Tu \neq u$, then there exists $n_0 \in \mathbb{N}_0$ such that $Tx_n \neq Tu$, for all $n \geq n_0$ (Indeed, if $x_{n+1} = Tx_n = Tu$ for infinite values of n, then uniqueness of the limit concludes that $Tu = u$). From (iii) and (2.3), we have

$$\begin{aligned}
\tau + F(d(Tx_n,Tu)) \quad &\leq \quad \tau + \alpha(x_n,u)F(d(Tx_n,Tu)) \\
&\leq \quad F(m(x_n,u) + LN_1(x_n,u)) \\
&\leq \quad F(m(x_n,u) + Ld(Tx_n,u)) \\
&= \quad F(m(x_n,u) + Ld(x_{n+1},u)))
\end{aligned}$$

And since F is continuous, as $n \to \infty$ we get

$$\tau + F(d(u,Tu)) \leq F(\lim_{n\to\infty}(m(x_n,u) + Ld(x_{n+1},u))), \tag{2.7}$$

where

$$m(x_n,u) = \max\left\{d(x_n,u), d(x_n,x_{n+1}), d(u,Tu), \frac{d(x_n,Tu)+d(u,x_{n+1})}{2}\right\}.$$

So,

$$\lim_{n\to\infty} m(x_n,u) = \max\left\{0,0,d(u,Tu), \frac{d(u,Tu)+0}{2}\right\} = d(u,Tu).$$

Also, we have

$$\lim_{n\to\infty} Ld(x_{n+1},u) = 0.$$

Therefore, from (2.7) we have

$$\tau + F(d(u,Tu)) \leq F(d(u,Tu)),$$

which is a contradiction as $\tau > 0$. So $d(u,Tu) = 0$, i.e., $Tu = u$. □

Example 2.2. Let $X = \{(0,0),(0,5),(5,0),(5,6)\}$ be endowed with the metric d defined by

$$d((x_1,x_2),(y_1,y_2)) = |x_1 - y_1| + |x_2 - y_2|.$$

Let T be self-mappings on X as follow:

$$T(x_1,x_2) = (\min\{x_1,x_2\},0).$$

Also, suppose that $\alpha(x_1,x_2) = L = 1$, $0 < \tau < 0.033$ and for $x \in (0,+\infty)$, $F(x) = \ln x$, and for all $\zeta \in \mathcal{Z}$, defined $\zeta(t,s) = \frac{11}{12}s - t$. Therefore all the hypothesis of Theorem 2.1 are verified.

The next result establishes a sufficient condition for uniqueness of fixed point.

Theorem 2.3. *Let (X,d) be a complete metric space and $T : X \to X$ be a mapping for which there exist $F \in \mathcal{F}$, $\tau > 0$, $L \geq 0$ and simulation function ζ such that $d(Tx,Ty) > 0$ implies that*

$$\zeta(\tau + \alpha(x,y)F(d(Tx,Ty)), F(m(x,y) + LN_2(x,y))) \geq 0, \tag{2.8}$$

where $m(x,y)$ is defined as in Theorem 2.1 and

$$N_2(x,y) = \min\{d(x,Tx), d(x,Ty), d(y,Tx)\}.$$

We further assume that $\alpha(x,y) \geq 1$ for each $x,y \in Fix(T)$. Then if T is satisfied the conditions (i), (ii) and (iii) of Theorem 2.1 and T or F is continuous then T has a unique fixed point.

Proof. By Theorem 2.1, T has a fixed point. Now, suppose that u and v are two fixed point of T. If $u \neq v$ then $d(Tu,Tv) > 0$. Also $\alpha(u,v) \geq 1$, because $u,v \in Fix(T)$, then by (2.8) and ($\zeta 2$)

$$\begin{aligned} 0 &\leq \zeta(\tau + \alpha(u,v)F(d(Tu,Tv)), F(m(u,v) + LN_2(u,v))) \\ &\leq F(m(u,v) + LN_2(u,v)) - (\tau + \alpha(u,v)F(d(Tu,Tv))). \end{aligned}$$

Therefore,

$$\tau + \alpha(u,v)F(d(Tu,Tv)) \leq F(m(u,v) + LN_2(u,v)). \tag{2.9}$$

Hence, (2.9) implies that

$$\begin{aligned} \tau + F(d(u,v)) &= \tau + F(d(Tu,Tv)) \\ &\leq \tau + \alpha(u,v)F(d(Tu,Tv)) \\ &\leq F(m(u,v) + LN_2(u,v)) \\ &\leq F(m(u,v) + Ld(u,Tu)) \\ &= F(m(u,v) + 0) \\ &= F(m(u,v)), \end{aligned}$$

where

$$\begin{aligned} m(u,v) &= \max\left\{d(u,v), d(u,Tu), d(v,Tv), \frac{d(u,Tv)+d(v,Tu)}{2}\right\} \\ &= \max\{d(u,v), 0, 0, \tfrac{d(u,v)+d(v,u)}{2}\} \\ &= d(u,v). \end{aligned}$$

So, we have

$$\tau + F(d(u,v)) \leq F(d(u,v)),$$

which is a contradiction, as $\tau > 0$. So, $u = v$. □

Theorem 2.4. *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping which there exist $F \in \mathcal{G}$, $\tau > 0$ and the simulation function ζ such that for all $x, y \in X$ with $Tx \neq Ty$ and $\frac{1}{2}d(x, Tx) \leq d(x, y)$ implies that*

$$\zeta(\tau + \alpha(x, y)F(d(Tx, Ty)), F(m(x, y))) \geq 0 \tag{2.10}$$

where $m(x, y)$ is defined as in Theorem 2.1, satisfying the following conditions:

- **(i):** *T is α-admissible,*
- **(ii):** *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$,*
- **(iii):** *if $\{x_n\}$ is a sequence in X such that $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}_0$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}_0$,*
- **(iv):** *T has the K-property.*

Then, T has a fixed point in X.

Proof. Let $x_0 \in X$ be such that $\alpha(x_0, Tx_0) \geq 1$. For any $n \in \mathbb{N}_0$, define

$$x_{n+1} = T(x_n).$$

Since T is α-admissible, one can easily obtain that

$$\alpha(x_n, x_{n+1}) \geq 1, \ \forall n \in \mathbb{N}_0. \tag{2.11}$$

If $x_{n_0+1} = x_{n_0}$ for some $n_0 \in \mathbb{N}_0$, then x_{n_0} is a fixed point of T. So, we can assume that $x_{n+1} \neq x_n$ for each $n \in \mathbb{N}_0$, i.e., $d(x_n, x_{n+1}) > 0$ and so

$$\frac{1}{2}d(x_n, Tx_n) = \frac{1}{2}d(x_n, x_{n+1}) < d(x_n, x_{n+1}). \tag{2.12}$$

Now from (2.10) and $(\zeta 2)$, there exist $F \in \mathcal{G}$ and $\tau > 0$ such that if $d(Tx, Ty) > 0$, then $\frac{1}{2}d(x, Tx) \leq d(x, y)$ implies that

$$\begin{aligned} 0 &\leq \zeta(\tau + \alpha(x, y)F(d(Tx, Ty)), F(m(x, y))) \\ &\leq F(m(x, y)) - (\tau + \alpha(x, y)F(d(Tx, Ty))). \end{aligned}$$

Then,

$$\tfrac{1}{2}d(x, Tx) \leq d(x, y) \Longrightarrow \tau + \alpha(x, y)F(d(Tx, Ty)) \leq F(m(x, y)), \tag{2.13}$$

where $m(x, y)$ is defined as in Theorem 2.1. Therefore, by (2.12) and (2.13)

$$\begin{aligned} \tau + F(d(Tx_n, Tx_{n+1})) &\leq \tau + \alpha(x_n, x_{n+1})F(d(Tx_n, Tx_{n+1})) \\ &\leq F(m(x_n, x_{n+1})), \end{aligned} \tag{2.14}$$

in which

$$\begin{aligned} m(x_n, x_{n+1}) &= \max\Big\{d(x_n, x_{n+1}), d(x_n, Tx_n), d(x_{n+1}, Tx_{n+1}), \\ &\qquad \tfrac{d(x_n, Tx_{n+1}) + d(x_{n+1}, Tx_n)}{2}\Big\} \\ &= \max\Big\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \tfrac{d(x_n, x_{n+2})}{2}\Big\} \\ &\leq \max\Big\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \tfrac{d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})}{2}\Big\} \\ &\leq \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}. \end{aligned}$$

Now, if $d(x_{n_0+1}, x_{n_0+2}) \geq d(x_{n_0}, x_{n_0+1})$ for some $n_0 \in \mathbb{N}_0$, then

$$m(x_{n_0}, x_{n_0+1}) \leq d(x_{n_0+1}, x_{n_0+2}),$$

and since F is strictly increasing,

$$F(m(x_{n_0}, x_{n_0+1})) \leq F(d(x_{n_0+1}, x_{n_0+2})).$$

Therefore, by (2.14)

$$\tau + F(d(x_{n_0+1}, x_{n_0+2})) \leq F(d(x_{n_0+1}, x_{n_0+2})).$$

So, $\tau \leq 0$, which is a contradiction. Consequently,

$$d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1}),\ \forall n \in \mathbb{N}_0. \tag{2.15}$$

Therefore,

$$m(x_n, x_{n+1}) \leq d(x_n, x_{n+1}),\ \forall n \in \mathbb{N}_0. \tag{2.16}$$

So, from (2.14) and (2.15) one can obtain that

$$\tau + F(d(x_{n+1}, x_{n+2})) \leq F(d(x_n, x_{n+1})),$$

or,

$$F(d(x_{n+1}, x_{n+2})) \leq F(d(x_n, x_{n+1})) - \tau.$$

In general, one can get

$$F(d(x_{n+1}, x_{n+2})) \leq F(d(x_0, x_1)) - n\tau.$$

Hence,

$$\lim_{n\to\infty} F(d(x_n, x_{n+1})) = -\infty,$$

which together with (G2) and Lemma 1.4, gives

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

Now, we claim that $\{x_n\}$ is a Cauchy sequence. If it is not true, then by Lemma 1.5, there exists $\epsilon_0 > 0$ and two sequences of positive integers $\{n_k\}$ and $\{m_k\}$ with $n_k > m_k > k$ such that $d(x_{m_k}, x_{n_k}) > \epsilon_0$, $d(x_{m_k}, x_{n_k-1}) < \epsilon_0$ and

$(L1)$ $\lim_{k\to\infty} d(x_{n_k}, x_{m_k}) = \epsilon_0$,
$(L2)$ $\lim_{k\to\infty} d(x_{n_k}, x_{m_k-1}) = \epsilon_0$,
$(L3)$ $\lim_{k\to\infty} d(x_{n_k+1}, x_{m_k}) = \epsilon_0$,
$(L4)$ $\lim_{k\to\infty} d(x_{n_k+1}, x_{m_k-1}) = \epsilon_0$.

Therefore, with notice to definition of $m(x, y)$ we have:

$$\begin{aligned}\lim_{k\to\infty} m(x_{n_k}, x_{m_k-1}) &= \lim_{k\to\infty} \max\Big\{ d(x_{n_k}, x_{m_k-1}), d(x_{n_k}, x_{n_k+1}), \\ &\qquad d(x_{m_k-1}, x_{m_k}), \tfrac{d(x_{n_k}, x_{m_k}) + d(x_{m_k-1}, x_{n_k+1}))}{2} \Big\} \\ &= \max\{\epsilon_0, 0, 0, \tfrac{\epsilon_0+\epsilon_0}{2}\} \\ &= \epsilon_0.\end{aligned}$$

So,

$$\lim_{k\to\infty} m(x_{n_k}, x_{m_k-1}) = \epsilon_0. \tag{2.17}$$

On the other hand, since $\lim_{k\to\infty} d(x_{n_k}, x_{m_k-1}) = \epsilon_0 > 0$, and $\lim_{k\to\infty} d(x_{n_k}, x_{n_k+1}) = 0$, with considering a subsequence if it is needed, one can assumed that, there exist $k_1 \in \mathbb{N}$ such that for any $k > k_1$ and $n_k > m_k > k$

$$d(x_{n_k}, x_{n_k+1}) \leq d(x_{n_k}, x_{m_k-1}).$$

So, it is clear that for all $k > k_1$ and $n_k > m_k > k$,

$$\frac{1}{2} d(x_{n_k}, Tx_{n_k}) = \frac{1}{2} d(x_{n_k}, x_{n_k+1}) < d(x_{n_k}, x_{m_k-1}). \tag{2.18}$$

Also, using the K-property, there exist $k_2 \in \mathbb{N}$ such that

$$\alpha(x_{n_k}, x_{m_k-1}) \geq 1,\ \forall k > k_2. \tag{2.19}$$

Let $k \geq \max\{k_1, k_2\}$, then from (2.19) and (2.13) we have

$$\begin{aligned}\tau + F(d(Tx_{n_k}, x_{m_k-1})) &\leq \tau + \alpha(x_{n_k}, x_{m_k-1})F(d(Tx_{n_k}, Tx_{m_k-1})) \\ &\leq F(m(x_{n_k}, x_{m_k-1})).\end{aligned}$$

Letting $n \to \infty$, since F is continuous, by (L1) and (2.17) we have

$$\tau + F(\epsilon_0) \leq F(\epsilon_0),$$

which is a contradiction, as $\tau > 0$. Consequently, $\{x_n\}$ is a Cauchy sequence in the complete metric space X. So there exists $u \in X$ such that $x_n \to u$, as $n \to \infty$. To complete the proof, we show that u is a fixed point of T. At first, we claim that for all $n \geq 0$

$$\frac{1}{2}d(x_n, x_{n+1}) \leq d(x_n, u) \text{ or } \frac{1}{2}d(x_{n+1}, x_{n+2}) \leq d(x_{n+1}, u). \tag{2.20}$$

In fact, If for some $n_0 \geq 0$, both of them are false then we will have

$$\frac{1}{2}d(x_{n_0}, x_{n_0+1}) > d(x_{n_0}, u) \text{ and } \frac{1}{2}d(x_{n_0+1}, x_{n_0+2}) > d(x_{n_0+1}, u).$$

So, with notice to (2.15) we have

$$\begin{aligned}d(x_{n_0}, x_{n_0+1}) &\leq d(x_{n_0}, u) + d(u, x_{n_0+1}) \\ &< \tfrac{1}{2}d(x_{n_0}, x_{n_0+1}) + \tfrac{1}{2}d(x_{n_0+1}, x_{n_0+2}) \\ &\leq \tfrac{1}{2}d(x_{n_0}, x_{n_0+1}) + \tfrac{1}{2}d(x_{n_0}, x_{n_0+1}) \\ &= d(x_{n_0}, x_{n_0+1}).\end{aligned}$$

Which is a contradiction and the claim is proved.
Well, let us begin with the first part of (2.20), i.e. suppose that

$$\frac{1}{2}d(x_n, x_{n+1}) \leq d(x_n, u),$$

and in contrary, assume that $Tu \neq u$. Without lose of generality, one can assume that $Tx_n \neq Tu$, for all $n \in \mathbb{N}_0$. (Indeed, if $x_{n+1} = Tx_n = Tu$ for infinite values of n, then uniqueness of the limit concludes that $Tu = u$).
Then, from (2.13) and (iii) we get

$$\begin{aligned}\tau + F(d(x_{n+1}, Tu)) &= \tau + F(d(Tx_n, Tu)) \\ &\leq \tau + \alpha(x_n, u)F(d(Tx_n, Tu)) \\ &\leq F(m(x_n, u)),\end{aligned}$$

and since F is continuous on $(o, +\infty)$, and $d(u, Tu) > 0$, as $n \to \infty$, we get

$$\tau + F(d(u, Tu)) \leq F(\lim_{n\to\infty}(m(x_n, u)). \tag{2.21}$$

But

$$m(x_n, u) = \max\left\{d(x_n, u), d(x_n, x_{n+1}), d(u, Tu), \frac{d(x_n, Tu) + d(u, x_{n+1})}{2}\right\}.$$

So we have

$$\lim_{n\to\infty} m(x_n, u) = \max\{0, 0, d(u, Tu), \frac{d(u, Tu) + 0}{2}\} = d(u, Tu).$$

Therefore, if $d(u, Tu) \neq 0$, then from (2.21) we have

$$\tau + F(d(u, Tu)) \leq F(d(u, Tu)),$$

which is a contradiction, as $\tau > 0$. So $d(u, Tu) = 0$, i.e. $Tu = u$. Finally, if we assume that the second part of (2.20) is true, i.e.

$$\frac{1}{2}d(x_{n+1}, x_{n+2}) \leq d(x_{n+1}, u).$$

Then, as the same manner, we can prove that $d(u, Tu) = 0$, i.e. $Tu = u$. □

Theorem 2.5. *Suppose that all the conditions of Theorem 2.4 are satisfied. In addition, assume that $\alpha(x,y) \geq 1$, for all $x, y \in Fix(T)$. Then T has a unique fixed point.*

Proof. Suppose that u and v are two fixed point of T. If $u \neq v$ then $d(Tu, Tv) > 0$. Also $\alpha(u,v) \geq 1$, because $u, v \in Fix(T)$. Also, it is clear that $\frac{1}{2}d(u,Tu) = 0 < d(u,v)$. Hence, (2.13) implies that

$$\begin{aligned} \tau + F(d(u,v)) &= \tau + F(d(Tu,Tv)) \\ &\leq \tau + \alpha(u,v)F(d(Tu,Tv)) \\ &\leq F(m(u,v)), \end{aligned}$$

where

$$\begin{aligned} m(u,v) &= \max\left\{d(u,v), d(u,Tu), d(v,Tv), \tfrac{d(u,Tv)+d(v,Tu)}{2}\right\} \\ &= \max\{d(u,v), 0, 0, \tfrac{d(u,v)+d(v,u)}{2}\} \\ &= d(u,v). \end{aligned}$$

So, we have

$$\tau + F(d(u,v)) \leq F(d(u,v)),$$

which is a contradiction, as $\tau > 0$. So $u = v$. □

In the next result we obtain a new version of Theorem 2.6 of [6].

Corollary 2.6. *Let (X,d) be a complete metric space and $T : X \to X$ be mapping such that for all $x, y \in X$, $\dfrac{1}{2}d(x,Tx) \leq d(x,y)$ implies that*

$$\zeta(d(Tx,Ty), m(x,y)) \geq 0,$$

where $\zeta \in \mathcal{Z}$ and $m(x,y)$ is defined as in Theorem 2.1. In addition, assume that $\alpha(x,y) \geq 1$ and condition (2.10) is true. Then T has a unique fixed point.

Proof. Since $\alpha(x,y) \geq 1$, it follows from condition (2.10) that

$$\begin{aligned} F(d(Tx,Ty)) &\leq \tau + F(d(Tx,Ty)) \\ &\leq \tau + \alpha(x,y)F(d(Tx,Ty)) \\ &\leq F(m(x,y)) \end{aligned}$$

Then,

$$F(d(Tx,Ty)) \leq F(m(x,y)).$$

Now, since F is strictly increasing, so $d(Tx,Ty) \leq m(x,y)$. Hence all the hypothesis of Theorem 2.6 of [6] is right. Therefore, the desired result is obtained. □

Example 2.7. Let $X = \{0,1,2\}$ be endowed with the metric d defined by $d(x,y) = |x-y|$, and $T : X \to X$ is defined as follows:

$$T(1) = T(2) = 1,\ T(0) = 2.$$

Furthermore, suppose that $\alpha(x,y) = 1$ for all $x, y \in X$, $0 < \tau \leq \ln 2$ and for $x \in (0,\infty)$, $F(x) = \ln x$, and for all $\zeta \in \mathcal{Z}$, defined $\zeta(t,s) = \frac{11}{12}s - t$. Therefore all the hypothesis of Theorem 2.5 are verified. Hence $u = 1$ is the unique fixed point of T.

3. Conclusion

The existence and uniqueness of a fixed point of almost contractions via simulation functions in metric spaces are investigated. Some examples and applications to illustrate the reality of our generalizations and usability the results are given. One can consider the article as the extended simulation version of the paper published in Carpathian Mathematical Publications 11 (2) (2019).

Statements and Declarations

The authors declare that they have no conflict of interest, and the manuscript has no associated data.

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