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ON A FEEDBACK CONTROL SYSTEM GOVERNED BY A FRACTIONAL SEMILINEAR DIFFERENTIAL INCLUSION WITH DELAY AND A SWEEPING PROCESS

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Abstract. We consider a controllability problem for a feedback control system governed by a fractional semilinear differential inclusion with delay and a sweeping process in a Hilbert space. We define the multioperator whose fixed points are solutions of the problem. By applying the methods of the fixed point theory for condensing multimaps we study the properties of this multioperator, in particular, we prove that under certain conditions it is condensing w.r.t. an appropriate measure of noncompactness. This allows to present the controllability principle.

Keywords. Controllability, Caputo fractional derivative, Differential inclusion, Sweeping process, Controllability, Fixed point, Multivalued map, Condensing multimap, Measure of noncompactness.

1. Introduction

In modern mathematics, one of the most important areas is control theory. This is primarily due to numerous applications in physics, chemistry and engineering associated with the modeling of various types of processes and phenomena. From a mathematical point of view, the complexity of such models lies in the need to use the theory of infinite-dimensional spaces. Research in this direction is relevant and is being carried out by a large number of scientists around the world (see, e.g., the surveys [3, 26] and the references therein).

Recently various controllability results were obtained for systems which can be described in terms of semilinear differential and functional differential inclusions in infinite-dimensional Banach spaces (see, among others, [5, 6, 7, 9, 14, 15, 21, 24] and the references therein). It should be mentioned that in the works [6, 7, 21, 24] it was not supposed that the semigroup generated by the linear part of a system is compact. It is known (see [31, 32]) that this compactness condition in the infinite-dimensional case creates some difficulties in the investigation of the controllability problem.

In the last years the study of the controllability problem was extended to systems governed by differential equations and inclusions of a fractional order (see, e.g., [1, 2, 10, 27, 34, 36, 37] and the references therein). In its essential part, it is caused by interesting and important applications which fractional systems find in physics, hydrodynamics, geophysics, engineering, biology, economics and other contemporary branches of natural sciences (see, e.g., [4, 11, 16, 19, 25, 28, 36]).

The simulation of processes in feedback control systems by means of differential inclusions and variational inequalities of various types in finite-dimensional and infinite-dimensional spaces is an actual problem of contemporary mathematics. In particular, the investigation of control systems whose dynamics is described by some differential or functional-differential equations or inclusions with a control parameter in an infinite-dimensional Banach space is very relevant. In many cases, the feedback constraints imposed on the choice of control are considered as solutions of so called sweeping processes in

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Hilbert spaces depending on the state of the system. Fundamental results on the existence, uniqueness and continuous dependence of solutions of sweeping processes were obtained by M. Kunze and M. D. P. Monteiro Marques [20, 22], M. Valadier [33], C. Castaing and M. Monteiro Marques [8], J. F. Edmond and L. Thibault [13], A. A. Tolstonogov [29, 30].

Let E be a Banach space and H be a Hilbert space. We consider controllability problem for a system governed by the following differential inclusion and the sweeping process

$$ {}^{C}D_0^{\alpha}x(t) \in Ax(t) + F(t, x_t, x(t), y(t)) + Bu(t),\ \ t \in [0,T], \tag{1.1}$$

$$ x(s) = \vartheta(s),\ \ s \in [-h, 0], \tag{1.2}$$

$$ -y'(t) \in N_{C(t)}(y(t)) + g(t, x(t), y(t)) + \rho y(t),\ \ t \in [0,T], \tag{1.3}$$

$$ y(0) = y_0 \in C(0), \tag{1.4}$$

$$ x(T) = x_1, \tag{1.5}$$

where ${}^{C}D_0^{\alpha},\ 0 < \alpha < 1$, is the Caputo fractional derivative, $A : D(A) \subset E \to E$ is a linear closed operator generating an uniformly bounded C_0−semigroup $\{\mathcal{U}(t), t \geq 0\}$ in the space E, $F\colon [0,T] \times C([-h,0];E) \times E \times H \multimap E$ is a multivalued nonlinearity and the function x_t describes the prehistory of the solution at the moment $t \in [0,T]$, i.e., $x_t(s) = x(t+s), s \in [-h,0], 0 < h < T$. A control function $u(\cdot)$ belongs to the space $L^{\infty}([0,T];U)$, where U is a Banach space of controls, $B : U \to E$ is a bounded linear operator. Here ρ is a positive number, $C : [0,T] \multimap H$ is a multimap with closed convex values, $N_{C(t)}(y)$ denotes the normal cone defined for a closed convex set $C(t) \subset H$ as

$$ N_{C(t)}(y) = \begin{cases} \{\xi \in H : \langle \xi, c - y\rangle \leq 0 \text{ for all } c \in C(t)\}, & \text{if } y \in C(t), \\ \emptyset, & \text{if } y \notin C(t), \end{cases} \tag{1.6}$$

and function $g : [0,T] \times E \times H \to H$ is a nonlinear map, and $\vartheta \in C([-h,0];E)$, $x_1 \in E, y_0 \in H$.

The controllability problem which we study in this paper may be formulated in the following way: for a given ϑ, x_1 we will consider a solution $x \in C([-h,T];E), y \in C([0,T];H)$ of the above system (1.1)-(1.4) and a control $u \in L^{\infty}([0,T];U)$ such that conditions (1.2) and (1.5) are satisfied.

To solve this problem, we use the methods of contemporary mathematics, which are based on the fixed points theory.

2. Preliminaries

2.1. **The fractional integral and Caputo fractional derivative.** For the considering of the main problem we need the following notions from fractional calculus (see, e.g., monographs [19, 28]).

Let $\mathcal{E}$ be a Banach space.

Definition 2.1. The fractional integral of an order $\alpha > 0$ of a function $g : [0,T] \to \mathcal{E}$ is the function $I_0^{\alpha}g$ of the following form:

$$ I_0^{\alpha}g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)\, ds, $$

where Γ is the Euler gamma function

$$ \Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} dx. $$

Definition 2.2. The Caputo fractional derivative of an order $\alpha \geq 0$ of a function $g \in C^n([0,T];\mathcal{E})$ is the function ${}^{C}D_0^{\alpha}g$ of the following form:

$$ {}^{C}D_0^{\alpha}g(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} g^{(n)}(s)\, ds, $$

where n and α are related by equality $n = [\alpha] + 1$.

2.2. **The measure of noncompactness and multivalued maps.** Let Y be a metric space. We will use the following notation:

- $P(Y)$ is the collection of all nonempty subsets of Y;
- $Pb(Y)$ is the collection of all nonempty bounded subsets of Y;
- $K(Y)$ is the collection of all compact subsets of Y.

If Y is a normed space the symbol $Kv(Y)$ will denote the collection of all nonempty convex compact subsets of Y.

Definition 2.3. (See, e.g., [18, 23]). Let $\mathcal{E}$ be a Banach space, $(\mathcal{A}, \geq)$ be a partially ordered set. A function $\beta : Pb(\mathcal{E}) \to \mathcal{A}$ is called the measure of noncompactness (MNC) in $\mathcal{E}$ if for each $\Omega \in Pb(\mathcal{E})$ we have:

$$\beta(\overline{\text{co}}\, \Omega) = \beta(\Omega),$$

where $\overline{\text{co}}\, \Omega$ denotes the closure of the convex hull of Ω.

A measure of noncompactness β is called:

1) monotone if for each $\Omega_0, \Omega_1 \in Pb(\mathcal{E})$, $\Omega_0 \subseteq \Omega_1$ implies $\beta(\Omega_0) \leq \beta(\Omega_1)$;
2) nonsingular if for each $a \in \mathcal{E}$ and each $\Omega \in Pb(\mathcal{E})$ we have $\beta(\{a\} \cup \Omega) = \beta(\Omega)$;

If $\mathcal{A}$ is a cone in a Banach space, the MNC β is called:

4) regular if $\beta(\Omega) = 0$ is equivalent to the relative compactness of $\Omega \in Pb(\mathcal{E})$;
5) real if $\mathcal{A}$ is the set of all real numbers $\mathbb{R}$ with the natural ordering;
6) algebraically semiadditive if $\beta(\Omega_0 + \Omega_1) \leq \beta(\Omega_0) + \beta(\Omega_1)$ for every $\Omega_0, \Omega_1 \in Pb(\mathcal{E})$.

As an example of a real MNC obeying all above properties, we can consider the Hausdorff MNC $\chi(\Omega)$:

$$\chi(\Omega) = \inf\{\varepsilon > 0, \text{for which } \Omega \text{ has a finite } \varepsilon\text{-net in } \mathcal{E}\ \}.$$

Notice that the Hausdorff MNC satisfies the semi-homogeneity condition:

$$\chi(\lambda\Omega) = |\lambda|\chi(\Omega),$$

for every $\lambda \in \mathbb{R}$ and $\Omega \in P(\mathcal{E})$.

Recall that the norm of a set $\mathcal{M} \in Pb(\mathcal{E})$ is defined by the formula:

$$\|\mathcal{M}\| = \sup_{x \in \mathcal{M}} \|x\|_{\mathcal{E}}$$

Definition 2.4 (See, e.g., [18])**.** Let X be a closed subset of $\mathcal{E}$; β a MNC in $\mathcal{E}$. A multivalued map (multimap) $\mathcal{F} : X \to K(\mathcal{E})$ is called condensing w.r.t. β (or β-condensing) if for every $\Omega \in Pb(X)$ which is not relatively compact we have:

$$\beta(F(\Omega)) \ngeq \beta(\Omega).$$

Definition 2.5 (See, e.g., [18, 23])**.** Let X and Y be metric spaces. A multivalued map (multimap) $\mathcal{F} : X \to P(Y)$ is called:

(i) upper semicontinuous (u.s.c.) if $\mathcal{F}^{-1}(V) = \{x \in X : \mathcal{F}(x) \subset V\}$ is an open subset of X for each open subset $V \subset Y$;
(ii) closed if its graph $\Gamma_{\mathcal{F}} = \{(x, y) : y \in \mathcal{F}(x)\}$ is the closed subset of $X \times Y$;
(iii) compact if its range $\mathcal{F}(X)$ is a relatively compact subset of Y;
(iv) quasicompact if its restriction to each compact subset $A \subset X$ is compact.

We will need the following assertion.

Lemma 2.6. *If $\mathcal{F} : X \to K(Y)$ is a closed quasicompact multimap then $\mathcal{F}$ is u.s.c.*

Theorem 2.7. *(Cf. [18], Corollary 3.3.1) Let $\mathcal{M}$ be a convex closed subset of $\mathcal{E}$ and $\mathcal{F} : \mathcal{M} \to Kv(\mathcal{M})$ a u.s.c. β–condensing multimap, where β is a nonsingular MNC in $\mathcal{E}$. Then the fixed point set $\operatorname{Fix} \mathcal{F} = \{x : x \in \mathcal{F}(x)\}$ is non-empty.*

Recall some notions (see, e.g., [18, 23]). Let E be a Banach space.

Definition 2.8. *For a given $1 \leq p \leq \infty$, a multifunction $G : [0, T] \to K(E)$ is called:*

- L^p–integrable *if it admits an L^p–Bochner integrable selection, i.e., there exists a function $g \in L^p((0, T); E)$ such that $g(t) \in G(t)$ for a.e. $t \in [0, T]$;*
- L^p–integrably bounded *if there exists a function $\xi \in L^p((0, T))$ such that*

$$\|G(t)\| \leq \xi(t)$$

for a.e. $t \in [0, T]$.

The set of all L^p–integrable selections of a multifunction $G : [0, T] \to K(E)$ is denoted by $\mathcal{S}^p_G$.

2.3. **Sweeping process in a Hilbert space.** Let M be a closed nonempty subset of a Hilbert space H and $x \in H$, then a distance from x to M, denoted by $d(x, M)$, is defined as

$$d(x, M) = \inf\{\|x - z\| | \, z \in M\},$$

and the set of nearest points to x in M is given as

$$pr_M(x) = \{z \in M | \, d(x, M) = \|x - z\|\}.$$

If $z \in pr_M x$ and $\alpha \geq 0$, then the vector $\alpha(x - z)$ is called a proximal normal to M at z. The set of all such vectors forms a cone called the proximal normal cone to M at z. It is denoted by $N^P_M(z)$. The limiting normal cone, denoted by $N^L_M(z)$, is defined as

$$N^L_M(z) = \{\eta \in H | \, \eta_n \rightharpoonup \eta, \, \eta_n \in N^P_M(z_n), \, z_n \to z\}.$$

For a fixed $r > 0$, the set M is said to be r-prox-regular (or uniformly prox-regular with constant $\frac{1}{r}$), if for any $z \in M$ and any $\eta \in N^L_M(z)$ such that $\|\eta\| < 1$, one has $z = pr_M(z + r\eta)$. If M is r-prox-regular, then the following holds (see [13]):

(i) for each $z \in M$, all the normal cones defined above coincide. In such a case, they will be denoted by $N_M(z)$;

(ii) for each $z \in H$ such that $d(z, M) < r$, the set $pr_M(z)$ is a singleton.

In the sequel, we will use the following notation. The collection of all nonempty closed subsets of H will be denoted by $C(H)$,

$$Cb(H) = \{A \in C(H) \colon \, A \text{ is bounded}\};$$

$$Cv(H) = \{A \in C(H) \colon \, A \text{ is convex}\}.$$

Let a multifunction $C : [0, T] \to C(H)$ be such that

$(H1)$ for every $t \in [0, T]$, the set $C(t)$ is an r-prox-regular;

$(H2)$ $C(t)$ varies in an absolutely continuous way, i.e., there exists an absolutely continuous function $\vartheta : [0, T] \to \mathbb{R}$ such that for each $x \in H$ and $s, t \in [0, T]$:

$$|d(x, C(t)) - d(x, C(s))| \leq |\vartheta(t) - \vartheta(s)|.$$

Theorem 2.9. *(see [13]) Assume that the multifunction $C(\cdot)$ satisfies $(H1)$ and $(H2)$. Let $h : [0, T] \to H$ be an integrable map. Then, for each $\eta_0 \in C(0)$, the sweeping process with perturbation*

$$-y'(t) \in N_{C(t)}(y(t)) + h(t) \;\; \text{a.e. } t \in [0, T], \tag{2.1}$$

$$y(0) = \eta_0, \tag{2.2}$$

has an unique absolutely continuous solution y. Moreover, the following inequality holds true

$$\|y'(t) + h(t)\| \le \|h(t)\| + |\vartheta'(t)| \text{ a.e. } t \in [a, b]. \tag{2.3}$$

Theorem 2.10. *(see [13]) Assume that the multifunction $C(\cdot)$ satisfies $(H1)$ and $(H2)$. Let $f : [0, T] \times H \to H$ be a map such that*

- (i) *for each $x \in H$ the function $f(\cdot, x) : [0, T] \to H$ is measurable;*
- (ii) *for every $\delta > 0$ there exists a non-negative function $k_\delta \in L^1[0, T]$ such that for each $(x, y) \in B_\delta(0) \times B_\delta(0)$ we have*

$$\|f(t, x) - f(t, y)\| \le k_\delta(t)\|x - y\|, \text{ a.e. } t \in [0, T];$$

- (iii) *there exists a non-negative function $\varsigma \in L^1[0, T]$ such that for each $x \in \cup_{s\in[0,T]} C(s)$:*

$$\|f(t, x)\| \le \varsigma(t)(1 + \|x\|), \text{ a.e. } t \in [0, T].$$

Then, for each $\eta_0 \in C(0)$, the following perturbed sweeping process

$$-y'(t) \in N_{C(t)}(y(t)) + f(t, y(t)), \text{ a.e. } t \in [0, T], \tag{2.4}$$

$$y(0) = \eta_0, \tag{2.5}$$

has a unique absolutely continuous solution y.

For a multifunction $C : [0, T] \to Cv(H)$ consider the following sweeping process with perturbation

$$-y'(t) \in N_{C(t)}(y(t)) + h(t) + \rho y(t), \tag{2.6}$$

where $h : [0, T] \to H$ is a bounded measurable function and $\rho > 0$.

We will assume that the multifunction C satisfies the following properties

$(H2')$ C is Lipschitz, i.e., there exists $L_C > 0$ such that for all $t_1, t_2 \in [0, T]$ we have

$$d_H(C(t_1), C(t_2)) \le L_C|t_1 - t_2|, \tag{2.7}$$

where the Hausdorff distance $d_H(C_1, C_2)$ between two closed sets $C_1, C_2 \subset H$ is defined as

$$d_H(C_1, C_2) = \max\{\sup_{a\in C_2} d(a, C_1), \sup_{b\in C_1} d(b, C_2)\};$$

$(H3)$ the set $\bigcup_{t\in[0,T]} C(t)$ is a relatively compact.

Notice that condition $(H2')$ is a particular case of $(H2)$, when $v(t) = L_C t$.

Under condition $(H2')$ for initial condition $y(0) \in C(0)$, the sweeping process (2.6) admits a unique absolutely continuous solution $y(t)$ satisfying (2.6) for almost all $t \in [0, T]$ (Theorem 2.10).

Let the map g satisfy the following conditions:

- (g1) for all $(x, y) \in E \times H$, the function $g(\cdot, x, y) : [0, T] \to E$ is measurable map;
- (g2) there exist constants $m_1, m_2 > 0$, such that for all $t \in \mathbb{R}$ and $x_1, x_2 \in E$, $y_1, y_2 \in H$ we have

$$\|g(t, x_1, y_1) - g(t, x_1, y_1)\|_H \le m_1\|x_1 - x_2\|_E + m_2\|y_1 - y_2\|_H;$$

- (g3) there exists a function $\sigma \in L^1_+([0, T])$ such that

$$\|g(t, x, y)\|_H \le \sigma(t)(1 + \|x\|_E + \|y\|_H), \text{ for a.e. } t \in [0, T].$$

From Theorems 2.9, 2.10 it follows that under conditions $(H1), (H2'), (H3)$, $(g1) - (g3)$ for each $x \in C([-h, T]; E)$ problem (1.3)-(1.4) has a unique solution $y_x \in C([0, T]; H)$.

3. Main Result

Let E be a separable Banach space and $\mathcal{C} := C([-h,0];E)$. We will suppose that the following assumptions hold true.

(A) the operator $A : D(A) \subset E \to E$ is an infinitesimal generator of an uniformly bounded C_0-semigroup $\{\mathcal{U}(t)\}_{t\geq 0}$ of linear operators in E. Denote by $M = \sup\{\|\mathcal{U}(t)\|\,; t \geq 0\}$.

The multimap F satisfy the following conditions:

$(F1)$ for each $(\xi, x, y) \in \mathcal{C} \times E \times H$ the multifunction $F(\cdot, \xi, x, y) : [0,T] \to Kv(E)$ admits a measurable selection;

$(F2)$ for a.e. $t \in [0,T]$ the multimap $F(t,\cdot,\cdot,\cdot) : \mathcal{C} \times E \times H \to Kv(E)$ is u.s.c.;

$(F3)$ for each $r > 0$ there exists a function $\omega_r \in L^\infty_+([0,T])$ such that

$$\|F(t,\xi,x,y)\|_E \leq \omega_r(t), \text{ a.e. } t \in [0,T]$$

for all $(\xi,x,y) \in \mathcal{C} \times E \times H$, $\|\xi\|_{\mathcal{C}} \leq r$ $\|x\|_E \leq r$, $\|y\|_H \leq r$;

$(F4)$ there exists a function $\mu \in L^\infty_+([0,T])$ such that for each bounded set $\Omega \subset E$, $\Delta \subset \mathcal{C}$ and every $y \in H$ we have:

$$\chi_E(F(t,\Delta,\Omega,y)) \leq \mu(t)(\chi_E(\Omega) + \varphi(\Delta)), \text{ a.e. } t \in [0,T],$$

where $\varphi(\Delta) = \sup_{s\in[-h,0]} \chi_E(\Delta(s))$, χ_E is the Hausdorff MNC in E, $\Delta(s) = \{y(s) : y \in \Delta\}$.

For a given $x \in C([-h,T];E)$ consider the multifunction

$$\Phi : [0,T] \to Kv(E), \qquad \Phi(t) = F(t, x_t, x(t), y_x(t)).$$

Let $\mathcal{D} \subset C([0,T];E)$ be a convex closed subset given as

$$\mathcal{D} = \{\xi \in C([0,T];E), \xi(0) = \vartheta(0)\}$$

and, for a given $\xi \in \mathcal{D}$, define a function $\xi[\vartheta] \in C([-h,T];E)$ by

$$\xi[\vartheta](t) = \begin{cases} \vartheta(t), & t \in [-h,0], \\ \xi(t), & t \in [0,T]. \end{cases}$$

From above conditions $(F1)$–$(F3)$, it follows (see, e.g., [18] Theorem 1.3.5) that the multifunction Φ is L^∞–integrable and, therefore, the superposition multioperator $\mathcal{P}^\infty_F : \mathcal{D} \to P(L^\infty([0,T];E))$ can be defined in the following way:

$$\mathcal{P}^\infty_F(x) = \{f \in L^\infty([0,T];E) : f(t) \in F(t, x[\vartheta]_t, x(t), y_x(t)) \text{ a.e. } t \in [0,T]\}.$$

Following monograph [18], we give the next definition of a mild solution to problem (1.1)-(1.2).

Definition 3.1. A mild solution to problem (1.1)-(1.2) on the interval $[-h,T]$ is a function $x \in C([-h,T];E)$ of the following form:

$$x(t) = \begin{cases} \vartheta(t), & t \in [-h,0], \\ \mathcal{G}(t)\vartheta(0) + \int_0^t (t-s)^{\alpha-1}\mathcal{T}(t-s)f(s)ds + \\ + \int_0^t (t-s)^{\alpha-1}\mathcal{T}(t-s)Bu(s)ds, & t \in [0,T], \end{cases}$$

where

$$\mathcal{G}(t) = \int_0^\infty \xi_\alpha(\theta)\mathcal{U}(t^\alpha\theta)d\theta, \qquad \mathcal{T}(t) = \alpha\int_0^\infty \theta\xi_\alpha(\theta)\mathcal{U}(t^\alpha\theta)d\theta,$$

$$\xi_\alpha(\theta) = \frac{1}{\alpha}\theta^{-1-\frac{1}{\alpha}}\Psi_\alpha(\theta^{-1/\alpha}),$$

$$\Psi_\alpha(\theta) = \frac{1}{\pi}\sum_{n=1}^\infty (-1)^{n-1}\theta^{-\alpha n-1}\frac{\Gamma(n\alpha+1)}{n!}\sin(n\pi\alpha), \qquad \theta \in \mathbb{R}^+,$$

and $f \in \mathcal{P}^\infty_F(x)$, $u \in L^\infty([0,T];U)$.

Remark 3.2. $\xi_\alpha(\theta) \geq 0,\ \int_0^\infty \xi_\alpha(\theta)\,d\theta = 1,\ \int_0^\infty \theta\xi_\alpha(\theta)\,d\theta = \frac{1}{\Gamma(\alpha+1)}$.

Lemma 3.3. *(see [36]) The operators $\mathcal{G}$ and $\mathcal{T}$ obey the following properties:*

(1) *for each $t \in [0,T]$, $\mathcal{G}(t)$ and $\mathcal{T}(t)$ are bounded linear operators and, moreover,*

$$\|\mathcal{G}(t)x\|_E \leq M\,\|x\|_E\,; \qquad \|\mathcal{T}(t)x\|_E \leq \frac{M}{\Gamma(\alpha)}\,\|x\|_E\,;$$

(2) *the operator functions $\mathcal{G}(t)$ and $\mathcal{T}(t)$ are strongly continuous for $t \in [0,T]$.*

Towards our goal we will suppose the usual assumption on the controllability of the corresponding linear problem. More exactly, we assume that the linear controllability operator $W : L^\infty([0,T];U) \to E$ given by

$$Wu = \int_0^T (T-s)^{\alpha-1}\mathcal{T}(T-s)Bu(s)ds$$

has a bounded right inverse $W^{-1} : E \to L^\infty([0,T];U)$ (cf. [3]).

We will assume that the operator W^{-1} satisfies the following regularity condition:

(W) there exists a function $\gamma \in L_+^\infty([0,T])$ such that for each bounded set $\Omega \subset E$ we have:

$$\chi_U\left(W^{-1}(\Omega)(t)\right) \leq \gamma(t)\chi_E(\Omega) \text{ for a.e. } t \in [0,T],$$

where χ_U is the Hausdorff MNC in U.

Let M_1, M_2 be positive constants such that

$$\|B\| \leq M_1, \quad \left\|W^{-1}\right\| \leq M_2.$$

We will need the operator $S : L^\infty([0,T];E) \to C([0,T];E)$,

$$S(f)(t) = \int_0^t (t-s)^{\alpha-1}\mathcal{T}(t-s)f(s)ds \\ + \int_0^t (t-s)^{\alpha-1}\mathcal{T}(t-s)\left[BW^{-1}\left(x_1 - \mathcal{G}(T)\vartheta(0) - \int_0^T (T-\tau)^{\alpha-1}\mathcal{T}(T-\tau)f(\tau)d\tau\right)(s)\right]ds.$$

Let us represent the operator S in the form $S(f) = S_1(f) + S_2(f)$, where

$$S_1(f)(t) = \int_0^t (t-s)^{\alpha-1}\mathcal{T}(t-s)f(s)ds,$$

$$S_2(f)(t) = \int_0^t (t-s)^{\alpha-1}\mathcal{T}(t-s)\left[BW^{-1}\left(x_1 - \mathcal{G}(T)\vartheta(0) - \int_0^T (T-\tau)^{\alpha-1}\mathcal{T}(T-\tau)f(\tau)d\tau\right)(s)\right]ds.$$

Lemma 3.4. *(see [17]) The operator S_1 obeys the following properties:*

(S_1) *for $1/\alpha < p < \infty$, there exists a constant $C \geq 0$ such that*

$$\|S_1(\xi)(t) - S_1(\eta)(t)\|_E^p \leq C^p \int_0^t \|\xi(s) - \eta(s)\|_E^p\,ds, \qquad \xi,\eta \in L^p([0,T];E);$$

(S_2) *for every compact set $K \subset E$ and bounded sequence $\{\eta_n\} \subset L^\infty([0,T];E)$ such that $\{\eta_n(t)\} \subset K$ for a.e. $t \in [0,T]$, the weak convergence $\eta_n \rightharpoonup \eta_0$ in $L^1([0,T];E)$ implies the convergence $S_1(\eta_n) \to S_1(\eta_0)$ in $C([0,T];E)$.*

We need the following assertions which follows from [17].

Lemma 3.5. *. Let Δ be a bounded subset of $L^\infty([0,T];E)$ such that*

$$\chi_E(\Delta(t)) \leq \kappa(t) \quad \text{a.e. } t \in [0,T],$$

where $\kappa \in L^\infty[0,T]$. Then

$$\chi_E(\{S_1\Delta(t)\}) \leq \frac{2M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\kappa(s)\,ds.$$

Lemma 3.6. *. Let $\Omega \subset C([0,T];E)$ ia a nonempty bounded set and $\Omega(t)$ is relatively compact set in E for each $t \in [0,T]$. Then the set*

$$\left\{ S_1 \circ \mathcal{P}_F^{\infty}(\Omega)(t) = \int_0^t (t-s)^{\alpha-1} \mathcal{T}(t-s) f(s) ds : f \in \mathcal{P}_F^{\infty}(\Omega) \right\}$$

is equicontinuity.

Let us mention that the operator S_2 may be represented in the form:

$$S_2(f) = S_1(BW^{-1}(x_1 - \mathcal{G}(t)x_0 - \Pi S_1(f))), \tag{3.1}$$

where $\Pi : C([0,T];E) \to E, \quad \Pi x = x(T)$ is a bounded linear operator. Taking into account that W^{-1}, B, and S_1 are bounded linear operators, we conclude that the assertions of Lemmas 3.4, 3.5 and 3.6 are true for the operator S_2 and hence for the operator S.

Now, consider the multioperator $G\colon \mathcal{D} \to P(\mathcal{D})$ given as

$$G(x) = j + S \circ \mathcal{P}_F^1(x),$$

where $j(t) = \mathcal{G}(t)\vartheta(0),\ t \in [0,T]$.

It is clear that a function $x \in \mathcal{D}$ is a fixed point of the multioperator G if and only if the function $x[\vartheta] \in C([-h,T];E)$ is a mild solution of controllability problem (1.1), (1.2), (1.5). So, our problem is reduced to the sourcing of a fixed point of the operator G.

Theorem 3.7. *The multioperator G is u.s.c.*

Proof. From the representation (3.1) and properties of multivalued maps (see, e.g., [18]) it follows that it is sufficient to prove the assertion for the multimap $S_1 \circ \mathcal{P}_F^{\infty}$.

Let us show that the multimap $S_1 \circ \mathcal{P}_F^{\infty}$ is quasicompact. Take a nonempty compact set $A \subset \mathcal{D}$ and consider any sequence $\{\xi_n\} \subset S_1 \circ \mathcal{P}_F^{\infty}(A)$. We have $\xi_n = S_1(f_n)$, where $f_n \in \mathcal{P}_{x_n}^{\infty}$ for a certain sequence $\{x_n\} \subset A$. We assume, w.l.o.g., that $x_n \to x_0 \in A$. From condition $(F4)$ it follows that the sequence $\{f_n(t)\} \subset E$ is relatively compact for a.e. $t \in [0,T]$ and hence the sequence $\{f_n\}_{n=1}^{\infty}$ is L^1-semicompact. By the Diestel criterion of weak relative compactness (see [12]), we can assume that $f_{n_k} \overset{L^1}{\rightharpoonup} f_0$ for some subsequence $\{f_{n_k}\}$. By the known property of weak closedness of the superposition multioperator (see [18], Lemma 5.1.1) we get $f_0 \in \mathcal{P}_F^{\infty}(x_0)$. Now, applying Lemma 3.4 we obtain for the corresponding subsequence $\xi_{n_k} \to \xi_0 = S_1(f_0) \in S_1 \circ \mathcal{P}_F^{\infty}(x_0)$.

The closedness of the multioperator $S_1 \circ \mathcal{P}_F^{\infty}$ can be proved via similar arguments and then the assertion follows from Lemma 2.6. □

Let us consider conditions under which the operator G is condensing. Introduce in the space $C([0,T];E)$ the measure of noncompactness

$$\nu : Pb(C([0,T];E)) \to \mathbb{R}_+^2$$

with the values in the cone $\mathbb{R}_+^2$ defined as

$$\nu(\Omega) = (\varphi(\Omega), mod_C(\Omega)),$$

where $\varphi(\Omega)$ is the module of fiber noncompactness

$$\varphi(\Omega) = \sup_{t \in [0,T]} \chi_E(\{x(t) : x \in \Omega\})$$

and the second component is the equicontinuity module which is given as

$$mod_C(\Omega) = \lim_{\delta \to 0} \sup_{x \in \Omega} \max_{|t_1 - t_2| \le \delta} \|x(t_1) - x(t_2)\|.$$

It is known (see [18]) that the MNC ν is monotone, nonsingular, algebraically semiadditive, and regular.

Now, let us formulate the following conditions under which the operator G is condensing.

Theorem 3.8. *Under conditions (A), $(F1)$ - $(F4)$, $(H1)$, $(H2')$, $(H3)$, (W) and condition*

(C) $$c := \frac{4MT^{\alpha}}{\Gamma(1+\alpha)} \|\mu\|_{\infty} \left(1 + \frac{2MM_1T^{\alpha}}{\Gamma(1+\alpha)} \|\gamma\|_{\infty}\right) < 1$$

the operator G is ν-condensing.

Proof. Let $\Omega \subset \mathcal{D}$ be a nonempty bounded set, $\|\Omega\| \leq r_\Omega$, with $r_\Omega > \|\vartheta\|_{\mathcal{C}}$ and $y_x \in C([0,T];H)$ be a solution of problem (1.3) - (1.4) determined by a function $x \in \mathcal{D}$. Suppose

$$\nu(G(\Omega)) \geq \nu(\Omega). \tag{3.2}$$

We will show that Ω is a relatively compact set.

Since the MNC ν is nonsingular we have

$$\nu(S \circ \mathcal{P}_F^\infty(\Omega)) \geq \nu(\Omega). \tag{3.3}$$

From (3.3) it follows that

$$\varphi(S \circ \mathcal{P}_F^\infty(\Omega)) \geq \varphi(\Omega). \tag{3.4}$$

Applying regularity condition $(F4)$ we will have for $0 \leq s \leq T$ the following estimate:

$$\begin{aligned}\chi_E(\mathcal{P}_F^\infty(\Omega)(s)) =& \chi_E(\{f(s) : f \in \mathcal{P}_F^\infty(\Omega)\}) \\ \leq& \mu(s) \cdot \Big(\chi_E(\{x(s) : x \in \Omega\}) + \varphi(\{x[\vartheta]_s : x \in \Omega\})\Big) \\ =& \mu(s) \cdot \Big(\chi_E(\{x(s) : x \in \Omega\}) + \sup_{\tau \in [0,s]} \chi_E(\{x(\tau) : x \in \Omega\})\Big) \\ \leq& 2\mu(s)\varphi(\Omega).\end{aligned}$$

Then, by using Lemma 3.5 we get

$$\chi_E\left(S_1 \circ \mathcal{P}_F^\infty(\Omega)\right)(t) \leq \frac{4M}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \mu(s) ds \cdot \varphi(\Omega) \leq \frac{4MT^\alpha}{\Gamma(1+\alpha)} \|\mu\|_\infty \cdot \varphi(\Omega).$$

Further, we have the estimate

$$\begin{aligned}&\chi_E\left(\left\{BW^{-1}\left(x_1 - \mathcal{G}(T)\vartheta(0) - \int_0^T (T-\tau)^{\alpha-1}\mathcal{T}(T-\tau)f(\tau)d\tau\right)(s) : f \in \mathcal{P}_F^\infty(\Omega)\right\}\right) \\ \leq& M_1\gamma(s)\chi_E\left(\left\{\left(\int_0^T (T-\tau)^{\alpha-1}\mathcal{T}(T-\tau)f(\tau)d\tau\right)(s) : f \in \mathcal{P}_F^\infty(\Omega)\right\}\right) \\ \leq& M_1\,\gamma(s)\,\frac{4MT^\alpha}{\Gamma(1+\alpha)}\|\mu\|_\infty \cdot \varphi(\Omega) \\ \leq& \frac{4MM_1T^\alpha}{\Gamma(1+\alpha)}\|\mu\|_\infty\|\gamma\|_\infty \cdot \varphi(\Omega).\end{aligned}$$

By using this estimate, we obtain

$$\begin{aligned}&\chi_E\left(S_2 \circ \mathcal{P}_F^\infty(\Omega)\right)(t) \\ \leq& \frac{2M}{\Gamma(\alpha)} \times \int_0^t (t-s)^{\alpha-1}\chi_E\left(\left\{BW^{-1}\left(x_1 - \mathcal{G}(T)\vartheta(0) - \int_0^T (T-\tau)^{\alpha-1}\mathcal{T}(T-\tau)f(\tau)d\tau\right)(s) : f \in \mathcal{P}_F^\infty(\Omega)\right\}\right) ds \\ \leq& \frac{8M^2M_1T^{2\alpha}}{\Gamma^2(1+\alpha)}\|\mu\|_\infty\|\gamma\|_\infty \cdot \varphi(\Omega).\end{aligned}$$

Therefore we get

$$\begin{aligned}\chi_E\left(S \circ \mathcal{P}_F^\infty(\Omega(t))\right) \leq& \frac{4MT^\alpha}{\Gamma(1+\alpha)}\|\mu\|_\infty \cdot \varphi(\Omega) + \frac{8M^2M_1T^{2\alpha}}{\Gamma^2(1+\alpha)}\|\mu\|_\infty\|\gamma\|_\infty \cdot \varphi(\Omega) \\ =& \frac{4MT^\alpha}{\Gamma(1+\alpha)}\|\mu\|_\infty\left(1 + \frac{2MM_1T^\alpha}{\Gamma(1+\alpha)}\|\gamma\|_\infty\right) \cdot \varphi(\Omega) \\ =& k\varphi(\Omega)\end{aligned}$$

Hence

$$\varphi\left(S \circ \mathcal{P}_F^\infty(\Omega)\right) \leq c\varphi(\Omega), \tag{3.5}$$

with $c < 1$.

Comparing inequalities (3.4) and (3.5) we have

$$\varphi(\Omega) \leq c\varphi(\Omega),$$

since $c < 1$, then

$$\varphi(\Omega) = 0.$$

Now, from (3.3) we obtain

$$mod_C(S \circ \mathcal{P}_F^{infty}(\Omega)) \geq mod_C(\Omega). \tag{3.6}$$

At the same time, according to Lemma 3.6, we can assert that the set $S \circ \mathcal{P}_F^{\infty}(\Omega)$ is equicontinuous, therefore

$$mod_C(S \circ \mathcal{P}_F^{\infty}(\Omega)) = 0.$$

From inequality (3.6) it follows that $mod_C\,(\Omega) = 0$ that yields $\nu(\Omega) = (0,0)$ implying the relative compactness of Ω. The assertion is proved. □

Theorem 3.9. *Under assumptions* (A), $(F1)$, $(F2)$, $(F4)$, $(H1)$, $(H2')$, $(H3)$, $(g1) - (g3)$, (W), (C), *suppose that condition* $(F3)$ *has the following form:*

$(F3')$ *there exists a sequence of functions* $\{\omega_n\} \subset L_+^\infty[0,T]$ *such that*

$$\sup_{\|x\|_E \leq n, \|y\|_H \leq n, \|\zeta\|_{\mathcal{C}} \leq n} \|F(t,\zeta,x,y)\| \leq w_n(t) \text{ for a.e. } t \in [0,T], \ n = 1,2,...$$

and

$$\lim_{n\to\infty} \inf \frac{1}{n}\|\omega_n\|_\infty = 0.$$

Then controllability problem (1.1)-(1.5) *has a solution.*

Proof. We will show that there exists a closed ball $B_R \subset C([0,T];E)$ centered at the origin of a sufficiently large radius $R > \|\vartheta\|_{\mathcal{C}}$ such that $G(B_R^{\mathcal{D}}) \subset B_R^{\mathcal{D}}$, where $B_R^{\mathcal{D}} = B_R \cap \mathcal{D}$.

Towards this goal, notice that if $n > \|\vartheta\|_{\mathcal{C}}$ then obviously condition $x(\cdot) \in \mathcal{D}$, $\|x\|_C \leq n$ implies $\|x[\vartheta]_t\|_{\mathcal{C}} \leq n$ for all $t \in [0,T]$ and hence, for such n we get the following estimate: if $z_n \in G(x_n)$, $\|x_n\|_C \leq n$, then

$$\|z_n\|_C \leq M\|\vartheta(0)\|_E + \frac{MT^\alpha}{\Gamma(1+\alpha)}\|\omega_n\|_\infty + \frac{MM_1M_2T^\alpha}{\Gamma(1+\alpha)}\Big(\|x_1\|_E + M\|\vartheta(0)\|_E + \frac{MT^\alpha}{\Gamma(1+\alpha)}\|\omega_n\|_\infty\Big). \tag{3.7}$$

For convenience, rewrite this estimate as

$$\|z_n\|_C \leq M\|\vartheta(0)\|_E + C_1\|\omega_n\|_\infty + C_2\Big(\|x_1\|_E + M\|\vartheta(0)\|_E + C_1\|\omega_n\|_\infty\Big), \tag{3.8}$$

where

$$C_1 = \frac{MT^\alpha}{\Gamma(1+\alpha)}, \quad C_2 = \frac{MM_1M_2T^\alpha}{\Gamma(1+\alpha)}. \tag{3.9}$$

Now, supposing the contrary to our assertion, we will have sequences $\{x_n\}$, $\{z_n\} \subset \mathcal{D}$, such that $z_n \in G(x_n)$, $\|x_n\|_C \leq n$ but $\|z_n\|_C > n$ for all sufficiently large n.

But then, applying (3.7), for all such n we have

$$1 < \frac{\|z_n\|_C}{n} \leq \frac{1}{n}M\|\vartheta(0)\|_E + C_1\frac{\|\omega_n\|_\infty}{n} + C_2\Big(\frac{1}{n}\|x_1\|_E + \frac{1}{n}M\|\vartheta(0)\|_E + C_1\frac{\|\omega_n\|_\infty}{n}\Big),$$

giving the contradiction.

It remains to apply Theorem 2.7. □

4. Conclusion

In this paper we studied the controllability for a system governed by a fractional semilinear order functional differential inclusion with delay and a sweeping process in a Hilbert space. We defined the multivalued operator whose fixed points are generating solutions of the problem. By using the methods of fractional analysis and the fixed point theory for condensing multivalued maps we studied the properties of this multioperator, in particular, we demonstrate that under certain conditions it is condensing w.r.t. an appropriate measure of noncompactness. This allows to present the controllability principle as the main result of the present paper.

Statements and Declarations

The authors declare that they have no conflict of interest, and the manuscript has no associated data.

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