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ARCWISE CONNECTEDNESS OF SOLUTION SETS IN SET OPTIMIZATION VIA SET LESS ORDER

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Abstract. This paper investigates the arcwise connectedness of approximate solution sets for set optimization problems under the set less order relation via a scalarization approach. First, we establish the continuity of a novel nonlinear scalarization function constructed based on the set less criterion. Next, scalarization properties of the approximate solution sets are rigorously derived. Finally, we derive arcwise connectedness of weak approximate solutions for set optimization problems. These results enrich the exploration of scalarization and connectedness under set less order relations and provide a theoretical foundation for subsequent research.

Keywords. Set optimization, Arcwise connectedness, Scalarization, Set less relation.

1. Introduction

In recent years, set optimization problems have garnered increasing attention as a fascinating research area, primarily due to the prevalence of set valued mappings in numerous practical optimization scenarios. These problems hold profound implications across both pure and applied mathematics, spanning diverse fields such as uncertain optimization, mathematical economics, viability theory, image processing, variational inequalities and fuzzy optimization. Among others, for a more detailed exploration of set optimization problems, we refer the reader to [16, 19].

The vector method is primarily concerned with finding effective solutions to image sets. And the set technique is based on contrasting the image sets through the set order relations. To compare sets, numerous set order relations have been employed in existing literature [13, 15, 16, 18], including the Minkowski order relation, the upper set less order relation, the lower set less order relation, and the set less order relation. Among these, the set less relation is regarded as one of the most crucial due to its significant role in practical scenarios. Inspired by this research trend, the present study focuses on set optimization problems using the set less relation as the theoretical framework.

The scalarization method is a common approach in optimization problems because it typically transforms complex problems into simpler ones. There are generally two types of scalarization functions: linear scalarization functions and nonlinear scalarization functions. The Gerstewitz function [5] and the signed distance function [11] are two well-known nonlinear scalarization functions that are available in the literature. Meanwhile, as well as their many extensions, please refer to references [3, 4, 14, 17] for details.

As we all know, studying the properties of solution sets is one of the important issues in optimization. Like continuity, well-posedness and connectedness [20, 21, 22]. Among this, connectedness plays

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an important role, as it allows continuous movement from one solution to another. In recent years, many scholars have conducted research on the connectedness of solutions to set optimization problems. In 2019, Han [10] studied the path connectedness of the l-minimal solutions for SOP using the linear scalarization technique. Later in [23], Peng studied the connectedness and path-connectedness of solution sets for weak generalized symmetric Ky Fan inequality problems with respect to addition-invariant set. In [8], Han established the connectedness of the weak p-minimal solutions for SOP by means of the scalarization approach. The connectedness of solution sets for generalized vector equilibrium problems via free-disposal sets in complete metric space is discussed in [25] by Shao. In [9], Han studied the connectedness of the approximate solutions of the SOP using the generalized Gerstewitz function. However, there are no articles studying the arcwise connectedness of solution sets through set less order relations, as there is too little research on scalarization under set less order relations.

Recently, Anh in [1] introduced a scalar function which useful for comparison sets by the set less relation and applied to investigate optimality conditions and representations for solutions of set optimization problems. Naturally, it is necessary to explore the continuity of this function in [1] which is neccessary for studying the stability of set optimization problems via scalarization method. The second aim of this paper is to establish the arcwise connectedness of the approximate solution sets in set optimization problems by using the new scalarization function.

This paper is organized as follows. In Section 2, we recall some definitions and properties needed in what follows. In Section 3, we study the scalarization results for the solution sets. In Section 4, the arcwise connectedness of the solution sets is studied. Finally, a summary of the conclusions is presented in Section 5.

2. Preliminaries

Throughout this paper, unless otherwise specified, let X,Y be two normed vector spaces. Assume that $C \subset Y$ as $C \neq Y$ is a pointed, closed and convex cone with int$C \neq \emptyset$. We denote by int A, cl A the topological interior, the topological closure of A, respectively. Let $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$ and $\mathbb{R}^0_+ = \{x \in \mathbb{R} : x > 0\}$ We denote by B_Y the closed unit ball in Y. Let A and B be two nonempty subsets of Y.

For the above cone C, we have known the lower set less relation, upper set less relation and set less relation (see [13, 16, 18]) respectively by

$$A \preccurlyeq^l B \Leftrightarrow B \subseteq A + C,$$
$$A \preccurlyeq^u B \Leftrightarrow A \subseteq B - C,$$
$$A \preccurlyeq^s B \Leftrightarrow A \preccurlyeq^l B \text{ and } A \preccurlyeq^u B.$$

Let $e \in \operatorname{int} C$. For $\varepsilon \geq 0$, the ε-lower relation $\preccurlyeq^l_\varepsilon$ and the weak ε-lower relation $\prec^l_\varepsilon$ in [2] are defined by

$$B \subseteq A + C + \varepsilon e,$$
$$B \subseteq A + \mathrm{intC} + \varepsilon \mathrm{e}.$$

Motivated by the above definitions, let $e \in \operatorname{int} C$ and $\varepsilon \geq 0$, we give the next notion of ε-set less relation $\preccurlyeq^s_\varepsilon$ and the weak ε-set less relation $\prec^s_\varepsilon$:

$$A \preccurlyeq^s_\varepsilon B \text{ if and only if } B \subseteq A + C + \varepsilon e \text{ and } A \subseteq B - C - \varepsilon e,$$
$$A \prec^s_\varepsilon B \text{ if and only if } B \subseteq A + \mathrm{intC} + \varepsilon \mathrm{e} \text{ and } \mathrm{A} \subseteq \mathrm{B} - \mathrm{intC} - \varepsilon \mathrm{e}.$$

Let $P(Y)$ be the family of all nonempty subsets of Y. It is said that a set $A \in P(Y)$ is C-proper if $A + C \neq Y$, C-convex if $A + C$ is a convex set, C-bounded if for any neighbourhood U of $0 \in Y$ there exists $t > 0$ such that $A \subseteq tU + C$ and C-compact if any cover of A of the form $\{U_\alpha + C\}_{\alpha \in I}$, where U_α is open for any $\alpha \in I$, admits a finite subcover.

Let $F : X \rightrightarrows Y$ be a set-valued mapping. Let M be a nonempty subset of X. Let us consider the following set optimization problem (SOP):

$$\min\{F(x) : x \in M\}.$$

Remark 2.1. [12] Let A be a nonempty subset of Y and $a \in A$. We say that a is a minimal point of A with respect to C, denoted by $a \in \mathrm{Min}(A)$, if $(A - a) \cap (-C) = \{0\}$. It follows from Corollary 3.8 of [12] that if A is a compact and nonempty set, then $\mathrm{Min}(A) \neq \emptyset$.

Definition 2.2. For $\varepsilon \geq 0$, $\bar{x} \in M$ is said to be

(i) a C-s-minimal solution of (SOP) if $x \in M$ and $F(x) \preccurlyeq^s F(\bar{x})$ implies $F(\bar{x}) \preccurlyeq^s F(x)$.
(ii) a weak C-s-minimal solution of (SOP) if $x \in M$ and $F(x) \prec^s F(\bar{x})$ implies $F(\bar{x}) \prec^s F(x)$.
(iii) a C-s-minimal approximate solution of (SOP) if $x \in M$ and $F(x) \preccurlyeq^s_\varepsilon F(\bar{x})$ implies $F(\bar{x}) \preccurlyeq^s_\varepsilon F(x)$.
(iv) a weak C-s-minimal approximate solution of (SOP) if $x \in M$ and $F(x) \prec^s_\varepsilon F(\bar{x})$ implies $F(\bar{x}) \prec^s_\varepsilon F(x)$.

Let $E_s(F)$, $W_s(F)$, $E_s(\varepsilon, F)$ and $W_s(\varepsilon, F)$ denote the C-s-minimal solution of (SOP), the weak C-s-minimal solution of (SOP), the C-s-minimal approximate solution of (SOP) and the weak C-s-minimal approximate solution of (SOP).

Definition 2.3. [7] Let X and Y be two topological vector spaces and C be a cone of Y. A set-valued mapping $F : X \rightrightarrows Y$ is said to be

(i) upper semicontinuous (u.s.c.) at $x_0 \in X$, if for any open set $V \subseteq Y$ with $F(x_0) \subseteq V$, there exists a neighbourhood U of x_0 in X such that $F(x) \subseteq V$ for all $x \in U$.
(ii) lower semicontinuous (l.s.c.) at $x_0 \in X$, if for any open set $V \subseteq Y$ with $F(x_0) \cap V \neq \emptyset$, there exists a neighbourhood U of x_0 in X such that $F(x) \cap V \neq \emptyset$ for all $x \in U$.
(iii) C-upper semicontinuous (C-u.s.c.) at $x_0 \in X$, if for any open set $V \subseteq Y$ with $F(x_0) \subseteq V$, there exists a neighbourhood U of x_0 in X such that $F(x) \subseteq V + C$ for all $x \in U$.
(iv) C-lower semicontinuous (C-l.s.c.) at $x_0 \in X$, if for any $y \in F(x_0)$ and any neighbourhood V of 0_Y, there exists a neighbourhood $U(x_0)$ of x_0 such that

$$F(x) \cap (y + V - C) \neq \emptyset, \ \forall x \in U(x_0).$$

Definition 2.4. [24] Let X and Y be two topological vector spaces and C is a closed, convex and pointed cone with nonempty interior. A set-valued mapping $F : M \rightrightarrows Y$ is said to be

(i) outer semicontinuous (osc) at $x_0 \in X$, if

$$\limsup_{x \to x_0} F(x) \subseteq F(x_0).$$

(ii) inner semicontinuous (isc) at $x_0 \in X$, if

$$F(x_0) \subseteq \liminf_{x \to x_0} F(x).$$

(iii) C-outer semicontinuous (C-osc) at $x_0 \in X$, if $F + C$ is outer semicontinuous at x_0.
(iv) C-inner semicontinuous (C-isc) at $x_0 \in X$, if $F + C$ is inner semicontinuous at x_0.

Definition 2.5. [16] Let M be a nonempty convex subset of X, $F : M \rightrightarrows Y$ be a set-valued mapping, C is a closed, convex and pointed cone with nonempty interior. We say that F is C-convex on M if for each $x_1, x_2 \in M$ and $\lambda \in [0, 1]$, we have

$$F(\lambda x_1 + (1 - \lambda)x_2) \preccurlyeq^s \lambda F(x_1) + (1 - \lambda)F(x_2).$$

Lemma 2.6. *Assume that $x_0 \in M$ and $F(x_0)$ is C-proper and C-closed.*

(i) *For $\varepsilon > 0$, $x_0 \in E_s(\varepsilon, F)$ if and only if there is no $y \in M$ satisfying $F(y) \preccurlyeq^s_\varepsilon F(x_0)$.*
(ii) *The inclusion $x_0 \in W_s(\varepsilon, F)$ if and only if there is no $y \in M$ satisfying $F(y) \prec^s_\varepsilon F(x_0)$.*

Proof. (i) If there is no $x \in M$ satisfying $F(x) \preccurlyeq_{\varepsilon}^{s} F(x_0)$, then $x_0 \in W_s(\varepsilon, F)$ by definition. Now, we prove the "only if" part. Assume on the contrary that there exists $x \in M$ satisfying $F(x) \preccurlyeq_{\varepsilon}^{s} F(x_0)$. Then $F(x_0) \preccurlyeq_{\varepsilon}^{s} F(x)$ because $x_0 \in E_s(\varepsilon, F)$. Using the above relations we get

$$F(x_0) \subseteq F(x) + \mathrm{intC} + \varepsilon e \subseteq \mathrm{F(x_0) + intC + intC + \varepsilon e \subset F(x_0) + intC}. \tag{2.1}$$

This implies that

$$F(x_0) + C \subset F(x_0) + C + \mathrm{intC} = \mathrm{F(x_0) + intC}. \tag{2.2}$$

It is impossible as $F(x_0)$ is C-proper and C-closed.

The proof of (ii) is similar to the proof of (i) and so we omit it here. □

Definition 2.7. [1] For $e \in \mathrm{int}C$, we consider a function $\varphi_e : P(Y) \times P(Y) \to \mathbb{R} \cup \{\pm\infty\}$ defined by

$$\varphi_e(A, B) = \inf\{\lambda \in \mathbb{R} \mid A \preccurlyeq^s \lambda e + B\} \quad \forall A, B \in P(Y),$$

we define $\psi : M \times M \to \mathbb{R} \cup \{\pm\infty\}$ by

$$\psi(x, y) = \varphi_e(F(x), F(y)), \quad \forall (x, y) \in M \times M.$$

Lemma 2.8. [1] *Let A, B, D be given in $P(Y)$ and $\lambda \in \mathbb{R}$. The function φ_e has following properties:*

(i) $\varphi_e(A + \lambda e, B) = \varphi_e(A, B) + \lambda$.

(ii) *If $\varphi_e(A, B) < \lambda$, then $A \prec^s \lambda e + B$. The converse statement holds if A is $-C$-compact and B is C-compact.*

(iii) *If A is $-C$-compact, B is C-compact and D is C-bounded, then*

$$A \prec^s B \implies \varphi_e(A, D) < \varphi_e(B, D).$$

(iv) *If A is C-closed, B is $-C$-closed and $\varphi_e(A, B) \in \mathbb{R}$. Then*

$$A \preccurlyeq^s \varphi_e(A, B)e + B.$$

Lemma 2.9. *Let A, B be given in $P(Y)$ and $\lambda \in \mathbb{R}$. If A is C-closed, B is $-C$-closed and $\varphi_e(A, B) \in \mathbb{R}$, $\varphi_e(A, B) \leq \lambda$ if and only if $A \preccurlyeq^s \lambda e + B$.*

Proof. If $\varphi_e(A, B) < \lambda$, we have $A \prec^s \lambda e + B$ by Lemma 2.8(ii). Then $A \preccurlyeq^s \lambda e + B$ holds. When $\varphi_e(A, B) = \lambda$, we get $\varphi_e(A, B) = \lambda$ from Lemma 2.8(iv). Conversely, if $\varphi_e(A, B) \leq \lambda$, then A $\preccurlyeq^s \lambda e + B$ by definition. □

Lemma 2.10. [1] *Let A be nonempty C-convex and B be nonempty $-C$-convex. For $A_1, A_2, B_1, B_2 \in P(Y)$ and $t \in [0, 1]$, we have the following statements.*

(i) $\varphi_e((1 - t)A_1 + tA_2, B) \leq (1 - t)\varphi_e(A_1, B) + t\varphi_e(A_2, B)$.

(ii) $\varphi_e(A, (1 - t)B_1 + tB_2) \leq (1 - t)\varphi_e(A, B_1) + t\varphi_e(A, B_2)$.

Lemma 2.11. *If F is $\pm C$-outer semicontinuous and $\pm C$-inner semicontinuous on M with nonempty and $\pm C$-compact values, then ψ is continuous on $M \times M$.*

Proof. **Step 1: Lower semicontinuity of ψ**

To show ψ is lower semicontinuous, fix $r \in \mathbb{R}$ and define

$$L = \{(x, y) \in M \times M \mid \psi(x, y) \leq r\}.$$

We prove that L is closed. Let $\{(x_n, y_n)\} \subseteq L$ with $(x_n, y_n) \to (x_0, y_0)$. By Lemma 2.9,

$$F(x_n) \subseteq re + F(y_n) - C. \tag{2.3}$$

Take any $z_0 \in F(x_0)$. By the $-C$-inner semicontinuity of F at x_0,

$$z_0 \in \liminf_{x \to x_0}(F(x) - C).$$

Thus, for $x_n \to x_0$, there exist $z_n \in F(x_n) - C$ such that $z_n \to z_0$. For sufficiently large n and any $\varepsilon > 0$,

$$z_0 \in z_n + \varepsilon e \subseteq F(x_n) - C + \varepsilon e. \tag{2.4}$$

Combining (2.3) and (2.4), for large n,

$$z_0 \in F(y_n) + (r + \varepsilon)e - C.$$

Hence,

$$z_0 - (r + \varepsilon)e \in \limsup_{y_n \to y_0} (F(y_n) - C).$$

By the $-C$-outer semicontinuity of F at y_0,

$$z_0 - (r + \varepsilon)e \in F(y_0) - C.$$

Since $F(y_0) - C$ is closed, letting $\varepsilon \to 0$, we get $z_0 \in F(y_0) - C + re$. Then

$$F(x_0) \subseteq F(y_0) - C + re,$$

and similarly $F(y_0) + re \subseteq F(x_0) + C$. By Lemma 2.9, $(x_0, y_0) \in L$, proving L is closed.

Step 2: Upper semicontinuity of ψ
Define for $r \in \mathbb{R}$,

$$L' = \{(x, y) \in M \times M \mid \psi(x, y) \geq r\}.$$

Suppose $\{(x_n, y_n)\} \subseteq L'$ with $(x_n, y_n) \to (x_0, y_0)$. Assume $(x_0, y_0) \notin L'$. Then,

$$F(x_0) \subseteq re + F(y_0) - \operatorname{int} C. \tag{2.5}$$

By the $-C$-outer semicontinuity of F at x_0, we have

$$\limsup_{x_n \to x_0} (F(x_n) - C) \subseteq F(x_0) - C. \tag{2.6}$$

By the $-C$-inner semicontinuity of F at y_0, we have

$$F(y_0) - C \subseteq \liminf_{y_n \to y_0} (F(y_n) - C). \tag{2.7}$$

It follows from (2.6) and (2.7), for sufficiently large n, we have

$$F(x_n) - C \subseteq F(x_0) - C + \varepsilon B_Y,$$
$$F(y_0) - C - \varepsilon B_Y \subseteq F(y_n) - C,$$

where B_Y is the closed unit ball in Y. Combining this with (2.5), we obtain

$$F(x_n) \subseteq re + F(y_n) - \operatorname{int} C,$$

implying $\psi(x_n, y_n) < r$. This contradicts $\{(x_n, y_n)\} \subseteq L'$. Hence, $(x_0, y_0) \in L'$, proving L' is closed. This completes the proof. □

Lemma 2.12. [6] *Let X be a paracompact Hausdorff arcwise connected space and let Y be a Banach space. Assume that*

(i) *$F : X \rightrightarrows Y$ is a lower semicontinuous set-valued mapping.*
(ii) *for each $x \in X$, $F(x)$ is nonempty, closed and convex.*

Then, $F(X)$ is an arcwise connected set.

3. Scalarization Results for The Solution Sets

In this section, we derive some scalarization results for the sets of weak s-minimal approximate solutions for set optimization problem.

We define the set-valued mapping $\Gamma : M \to P(M)$ as follows:

$$\Gamma(\varepsilon, x) = \{u \in M : \psi(y,x) + \varepsilon \geq \psi(u,x),\ \forall y \in M\}, \quad \forall(\varepsilon, x) \in \mathbb{R}_+ \times M$$
$$= \{u \in M : \varphi_e(F(y), F(x)) + \varepsilon \geq \varphi_e(F(u), F(x)),\ \forall y \in M\}$$

Theorem 3.1. *Let $\varepsilon \geq 0$. Assume that $F(x)$ is C-compact for any $x \in M$. Then*

$$W_s(\varepsilon, F) = \bigcup_{x\in M} \Gamma(\varepsilon, x).$$

Proof. Let $x^* \in W_s(\varepsilon, F)$. By Lemma 2.6, there exists no $x_0 \in M$ such that

$$F(x_0) \prec^s_\varepsilon F(x^*),$$

or equivalently,

$$F(x_0) + \varepsilon e \prec^s F(x^*).$$

Applying Lemma 2.8, we have

$$\varphi_e\big(F(x_0) + \varepsilon e, F(x^*)\big) < \varphi_e\big(F(x^*), F(x^*)\big),$$

this implies that there is no $x_0 \in M$ such that

$$\varphi_e\big(F(x_0), F(x^*)\big) + \varepsilon < \varphi_e\big(F(x^*), F(x^*)\big).$$

Negating this inequality, we obtain for all $x \in M$,

$$\varphi_e\big(F(x), F(x^*)\big) + \varepsilon \geq \varphi_e\big(F(x^*), F(x^*)\big).$$

Thus, $x^* \in \Gamma(\varepsilon, x^*) \subseteq \bigcup_{x\in M} \Gamma(\varepsilon, x)$.

Conversely, Let $\bar{v} \in \bigcup_{x\in M} \Gamma(\varepsilon, x)$. Then there exists $\bar{x} \in M$ such that $\bar{v} \in \Gamma(\varepsilon, \bar{x})$, i.e.,

$$\varphi_e(F(y), F(\bar{x})) + \varepsilon \geq \varphi_e(F(\bar{v}), F(\bar{x})), \forall y \in M. \tag{3.1}$$

Suppose for contradiction that $\bar{v} \notin W_s(\varepsilon, F)$. By Lemma 2.6, there exists $\bar{y} \in M$ such that

$$F(\bar{y}) \prec^s_\varepsilon F(\bar{v}), \quad \text{or equivalently,} \quad F(\bar{y}) + \varepsilon e \prec^s F(\bar{v}).$$

Together with 2.8, this implies

$$\varphi_e(F(\bar{y}), F(\bar{x})) + \varepsilon < \varphi_e(F(\bar{v}), F(\bar{x})),$$

which directly contradicts (3.1). Hence, $\bar{v} \in W_s(\varepsilon, F)$. The proof is completed. □

Theorem 3.2. *Let M is nonempty and compact and $x \in M$. Assume that F is $\pm C$-outer semicontinuous and $\pm C$-inner semicontinuous on M with nonempty and C-compact values. Then $\Gamma(\varepsilon, x) \neq \emptyset$ for any $\varepsilon \geq 0$.*

Proof. It follows from Lemma 2.11 that $\psi(\cdot, x)$ is continuous on M. Since M is nonempty and compact, it is easy to see that $\Gamma(0, x) \neq \emptyset$ by Weierstrass theorem. For any $\varepsilon \geq 0$, due to $\Gamma(0, x) \subseteq \Gamma(\varepsilon, x)$, we have $\Gamma(\varepsilon, x) \neq \emptyset$. □

Theorem 3.3. *Assume that F is C-convex on M and $\Gamma(0, x) \neq \emptyset$. Then $\Gamma(\cdot, x)$ is l.s.c. on $\mathbb{R}^0_+$.*

Proof. Suppose to the contrary that there exists $\varepsilon_0 \in \mathbb{R}_+^0$ such that $\Gamma(\cdot, x)$ is not l.s.c. at ε_0. Then there exist $v_0 \in \Gamma(\varepsilon_0, x)$, a neighbourhood W_0 of $0 \in X$ and a sequence $\{\varepsilon_n\} \subseteq \mathbb{R}_+$ with $\varepsilon_n \to \varepsilon_0$ such that

$$(v_0 + W_0) \cap \Gamma(\varepsilon_n, x) = \emptyset, \quad \forall n \in \mathbb{N}. \tag{3.2}$$

Suppose that $\varepsilon_0 \leq \varepsilon_n$. Then $v_0 \in \Gamma(\varepsilon_0, x) \subseteq \Gamma(\varepsilon_n, x)$, which contradicts (3.2). Hence, we get that $\varepsilon_0 > \varepsilon_n$ for any $n \in \mathbb{N}$. Due to $\Gamma(0, x) \neq \emptyset$, let $v' \in \Gamma(0, x)$, and so

$$\varphi_e(F(y), F(x)) \geq \varphi_e(F(v'), F(x)), \quad \forall y \in M. \tag{3.3}$$

It follows from $v_0 \in \Gamma(\varepsilon_0, x)$ that

$$\varphi_e(F(y), F(x)) + \varepsilon_0 \geq \varphi_e(F(v_0), F(x)), \quad \forall y \in M. \tag{3.4}$$

Since $\varepsilon_n \to \varepsilon_0$, there exists $n_0 \in \mathbb{N}$ such that

$$v_\varepsilon := \frac{\varepsilon_{n_0}}{\varepsilon_0} v_0 + \frac{\varepsilon_0 - \varepsilon_{n_0}}{\varepsilon_0} v' = v_0 + \frac{\varepsilon_0 - \varepsilon_{n_0}}{\varepsilon_0}(v' - v_0) \in v_0 + W_0. \tag{3.5}$$

We conclude from Lemma 2.10 that

$$\frac{\varepsilon_0 - \varepsilon_{n_0}}{\varepsilon_0} \varphi_e(F(v'), F(x)) + \frac{\varepsilon_{n_0}}{\varepsilon_0} \varphi_e(F(v_0), F(x)) \geq \varphi_e(F(v_\varepsilon), F(x)). \tag{3.6}$$

Thanks to (3.3), (3.4) and (3.6), we have

$$\varphi_e(F(y), F(x)) + \varepsilon_{n_0} \geq \varphi_e(F(v_\varepsilon), F(x)), \quad \forall y \in M,$$

which yields $v_\varepsilon \in \Gamma(\varepsilon_{n_0}, x)$. This together with (3.5) implies that

$$v_\varepsilon \in (v_0 + W_0) \cap \Gamma(\varepsilon_{n_0}, x),$$

which contradicts (3.2). This completes the proof. □

Theorem 3.4. *Assume that C is nonempty and convex, F is C-convex on M. Then $\Gamma(\varepsilon, x)$ is convex for any $(\varepsilon, x) \in \mathbb{R}_+ \times M$.*

Proof. Let $t \in [0, 1]$ and $u_1, u_2 \in \Gamma(\varepsilon, x)$. Then, for any $y \in M$, we have

$$t(\psi(y, x) + \varepsilon) \geq t\psi(u_1, x), \tag{3.7}$$

$$(1 - t)(\psi(y, x) + \varepsilon) \geq (1 - t)\psi(u_2, x). \tag{3.8}$$

Combining (3.7) and (3.8) , we get that

$$\psi(y, x) + \varepsilon \geq t\psi(u_1, x) + (1 - t)\psi(u_2, x).$$

By Lemma 2.10, we have

$$t\psi(u_1, x) + (1 - t)\psi(u_2, x) \geq \psi(tu_1 + (1 - t)u_2, x).$$

Thus,

$$\psi(y, x) + \varepsilon \geq \psi(tu_1 + (1 - t)u_2, x),$$

which implies $tu_1 + (1 - t)u_2 \in \Gamma(\varepsilon, x)$. Therefore, $\Gamma(\varepsilon, x)$ is convex. □

4. Arcwise Connectedness of The Solution Sets

Theorem 4.1. *Let $\varepsilon_0 > 0$, X be a Banach space and M be a nonempty, convex and compact subset of X. Assume that F is $\pm C$-outer semicontinuous, $\pm C$-inner semicontinuous and C-convex on M with nonempty $\pm C$-compact values and $x \in M$. Then*

(i) $\Gamma(\varepsilon_0, x)$ *is closed.*
(ii) $\Gamma(\varepsilon_0, \cdot)$ *is l.s.c. on M.*

Proof. (i) Let any $\{v_n\} \subseteq \Gamma(\varepsilon_0, x)$ with $v_n \to v_0$, it suffices to prove that $v_0 \in \Gamma(\varepsilon_0, x)$. It is clear that $v_0 \in M$. By $\{v_n\} \subseteq \Gamma(\varepsilon_0, x)$, one has

$$\psi(y, x) + \varepsilon_0 \geq \psi(v_n, x), \quad \forall y \in M. \tag{4.1}$$

It follows from Lemma 2.11 that ψ is continuous on $M \times M$. Combining this with (4.1), we get

$$\psi(y, x) + \varepsilon_0 \geq \psi(v_0, x), \quad \forall y \in M,$$

which implies $v_0 \in \Gamma(\varepsilon_0, x)$.

(ii) Suppose to the contrary that there exists $x_0 \in M$ such that $\Gamma(\varepsilon_0, \cdot)$ is not l.s.c. at x_0. Then there exist $v_0 \in \Gamma(\varepsilon_0, x_0)$, a neighbourhood W_0 of $0 \in X$ and a sequence $\{x_n\} \subseteq M$ with $x_n \to x_0$ such that

$$(v_0 + W_0) \cap \Gamma(\varepsilon_0, x_n) = \emptyset, \quad \forall n \in \mathbb{N}. \tag{4.2}$$

In view of Theorem 3.2, we have $\Gamma(0, x_0) \neq \emptyset$. It follows from Theorem 3.3 that $\Gamma(\cdot, x_0)$ is l.s.c. at ε_0. For the above $v_0 \in \Gamma(\varepsilon_0, x_0)$ and the neighbourhood W_0 of $0 \in X$, there exists a neighbourhood $U(\varepsilon_0)$ of ε_0 such that

$$(v_0 + W_0) \cap \Gamma(\varepsilon, x_0) \neq \emptyset, \quad \forall \varepsilon \in U(\varepsilon_0).$$

Choose $\varepsilon' \in U(\varepsilon_0)$ with $0 < \varepsilon' < \varepsilon_0$. Then $(v_0 + W_0) \cap \Gamma(\varepsilon', x_0) \neq \emptyset$. Let

$$v' \in (v_0 + W_0) \cap \Gamma(\varepsilon', x_0). \tag{4.3}$$

Due to (4.3), we have

$$\varphi_e(F(y), F(x_0)) + \varepsilon' \geq \varphi_e(F(v'), F(x_0)), \quad \forall y \in M. \tag{4.4}$$

It is claimed that there exists $n_0 \in \mathbb{N}$ such that

$$v' \in \Gamma(\varepsilon_0, x_n), \quad \forall n \geq n_0. \tag{4.5}$$

Indeed, if not, then for any $n \in \mathbb{N}$, there exists $n_k \geq n$ such that $v' \notin \Gamma(\varepsilon_0, x_{n_k})$. Without loss of generality, we assume that $v' \notin \Gamma(\varepsilon_0, x_n)$ for any $n \in \mathbb{N}$. Then there is $y_n \in M$ such that

$$\varphi_e(F(y_n), F(x_n)) + \varepsilon_0 < \varphi_e(F(v'), F(x_n)). \tag{4.6}$$

Since M is compact, without loss of generality, we assume that $y_n \to y_0 \in M$. It follows from Lemma 2.11 that ψ is continuous on $M \times M$. Combining this with (4.6), we get

$$\varphi_e(F(y_0), F(x_0)) + \varepsilon' < \varphi_e(F(y_0), F(x_0)) + \varepsilon_0 \leq \varphi_e(F(v'), F(x_0)),$$

which contradicts (4.4). Therefore, (4.5) holds. This together with (4.3) implies that

$$v' \in (v_0 + W_0) \cap \Gamma(\varepsilon_0, x_n), \quad \forall n \geq n_0,$$

which contradicts (4.2). □

Theorem 4.2. *Let $\varepsilon_0 > 0$, X be a Banach space and M be a nonempty, convex and compact subset of X. Assume that F is $\pm C$-outer semicontinuous, $\pm C$-inner semicontinuous and C-convex on M with nonempty $\pm C$-compact values and $x \in M$. Then $W_s(\varepsilon_0, F)$ is arcwise connected.*

Proof. By Theorem 4.1(i), Theorem 3.2 and 3.4, we can know that $\Gamma(\varepsilon_0, x)$ is nonempty, convex and closed for any $x \in M$. It follows from Theorem 3.2(ii) that $\Gamma(\varepsilon_0, \cdot)$ is l.s.c. on M. By Theorem 3.1, we have

$$\bigcup_{x \in M} \Gamma(\varepsilon_0, x) = W_s(\varepsilon_0, F).$$

Therefore, we conclude from Lemma 2.12 that $W_s(\varepsilon_0, F)$ is arcwise connected. This completes the proof. □

Example 4.3. Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $M = [-5, 5]$ and $C = \mathbb{R}^2_+$. Let $e_0 = (1, 1) \in \text{int}\, C$ and $\varepsilon_0 = 1$. The set-valued mapping $F : X \rightrightarrows Y$ is defined as follows:

$$F(x) = (4x^2, 2x^2 - 8x + 10) + B_Y + C, \quad x \in X.$$

$M = [-5, 5]$ is a closed interval in $\mathbb{R}$, and hence M is nonempty, convex, and compact. Meanwhile, it is easy to see that F is $\pm C$-outer semicontinuous and $\pm C$-inner semicontinuous. B_Y is compact, and C is closed. Then $B_Y + C$ is closed and bounded in Y, and so F is C-compact. Since $f(x) = 4x^2$ and $g(x) = 2x^2 - 8x + 10$ are convex functions, Thus, $F(x)$ is C-convex.

All conditions of Theorem 4.2 are satisfied. Therefore, $W_s(\varepsilon_0, F)$ is arcwise connected. For instance, $x = 2$ and $x = 3$ belong to $W_s(\varepsilon_0, F)$. This together with the arcwise connectedness of $W_s(\varepsilon_0, F)$ implies that $[2, 3] \subseteq W_s(\varepsilon_0, F)$.

5. Conclusion

In this paper, we first obtained the continuity of the new scalarization function. Subsequently, we studied the properties of solutions using the scalarization function, such as convexity and lower semicontinuity. Finally, we established the arcwise connectedness of the weak approximate solutions and provided numerical examples to verify the validity of the theorem. In the future, we will use this scalarizing function to explore other stability aspects of the solution sets in set optimization, such as well-posedness and Painlevé-Kuratowski convergence.

Statements and Declarations

The authors declare that they have no conflict of interest, and the manuscript has no associated data.

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