APPLICABLE NONLINEAR ANALYSIS
Volume 2 (2025), No. 2, 91–96
https://doi.org/10.69829/apna-025-0202-ta03
Tulipa Opera Scholarum

ATTAINMENT OF THE MINIMAL DISPLACEMENT FOR NON-EXPANSIVE OPERATORS IN HILBERT SPACES

BIAGIO RICCERI[1,*]

[1] *Department of Mathematics and Informatics, University of Catania, Viale A. Doria 95125 Catania, Italy*

Abstract. Let H be a real Hilbert space and let $T : H \to H$ be a sequentially weakly continuous operator which is non-expansive with respect to some norm on H equivalent to $\|\cdot\|_H$. In this paper, we provide a sufficient condition ensuring the existence of a point $\tilde{x} \in H$ such that

$$\|\tilde{x} - T(\tilde{x})\|_H = \inf_{x\in H} \|x - T(x)\|_H.$$

Keywords. Minimal displacement, Non-expansive operator, Hilbert space, Minimax theorem.

1. Introduction and Statement of The Main Result

Let $(X, \|\cdot\|)$ be a real Banach space and C a (non-empty) subset of X. An operator $T : C \to X$ is said to be non-expansive if

$$\|T(x) - T(u)\| \le \|x - u\|$$

for all $x, u \in C$.

The intensive study of such operators is by now 60 years old. Indeed, in 1965 [3, 6, 7], it was proved that if X is a Hilbert space and C is closed, convex and bounded, then any non-expansive operator $T : C \to C$ has a fixed point, that is a point $x \in C$ such that $x = T(x)$. Notice that such a result is meaningful out of the two following situations: X is norm compact (in this case T, being continuous, would have a fixed point by Schauder theorem); T is weakly continuous (in this case, since X is weakly compact, T would have a fixed point by Tychonoff theorem).

Of course, non-expansiveness depends on the particular considered norm. So, the following serious problem was formulated: if $(X, \|\cdot\|)$ is a Hilbert space ($\|\cdot\|$ is the norm induced by the scalar product) and if $\|\cdot\|_1$ is any norm equivalent to $\|\cdot\|$, is it true that, for every closed, convex, bounded set $C \subset X$, any operator $T : C \to C$, non-expansive with respect to $\|\cdot\|_1$, has a fixed point ? Only very partial answers are known up to now: we refer to the recent paper [9] and to references therein.

Clearly, in the previous result, the boundedness of C is essential, as it is shown by the classical example $C = X$, $T(x) = x + w$, where $w \in X \setminus \{0\}$. Hence, when C is unbounded, to get the existence of fixed points for the non-expansive operator $T : C \to C$, more assumptions on T have necessarily to be required. In this connection, we refer to the very recent paper [4].

The number $\inf_{x\in C} \|x - T(x)\|$ is called the minimal displacement of T. Since [5], it has been considered in many papers dealing with the fixed point theory of non-expansive functions (see [1, 2, 8, 10], to mention a few).

The aim of this very short note is to give a contribution on the attainment of this number when $C = X$. Our result is as follows:

*Corresponding author.
E-mail address: biagio.ricceri@unict.it
2020 Mathematics Subject Classification: 47H09, 47H10.
Accepted: April 21, 2025.

Theorem 1.1. *Let $(X, \|\cdot\|_X)$ be a real Banach space, let H be a real Hilbert space, let $\Phi : X \to H$ be a linear homeomorphism onto H and let $\Psi : X \to H$ be a sequentially weakly continuous operator such that*

$$\{x \in X : \langle \Phi(x), \Psi(x)\rangle = c^2\} \subseteq \{x \in X : \|\Psi(x)\|_H \leq c\}$$

for some $c > \frac{\|\Psi(0)\|_H}{\sqrt{3}}$. Moreover, assume that there are a norm $\|\cdot\|_1$ on X, equivalent to $\|\cdot\|_X$, and a norm $\|\cdot\|_2$ on Y, equivalent to $\|\cdot\|_H$, such that

$$\|\Psi(x) - \Psi(u)\|_2 \leq \frac{1}{\alpha_\Phi}\|x - u\|_1$$

for all $x, u \in X$, where

$$\alpha_\Phi = \sup_{\|y\|_2 \leq 1} \|\Phi^{-1}(y)\|_1.$$

Then, there exists $\tilde{x} \in X$ such that

$$\|\Phi(\tilde{x}) - \Psi(\tilde{x})\|_H = \inf_{x \in X} \|\Phi(x) - \Psi(x)\|_H.$$

Of course, applying Theorem 1.1 with $X = H$, $\Phi = \mathrm{id}$ and $\|\cdot\|_1 = \|\cdot\|_2$, we get the following

Corollary 1.2. *Let H be a real Hilbert space and let $\Psi : H \to H$ be a sequentially weakly continuous operator. Assume that, for some $c > \frac{\|\Psi(0)\|_H}{\sqrt{3}}$, the following inclusion*

$$\{x \in H : \langle x, \Psi(x)\rangle = c^2\} \subseteq \{x \in H : \|\Psi(x)\|_H \leq c\}$$

holds. Moreover, assume that there exists an equivalent norm on H with respect to which Ψ is non-expansive.

Then, there exists $\tilde{x} \in H$ such that

$$\|\tilde{x} - \Psi(\tilde{x})\|_H = \inf_{x \in H} \|x - \Psi(x)\|_H.$$

Our proof of Theorem 1.1 is fully based on the use of a minimax result proved in [13].

The crucial assumption of Theorem 1.1 (that is the required inclusion) just comes from this particular approach. At present, we cannot indicate some intermediate condition ensuring that inclusion, but supposing that $\sup_{x \in H}\langle \Phi(x), \Psi(x)\rangle < +\infty$.

In particular, the present note aims to stimulate further studies in this direction.

2. Abstract Results and Proof of Theorem 1.1

We start proving the following

Theorem 2.1. *Let X be a topological space, let $I \subseteq \mathbf{R}$ be an interval, let $\{I_n\}$ be a non-decreasing sequence of compact intervals such that $I = \cup_{n \in \mathbf{N}} I_n$, and let $f : X \times I \to \mathbf{R}$ be a an upper semicontinous function such that $f(\cdot, \lambda)$ is continuous for all $\lambda \in I$. Moreover, assume that:*
(a_1) there exists a set $D \subset I$, dense in I, such that $f(\cdot, \lambda)$ is inf-connected for all $\lambda \in D$;
(a_2) for each $x \in X$ and each $n \in \mathbf{N}$, the set of all global maxima of the restriction to I_n of the function $f(x, \cdot)$ is connected;
(a_3) there exists (a possibly different) topology τ_1 one X such that the function $f(\cdot, \lambda)$ is τ_1 sequentially lower semicontinuous for all $\lambda \in I$ and sequentially τ_1 inf-compact for some $\lambda_0 \in I$;
(a_4) for some $x_0 \in X$, the function $f(x_0, \cdot)$ is sup-compact.

Then, there exist $\tilde{x} \in X$ and $\tilde{\lambda} \in I$ such that

$$f(\tilde{x}, \tilde{\lambda}) = \inf_{x \in X} f(x, \tilde{\lambda}) = \sup_{\lambda \in I} f(\tilde{x}, \lambda). \tag{2.1}$$

Proof. Fix $n \in \mathbf{N}$. Thanks to (a_1) and (a_2), we can apply Theorem 1.2 of [13]. Accordingly, we have

$$\sup_{I_n} \inf_X f = \inf_X \sup_{I_n} f.$$

Now, (a_3) allows us to apply Proposition 2.2 of [12], and so we obtain

$$\sup_{I} \inf_X f = \inf_X \sup_{I} f. \tag{2.2}$$

Consider the functions $\alpha : X \to]-\infty, +\infty]$ and $\beta : I \to [-\infty, +\infty[$ defined by

$$\alpha(x) = \sup_{\lambda \in I} f(x, \lambda)$$

for all $x \in X$,

$$\beta(\lambda) = \inf_{x \in X} f(x, \lambda)$$

for all $\lambda \in I$. Of course, by (a_3), α is τ_1 sequentially lower semicontinuous and τ_1 sequentially inf-compact, while, by (a_4), β is upper semicontinuous and sup-compact. Consequently, there exist $\tilde{x} \in X$ and $\tilde{\lambda} \in I$ such that

$$\alpha(\tilde{x}) = \inf_X \alpha \tag{2.3}$$

$$\beta(\tilde{\lambda}) = \sup_I \beta. \tag{2.4}$$

Now, (2.1) follows from (2.2), (2.3) and (2.4). □

From Theorem 2.1, we get

Theorem 2.2. *Let X, Y be two topological spaces, let $\varphi : Y \to \mathbf{R}$ be a continuous inf-connected function, let $I \subseteq \mathbf{R}$ be an interval, let $\{I_n\}$ be a non-decreasing sequence of compact intervals such that $I = \cup_{n \in \mathbf{N}} I_n$, and let $\gamma : I \to \mathbf{R}$, $g : X \times I \to Y$ be two continuous functions. Assume that:*
(b_1) there exists a set $D \subseteq I$, dense in I, such that, for each $\lambda \in I$, the function $g(\cdot, \lambda)$ is surjective and open, and there exists a set A_λ, dense in Y, such that the set $\{x \in X : g(x, \lambda) = y\}$ is connected for each $y \in A_\lambda$;
(b_2) for each $x \in X$ and each $n \in \mathbf{N}$, the set of all global maxima of the restriction to I_n of the function $\varphi(g(x, \cdot)) + \gamma(\cdot)$ is connected;
(b_3) there exists (a possibly different) topology τ_1 one X such that the function $\varphi(g(\cdot, \lambda))$ is τ_1 sequentially lower semicontinuous for all $\lambda \in I$ and sequentially τ_1 inf-compact for some $\lambda_0 \in I$;
(b_4) for some $x_0 \in X$, the function $\varphi(g(x_0, \cdot)) + \gamma(\cdot)$ is sup-compact.

Then, there exist $\tilde{x} \in X$ and $\tilde{\lambda} \in I$ such that

$$\varphi(g(\tilde{x}, \tilde{\lambda})) = \inf_{x \in X} \varphi(g(x, \tilde{\lambda})) = \sup_{\lambda \in I} (\varphi(g(\tilde{x}, \lambda)) + \gamma(\lambda)) - \gamma(\tilde{\lambda}).$$

Proof. Consider the continuous function $f : X \times I \to \mathbf{R}$ defined by

$$f(x, \lambda) = \varphi(g(x, \lambda)) + \gamma(\lambda)$$

for all $(x, \lambda) \in X \times I$. Fix $\lambda \in D$ and $r \in \mathbf{R}$ so that $f(x, \lambda) < r$ for some $x \in X$. We have

$$\{x \in X : f(x, \lambda) < r\} = g^{-1}(\varphi^{-1}(]-\infty, r - \gamma(\lambda)[), \lambda).$$

Notice that the set $\varphi^{-1}(]-\infty, r - \gamma(\lambda)[)$ is non-empty, open and connected, since φ is continuous and inf-connected. Since $\lambda \in D$, the multifunction $y \to g^{-1}(y, \lambda)$ is non-empty valued and lower semicontinuous in Y. Notice that the set $A_\lambda \cap \varphi^{-1}(]-\infty, r - \gamma(\lambda)[)$ is dense in $\varphi^{-1}(]-\infty, r - \gamma(\lambda)[)$. Consequently, by Proposition 5.6 of [11], the set $g^{-1}(\varphi^{-1}(]-\infty, r - \gamma(\lambda)[), \lambda)$ is connected. In other words, we have shown that the function $f(\cdot, \lambda)$ is inf-connected. So, f satisfies each assumption of Theorem 2.1, and the conclusion follows. □

Theorem 2.3. *Let X be a topological space, let Y be a set in a real inner product space H, with $0 \in Y$, such that its intersection with any open ball, centered at 0, is connected. Let $\Phi, \Psi : X \to H$ be two continuous operators satisfying the following conditions:*
(c_1) *for each $\lambda \in]-1,1[$, $(\Phi - \lambda\Psi)(X) = Y$, the function $\Phi - \lambda\Psi$ is open with respect to relative (norm) topology of Y, and there exists a set $A_\lambda \subseteq Y$, dense in Y, such that the set $(\Phi - \lambda\Psi)^{-1}(y)$ is connected for each $y \in A_\lambda$;*
(c_2) *there exists (a possibly different) topology τ_1 on X such that the function $\|\Phi(\cdot) - \lambda\Psi(\cdot)\|_H$ is τ_1 sequentially lower semicontinuous for all $\lambda \in [-1,1]$ and sequentially τ_1 inf-compact for some $\lambda_0 \in [-1,1]$;*
(c_3) *there exists $c > \frac{1}{\sqrt{3}} \inf_{x \in X} \|\Phi(x) + \Psi(x)\|$ such that*

$$\{x \in X : \langle \Phi(x), \Psi(x) \rangle = c^2\} \subseteq \{x \in X : \|\Psi(x)\|_H \le c\}.$$

Then, there exists $\tilde{x} \in X$ such that

$$\|\Phi(\tilde{x}) - \Psi(\tilde{x})\|_H = \inf_{x \in X} \|\Phi(x) - \Psi(x)\|_H.$$

Proof. We are going to apply Theorem 2.2 considering Y with the relative norm topology. We then take:

$$I = I_n = [-1,1],$$
$$\varphi(y) = \|y\|_H^2,$$
$$g(x,\lambda) = \Phi(x) - \lambda\Psi(x),$$
$$\gamma(\lambda) = c^2(2\lambda - \lambda^2).$$

Of course, (c_1) implies directly (b_1) (taking $D =]-1,1[$); (c_2) implies directly (b_3); (b_4) holds since I is compact. Finally, let us show that (b_2) holds. For all $x \in X$, $\lambda \in [-1,1]$, we have

$$\varphi(g(x,\lambda)) + \gamma(\lambda) = \|\Phi(x) - \lambda\Psi(x)\|_H^2 + c^2(2\lambda - \lambda^2)$$
$$= \|\Phi(x)\|_H^2 + (\|\Psi(x)\|_H^2 - c^2)\lambda^2 - 2(\langle \Phi(x), \Psi(x) \rangle - c^2)\lambda.$$

Fix $x \in X$. We distinguish two cases. First, assume that $\|\Psi(x)\|_H \le c$. In this case, the function $\varphi(g(x,\cdot)) + \gamma(\cdot)$ is concave in $[-1,1]$, and so the set of its global maxima is connected. Now, suppose that $\|\Psi(x)\|_H > c$. In this case, the function $\varphi(g(x,\cdot)) + \gamma(\cdot)$ is strictly convex in $[-1,1]$ and so it attains its maximum either at -1 or at 1. In view of (c_3), we have

$$\langle \Phi(x), \Psi(x) \rangle \neq c^2$$

and so

$$\varphi(g(x,-1)) + \gamma(-1) \neq \varphi(g(x,1)) + \gamma(1).$$

Consequently, the function $\varphi(g(x,\cdot)) + \gamma(\cdot)$ has a unique global maximum in $[-1,1]$. Now, Theorem 2.2 ensures the existence of $(\tilde{x}, \tilde{\lambda}) \in X \times [-1,1]$ such that

$$\|\Phi(\tilde{x}) - \tilde{\lambda}\Psi(\tilde{x})\|_H^2 = \inf_{x \in X} \|\Phi(x) - \tilde{\lambda}\Psi(x)\|_H^2 = \sup_{\lambda \in [-1,1]} (\|\Phi(\tilde{x}) - \tilde{\lambda}\Psi(\tilde{x})\|_H^2 + \gamma(\lambda)) - \gamma(\tilde{\lambda}). \quad (2.5)$$

We claim that $\tilde{\lambda} = 1$. Arguing by contradiction, suppose that $\tilde{\lambda} \in [-1,1[$. We distinguish two cases. First, assume that $\tilde{\lambda} \in]-1,1[$. Then, in view of (c_1), we would have $(\Phi - \tilde{\lambda}\Psi)(X) = Y$, and so

$$\inf_{x \in X} \|\Phi(x) - \tilde{\lambda}\Psi(x)\|_H = 0$$

since $0 \in Y$. Consequently, (2.5) would give

$$\gamma(1) = \sup_{\lambda \in [-1,1]} \gamma(\lambda) \le \sup_{\lambda \in [-1,1]} (\|\Phi(\tilde{x}) - \lambda\Psi(\tilde{x})\|_H^2 + \gamma(\lambda)) = \gamma(\tilde{\lambda}) < \gamma(1).$$

Now, assume that $\tilde{\lambda} = -1$. By the choice of c, we have

$$\|\Phi(\tilde{x}) + \Psi(\tilde{x})\|_H^2 + \gamma(-1) < 0.$$

But

$$\sup_{\lambda \in [-1,1]} (\|\Phi(\tilde{x}) - \lambda\Psi(\tilde{x})\|_H^2 + \gamma(\lambda) \geq \|\Phi(\tilde{x})\|_H^2 \geq 0$$

against (2.5). So, it remains proved that

$$\|\Phi(\tilde{x}) - \Psi(\tilde{x})\|_H = \inf_{x \in X} \|\Phi(x) - \Psi(x)\|_H$$

and the proof is complete. □

Remark 2.4. From the proof of Theorem 2.3, it follows that when the inclusion

$$\{x \in X : \|\Psi(x)\|_H > c\} \subseteq \{x \in X : \langle \Phi(x), \Psi(x) \rangle > c^2\}$$

holds, we have $\Psi(\tilde{x}) \leq c$.

Now, we can prove Theorem 1.1.

Proof. We are going to apply Theorem 2.3 taking $Y = H$. Fix $\lambda \in]-1, 1[$. Then, since Φ^{-1} is Lipschitzian (from $(Y, \|\cdot\|_2)$ to $(X, \|\cdot\|_1)$), with constant α_Φ, the operator $\Phi^{-1} \circ \lambda\Psi$ turns out to be a contraction on $(X, \|\cdot\|_1)$. In turn, this classically implies that the operator $\Phi - \lambda\Psi$ is a homeomorphism between $(X, \|\cdot\|_1)$ and $(Y, \|\cdot\|_2)$. Therefore, the operator $\Phi - \lambda\Psi$ is a homeomorphism between $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$, due to the equivalence of the norms, and so (c_1) is satisfied. Finally, consider (c_2). Since Ψ is assumed to be sequentially weakly continuous (and also Φ is so being linear and continuous), the function $\|\Phi - \lambda\Psi\|_H$ is sequentially weakly lower semicontinuous for all $\lambda \in \mathbf{R}$. Moreover, since Φ is a linear homeorphism between X and H, X is reflexive and we have

$$\lim_{\|x\|_X \to +\infty} \|\Phi(x)\|_H = +\infty.$$

Consequently, (c_2) is satisfied (with $\lambda_0 = 0$) taking the weak topology, and the conclusion follows. □

An interesting consequence of Theorem 1.1 is as follows:

Theorem 2.5. *Let the assumptions of Theorem 1.1 be satisfied. In addition, suppose that Ψ is C^1. Then, there exists $\tilde{x} \in X$ such that either $\Phi(\tilde{x}) = \Psi(\tilde{x})$ or $\Phi - \Psi'(\tilde{x})$ is not surjective.*

Proof. Since Ψ is C^1, the function $\|\Phi(\cdot) - \Psi(\cdot)\|_H^2$ is C^1 too and we have

$$\frac{d}{dx}(\|\Phi(x) - \Psi(x)\|_H^2)(u) = 2\langle \Phi(x) - \Psi(x), \Phi(u) - \Psi'(x)(u) \rangle$$

for all $x, u \in X$. Now, let $\tilde{x} \in X$ be a point satisfying the conclusion of Theorem 1.1. So, according to Fermat theorem, $\tilde{x}$ is a critical point of $\|\Phi(\cdot) - \Psi(\cdot)\|_H^2$. Therefore

$$2\langle \Phi(\tilde{x}) - \Psi(\tilde{x}), \Phi(u) - \Psi'(\tilde{x})(u) \rangle = 0$$

for all $u \in X$, and the conclusion follows. □

We conclude with a consequence of Corollary 1.2.

Theorem 2.6. *Let H be a real Hilbert space and let let $J : H \to \mathbf{R}$ be a C^1 functional such that J' is sequentially weakly continuous. Moreover, assume that:*
(d_1) there exists $\beta > 0$ such that

$$J(x) \leq \frac{1}{2}\|x\|^2 + \beta$$

for all $x \in H$;

(d_2) *there exists* $c > \frac{\|J'(0)\|_H}{\sqrt{3}}$ *such that*

$$\{x \in H : \langle x, J'(x) \rangle = c^2\} \subseteq \{x \in H : \|J'(x)\|_H \leq c\};$$

(d_3) *there exists an equivalent norm on* H *with respect to which* J' *is non-expansive.*

Then, J' *has a fixed point.*

Proof. It suffices to observe that, in view of (d_1), the functional $\frac{1}{2}\|\cdot\|^2 - J(\cdot)$ is bounded below. Then, from a classical consequence of Ekeland variational principle it follows that

$$\inf_{x \in H} \|x - J'(x)\|_H = 0$$

and so the conclusion follows directly from Corollary 1.2. □

Statements and Declarations

The author declares that he has no conflict of interest, and the manuscript has no associated data.

Acknowledgments

This work has been funded by the European Union - NextGenerationEU Mission 4 - Component 2 - Investment 1.1 under the Italian Ministry of University and Research (MUR) programme “PRIN 2022” - grant number 2022BCFHN2 - Advanced theoretical aspects in PDEs and their applications - CUP: E53D23005650006. The author has also been supported by the Gruppo Nazionale per l’Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM) and by the Università degli Studi di Catania, PIACERI 2024-2026, Linea di intervento 2, Progetto “PAFA”.

References

[1] A. I. Ban and S. G. Gal. On the minimal displacement of points under mappings. *Archivum Mathematicum* , 38:273-284, 2002.

[2] J. Borwein, S. Reich, and I. Shafrir. Krasnoselski-Mann iterations in normed spaces. *Canadian Mathematical Bulletin*, 35:21-28, 1992.

[3] F. E. Browder. Nonexpansive nonlinear operators in a Banach space. *Proceedings of the National Academy of Sciences of the United States of America* , 54:1041-1044, 1965.

[4] J. Garcia-Falset. Fixed points of continuous pseudocontractive mappings on convex, closed unbounded domains. *Proceedings of the American Mathematical Society*, 152:2863-2876, 2024.

[5] K. Goebel. On the minimal displacement of points under Lipschitzian mappings. *Pacific Journal of Mathematics*, 45:151-163,1973.

[6] D. Göhde. Zum Prinzip der kontraktiven Abbildung. *Mathematische Nachrichten*, 30:251-258, 1965.

[7] W. A. Kirk. A fixed point theorem for mappings which do not increase distances. *American Mathematical Monthly*, 72: 1004-1006, 1965.

[8] E. Matoušková and S. Reich. Reflexivity and approximate fixed points. *Studia Mathematica*, 159:403-415, 2003.

[9] F. Rambla-Barreno. Stability of the fixed point property for Hilbert spaces. *Journal of Mathematical Analysis and Applications*, 509(2):125849, 2022.

[10] S. Reich. On the asymptotic behavior of nonlinear semigroups and the range of accretive operators. *Journal of Mathematical Analysis and Applications*, 79:113-126, 1981.

[11] B. Ricceri. Nonlinear eigenvalue problems. In D. Y. Gao and D. Motreanu, editors, *Handbook of Nonconvex Analysis and Applications*, Pages 543-595, International Press, 2010.

[12] B. Ricceri. On a minimax theorem: an improvement, a new proof and an overview of its applications. *Minimax Theory and Its Applications*, 2:99-152, 2017.

[13] B. Ricceri. Minimax theorems in a fully non-convex setting. *Journal of Nonlinear and Variational Analysis*, 3:45-52, 2019.