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MULTIPLICITY OF SOLUTIONS FOR THE DISCRETE ROBIN PROBLEM INVOLVING THE $p(k)$-LAPLACE KIRCHHOFF TYPE EQUATIONS

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Abstract. In this paper, we establish results on the existence and multiplicity of solutions for a discrete Robin boundary value problem involving the variable exponent $p(k)$-Laplacian of Kirchhoff type in a finite-dimensional Banach space. Our approach relies on variational techniques combined with tools from critical point theory.

Keywords. Kirchhoff type equation, Discrete Robin problem, Multiple solutions, Variational methods, Critical point theory.

1. Introduction

In this work, we consider the following discrete Robin problem of $p(k)$-Laplace Kirchhoff type.

$$\begin{cases} -M(\zeta[u])[\triangle(a(k-1,|\triangle u(k-1)|)\triangle u(k-1)) - q(k)|u(k)|^{p(k)-2}u(k)] \\ \quad = \lambda f(k,u(k)), \quad k \in \mathbb{Z}(1,T) \\ \triangle u(0) = u(T+1) = 0, \end{cases} \tag{1.1}$$

where $T \geq 2$ is a fixed positive integer, $\mathbb{Z}(a,b)$ denotes the discrete interval $\{a, a+1, \dots, b-1, b\}$ with a and b integers such that $a < b$, $\triangle u(k) = u(k+1) - u(k)$ is the forward difference operator, $\lambda > 0$ is a real parameter and $f : \mathbb{Z}(1,T) \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Moreover, $a(k,\cdot), M : [0,\infty) \to [0,\infty)$ are two continuous functions for all $k \in \mathbb{Z}(0,T)$, $t \in [0,\infty)$ with the function $t \to M(t)$ nondecreasing, $\zeta[u]$ is a nonlocal term defined by the following relation

$$\zeta[u] = \sum_{k=1}^{T} \left(A_0(k-1, |\triangle u(k-1)|) + \frac{q(k)}{p(k)} |u(k)|^{p(k)} \right),$$

$A_0 : \mathbb{Z}(1,T) \times [0,\infty) \to [0,\infty)$ such that $A_0(k,t) = \displaystyle\int_0^t a(k,\xi)\xi \, d\xi$, the function $p : \mathbb{Z}(0,T) \to (1,\infty)$ is bounded. We denote it as short

$$p^+ := \max_{k \in \mathbb{Z}(0,T)} p(k), \quad p^- := \min_{k \in \mathbb{Z}(0,T)} p(k),$$

and the function $q : \mathbb{Z}(1,T) \to (1,\infty)$ is bounded such that

$$\overline{q} := \max_{k \in \mathbb{Z}(1,T)} q(k), \quad \underline{q} := \min_{k \in \mathbb{Z}(1,T)} q(k), \quad Q := \sum_{k=1}^{T} q(k).$$

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We consider in Problem (1.1) two different boundary conditions. Neumann boundary condition $(\triangle u(0) = 0)$ and Dirichlet boundary condition $(u(T+1) = 0)$. In the literature, the boundary condition considered in this paper is called a mixed or Robin boundary condition.
We also suppose that a and M satisfy the following conditions.

$(H1)$ $a_1 : \mathbb{Z}(0,T) \to [0,\infty)$ and there exists a constant $a_2 > 0$ such that
$$|a(k,|\xi|)\xi| \leq a_1(k) + a_2|\xi|^{p(k)-1},$$
for all $k \in \mathbb{Z}(0,T)$ and $\xi \in \mathbb{R}$.

$(H2)$ For all $k \in \mathbb{Z}(0,T)$ and $\xi > 0$, one has
$$0 \leq a(k,|\xi|)\xi^2 \leq p^+ \int_0^{|\xi|} a(k,s)s\,ds.$$

$(H3)$ There exists a positive constant c such that
$$\min\left\{a(k,|\xi|),\ |\xi|\frac{\partial a}{\partial \xi}(k,|\xi|) + a(k,|\xi|)\right\} \geq c|\xi|^{p(k)-2},$$
for all $k \in \mathbb{Z}(0,T)$ and $\xi \in \mathbb{R}$.

$(H4)$ $M : [0,\infty) \to [0,\infty)$ is continuous, nondecreasing and there exist two positive constant m_0 and m_1 such that
$$m_0 \leq M(t) \leq m_1 \ \text{ for } \ t > 0.$$

Remark 1. *As examples of functions A_0 and a satisfying the assumptions $(H1)$-$(H4)$, we can give the following.*

(i) *If we set*
$$M(A_0(k,|\xi|)) = \frac{1}{p(k)}|\xi|^{p(k)} \ \text{ and } \ M(t) = 1,$$
then
$$a(k,|\xi|) = |\xi|^{p(k)-2}, \ \text{ for all } \ (k,\xi) \in \mathbb{Z}(1,T) \times \mathbb{R}.$$

(ii) *Now, if we put*
$$M(A_0(k,|\xi|)) = a + \frac{b}{p(k)}\left[\left(1+|\xi|^2\right)^{\frac{p(k)}{2}} - 1\right] \ \text{ and } \ M(t) = a + bt,$$
then
$$a(k,|\xi|) = \left(1+|\xi|^2\right)^{\frac{p(k)-2}{2}}, \ \text{ for all } \ (k,\xi) \in \mathbb{Z}(1,T) \times \mathbb{R}.$$

The nonhomogeneous differential operator involved in (1.1), i.e.,
$$\triangle(a(k-1,|\triangle u(k-1)|)\triangle u(k-1)),$$
where a satisfies $(H1)$-$(H4)$ was recently analyzed by Moussa et *al* (see [42]). This operator generalizes the usual operators with variable exponent. In the particular case where
$$\triangle_{p(k-1)}u(k-1) := \triangle\left(|\triangle u(k-1)|^{p(k-1)-2}\triangle u(k-1)\right),$$
the operator involved in (1.1) is the standard $p(k)$-Laplace difference operator. The paper includes the case $p(k)$-mean curvature operator or $p(k)$-capillarity differential operator, such as
$$\triangle\left(\left(1+|\triangle u(k-1)|^2\right)^{\frac{p(k-1)-2}{2}}\triangle u(k-1)\right)$$
and
$$\triangle\left(\left(1+\frac{|\triangle u(k-1)|^{p(k-1)}}{\sqrt{1+|\triangle u(k-1)|^{2p(k-1)}}}\right)|\triangle u(k-1)|^{p(k-1)-2}\triangle u(k-1)\right)$$

The presence of the nonlocal term $\zeta[u]$ is an important feature of this paper. Problem (1.1) is related to the stationary version of the continuous Kirchhoff equation, established by Kirchhoff in 1876. To be more precise, the following model, which is called Kirchhoff equation, was introduced in [30],

$$\rho\frac{\partial^2 u}{\partial t^2}=\left(T_0+\frac{Ea}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2\,dx\right)\frac{\partial^2 u}{\partial x^2}, \tag{1.2}$$

where $\rho>0$ is the mass per unit length, T_0 is the base tension, E is the Young modulus, a is the area of cross section and L is the initial length of the string.

Equation (1.2) takes into account the change of the tension on the string, which is caused by the change of its length during the vibration. After that, several physicists also considered such equations for their research in the theory of nonlinear vibrations theoretically or experimentally (see [15, 16, 44, 46]). On the other hand, Kirchhoff's equation received great attention only after Lions in 1978 (see [34]) proposed an abstract framework of the problem related to the stationary analog of the equation of Kirchhoff type. In recent years, many authors have investigated Kirchhoff-type equations, we refer the readers to [4, 17]. We also refer the readers to the recent results of the discrete Kirchhoff type problems [18, 26, 27, 33, 41, 45, 47, 52, 53, 54] and the references therein. In the recent work [43], the present authors have dealt with the $p(k)$-Laplace Kirchhoff type equations by using the critical point theory and mountain pass theorem.

The nonlinear difference equations arise in various research fields. Discrete boundary value problems have received some attention because of related applications in elastic mechanics [58], electrorheological fluids [50, 51], and image restoration [19]. In recent years, many authors have studied the existence and multiplicity of solutions of discrete problems subject to various boundary value conditions by using different methods such as fixed point theory, the method of upper and lower solution techniques, Rabinowitz's global bifurcation theorem, etc. (see [2, 6, 25]). We refer to the recent monograph by Agarwal [1] and the papers [14, 38] for more details on difference equations and their applications. The studies for discrete $p(k)$-Laplacian problems have been extensively considered in many papers, see, for example, [5, 20, 21, 22, 31, 32, 37, 48]. When the discrete $p(k)$ is a constant, namely, the so-called discrete p-Laplacian operator, we refer the readers to [9, 12, 13] and the references therein. The discrete $p(k)$-Laplacian operator has more complicated nonlinearities than the discrete p-Laplacian operator, for example, it is not homogeneous. Recently, boundary value problems of difference equations with ϕ-Laplacian have received extensive attention from many researchers, see, for example, [35, 36, 49, 57].

Problem (1.1) can be seen as a discrete variant of the variable exponent anisotropic problem

$$\begin{cases}-\displaystyle\sum_{i=1}^N\frac{\partial}{\partial x_i}a_i\left(x,\left|\frac{\partial u}{\partial x_i}\right|\right)\frac{\partial u}{\partial x_i}+q(x)|u|^{p_i(x)-2}u=\lambda f(x,u)\ \text{ in }\ \Omega,\\ \dfrac{\partial u}{\partial n}=0\ \text{ on }\ \partial\Omega_1,\\ u=0\ \text{ on }\ \partial\Omega_2,\end{cases} \tag{1.3}$$

where $\Omega\subset\mathbb{R}^N$ $(N\geq 3)$ is a bounded domain with smooth boundary, $\partial\Omega_1\cup\partial\Omega_2=\partial\Omega$, $f\in C\left(\overline{\Omega}\times\mathbb{R},\mathbb{R}\right)$ is given function that satisfy some properties, $p_i(x)$ and $q(x)\geq 1$ are continuous functions on $\overline{\Omega}$ such that $1<p_i(x)<N$ and $\displaystyle\sum_{i=1}^N\frac{1}{p_i^-}>1$ for all $x\in\overline{\Omega}$ and all $i\in\mathbb{Z}[1,N]$, where $p_i^-:=\inf_{x\in\Omega}p_i(x)$, $\lambda>0$ is real number.

Recently, I. H. Kim and Y. H. Kim [29] studied problem (1.3) under homogeneous Dirichlet boundary condition ($u=0$ on $\partial\Omega$).

In a recent paper [40], by using variational methods, the present authors consider the multiplicity of solutions of the following discrete Robin problem with $p(k)$-Laplacian.

$$\begin{cases} -\triangle(a(k-1,|\triangle u(k-1)|)\triangle u(k-1)) = \lambda f(k,u(k)) + \mu g(k,u(k)), \quad k\in\mathbb{Z}(1,T) \\ \triangle u(0) = u(T+1) = 0. \end{cases} \tag{1.4}$$

Here, we will generalize this result.

Motivated by the above papers and the results in [3, 11, 24] and [28], this paper aims to establish the existence and multiplicity of problem solutions (1.1), via variational methods and critical point theory.

The organization of the paper is as follows. In Section 2, we establish the variational framework associated with problem (1.1). Some preliminary results are also stated in this section. In Section 3, we apply a result of Bonanno et *al.* (see [8]), to prove the existence of at least one nontrivial solution of problem (1.1). In Section 4, one uses a result of Bonanno and D'Aguí (see [7]), to prove the existence of at least two nontrivial solutions of problem (1.1). Finally, in Section 5, one uses a result of Bonanno (see [10]) to prove the existence of at least three solutions of problem (1.1).

2. Variational Framework and Preliminary Results

In this section, we establish a variational framework associated with problem (1.1). We consider the T-dimensional Banach space

$$S = \{u : \mathbb{Z}(0,T+1) \to \mathbb{R} \ \text{ such that } \ \triangle u(0) = u(T+1) = 0\}$$

endowed with the norm

$$\|u\| = \left(\sum_{k=1}^{T}\left(|\triangle u(k)|^{p^-} + q(k)|u(k)|^{p^-}\right)\right)^{1/p^-}.$$

On the space S, we also introduce the following norm

$$\|u\|_{p^+} = \left(\sum_{k=1}^{T}\left(|\triangle u(k)|^{p^+} + q(k)|u(k)|^{p^+}\right)\right)^{1/p^+}$$

and the Luxemburg norm

$$\|u\|_{p(\cdot)} = \inf\left\{\mu > 0 : \ \sum_{k=1}^{T}\left(\left|\frac{\triangle u(k)}{\mu}\right|^{p(k)} + q(k)\left|\frac{u(k)}{\mu}\right|^{p(k)}\right) \le 1\right\}.$$

We will use the following inequality

$$L\|u\| \le \|u\|_{p^+} \le 2^{\frac{p^+-p^-}{p^-p^+}} L\|u\|, \tag{2.1}$$

where $L = (2\max\{T,Q\})^{\frac{p^--p^+}{p^-p^+}}$. Indeed, by weighted Hölder's inequality (see [39]), we get

$$\begin{aligned} \sum_{k=1}^{T}|\triangle u(k)|^{p^-} &\le \left(\sum_{k=1}^{T}1^{\frac{p^+}{p^+-p^-}}\right)^{\frac{p^+-p^-}{p^+}}\left(\sum_{k=1}^{T}\left(|\triangle u(k)|^{p^-}\right)^{\frac{p^+}{p^-}}\right)^{\frac{p^-}{p^+}} \\ &\le T^{\frac{p^+-p^-}{p^+}}\left(\sum_{k=1}^{T}|\triangle u(k)|^{p^+}\right)^{\frac{p^-}{p^+}}. \end{aligned}$$

We obtain, in a similar way as before,

$$\sum_{k=1}^{T} q(k)\,|u(k)|^{p^-} \le Q^{\frac{p^+-p^-}{p^+}} \left(\sum_{k=1}^{T} q(k)|u(k)|^{p^+}\right)^{\frac{p^-}{p^+}}.$$

Combining the above inequalities with $\frac{p^-}{p^+} \le 1$, we obtain

$$\begin{aligned}
\|u\|^{p^-} &\le (\max\{T,Q\})^{\frac{p^+-p^-}{p^+}} \times \left(\left(\sum_{k=1}^{T} |\triangle u(k)|^{p^+}\right)^{\frac{p^-}{p^+}} + \left(\sum_{k=1}^{T} q(k)|u(k)|^{p^+}\right)^{\frac{p^-}{p^+}}\right)\\
&\le 2^{1-\frac{p^-}{p^+}} (\max\{T,Q\})^{\frac{p^+-p^-}{p^+}} \times \left(\sum_{k=1}^{T} |\triangle u(k)|^{p^+} + \sum_{k=1}^{T} q(k)|u(k)|^{p^+}\right)^{\frac{p^-}{p^+}} = L^{-p^-} \|u\|_{p^+}^{p^-}.
\end{aligned}$$

Consequently, $L\|u\| \le \|u\|_{p^+}$.

On the other hand, we get from the fact that $\frac{p^+}{p^-} \ge 1$, the following.

$$\begin{aligned}
\|u\|_{p^+}^{p^+} &\le (\max\{T,Q\})^{\frac{p^--p^+}{p^-}} \times \left(\left(\sum_{k=1}^{T} |\triangle u(k)|^{p^-}\right)^{\frac{p^+}{p^-}} + \left(\sum_{k=1}^{T} q(k)|u(k)|^{p^-}\right)^{\frac{p^+}{p^-}}\right)\\
&\le (\max\{T,Q\})^{\frac{p^--p^+}{p^-}} \times \left(\sum_{k=1}^{T} |\triangle u(k)|^{p^-} + \sum_{k=1}^{T} q(k)|u(k)|^{p^-}\right)^{\frac{p^+}{p^-}} = 2^{\frac{p^+-p^-}{p^-}} L^{p^+} \|u\|^{p^+}.
\end{aligned}$$

Consequently, $\|u\|_{p^+} \le 2^{\frac{p^+-p^-}{p^-p^+}} L\|u\|$. Therefore, we obtain that (2.1) holds.
We consider another norm in S, that is,

$$\|u\|_\infty := \max\{|u(k)| : k \in \mathbb{Z}(1,T)\}, \quad \text{for all} \quad u \in S.$$

For every $u \in S$ and $p^- > 1$, the following relation

$$\|u\|_\infty \le \kappa\|u\| \tag{2.2}$$

holds, where

$$\kappa := \begin{cases} T^{\frac{p^--1}{p^-}} & \text{if } 0 \le \underline{q} \le T^{1-p^-},\\ \underline{q}^{-1/p^-} & \text{if } \qquad \underline{q} \ge T^{1-p^-}. \end{cases}$$

Indeed, let $\tau \in \mathbb{Z}(1,T)$ such that

$$|u(\tau)| = \max_{k\in\mathbb{Z}(1,T)} |u(k)|.$$

For $0 \le \underline{q} \le T^{1-p^-}$, by Hölder inequality, we get

$$\begin{aligned}
\|u\|_\infty = |u(\tau)| &\le \sum_{k=1}^{T} |\triangle u(k)| + \sum_{k=1}^{T} |u(k)|\\
&\le T^{\frac{p^--1}{p^-}} \left(\sum_{k=1}^{T} |\triangle u(k)|^{p^-}\right)^{\frac{1}{p^-}} + T^{\frac{p^--1}{p^-}} \left(\sum_{k=1}^{T} |u(k)|^{p^-}\right)^{\frac{1}{p^-}}
\end{aligned}$$

$$\leq \; T^{\frac{p^{-}-1}{p^{-}}}\left(\left(\sum_{k=1}^{T}|\triangle u(k)|^{p^{-}}\right)^{\frac{1}{p^{-}}}+\left(\sum_{k=1}^{T}q(k)\,|u(k)|^{p^{-}}\right)^{\frac{1}{p^{-}}}\right)$$

$$\leq \; T^{\frac{p^{-}-1}{p^{-}}}\left(\sum_{k=1}^{T}\left(|\triangle u(k)|^{p^{-}}+q(k)|u(k)|^{p^{-}}\right)\right)^{\frac{1}{p^{-}}}=T^{\frac{p^{-}-1}{p^{-}}}\|u\|$$

and for $\underline{q}\geq T^{1-p^{-}}$, we obtain

$$\begin{aligned}\min_{k\in\mathbb{Z}(1,T)} q(k)|u(\tau)|^{p^{-}} &\leq \sum_{k=1}^{T}q(k)\,|u(k)|^{p^{-}}\\ &\leq \sum_{k=1}^{T}|\triangle u(k)|^{p^{-}}+\sum_{k=1}^{T}q(k)|u(k)|^{p^{-}}.\end{aligned}$$

Hence,

$$\|u\|_{\infty}\leq\frac{1}{\underline{q}^{1/p^{-}}}\|u\| \text{ for all } u\in S,$$

where $\underline{q}:=\min_{k\in\mathbb{Z}(1,T)}q(k)$. Therefore, we obtain that (2.2) holds.
Since on S, all norms are equivalent, there exist constants $0<K_1<K_2$ such that

$$K_1\|u\|_{p(\cdot)}\leq\|u\|\leq K_2\|u\|_{p(\cdot)}. \tag{2.3}$$

Now, let $\varphi:S\to\mathbb{R}$ be given by

$$\varphi(u)=\sum_{k=1}^{T}|\triangle u(k)|^{p(k)}. \tag{2.4}$$

It is easy to check that for all $u,u_n\in S$, the following properties hold.

$$\text{If } \|u\|_{p(\cdot)}>1, \text{ then } \|u\|_{p(\cdot)}^{p^{-}}\leq\varphi(u)+\sum_{k=1}^{T}q(k)|u(k)|^{p(k)}\leq\|u\|_{p(\cdot)}^{p^{+}}. \tag{2.5}$$

$$\text{If } \|u\|_{p(\cdot)}<1, \text{ then } \|u\|_{p(\cdot)}^{p^{+}}\leq\varphi(u)+\sum_{k=1}^{T}q(k)|u(k)|^{p(k)}\leq\|u\|_{p(\cdot)}^{p^{-}}. \tag{2.6}$$

Next, we introduce the functionals $\Phi,\Psi:S\to\mathbb{R}$ defined by

$$\Phi(u)=\widehat{M}(\zeta[u]), \tag{2.7}$$

$$\Psi(u)=\sum_{k=1}^{T}F(k,u(k)), \tag{2.8}$$

where $\widehat{M}(t)=\int_0^t M(\xi)\,d\xi$ and $F(k,t)=\int_0^t f(k,\xi)\,d\xi.$

The energy functional $I_\lambda:S\to\mathbb{R}$ corresponding to problem (1.1) is

$$I_\lambda(u)=\Phi(u)-\lambda\Psi(u), \text{ for all } u\in S. \tag{2.9}$$

Throughout this paper, we recall that $u\in S$ is a (weak) solution of problem (1.1) if

$$M\left(\zeta[u]\right)\sum_{k=1}^{T}\left[a(k,|\triangle u(k)|)\triangle u(k)\triangle v(k)+q(k)|u(k)|^{p(k)-2}u(k)v(k)\right]=\lambda\sum_{k=1}^{T}f(k,u(k))v(k) \tag{2.10}$$

for any $v \in S$.
It is easy to see that Φ and Ψ are two functionals of class $C^1(S,\mathbb{R})$ whose Gâteaux derivatives at the point $u \in S$ are given by

$$\left\langle \Phi'(u), v\right\rangle = M\left(\zeta[u]\right)\sum_{k=1}^{T}\left[a(k,|\triangle u(k)|)\triangle u(k)\triangle v(k) + q(k)|u(k)|^{p(k)-2}u(k)v(k)\right] \tag{2.11}$$

and

$$\left\langle \Psi'(u), v\right\rangle = \sum_{k=1}^{T} f(k,u(k))v(k), \tag{2.12}$$

for all $u, v \in S$.

From (2.11) and (2.12), we observe that I_λ is of class $C^1(S,\mathbb{R})$ and its derivative is given by

$$\left\langle I'_\lambda(u), v\right\rangle = \left\langle \Phi'(u), v\right\rangle - \lambda\left\langle \Psi'(u), v\right\rangle,$$

for all $u, v \in S$. Note that as $\triangle u(0) = u(T+1) = 0$, one has

$$\sum_{k=1}^{T} a(k,|\triangle u(k)|)\triangle u(k)\triangle v(k) = -\sum_{k=1}^{T}\triangle(a(k-1,|\triangle u(k-1)|)\triangle u(k-1))v(k)$$

and thus,

$$\begin{aligned}\left\langle I'_\lambda(u), v\right\rangle &= \sum_{k=1}^{T}\Big[-M(\zeta[u])[\triangle(a(k-1,|\triangle u(k-1)|)\triangle u(k-1))\\ &\qquad -q(k)|u(k)|^{p(k)-2}u(k)] - \lambda f(k,u(k))\Big]v(k).\end{aligned}$$

Hence, the critical points of I_λ are exactly the solutions of problem (1.1).
Now, we will use the following auxiliary result.

Lemma 1. *(i) Let $u \in S$ and $\|u\| > 1$. Then,*

$$\sum_{k=1}^{T}\left[|\triangle u(k)|^{p(k)} + q(k)|u(k)|^{p(k)}\right] \geq \|u\|^{p^-} - (1+\overline{q})T.$$

(ii) Let $u \in S$ and $\|u\| < 1$. Then,

$$\sum_{k=1}^{T}\left[|\triangle u(k)|^{p(k)} + q(k)|u(k)|^{p(k)}\right] \geq L^{p^+}\|u\|^{p^+}.$$

(iii) Let $u \in S$. Then,

$$\sum_{k=1}^{T}\left[|\triangle u(k)|^{p(k)} + q(k)|u(k)|^{p(k)}\right] \leq 2^{\frac{p^+-p^-}{p^-}}L^{p^+}\|u\|^{p^+} + (1+\overline{q})T.$$

Proof. Let $u \in S$ be fixed. By a similar argument as in [22], we define

$$\beta_k := \begin{cases} p^+ & \text{if } |\triangle u(k)| \leq 1\\ p^- & \text{if } |\triangle u(k)| > 1\end{cases} \qquad \text{and} \qquad \delta_k := \begin{cases} p^+ & \text{if } |u(k)| \leq 1\\ p^- & \text{if } |u(k)| > 1,\end{cases}$$

for each $k \in \mathbb{Z}(1,T)$.
(i) For $u \in S$ with $\|u\| > 1$, one has

$$\sum_{k=1}^{T}\left[|\triangle u(k)|^{p(k)} + q(k)|u(k)|^{p(k)}\right] \geq \sum_{k=1,\beta_k=p^+}^{T}|\triangle u(k)|^{p^+} + \sum_{k=1,\beta_k=p^-}^{T}|\triangle u(k)|^{p^-}$$

$$
\begin{aligned}
& + \sum_{k=1,\delta_k=p^+}^{T} q(k)\,|u(k)|^{p^+} + \sum_{k=1,\delta_k=p^-}^{T} q(k)\,|u(k)|^{p^-} \\
= & \sum_{k=1}^{T} |\triangle u(k)|^{p^-} - \sum_{k=1,\beta_k=p^+}^{T} \left(|\triangle u(k)|^{p^-} - |\triangle u(k)|^{p^+} \right) \\
& + \sum_{k=1}^{T} q(k)|u(k)|^{p^-} - \overline{q} \sum_{k=1,\delta_k=p^+}^{T} \left(|u(k)|^{p^-} - |u(k)|^{p^+} \right) \\
\geq & \sum_{k=1}^{T} |\triangle u(k)|^{p^-} - T + \sum_{k=1}^{T} q(k)|u(k)|^{p^-} - \overline{q}T \\
= & \|u\|^{p^-} - (1+\overline{q})T.
\end{aligned}
$$

(ii) As $|\triangle u(k)| < 1$ and $|u(k)| < 1$ for each $k \in \mathbb{Z}(1,T)$ since $\|u\| < 1$, we deduce that

$$
\sum_{k=1}^{T} |\triangle u(k)|^{p(k)} \geq \sum_{k=1}^{T} |\triangle u(k)|^{p^+} \quad \text{and} \quad \sum_{k=1}^{T} q(k)|u(k)|^{p(k)} \geq \sum_{k=1}^{T} q(k)|u(k)|^{p^+}.
$$

Hence, by the above inequalities and the relation (2.1), we obtain

$$
\begin{aligned}
\sum_{k=1}^{T} \left[|\triangle u(k)|^{p(k)} + q(k)|u(k)|^{p(k)} \right] & \geq \sum_{k=1}^{T} \left[|\triangle u(k)|^{p^+} + q(k)|u(k)|^{p^+} \right] \\
& = \|u\|_{p^+}^{p^+} \\
& \geq L^{p^+} \|u\|^{p^+}.
\end{aligned}
$$

(iii) Indeed, we deduce by relation (2.1) that

$$
\begin{aligned}
\sum_{k=1}^{T} \left[|\triangle u(k)|^{p(k)} + q(k)|u(k)|^{p(k)} \right] \leq & \sum_{k=1,\beta_k=p^-}^{T} |\triangle u(k)|^{p^+} + \sum_{k=1,\beta_k=p^+}^{T} |\triangle u(k)|^{p^-} \\
& + \sum_{k=1,\delta_k=p^-}^{T} q(k)\,|u(k)|^{p^+} + \sum_{k=1,\delta_k=p^+}^{T} q(k)\,|u(k)|^{p^-} \\
= & \sum_{k=1}^{T} |\triangle u(k)|^{p^+} + \sum_{k=1,\beta_k=p^+}^{T} \left(|\triangle u(k)|^{p^-} - |\triangle u(k)|^{p^+} \right) \\
& + \sum_{k=1}^{T} q(k)|u(k)|^{p^+} + \overline{q} \sum_{k=1,\delta_k=p^+}^{T} \left(|u(k)|^{p^-} - |u(k)|^{p^+} \right) \\
\leq & \sum_{k=1}^{T} |\triangle u(k)|^{p^+} + T + \sum_{k=1}^{T} q(k)|u(k)|^{p^+} + \overline{q}T \\
= & \|u\|_{p^+}^{p^+} + (1+\overline{q})T \leq 2^{\frac{p^+-p^-}{p^-}} L^{p^+} \|u\|^{p^+} + (1+\overline{q})T.
\end{aligned}
$$

Proposition 2.1. *Assume that the condition $(H3)$ is fulfilled. Then, the following estimate*

$$
\begin{aligned}
& \langle a(k,|u|)u - a(k,|v|)v, u-v \rangle \\
& \geq \begin{cases} c\left(|u|+|v|\right)^{p(k)-2} |u-v|^2 & \text{if } 1 < p(k) < 2 \\ 4^{2-p^+} c|u-v|^{p(k)} & \text{if } \quad p(k) \geq 2 \end{cases}
\end{aligned} \tag{2.13}
$$

holds true for all $u, v \in \mathbb{R}$ *and* $k \in \mathbb{Z}(1,T)$ *with* $(u,v) \neq (0,0)$.

Proof. Let $u, v \in \mathbb{R}$ such that $(u,v) \neq (0,0)$. Let us define $\psi(k,u) = a(k,|u|)u$. Let u in $\mathbb{R} \setminus \{0\}$. From $(H3)$, we obtain

$$\begin{aligned} \frac{\partial \psi(k,u)}{\partial u} &= |u|\frac{\partial a}{\partial u}(k,|u|) + a(k,|u|) \\ &\geq c|u|^{p(k)-2}. \end{aligned} \tag{2.14}$$

Note that

$$\psi(k,u) - \psi(k,v) = \int_0^1 \frac{\partial \psi(k, v + t(u-v))}{\partial u}(u-v)\,dt. \tag{2.15}$$

Let $k \in \mathbb{Z}(0,T)$ with $p(k) \geq 2$. Then, by (2.14) and (2.15), we get

$$\begin{aligned} \langle a(k,|u|)u - a(k,|v|)v, u-v\rangle &= \int_0^1 \frac{\partial \psi}{\partial u}(k, v+t(u-v))(u-v)(u-v)\,dt \\ &\geq \int_0^1 c|v+t(u-v)|^{p(k)-2}|u-v|^2\,dt. \end{aligned}$$

Without loss of generality, we may assume that $|u| \leq |v|$. Then, $|u-v| \leq |u| + |v| \leq 2|v|$. For any $t \in [0, 1/4]$, one has

$$\begin{aligned} |v| &\leq |v+t(u-v)| + t|u-v| \\ &\leq |v+t(u-v)| + \frac{1}{4}|u-v| \end{aligned}$$

and so

$$|v+t(u-v)| \geq |v| - \frac{1}{4}|u-v| \geq \frac{1}{4}|u-v|.$$

Therefore

$$\begin{aligned} \langle a(k,|u|)u - a(k,|v|)v, u-v\rangle &\geq \int_0^1 c|v+t(u-v)|^{p(k)-2}|u-v|^2\,dt \\ &\geq 4^{2-p^+}c|u-v|^{p(k)}. \end{aligned}$$

For $k \in \mathbb{Z}(0,T)$ with $1 < p(k) < 2$. As before, we deduce by $(H3)$ that, for all $u \in \mathbb{R} \setminus \{0\}$,

$$\begin{aligned} \frac{\partial \psi(k,u)}{\partial u} &= |u|\frac{\partial a}{\partial u}(k,|u|) + a(k,|u|) \\ &\geq c|u|^{p(k)-2}. \end{aligned}$$

Using the fact that $|tu + (1-t)v| \leq |u| + |v|$, we infer that

$$\begin{aligned} \langle a(k,|u|)u - a(k,|v|)v, u-v\rangle &\geq \int_0^1 c|v+t(u-v)|^{p(k)-2}|u-v|^2\,dt \\ &\geq c\left(|u|+|v|\right)^{p(k)-2}|u-v|^2. \end{aligned}$$

The proof of Proposition 2.1 is complete.

Lemma 2. *Assume that* $(H1)$ *and* $(H3)$*-*$(H4)$ *hold. Then, the operator* $\Phi' : S \to S^*$ *is strictly monotone on* S *and a mapping of type* (S_+)*, i.e., if* $u_n \rightharpoonup u$ *in* S *as* $n \to \infty$ *and* $\limsup_{n\to\infty}\langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle \leq 0$, *then* $u_n \to u$ *in* S *as* $n \to \infty$. *Here,* $\langle \cdot,\cdot\rangle$ *denotes the duality pairing between* S^* *and* S.

Proof. Firstly, we prove that Φ' is a strictly monotone operator.
We consider the functional $\phi : S \to \mathbb{R}$ defined by

$$\phi(u) = \zeta[u] = \sum_{k=1}^{T} \left(\int_0^{|\triangle u(k)|} a(k,s)s\,ds + \frac{q(k)}{p(k)}|u(k)|^{p(k)} \right), \quad \text{for all } u \in S.$$

Then, $\phi \in C^1(S,\mathbb{R})$ and its Gâteaux derivative at the point $u \in S$ is

$$\left\langle \phi'(u), v \right\rangle = \sum_{k=1}^{T} a(k, |\triangle u(k)|)\triangle u(k)\triangle v(k) + \sum_{k=1}^{T} q(k)|u(k)|^{p(k)-2}u(k)v(k),$$

for all $u, v \in S$.
For all $u, v \in S$ such that $u \neq v$, we obtain

$$\begin{aligned} \langle \phi'(u) - \phi'(v), u - v\rangle \ &= \ \sum_{k=1}^{T} \left(a(k, |\triangle u(k)|)\triangle u(k1) - a(k, |\triangle v(k)|)\triangle v(k)\right) \triangle(u-v)(k) \\ &\quad + \sum_{k=1}^{T} q(k) \left(|u(k)|^{p(k)-2}u(k) - |v(k)|^{p(k)-2}v(k) \right) (u(k) - v(k)) . \end{aligned}$$

By using (2.13) and taking into account the following well-known inequality, for any $\xi, \eta \in \mathbb{R}$,

$$\begin{aligned} &\left(|\xi|^{p(k)-2}\xi - |\eta|^{p(k)-2}\eta \right) (\xi - \eta) \\ &\geq \begin{cases} c_1 \left(|\xi| + |\eta|\right)^{p(k)-2} |\xi - \eta|^2 & \text{if } 1 < p(k) < 2, \\ c_2|\xi - \eta|^{p(k)} & \text{if } \quad p(k) \geq 2, \end{cases} \end{aligned} \tag{2.16}$$

we see that for all $u, v \in S$ with $u \neq v$,

$$\begin{aligned} &\langle \phi'(u) - \phi'(v), u - v\rangle \\ &\qquad \geq \begin{cases} \min\{c, c_1\} \displaystyle\sum_{k=1}^{T} \hat{u}(k)^{p(k)-2}|\triangle u(k) - \triangle v(k)|^2 > 0 & \text{if } 1 < p(k) < 2, \\ \min\left\{4^{2-p^+}c, c_2\right\} \displaystyle\sum_{k=1}^{T} |\triangle u(k) - \triangle v(k)|^{p(k)} > 0 & \text{if } \quad p(k) \geq 2, \end{cases} \end{aligned}$$

where $\hat{u}(k) = |\triangle u(k)| + |\triangle v(k)|$. This implies that ϕ' is strictly monotone. Thus, by Proposition 25.10 of [55], ϕ is strictly convex. On the other hand, since M is nondecreasing, $\widehat{M}$ is convex in $(0, \infty)$. Then, for all $u, v \in S$ with $u \neq v$ and each $s, t \in (0, 1)$ with $s + t = 1$, we have

$$\widehat{M}\left(\phi(su + tv)\right) < \widehat{M}\left(s\phi(u) + t\phi(v)\right) \leq s\widehat{M}\left(\phi(u)\right) + t\widehat{M}\left(\phi(v)\right).$$

Therefore, we obtain that Φ is strictly convex and thus Φ' is strictly monotone in S.
Now, we prove that the operator Φ' is of type (S_+). Let $\{u_n\} \subset S$ be a sequence such that $u_n \rightharpoonup u$ in S as $n \to \infty$ and

$$\limsup_{n\to\infty} \langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle \leq 0.$$

We will prove that $u_n \to u$ in S.
The above inequality and the strictly monotonicity of Φ' imply that

$$\langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle \to 0 \ \text{ as } \ n \to \infty.$$

So,

$$\langle \Phi'(u_n), u_n - u\rangle \to 0 \ \text{ as } \ n \to \infty,$$

which means that, as $n \to \infty$,

$$(\zeta[u_n])\left[\sum_{k=1}^{T} a(k,|\triangle u_n(k)|)\triangle u_n(k)\triangle(u_n-u)(k)+\sum_{k=1}^{T} q(k)|u_n(k)|^{p(k)-2}u_n(k)(u_n-u)(k)\right]\to 0. \quad (2.17)$$

Thus, by $(H1)$, (2.2) and Lemma $1(iii)$, we get that

$$\begin{aligned}
\zeta[u_n] &= \sum_{k=1}^{T}\left(\int_0^{|\triangle u_n(k)|} a(k,s)s\,ds+\frac{q(k)}{p(k)}|u_n(k)|^{p(k)}\right)\\
&\le \sum_{k=1}^{T} a_1(k)|\triangle u_n(k)|+\sum_{k=1}^{T}\frac{a_2}{p(k)}|\triangle u_n(k)|^{p(k)}+\sum_{k=1}^{T}\frac{q(k)}{p(k)}|u_n(k)|^{p(k)}\\
&\le \overline{a_1}\sum_{k=1}^{T}|\triangle u_n(k)|+\frac{\max\{1,a_2\}}{p^-}\left(2^{\frac{p^+-p^-}{p^-}}L^{p^+}\|u_n\|^{p^+}+(1+\overline{q})T\right)\\
&\le 2\overline{a_1}\kappa T\|u_n\|+\frac{\max\{1,a_2\}}{p^-}(1+\overline{q})T+\frac{2^{\frac{p^+-p^-}{p^-}}L^{p^+}\max\{1,a_2\}}{p^-}\|u_n\|^{p^+}.
\end{aligned}$$

So, we infer that $(\zeta[u_n])_{n\ge 1}$ is bounded.
Since M is continuous, up to a subsequence, there is $t_0 \ge 0$ such that

$$M(\zeta[u_n]) \to M(t_0) \ge m_0 \text{ as } n\to\infty. \quad (2.18)$$

Thus, it follows from (2.17) and (2.18) that, as $n\to\infty$,

$$\left[\sum_{k=1}^{T} a(k,|\triangle u_n(k)|)\triangle u_n(k)\triangle(u_n-u)(k)+\sum_{k=1}^{T} q(k)|u_n(k)|^{p(k)-2}u_n(k)(u_n-u)(k)\right]\to 0.$$

Since $u_n\to u$, one has $\sum_{k=1}^{T} a(k,|\triangle u_n(k)|)\triangle u_n(k)\triangle(u_n-u)(k)\to 0$ as $n\to\infty$. Then,

$$\sum_{k=1}^{T} q(k)|u_n(k)|^{p(k)-2}u_n(k)(u_n-u)(k)\to 0 \text{ as } n\to\infty \quad (2.19)$$

since q is bounded. Consequently,

$$\langle \phi'(u_n), u_n-u\rangle \to 0 \text{ as } n\to\infty.$$

So,

$$\langle \phi'(u_n)-\phi'(u), u_n-u\rangle \to 0 \text{ as } n\to\infty. \quad (2.20)$$

Now, we show that $\varphi(u_n-u)+\sum_{k=1}^{T} q(k)|u_n(k)-u(k)|^{p(k)}\to 0$ as $n\to\infty$. That is,

$$\sum_{k=1}^{T}|\triangle u_n(k)-\triangle u(k)|^{p(k)}\to 0 \text{ as } n\to\infty \quad (2.21)$$

and

$$\sum_{k=1}^{T} q(k)|u_n(k)-u(k)|^{p(k)}\to 0 \text{ as } n\to\infty. \quad (2.22)$$

Furthermore, by (2.16) and Proposition 2.1, we have

$$\langle \phi'(u_n)-\phi'(u), u_n-u\rangle \ge \begin{cases}\min\{c,c_1\}\sum_{k=1}^{T}\breve{u}(k)^{p(k)-2}|\triangle u_n(k)-\triangle u(k)|^2 & \text{if } 1<p(k)<2,\\ \min\left\{4^{2-p^+}c,c_2\right\}\sum_{k=1}^{T}|\triangle u_n(k)-\triangle u(k)|^{p(k)} & \text{if } p(k)\ge 2,\end{cases} \quad (2.23)$$

with $\breve{u}(k)=|\triangle u_n(k)|+|\triangle u(k)|$.

By using the discrete Hölder inequality (see [23]), we get

$$\begin{aligned}\sum_{k=1}^{T}|\triangle u_n(k)-\triangle u(k)|^{p(k)} &= \sum_{k=1}^{T}\breve{u}(k)^{\frac{p(k)(2-p(k))}{2}}\left(\breve{u}(k)^{\frac{p(k)(p(k)-2)}{2}}|\triangle u_n(k)-\triangle u(k)|^{p(k)}\right)\\ &\leq 2\|\breve{u}^{\frac{p(\cdot)(2-p(\cdot))}{2}}\|_{\frac{2}{2-p(\cdot)}}\|\breve{u}^{\frac{p(\cdot)(p(\cdot)-2)}{2}}|\triangle u_n(k)-\triangle u(k)|^{p(\cdot)}\|_{\frac{2}{p(\cdot)}}\\ &\leq 2\|\breve{u}\|_{p(\cdot)}^{\sigma}\left(\sum_{k=1}^{T}\breve{u}(k)^{p(k)-2}|\triangle u_n(k)-\triangle u(k)|^2\right)^{\gamma},\end{aligned}$$

where σ is either $p^-(2-\tilde{p})/2$ or $\tilde{p}(2-p^-)/2$ and γ is either $p^-/2$ or $\tilde{p}/2$ with $\tilde{p}=\sup_{\{k\in\mathbb{Z}:1<p(k)<2\}}p(k)$. Then, the above inequality combined with (2.20) and (2.23) imply that

$$\sum_{k=1}^{T}|\triangle u_n(k)-\triangle u(k)|^{p(k)}\to 0 \text{ as } n\to\infty. \tag{2.24}$$

Next, we will prove that $\sum_{k=1}^{T}q(k)|u_n(k)|^{p(k)-2}u_n(k)(u_n-u)(k)\to 0$. We first suppose that $k\in\mathbb{Z}(0,T)$ such that $p(k)\geq 2$. For any $u,u_n\in S$, we get by (2.16) that

$$\left(|u_n(k)|^{p(k)-2}u_n(k)-|u(k)|^{p(k)-2}u(k)\right)(u_n(k)-u(k))\geq c_2|u_n(k)-u(k)|^{p(k)}.$$

Thus, summing up k from 1 to T, we get

$$\sum_{k=1}^{T}q(k)|u_n(k)-u(k)|^{p(k)}\leq c_2\sum_{k=1}^{T}q(k)\left(|u_n(k)|^{p(k)-2}u_n(k)-|u(k)|^{p(k)-2}u(k)\right)(u_n(k)-u(k)). \tag{2.25}$$

From (2.19) and (2.25), we obtain

$$\lim_{n\to\infty}\sum_{k=1}^{T}q(k)|u_n(k)-u(k)|^{p(k)}=0. \tag{2.26}$$

Next, for $k\in\mathbb{Z}(0,T)$ such that $1<p(k)<2$, from (2.16), we see that

$$\begin{aligned}&\sum_{k=1}^{T}q(k)\left(|u_n(k)|^{p(k)-2}u_n(k)-|u(k)|^{p(k)-2}u(k)\right)(u_n(k)-u(k))\\ \geq\;& c_1\sum_{k=1}^{T}q(k)\breve{u}(k)^{p(k)-2}|u_n(k)-u(k)|^2,\end{aligned} \tag{2.27}$$

with $\breve{u}(k)=|u_n(k)|+|u(k)|$.
The discrete Hölder inequality (see [23]) implies that

$$\begin{aligned}\sum_{k=1}^{T}q(k)|u_n(k)-u(k)|^{p(k)} &= \sum_{k=1}^{T}q(k)\breve{u}(k)^{\frac{p(k)(2-p(k))}{2}}\left(\breve{u}(k)^{\frac{p(k)(p(k)-2)}{2}}|u_n(k)-u(k)|^{p(k)}\right)\\ &\leq 2\|q(k)\breve{u}^{\frac{p(\cdot)(2-p(\cdot))}{2}}\|_{\frac{2}{2-p(\cdot)}}\|q(k)\breve{u}^{\frac{p(\cdot)(p(\cdot)-2)}{2}}|u_n(k)-u(k)|^{p(\cdot)}\|_{\frac{2}{p(\cdot)}}\\ &\leq 2\|\breve{u}\|_{p(\cdot)}^{\nu}\left(\sum_{k=1}^{T}q(k)\hat{u}(k)^{p(k)-2}|u_n(k)-u(k)|^2\right)^{\varsigma},\end{aligned} \tag{2.28}$$

where ν is either $p^-(2-\hat{p})/2$ or $\hat{p}(2-p^-)/2$ and ς is either $p^-/2$ or $\hat{p}/2$ with

$$\hat{p}=\max_{\{k\in\mathbb{Z}(0,T):1<p(k)<2\}}p(k).$$

Then, using (2.19), (2.27) and (2.28), we deduce that

$$\lim_{n\to\infty}\sum_{k=1}^{T} q(k)|u_n(k)-u(k)|^{p(k)} = 0. \tag{2.29}$$

Relations (2.5), (2.6) combined with (2.24), (2.26) and (2.29) imply that $\|u_n - u\| \to 0$ as $n \to \infty$. Hence, Φ' is of type (S_+). The proof of Lemma 2 is complete.

Lemma 3. *Assume that the hypotheses $(H1)$ and $(H3)$-$(H4)$ are satisfied. Then, the functional $\Phi : S \to \mathbb{R}$ is weakly lower semicontinuous, i.e., $u_n \rightharpoonup u$ in S as $n \to \infty$ implies that $\Phi(u) \le \liminf_{n\to\infty} \Phi(u_n)$.*

Proof. Suppose that $u_n \rightharpoonup u$ in S as $n \to \infty$. Then, by (2.11) and Lemma 2, we obtain that Φ is convex (see[56, Proposition 42.6]) and thus for any $n \in \mathbb{N}$,

$$\Phi(u_n) \ge \Phi(u) + \langle \Phi'(u), u_n - u\rangle.$$

So,

$$\liminf_{n\to\infty} \Phi(u_n) \ge \Phi(u) + \liminf_{n\to\infty} \langle \Phi'(u), u_n - u\rangle = \Phi(u).$$

This proves that Φ is weakly lower semicontinuous, and the proof is complete.

3. Existence of At Least One Nontrivial Solution of (1.1)

In this section, one uses the following result due to Bonanno et *al.* (see [8]).

Theorem 1. [8] *Let X be a finite dimensional Banach space and let $I_\lambda : X \to \mathbb{R}$ be a function satisfying the following structure hypothesis.*

$(\hat{H})$ *$I_\lambda(u) = \Phi(u) - \lambda\Psi(u)$ for all $u \in X$, where $\Phi, \Psi : X \to \mathbb{R}$ are two functions of class C^1 on X with Φ coercive, such that*

$$\inf_X \Phi = \Phi(0) = \Psi(0) = 0,$$

and λ is a real positive parameter.

Then, let $r > 0$, for each $\lambda \in \left(0, \dfrac{r}{\sup\limits_{u\in\Phi^{-1}(0,r)} \Psi(u)}\right)$, the function I_λ admits at least a local minimum $\bar{u} \in X$ such that $\Phi(\bar{u}) < r$, $I_\lambda(\bar{u}) \le I_\lambda(u)$ for all $u \in \Phi^{-1}(0,r)$ and $I'_\lambda(\bar{u}) = 0$.

We have the following.

Theorem 2. *Let ε be a positive constant. Suppose that $f(k,0) \ne 0$ for each $k \in \mathbb{Z}(1,T)$. Then, for each*

$$\lambda \in \left(0, \frac{m_0 \min\{1,c\}}{p^+ \kappa^{p^*} K^{p^*}} \frac{\varepsilon^{p^*}}{\sum\limits_{k=1}^{T} \max\limits_{|t|\le\varepsilon} F(k,t)}\right),$$

problem (1.1) has at least one nontrivial solution $u \in S$ such that $\|u\|_\infty < \varepsilon$.

Proof. Let us apply Theorem 1 to problem (1.1). Take $X = S$ and put Φ, Ψ and I_λ as in (2.7), (2.8) and (2.9), respectively. We know that the functionals Φ and Ψ are of class C^1 and by the definitions of Φ and Ψ, one has $\inf_S \Phi = \Phi(0) = \Psi(0) = 0$. Moreover, Φ is coercive. Indeed, for $u \in S$ such that $\|u\| > 1$, Using (2.3)-(2.5) and $(H2)$-$(H4)$, we get

$$\Phi(u) \ge m_0 \int_0^{\zeta[u]} d\xi \ge \frac{m_0 \min\{1,c\}}{p^+ K_2^{p^-}} \|u\|^{p^-} \to \infty, \quad \text{as} \quad \|u\| \to \infty.$$

So, I_λ satisfies condition $(\hat{H})$ in Theorem 1. Note that the critical points of I_λ are exactly the solutions of problem (1.1).

Now, we denote

$$r := \frac{m_0 \min\{1, c\}}{p^+ \kappa^{p^*} K^{p^*}} \varepsilon^{p^*}$$

and we also use the following notations.

$$\alpha^{p*} := \begin{cases} \alpha^{p^-} & \text{if } \alpha > 1, \\ \alpha^{p^+} & \text{if } \alpha < 1 \end{cases} \quad \text{and} \quad K := \begin{cases} K_1 & \text{if } \|u\| < K_1, \\ K_2 & \text{if } \|u\| > K_2. \end{cases} \tag{3.1}$$

Let $u \in S$ such that $u \in \Phi^{-1}(0, r)$, by $(H2)$-$(H4)$, (2.2)-(2.6) and (3.1), it follows that

$$\|u\|_\infty \leq \kappa K \left(\frac{rp^+}{m_0 \min\{1, c\}} \right)^{1/p^*} = \varepsilon,$$

for all $u \in S$ such that

$$\|u\| \leq K \left(\frac{rp^+}{m_0 \min\{1, c\}} \right)^{1/p^*}.$$

Hence, we infer that

$$\sup_{u \in \Phi^{-1}(0,r)} \Psi(u) = \sup_{u \in \Phi^{-1}(0,r)} \sum_{k=1}^{T} F(k, u(k)) \leq \sum_{k=1}^{T} \max_{|t| \leq \varepsilon} F(k, t).$$

Thus, one has

$$\frac{\sup\limits_{u \in \Phi^{-1}(0,r)} \Psi(u)}{r} \leq \frac{p^+ \kappa^{p^*} K^{p^*}}{m_0 \min\{1, c\}} \frac{\sum\limits_{k=1}^{T} \max\limits_{|t| \leq \varepsilon} F(k, t)}{\varepsilon^{p^*}} < \frac{1}{\lambda}.$$

Consequently, owing to Theorem 1, for every

$$\lambda \in \left(0, \frac{m_0 \min\{1, c\}}{p^+ \kappa^{p^*} K^{p^*}} \frac{\varepsilon^{p^*}}{\sum\limits_{k=1}^{T} \max\limits_{|t| \leq \varepsilon} F(k, t)} \right) \subset \left(0, \frac{r}{\sup\limits_{u \in \Phi^{-1}(0,r)} \Psi(u)} \right),$$

the functional I_λ admits one critical point $u \in S$ such that $\Phi(u) < r$ and therefore u is a nontrivial solution of (1.1) such that $\|u\|_\infty \leq \varepsilon$.
We have the following.

Theorem 3. *Assume that there exist $\varepsilon, b_n > 0$ with $\lim\limits_{n \to \infty} b_n = 0$ such that*

$$\left(\frac{m_1 p^+ (2\overline{a_1} p^- b_n + b_n^{p^-} \max\{1, a_2\}(2 + Q))}{m_0 p^- \min\{1, c\}} \right)^{1/p^*} \kappa K < \varepsilon, \tag{3.2}$$

for each $n \in \mathbb{Z}(1, T)$ and

$$\limsup_{|t| \to 0} \frac{\sum\limits_{k=1}^{T} F(k, t)}{|t|^{p^-}} = \infty \ \text{ for any } \ k \in \mathbb{Z}(1, T). \tag{3.3}$$

Then, for every

$$\lambda \in \left(0, \frac{m_0 \min\{1, c\}}{p^+ \kappa^{p^*} K^{p^*}} \frac{\varepsilon^{p^*}}{\sum\limits_{k=1}^{T} F(k, \varepsilon)} \right),$$

problem (1.1) has at least one nontrivial solution $u \in S$ such that $\|u\|_\infty < \varepsilon$.

Proof. The functionals Φ in (2.7) and Ψ given by (2.8) satisfy the regularity conditions in Theorem 1. Similarly to the proof of Theorem 2, setting

$$r := \frac{m_0 \min\{1, c\}}{p^+ \kappa^{p^*} K^{p^*}} \varepsilon^{p^*}. \tag{3.4}$$

For all $u \in S$ such that $\Phi(u) < r$, we obtain

$$\|u\|_\infty \leq \varepsilon.$$

Hence, we get

$$\sup_{u \in \Phi^{-1}(0,r)} \Psi(u) = \sup_{u \in \Phi^{-1}(0,r)} \sum_{k=1}^{T} F(k, u(k)) \leq \sum_{k=1}^{T} \max_{|t| \leq \varepsilon} F(k, t) = \sum_{k=1}^{T} F(k, \varepsilon).$$

Then, one has

$$\frac{\sup\limits_{u \in \Phi^{-1}(0,r)} \Psi(u)}{r} \leq \frac{p^+ \kappa^{p^*} K^{p^*}}{m_0 \min\{1, c\}} \frac{\sum\limits_{k=1}^{T} F(k, \varepsilon)}{\varepsilon^{p^*}}.$$

Therefore, from Theorem 1, for each

$$\lambda < \frac{m_0 \min\{1, c\}}{p^+ \kappa^{p^*} K^{p^*}} \frac{\varepsilon^{p^*}}{\sum\limits_{k=1}^{T} F(k, \varepsilon)} \leq \frac{r}{\sup\limits_{u \in \Phi^{-1}(0,r)} \Psi(u)},$$

the problem (1.1) admits a solution $u_\lambda \in S$ which is a global minimum of the restriction of the functional I_λ to $\Phi^{-1}(-\infty, r)$.

Now, we prove that u_λ is nontrivial. Indeed, choose a real number

$$\rho > \frac{m_1 \max\{1, a_2\}(2 + Q)}{\lambda p^-}.$$

Since $\limsup\limits_{|t| \to 0} \dfrac{\sum_{k=1}^{T} F(k, t)}{|t|^{p^-}} = \infty$, there exists a sequence $\{b_n\}$ of positive numbers with $b_n \in (0, 1)$ and $b_n \to 0$ as $n \to \infty$ such that

$$\lim_{n \to \infty} \frac{\sum\limits_{k=1}^{T} F(k, b_n)}{|b_n|^{p^-}} = \infty.$$

Hence, there exists $N \in \mathbb{N}$ such that for any $n > N$, one has

$$\sum_{k=1}^{T} F(k, b_n) \geq \rho |b_n|^{p^-}.$$

We define a sequence $\{v_n\}$ in S by $v_n(k) = b_n$ for every $k \in \mathbb{Z}(1, T)$ and $\triangle v_n(0) = v_n(T+1) = 0$. It is easy to see that

$$\begin{aligned} \Phi(v_n) &\leq m_1 \left(2\overline{a_1} b_n + \frac{1}{p^-} \max\{1, a_2\} \left(b_n^{p(0)} + b_n^{p(T)} + \sum_{k=1}^{T} q(k) b_n^{p(k)} \right) \right) \\ &\leq m_1 \left(2\overline{a_1} b_n + \frac{b_n^{p^-} \max\{1, a_2\}(2 + Q)}{p^-} \right). \end{aligned}$$

Moreover, one has

$$\Psi(v_n) = \sum_{k=1}^{T} F(k, b_n) \geq \rho |b_n|^{p^-}.$$

Hence, from the condition (3.2), we get

$$\Phi(v_n) < r.$$

Therefore, we obtain

$$\frac{\Psi(v_n)}{\Phi(v_n)} \geq \frac{p^-\rho|b_n|^{p^-}}{m_1(2\overline{a_1}p^- b_n + b_n^{p^-}\max\{1,a_2\}(2+Q))} \to \infty \text{ as } n \to \infty.$$

Then, there exists a sequence $\{w_n\}$ in S with $w_n \to 0$ as $n \to \infty$ such that, for n sufficiently large, one has

$$w_n \in \Phi^{-1}(-\infty, r) \text{ and } \Phi(w_n)/\Psi(w_n) < \lambda$$

which implies that

$$I_\lambda(w_n) = \Phi(w_n) - \lambda\Psi(w_n) < 0.$$

Note that u_λ is a global minimum of the restriction of I_λ to $\Phi^{-1}(-\infty, r)$, then one has

$$I_\lambda(u_\lambda) \leq I_\lambda(w_n) < 0 = I_\lambda(0).$$

Therefore, u_λ is nontrivial.
Finally, we show that $\|u_\lambda\|_\infty < \varepsilon$.
Note that $\Phi(u_\lambda) < r$, then by $(H2)$-$(H4)$, (2.3)-(2.6) and (3.1), one has

$$\begin{aligned} r > \Phi(u_\lambda) &\geq \frac{m_0 \min\{1,c\}\left(\varphi(u_\lambda) + \sum_{k=1}^{T} q(k)|u_\lambda(k)|^{p(k)}\right)}{p^+} \\ &\geq \frac{m_0 \min\{1,c\}\|u_\lambda\|_{p(\cdot)}^{p^*}}{p^+} \geq \frac{m_0 \min\{1,c\}\|u_\lambda\|^{p^*}}{p^+ K^{p^*}}, \end{aligned}$$

which implies that

$$\|u_\lambda\| < K\left(\frac{rp^+}{m_0 \min\{1,c\}}\right)^{1/p^*}. \tag{3.5}$$

Thus, by (2.2), (3.4) and (3.5), one has

$$\|u_\lambda\|_\infty < \varepsilon.$$

The proof is thus complete.

4. Existence of At Least Two Nontrivial Solutions of (1.1)

In this section, one uses the following theorem due to Bonanno and D'Aguí (see [7]).

Theorem 4. [7] *Let X be a real finite dimensional Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that $\inf_{u \in X} \Phi(u) = \Phi(0) = \Psi(0) = 0$. Assume that there exist $r \in \mathbb{R}$ and $\tilde{u} \in X$, with $0 < \Phi(\tilde{u}) < r$, such that*

- (i) $\sigma = \dfrac{1}{r} \sup_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u) < \dfrac{\Psi(\tilde{u})}{\Phi(\tilde{u})} = \rho,$
- (ii) *for each $\lambda \in \Lambda := (\frac{1}{\rho}, \frac{1}{\sigma})$, the functional $I_\lambda := \Phi - \lambda\Psi$ satisfies the (PS)-condition and it is unbounded from below.*

Then, for each $\lambda \in \Lambda$, the functional I_λ admits at least two non-zero critical points $u_{\lambda,1}, u_{\lambda,2}$ such that $I(u_{\lambda,1}) < 0 < I(u_{\lambda,2})$.

We denote

$$\lambda^* := \frac{(2T)^{\frac{p^+-p^-}{p^-}} L^{p^+} m_1 \max\{1,a_2\}\left(2^{p^-} + \overline{q}\right)^{p^+/p^-}}{(\sigma - \varepsilon)p^-},$$

where

$$0<\varepsilon<\sigma-\frac{(2T)^{\frac{p^+-p^-}{p^-}}L^{p^+}m_1\max\{1,a_2\}\left(2^{p^-}+\overline{q}\right)^{p^+/p^-}}{\lambda p^-}$$

and we make the following additional assumptions.

$(H5)$ There exists σ with $\sigma>\dfrac{(2T)^{\frac{p^+-p^-}{p^-}}L^{p^+}m_1\max\{1,a_2\}\left(2^{p^-}+\overline{q}\right)^{p^+/p^-}}{\lambda p^-}$ such that

$$\liminf_{|t|\to\infty}\frac{F(k,t)}{|t|^{p^+}}\geq\sigma \quad \text{for any} \quad k\in\mathbb{Z}(1,T).$$

$(H6)$ $\displaystyle\sum_{k=1}^{T}\max_{|t|\leq M}|f(k,t)|<\infty$ for all $M>0$.

Lemma 4. *Suppose that $(H1)$, $(H4)$-$(H6)$ hold. Then, for any $\lambda>\lambda^*$, the functional I_λ defined in (2.9) is unbounded from below and satisfies the (PS)-condition.*

Proof. Let us fix $\lambda>\lambda^*$ and let

$$0<\varepsilon<\sigma-\frac{(2T)^{\frac{p^+-p^-}{p^-}}L^{p^+}m_1\max\{1,a_2\}\left(2^{p^-}+\overline{q}\right)^{p^+/p^-}}{\lambda p^-}.$$

Since $\displaystyle\liminf_{|t|\to\infty}\frac{F(k,t)}{|t|^{p^+}}\geq\sigma$ for any $k\in\mathbb{Z}(1,T)$, there exists $\tau>0$ such that

$$F(k,t)\geq(\sigma-\varepsilon)|t|^{p^+} \quad \text{for all} \quad k\in\mathbb{Z}(1,T) \quad \text{and all} \quad t\in\mathbb{R} \quad \text{with } |t|>\tau.$$

Moreover, since $t\to F(k,t)-(\sigma-\varepsilon)|t|^{p^+}$ is continuous on $[-\tau,\tau]$, there exists $C_\tau>0$ such that

$$F(k,t)-(\sigma-\varepsilon)|t|^{p^+}\geq -C_\tau \quad \text{for all } k\in\mathbb{Z}(1,T) \text{ and all } t\in[-\tau,\tau]\,.$$

Therefore, we get

$$F(k,t)\geq(\sigma-\varepsilon)|t|^{p^+}-C_\tau \quad \text{for all } (k,t)\in\mathbb{Z}(1,T)\times\mathbb{R}. \tag{4.1}$$

Thus, by (4.1), $(H1)$ and Lemma $1(iii)$, one has

$$\begin{aligned}
I_\lambda(u) &\leq m_1\int_0^{\zeta[u]}d\xi-\lambda\sum_{k=1}^{T}\left((\sigma-\varepsilon)|u(k)|^{p^+}-C_\tau\right)\\
&\leq m_1\left(2\overline{a_1}\sum_{k=1}^{T}|u(k)|+\frac{\max\{1,a_2\}}{p^-}\left(2^{\frac{p^+-p^-}{p^-}}L^{p^+}\|u\|^{p^+}+(1+\overline{q})T\right)\right)\\
&\quad -\lambda(\sigma-\varepsilon)\sum_{k=1}^{T}|u(k)|^{p^+}+\lambda C_\tau T.
\end{aligned} \tag{4.2}$$

Since

$$\begin{aligned}
\|u\|^{p^-} &\leq 2^{p^-}\sum_{k=1}^{T}|u(k)|^{p^-}+\overline{q}\sum_{k=1}^{T}|u(k)|^{p^-}\\
&\leq \left(2^{p^-}+\overline{q}\right)T^{\frac{p^+-p^-}{p^+}}\left(\sum_{k=1}^{T}|u(k)|^{p^+}\right)^{p^-/p^+},
\end{aligned}$$

then,

$$\sum_{k=1}^{T}|u(k)|^{p^+}\geq\frac{\|u\|^{p^+}}{T^{\frac{p^+-p^-}{p^-}}\left(2^{p^-}+\overline{q}\right)^{p^+/p^-}}. \tag{4.3}$$

From (2.2), (4.2) and (4.3), we see that

$$\begin{aligned} I_\lambda(u) &\leq 2m_1\overline{a_1}T\kappa\|u\| + \left(\frac{m_1 \max\{1,a_2\} 2^{\frac{p^+-p^-}{p^-}} L^{p^+}}{p^-} - \lambda \frac{\sigma-\varepsilon}{T^{\frac{p^+-p^-}{p^-}}\left(2^{p^-}+\overline{q}\right)^{p^+/p^-}}\right)\|u\|^{p^+} \\ &\quad + \frac{m_1\max\{1,a_2\}(1+\overline{q})T}{p^-} + \lambda C_\tau T \\ &= \frac{(\sigma-\varepsilon)(\lambda^*-\lambda)}{T^{\frac{p^+-p^-}{p^-}}\left(2^{p^-}+\overline{q}\right)^{p^+/p^-}}\|u\|^{p^+} + 2m_1\overline{a_1}T\kappa\|u\| + \frac{m_1\max\{1,a_2\}(1+\overline{q})T}{p^-} + \lambda C_\tau T. \end{aligned}$$

Since $\lambda^*-\lambda<0$, then $I_\lambda(u)\to-\infty$ whenever $\|u\|\to\infty$. Hence, I_λ is unbounded from below.

To see that the (PS)-condition is fulfilled, notice that $J_\lambda=-I_\lambda$. We see that J_λ is coercive. Let us take a sequence $\{u_n\}\subset S$ such that $\{J_\lambda(u_n)\}$ is bounded and $\|J'_\lambda(u_n)\|_{S^*}\to 0$ as $n\to\infty$. Since J_λ is coercive, $\{u_n\}$ is bounded in S. Hence, by passing to a subsequence, we may assume that there exists $u\in S$ such that $u_n\rightharpoonup u$ weakly in S. Now, we will prove that $u_n\to u$ in S. Note that $I_\lambda=\Phi-\lambda\Psi$, one has

$$\langle\Phi'(u_n),u_n-u\rangle = -\langle J'_\lambda(u_n),u_n-u\rangle + \lambda\langle\Psi'(u_n),u_n-u\rangle.$$

Since $\|J'_\lambda(u_n)\|_{S^*}\to 0$ and $\{u_n-u\}$ is bounded in S, then by the inequality

$$|-\langle J'_\lambda(u_n),u_n-u\rangle| \leq \|J'_\lambda(u_n)\|_{S^*}\|u_n-u\|$$

we get

$$-\langle J'_\lambda(u_n),u_n-u\rangle\to 0 \quad\text{as}\quad n\to\infty.$$

Moreover, from $(H6)$, there exists $C>0$ such that

$$|\langle\Psi'(u_n),u_n-u\rangle| \leq \sum_{k=1}^{T}\max_{|t|\leq M}|f(k,t)|\,\|u_n-u\| \leq C\|u_n-u\|\to 0, \quad\text{as}\quad n\to\infty.$$

Therefore,

$$\limsup_{n\to\infty}\langle\Phi'(u_n),u_n-u\rangle\leq 0.$$

Then,

$$\limsup_{n\to\infty}\langle\Phi'(u_n)-\Phi'(u),u_n-u\rangle\leq 0.$$

By Lemma 2, one has $u_n\to u$ in S as $n\to\infty$. Thus, I_λ satisfies the (PS)-condition. The proof of Lemma 4 is complete.

Theorem 5. *Assume that there exist two positive constants d and ε with*

$$\varepsilon > \kappa K\left(\frac{m_1p^+}{m_0p^-}\right)^{1/p^*}\left(\frac{2\overline{a_1}dp^- + d^{p^*}\max\{1,a_2\}(2+Q)}{\min\{1,c\}}\right)^{\frac{1}{p^*}} \tag{4.4}$$

such that

$$\frac{\sum_{k=1}^{T}\max_{|t|\leq\varepsilon}F(k,t)}{\varepsilon^{p^*}} < \frac{m_0\min\{1,c\}}{p^+\kappa^{p^*}K^{p^*}}\min\left\{\frac{p^-\sum_{k=1}^{T}F(k,d)}{m_1(2\overline{a_1}dp^-+d^{p^*}\max\{1,a_2\}(2+Q))}, \frac{1}{\lambda^*}\right\}. \tag{4.5}$$

Then, for any

$$\lambda\in\left(\max\left\{\frac{m_1(2\overline{a_1}dp^-+d^{p^*}\max\{1,a_2\}(2+Q))}{p^-\sum_{k=1}^{T}F(k,d)}, \lambda^*\right\}, \frac{\frac{m_0\min\{1,c\}}{p^+\kappa^{p^*}K^{p^*}}\varepsilon^{p^*}}{\sum_{k=1}^{T}\max_{|t|\leq\varepsilon}F(k,t)}\right),$$

problem (1.1) *admits at least two nontrivial solutions.*

Proof. We are going to apply Theorem 4. Take $X = S$ and put Φ, Ψ, I_λ as in (2.7), (2.8) and (2.9), respectively. We know that Φ and Ψ are continuously Gâteaux differentiable. Besides, by the definitions of Φ and Ψ, one has $\inf_{u\in S} \Phi(u) = \Phi(0) = \Psi(0) = 0$. Thus, the regularity assumptions required on Φ and Ψ are verified. Note that the critical points of I_λ are exactly the solutions of problem (1.1). Put

$$r = \frac{m_0 \min\{1, c\}}{p^+ \kappa^{p^*} K^{p^*}} \varepsilon^{p^*}.$$

For all $u \in S$ such that $\Phi(u) < r$, by $(H2)$-$(H4)$, (2.3)-(2.6) and (3.1), one has

$$r > \Phi(u) = \widehat{M}(\zeta[u]) \geq m_0 \int_0^{\zeta[u]} d\xi \geq \frac{m_0 \min\{1, c\}}{p^+ K^{p^*}} \|u\|^{p^*},$$

which implies that

$$\|u\| \leq K \left(\frac{rp^+}{m_0 \min\{1, c\}} \right)^{1/p^*}.$$

From (2.2), we obtain

$$\|u\|_\infty \leq \kappa \|u\| \leq \kappa K \left(\frac{rp^+}{m_0 \min\{1, c\}} \right)^{1/p^*} = \varepsilon.$$

Hence, we have

$$\sup_{u\in\Phi^{-1}(-\infty,r)} \Psi(u) = \sup_{u\in\Phi^{-1}(-\infty,r)} \sum_{k=1}^{T} F(k, u(k)) \leq \sum_{k=1}^{T} \max_{|t|\leq\varepsilon} F(k, t).$$

Therefore, we obtain

$$\frac{\sup_{u\in\Phi^{-1}(-\infty,r)} \Psi(u)}{r} \leq \frac{p^+ \kappa^{p^*} K^{p^*}}{m_0 \min\{1, c\}} \frac{\sum_{k=1}^{T} \max_{|t|\leq\varepsilon} F(k, t)}{\varepsilon^{p^*}}. \tag{4.6}$$

On the other hand, pick $\tilde{u} \in S$, defined as follows.

$$\tilde{u}(k) := \begin{cases} d & \text{if } k \in \mathbb{Z}(1, T), \\ 0 & \text{otherwise.} \end{cases} \tag{4.7}$$

It is easy to verify that

$$\frac{m_0 \min\{1, c\} d^{p^*}}{p^+} (2 + Q) \leq \Phi(\tilde{u}) \leq m_1 \left(2\overline{a_1} d + \frac{d^{p^*} \max\{1, a_2\} (2 + Q)}{p^-} \right) \tag{4.8}$$

and

$$\Psi(\tilde{u}) = \sum_{k=1}^{T} F(k, \tilde{u}(k)) = \sum_{k=1}^{T} F(k, d). \tag{4.9}$$

From the condition (4.4), we obtain $0 < \Phi(\tilde{u}) < r$. Moreover, we have

$$\frac{\Psi(\tilde{u})}{\Phi(\tilde{u})} \geq \frac{p^- \sum_{k=1}^{T} F(k, d)}{m_1(2\overline{a_1} dp^- + d^{p^*} \max\{1, a_2\}(2 + Q))}. \tag{4.10}$$

So, it follows from (4.5), (4.6) and (4.10) that

$$\sigma = \frac{1}{r} \sup_{u\in\Phi^{-1}(]-\infty,r])} \Psi(u) < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})} = \rho.$$

Therefore, the assumption (i) of Theorem 4 is satisfied. Taking into account Lemma 4, all the conditions of Theorem 4 are verified. Thus, from Theorem 4, for each $\lambda \in \Lambda$, the functional I_λ has at least two non-zero critical points $u_{\lambda,1}, u_{\lambda,2} \in S$ such that $I(u_{\lambda,1}) < 0 < I(u_{\lambda,2})$ for all $\lambda \in \Lambda$, which are nontrivial solutions of problem (1.1).

5. Existence of At Least Three Solutions of (1.1)

In this section, one uses the following theorem due to Bonanno (see [10]).

Theorem 6. [10] *Let X be a separable and reflexive real Banach space, $\Phi : X \to \mathbb{R}$ a nonnegative continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume that there exists $u_0 \in X$ such that $\Phi(u_0) = \Psi(u_0) = 0$ and that*

(i) $\lim_{\|u\|\to\infty} (\Phi(u) - \lambda\Psi(u)) = \infty$ *for all* $\lambda \in (0, \infty)$;

(ii) *there are* $r > 0$, $\tilde{u} \in X$ *such that*
$$r < \Phi(\tilde{u});$$

(iii) $\sup_{u\in\Phi^{-1}(-\infty,r)} \Psi(u) < (r/(r + \Phi(\tilde{u})))\, \Psi(\tilde{u})$.

Then, for each

$$\lambda \in \Lambda := \left(\frac{\Phi(\tilde{u})}{\Psi(\tilde{u}) - \sup_{u\in\Phi^{-1}(-\infty,r)} \Psi(u)}, \frac{r}{\sup_{u\in\Phi^{-1}(-\infty,r)} \Psi(u)} \right), \tag{5.1}$$

the equation

$$\Phi'(u) - \lambda\Psi'(u) = 0 \tag{5.2}$$

has at least three solutions in X and, moreover, for each $h > 1$, there exists an open interval

$$\bar{\Lambda} \subseteq \left(0, \frac{hr}{r\left(\Psi(\tilde{u})/\Phi(\tilde{u})\right) - \sup_{u\in\Phi^{-1}(-\infty,r)} \Psi(u)}\right) \tag{5.3}$$

and a positive real number ν such that, for each $\lambda \in \bar{\Lambda}$, (5.2) has at least three solutions in X whose norms are less than ν.

Theorem 7. *Assume that there exist constants $\varepsilon, d, \delta > 0$ with $\varepsilon < K\kappa(2 + Q)^{1/p^*} d$ and $\delta < \dfrac{m_0 \min\{1, c\}}{p^+\lambda T K_2^{p^-} \kappa^{p^-}}$ such that*

(a_1) $\displaystyle\sum_{k=1}^{T} \max_{|t|\le\varepsilon} F(k, t) < \frac{m_0 p^- \min\{1, c\}\varepsilon^{p^*} \sum_{k=1}^{T} F(k, d)}{m_0 p^- \min\{1, c\}\varepsilon^{p^*} + m_1 p^+ K^{p^*}\kappa^{p^*}(2\overline{a_1}p^- d + d^{p^*} \max\{1, a_2\}(2 + Q))}$;

(a_2) $\displaystyle\limsup_{|t|\to\infty} \frac{F(k, t)}{|t|^{p^-}} \le \delta$ *for any* $k \in \mathbb{Z}(1, T)$.

Further, set

$$\begin{aligned}\psi_1 &= \frac{p^+ K^{p^*}\kappa^{p^*} \sum_{k=1}^{T} \max_{|t|\le\varepsilon} F(k, t)}{m_0 \min\{1, c\}\varepsilon^{p^*}},\\ \psi_2 &= \frac{p^-\left[\sum_{k=1}^{T} F(k, d) - \sum_{k=1}^{T} \max_{|t|\le\varepsilon} F(k, t)\right]}{m_1(2\overline{a_1}p^- d + d^{p^*} \max\{1, a_2\}(2 + Q))}\end{aligned}$$

and for every $h > 1$,

$$(a_3) \quad \tilde{a} = \frac{m_0 \min\{1, c\}(2+Q)h(\varepsilon d)^{p^*}}{p^+ \varepsilon^{p^*} \sum_{k=1}^{T} F(k,d) - K^{p^*} \kappa^{p^*} p^+ d^{p^*} (2+Q) \sum_{k=1}^{T} \max_{|t| \le \varepsilon} F(k,t)}.$$

Then, for each

$$\lambda \in \left(\frac{1}{\psi_2}, \frac{1}{\psi_1} \right),$$

the problem (1.1) *has at least three solutions in S and moreover, for every $h > 1$, there exists an open interval $\bar{\Lambda} \subseteq (0, \tilde{a})$ and a positive real number ν such that for each $\lambda \in \bar{\Lambda}$, the problem* (1.1) *has at least three solutions in S, whose norms in S are less than ν.*

Proof. We shall apply Theorem 6. Take $X = S$ and put Φ, Ψ, I_λ as in (2.7), (2.8) and (2.9), respectively. We know that Φ and Ψ are two continuously Gâteaux differentiable functionals. On the other hand, by Lemma 3, Φ is sequentially weakly lower semicontinuous. Now, we will prove that Φ' admits a continuous inverse on S^*. By Lemma 2, Φ' is strictly monotone, which yields that Φ' is an injection. Moreover, it follows from $(H3)$-$(H4)$ and Lemma 1 that

$$\begin{aligned} \langle \Phi'(u), u \rangle &\ge m_0 \min\{1, c\} \sum_{k=1}^{T} \left[|\triangle u(k)|^{p(k)} + q(k)|u(k)|^{p(k)} \right] \\ &\ge m_0 \min\{1, c\} \left(C_1 \min\left\{ \|u\|^{p^-}, \|u\|^{p^+} \right\} - C_2 \right) \to \infty \text{ as } \|u\| \to \infty, \end{aligned}$$

where C_1 and C_2 are positive constants. Hence, by Minty-Browder theorem (see [55]), Φ' is onto. Therefore, $(\Phi')^{-1} : S^* \to S$ exists. Now, we prove that $(\Phi')^{-1}$ is continuous. Let $v, v_n \in S^*$ such that $v_n \to v$ in S^*. Therefore, there exist unique $u, u_n \in S$ such that

$$\Phi'(u) = v \text{ and } \Phi'(u_n) = v_n \text{ for all } n \in \mathbb{N}.$$

The above relations imply that

$$\langle \Phi'(u) - \Phi'(u_n), u - u_n \rangle = \langle v - v_n, u - u_n \rangle \text{ for all } n \in \mathbb{N}.$$

Since $v_n \to v$ in S^*, by the inequality

$$|\langle \Phi'(v) - \Phi'(v_n), v - v_n \rangle| = |\langle u - u_n, v - v_n \rangle| \le \|u - u_n\|_{S^*} \|v - v_n\|,$$

we infer that

$$\langle \Phi'(u) - \Phi'(u_n), u - u_n \rangle \to 0, \text{ as } n \to \infty$$

and so

$$\limsup_{n \to \infty} \langle \Phi'(u) - \Phi'(u_n), u - u_n \rangle \le 0.$$

From Lemma 2, we have $u_n \to u$ as $n \to \infty$. Hence, $(\Phi')^{-1}$ is continuous.

Now, we show that Ψ' is compact. Suppose that $\{u_n\} \subset S$ such that $u_n \to u$ in S as $n \to \infty$. Note that f is continuous, then by (2.8), one has $\Psi(u_n) \to \Psi(u)$ as $n \to \infty$. So, Ψ' is compact.
Then, Φ and Ψ satisfy all regularity assumptions requested in Theorem 6. Next, choose $u_0(k) = 0$ for each $k \in \mathbb{Z}(0, T+1)$, it is easy to see that $u_0 \in S$ and $\Phi(u_0) = \Psi(u_0) = 0$. Note that the critical points of I_λ are exactly the solutions of problem (1.1). Now, we prove that

$$\lim_{\|u\| \to \infty} (\Phi(u) - \lambda \Psi(u)) = \infty.$$

Let us fix $\lambda > 0$ and let ε_∞ satisfy

$$0 < \varepsilon_\infty < \frac{m_0 \min\{1, c\}}{p^+ K_2^{p^-} \lambda T (2T+2)^{p^- - 1}} - \delta.$$

From $\limsup_{|t|\to\infty} \dfrac{F(k,t)}{|t|^{p^-}} \le \delta$, there exists $\varrho > 0$ such that

$$F(k,t) \le (\delta + \varepsilon_\infty)|t|^{p^-} \text{ for all } (k,|t|) \in \mathbb{Z}(1,T) \times (\varrho,\infty).$$

Since $t \to F(k,t) - (\delta + \varepsilon_\infty)|t|^{p^-}$ is continuous on $[-\varrho,\varrho]$, there exists $C_\varrho > 0$ such that

$$F(k,t) - (\delta + \varepsilon_\infty)|t|^{p^-} \le C_\varrho \text{ for all } (k,t) \in \mathbb{Z}(1,T) \times (-\varrho,\varrho).$$

Hence, we obtain

$$F(k,t) \le (\delta + \varepsilon_\infty)|t|^{p^-} + C_\varrho \text{ for all } (k,t) \in \mathbb{Z}(1,T) \times \mathbb{R}.$$

For $u \in S$ with $\|u\| > 1$, the above inequality and by $(H2)$-$(H4)$ and (2.2)-(2.5), we obtain

$$\begin{aligned} I_\lambda(u) &= \widehat{M}(\zeta[u]) - \lambda \sum_{k=1}^{T} F(k,u(k)) \\ &\ge m_0 \int_0^{\zeta[u]} d\xi - \lambda(\delta + \varepsilon_\infty) \sum_{k=1}^{T} \|u\|_\infty^{p^-} - C_\varrho \lambda T \\ &\ge \left(\frac{m_0 \min\{1,c\}}{p^+ K_2^{p^-}} - \lambda(\delta + \varepsilon_\infty) T \kappa^{p^-} \right) \|u\|^{p^-} - C_\varrho \lambda T \to \infty, \end{aligned}$$

as $\|u\| \to \infty$. Then, condition (i) of Theorem 6 is satisfied.

Now, put

$$r = \frac{m_0 \min\{1,c\}}{p^+ \kappa^{p^*} K^{p^*}} \varepsilon^{p^*}.$$

For all $u \in S$ such that $\Phi(u) < r$, by $(H2)$-$(H4)$, (2.3)-(2.6) and (3.1), we have

$$\begin{aligned} r > \Phi(u) &\ge \frac{m_0 \min\{1,c\}}{p^+} \left(\varphi(u) + \sum_{k=1}^{T} q(k)|u(k)|^{p(k)} \right) \\ &\ge \frac{m_0 \min\{1,c\}}{p^+} \|u\|_{p(\cdot)}^{p^*} \ge \frac{m_0 \min\{1,c\}}{p^+ K^{p^*}} \|u\|^{p^*} \end{aligned}$$

and so

$$\|u\| \le K \left(\frac{r p^+}{m_0 \min\{1,c\}} \right)^{1/p^*}.$$

By (2.2), we get

$$\|u\|_\infty \le \kappa \|u\| \le \kappa K \left(\frac{r p^+}{m_0 \min\{1,c\}} \right)^{1/p^*} = \varepsilon.$$

The definition of r and the above inequality ensure that

$$\Phi^{-1}(-\infty,r) \subseteq \{u \in S \text{ such that } \|u\|_\infty \le \varepsilon\}.$$

Therefore, we obtain

$$\sup_{u \in \Phi^{-1}(-\infty,r)} \Psi(u) = \sup_{u \in \Phi^{-1}(-\infty,r)} \sum_{k=1}^{T} F(k,u(k)) \le \sum_{k=1}^{T} \max_{|t| \le \varepsilon} F(k,t). \tag{5.4}$$

Similarly as in the proof of Theorem 5, we denote $\tilde{u}(k)$, $\Phi(\tilde{u})$, $\Psi(\tilde{u})$ as in (4.7), (4.8) and (4.9), respectively. From $\varepsilon < K\kappa(2+Q)^{1/p^*} d$, we get

$$\Phi(\tilde{u}) > r,$$

that is, condition (ii) of Theorem 6.
Moreover, one has

$$\frac{r\Psi(\tilde{u})}{r+\Phi(\tilde{u})} \geq \frac{m_0 p^- \min\{1,c\}\varepsilon^{p^*}\sum_{k=1}^{T}F(k,d)}{m_0 p^- \min\{1,c\}\varepsilon^{p^*} + m_1 p^+ K^{p^*}\kappa^{p^*}(2\overline{a_1}p^- d + d^{p^*}\max\{1,a_2\}(2+Q))}. \tag{5.5}$$

By virtue of (5.4) and (5.5), taking into account (a_1), we have

$$\sup_{u\in\Phi^{-1}(-\infty,r)}\Psi(u) < \frac{r\Psi(\tilde{u})}{r+\Phi(\tilde{u})}.$$

Therefore, condition (iii) of Theorem 6 is satisfied. Furthermore, we have

$$\frac{\Phi(\tilde{u})}{\Psi(\tilde{u}) - \sup_{u\in\Phi^{-1}(-\infty,r)}\Psi(u)} \leq \frac{m_1(2\overline{a_1}p^- d + d^{p^*}\max\{1,a_2\}(2+Q))}{p^-\left[\sum_{k=1}^{T}F(k,d) - \sum_{k=1}^{T}\max_{|t|\leq\varepsilon}F(k,t)\right]} = \frac{1}{\psi_2},$$

$$\frac{r}{\sup_{u\in\Phi^{-1}(-\infty,r)}\Psi(u)} \geq \frac{m_0\min\{1,c\}\varepsilon^{p^*}}{p^+ K^{p^*}\kappa^{p^*}\sum_{k=1}^{T}\max_{|t|\leq\varepsilon}F(k,t)} = \frac{1}{\psi_1}.$$

Hence, by (a_1), we get $\psi_1 < \psi_2$. Thus, for each $\lambda \in \left(\frac{1}{\psi_2}, \frac{1}{\psi_1}\right)$, the problem (1.1) has at least three solutions in S. On the other hand, one has

$$\frac{hr}{r\frac{\Psi(\tilde{u})}{\Phi(\tilde{u})} - \sup_{\Phi(u)<r}\Psi(u)} \leq \frac{hr}{\frac{m_0\min\{1,c\}\varepsilon^{p^*}p^+\sum_{k=1}^{T}F(k,d)}{p^+K^{p^*}\kappa^{p^*}m_0\min\{1,c\}d^{p^*}(2+Q)} - \sum_{k=1}^{T}\max_{|t|\leq\varepsilon}F(k,t)} = \tilde{a},$$

with $\tilde{a}$ given in (a_3).

So, using Theorem 6, for each $h > 1$, there exist an open interval $\bar{\Lambda} \subseteq [0,\tilde{a}]$ and a positive real number ν such that for any $\lambda \in \bar{\Lambda}$, the equation

$$\Phi'(u) - \lambda\Psi'(u) = 0$$

admits at least three solutions in S. Therefore, problem (1.1) admits at least three solutions in S, whose norms are less than ν. The proof of Theorem 7 is complete.

Statements and Declarations

The authors declare that they have no conflict of interest, and the manuscript has no associated data.

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