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QUASI-AVERAGED AND QUASI-DOUBLE AVERAGED MAPPINGS

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Abstract. We introduce the concept of quasi-averaged mappings within quasi-normed linear spaces. Employing these class of mappings, we derive fixed point theorems and demonstrate convergence results for the Krasnoselskij iteration method associated to various types of enriched contractions. This comprehensive analysis encompasses a range of contraction types, such as Chatterjea, Kannan, Bianchini, Ćirić, Hardy-Rogers contractions, and enriched almost contractions in quasi-Banach spaces. In addition, we present the definition of three distinct types of quasi-double averaged mappings associated with weakly enriched contraction mappings. Moreover, we emphasize the presence of fixed points within these mappings, underlining both their existence and uniqueness. Some illustrative examples are furnished to support our theoretical results and effectual generalization. Finally, we present sufficient conditions guaranteeing the equivalence between the set of fixed points of the quasi-double averaged mapping associated with a weakly enriched mapping and that of the weakly enriched contraction mapping itself.

Keywords. Fixed point, Quasi-normed linear space, Quasi-averaged mapping, Quasi-double averaged mapping, Weakly enriched contraction mapping.

1. Introduction

Throughout the course of this paper, we employ the following notations:

$\mathcal{F}(T)$	The set of points that are fixed under the map T.
$QNLS$	Quasi-Normed linear space.
QBS	Quasi-Banach space.
$\mathbb{N}_0$	$\mathbb{N} \cup \{0\}$.
$\mathbb{R}^+$	The set of nonnegative real numbers.
LHS	Left-hand side.
RHS	Right-hand side.
I	Identity mapping.

Fixed point theory caters crucial resources for nonlinear analysis, particularly in examining the presence and approximation of solutions for a range of nonlinear problems in variational inequalities, nonlinear functional equations, optimization and equilibrium problems, etc. Picard operators are irreplaceable gems in nonlinear analysis, specifically within the realm of mappings eloquently labeled as Banach contractions in our mathematical domain and its intersecting areas. Originating from Banach's pioneering work in [2] within the context of a Banach space, this class of operators has emerged as a cornerstone. Cacciopoli, in a noteworthy extension illuminated in [9], gracefully expanded Banach's

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contraction theorem to encompass the elegance of complete metric spaces. The robust connection between the theories of metric spaces and normed spaces has given rise to several foundational concepts initially grounded in metric spaces, later evolving to accommodate normed structures, and vice-versa.

In 2020, Berinde and Păcurar [3] introduced a groundbreaking concept—the extended Banach contractive condition on normed spaces, elegantly termed as enriched contraction mappings in Banach spaces. This innovation encompasses enriched Banach contractions, where the notion of Picard-Banach contractions is seamlessly integrated as one of its distinctive cases. They extended the Chatterjea contractions, Kannan contractive condition, Bianchini mappings in normed spaces called as enriched Chatterjea mappings [4], enriched Kannan contractions [5], enriched Bianchini mappings [5] respectively. They also presented an approach to address problems on split feasibility as well as variational inequalities. These studies are distinguished by two notable features. Firstly, the space under consideration is a Banach space instead of some generalised version of a metric space. Secondly, the chosen iterative scheme is the Krasnoselskij iterative scheme, diverging from the conventional Picard iteration scheme. They provided a demonstration establishing both the existence and exclusivity of fixed points for these enriched mappings within the context of Banach spaces.

Continuing the exploration of enriched contractions within the framework of Banach spaces, Nithiara -yaphaks and Sintunavarat [17] introduced the notion of weak enriched contraction mappings, and proposed a novel extension of an averaged mapping, termed the double averaged mapping. The averaged mapping is created by a convex combination of a self-map T and the identity operator I on a normed linear space using a constant $\lambda \in (0, 1]$. This idea was extended to the double-averaged mapping which contains a quadratic term T^2 and a constant λ is separated into two nonnegative constants β_1 and β_2. They noted that a contraction mapping enhanced with enrichment is Lipschitz continuous, thereby implying its continuity. Consequently, any mapping that is discontinuous cannot be considered as an enriched contraction mapping. However, this limitation is addressed by weak enriched contraction mappings, as there are mappings that are discontinuous that still qualify as weak enriched contraction mappings. The duo also formulated specific conditions to demonstrate that the set of invariant points of a double averaged mapping associated with a weakly enriched contraction mapping is equivalent to that of the original map.

In this manuscript, we introduce the concept of quasi-averaged mappings, a pivotal notion in establishing fixed point theorems for various enriched mappings in Quasi-Banach spaces (QBSs). The establishment of these propositions and theorems is accomplished using the nonconventional iterative approach known as the Krasnoselskij iterative scheme. The fundamental distinction between a norm and a quasi-norm lies in their continuity properties: the former is uniformly continuous whereas the latter is not even continuous, as a result the topologies differ. For each type of mappings, we establish fixed point theorems in a QBS, where suitable conditions on the parameters (dependence on the quasi-index q of QNLS $(\mathcal{Z}, \|\cdot\|, q)$) are imposed in various cases. Specifically, we introduce three types of quasi-double averaged mappings denoted as $\mathcal{T}_{\beta_1 q, \beta_2}$, $\mathcal{T}_{\beta_1, \beta_2 q}$, and $\mathcal{T}_{\beta_1 q, \beta_2 q}$, and establish the existence of fixed points for each of these cases. We derive sufficient conditions that demonstrate the equivalence between the set of fixed points of $\mathcal{T}_{\beta_1, \beta_2}$ and the set of invariant points of the self-map T, where T operates on a closed convex subset $\mathfrak{C}$ of a quasi-normed space $(\mathcal{Z}, \|\cdot\|, q)$. The exploration of these mathematical constructs contributes to a deeper understanding of convergence patterns and fixed point properties within the diverse scope of enriched contraction theories. For some recent papers, one can consult [28, 29].

This paper is organized as follows. In Section 2, preliminaries and a clear problem statement are provided. Write an outline or delineation of the paper here.

2. Preliminaries

Definition 2.1 (QNLS)**.** [15] Suppose that $q \geq 1$ is a given real, and $\mathcal{Z}$ is a vector space over a field $\mathbb{L}$ $(= \mathbb{R}$ or $\mathbb{C})$. A functional $\|\cdot\|\colon \mathcal{Z} \to [0, \infty)$ is said to be a quasi-norm if the following axioms are satisfied:

Q1. $\|a\| = 0$ if and only if $a = 0$,
Q2. $\|\psi a\| = |\psi|\|a\|$, for all $\psi \in \mathbb{L},\ a \in \mathcal{Z}$,
Q3. $\|a + b\| \leq q\,(\|a\| + \|b\|\,)$, for all $a,\ b \in \mathcal{Z}$.

Then, $(\mathcal{Z}, \|\cdot\|, q)$ is called a quasi-normed linear space.

Example 2.2. [19] We know $\mathcal{Z} = \mathbb{R}^2$ is a real vector space. For $a = (a_1, a_2) \in \mathcal{Z}$, we define

$$\|a\| = (\sqrt{|a_1|} + \sqrt{|a_2|})^2.$$

Then, $(\mathcal{Z}, \|\cdot\|, 2)$ is a QNLS. We should note that it is not a normed linear space. Specifically, when we consider normed linear spaces, the symbol $\|\cdot\|$ doesn't function as a typical norm.

Definition 2.3 (QBS)**.** [14] Let $(\mathcal{Z}, \|\cdot\|, q)$ be a QNLS. If the quasi-metric (b-metric) generated by the quasi-norm is complete, then the space $(\mathcal{Z}, \|\cdot\|, q)$ is called a QBS.

Lemma 2.4. [16] *In a b-metric space denoted as $(\mathcal{Z}, d, q)$ with a constant $q \geq 1$, a sequence $(z_n)_{n\in\mathbb{N}}$ is Cauchy if, for every natural number n, there exists a nonnegative constant $\kappa < 1$ such that the inequality $d(z_{n+1}, z_n) \leq \kappa d(z_n, z_{n-1})$ holds.*

In a QNLS, a quasi-norm induces a distance, called b-metric. Therefore, the following is an immediate result from Lemma 2.4.

Lemma 2.5. *A sequence $(z_n)_{n\in\mathbb{N}}$ composed of elements from a QNLS $(\mathcal{Z}, \|\cdot\|, q)$ is a Cauchy sequence if there exists a non-negative constant $\kappa < 1$ such that for every natural number n, the following inequality holds:*

$$\|z_{n+1} - z_n\| \leq \kappa\|z_n - z_{n-1}\|.$$

Definition 2.6 (Averaged Mapping)**.** [1] Suppose that T is a self-map on $\mathcal{Z}$, where $(\mathcal{Z}, \|\cdot\|)$ is a normed linear space. Then the the average mapping $\mathcal{S}_\lambda\colon \mathcal{Z} \to \mathcal{Z}$ is given by

$$\mathcal{S}_\lambda z = (1 - \lambda)z + \lambda T z \quad \text{for all } z \in \mathcal{Z},\ \lambda \in (0, 1].$$

The averaged mapping is created by taking a convex combination of two maps namely the identity mapping I and T using a constant $\lambda \in (0, 1]$.

Remark 2.7. The averaged mapping possesses the characteristic that its collection of fixed points, denoted as $\mathcal{F}(\mathcal{S}_\lambda)$, coincide with the set of points that are invariant or fixed with respect to the original mapping T, expressed as $\mathcal{F}(T)$.

Definition 2.8 (Double-averaged mapping)**.** [17] In a normed linear space $\mathcal{Z}$, a double averaged mapping is defined by considering the identity mapping I and and a self-mapping as

$$T_{\beta_1,\beta_2} := (1 - \beta_1 - \beta_2)I + \beta_1 T + \beta_2 T^2,$$

where $\beta_1 > 0,\ \beta_2 \geq 0$ and $\beta_1 + \beta_2 \in (0, 1]$.

Remark 2.9. The following points connect relationships between the averaged and double-averaged mappings:

- For $\beta_1 = \lambda,\ \beta_2 = 0,\ T_{\beta_1,0}$ reduces to $\mathcal{S}_\lambda$.
- $\mathcal{F}(T) \subseteq \mathcal{F}((T_{\beta_1,\beta_2}))$. This inclusion can be strict.

In the following section, we establish fixed point theorems for different enriched contraction mappings within a QBS framework, where the parameters of these contractions depend on the quasi-index q.

3. Main Results

Definition 3.1 (Quasi-averaged mapping)**.** Suppose that T is a self-map on a QNLS $\mathcal{Z}$ and $\lambda \in (0, 1)$. The quasi-averaged mapping $\mathcal{S}_\lambda$ is defined as

$$\mathcal{S}_\lambda(z) = (1 - \lambda q)z + \lambda q T z \quad \text{for all } z \in \mathcal{Z}. \tag{3.1}$$

Remark 3.2. $\mathcal{F}(\mathcal{S}_\lambda) = \mathcal{F}(T)$.

3.1. **Enriched contraction.**

Definition 3.3 (Enriched contraction)**.** Let $(\mathcal{Z}, \|\cdot\|, q)$ be a QNLS. A mapping $T\colon \mathcal{Z} \to \mathcal{Z}$ is referred to as an enriched contraction if we can find $g \in [0, \infty)$ and $\tau \in [0, g + 1)$ so that the following inequality is true:

$$\|g(y - z) + Ty - Tz\| \le \tau \|y - z\|, \text{ for all } y,\ z \in \mathcal{Z}. \tag{3.2}$$

To designate the scalars g and τ associated to (3.2), we name T as an enriched contraction with parameters g and τ.

Example 3.4. Let $\mathcal{Z} = l^{\frac{1}{2}}$ be endowed with quasi-norm given by

$$\|z\| = \Big(\sum_{j=1}^{\infty} |z_j|^{\frac{1}{2}}\Big)^2,\ z \in l^{\frac{1}{2}}.$$

Let $u = (1, \frac{1}{2^4}, \frac{1}{3^4}, \ldots) \in l^{\frac{1}{2}}$ be a fixed vector and $T\colon \mathcal{Z} \to \mathcal{Z}$ be defined by

$$Tz = u - z, \text{ for all } z \in \mathcal{Z}.$$

Then, $\|Tz_1 - Tz_2\| = \|z_1 - z_2\|$. This means that T is a non-expansive mapping. In particular, it is an isometry.

Demonstrating that T is an enriched contraction in $l^{\frac{1}{2}}$ is equivalent to establishing the following inequality:

$$\|(g - 1)(z_1 - z_2)\| \le \tau \|z_1 - z_2\|, \text{ for all } z_1,\ z_2 \in l^{\frac{1}{2}}.$$

The above inequality is true for all $z_1,\ z_2 \in l^{\frac{1}{2}}$ if we choose $g \ge 1$ and $\tau = g - 1$. Therefore, for any $g \in [1, \infty),\ T$ is an enriched contraction with parameters g and $g - 1$. Note that T has only one fixed point.

Remark 3.5. (1) For $g = 0$, we get a contraction T with parameters 0 and θ, where $\tau \in [0, 1)$.

(2) Following Example 3.4, let us show that T is not considered a contraction. If it were a contraction, we must find a constant $\tau \in [0, 1)$ satisfying

$$\|z_1 - z_2\| \le \tau \|z_1 - z_2\| \text{ for all } z_1,\ z_2 \in l^{\frac{1}{2}}.$$

For any pair of distinct $z_1,\ z_2$, we have $1 \le \tau < 1$, which is a contradiction.

(3) In a QBS, a contraction is always an enriched contraction, although the converse need not hold.

(4) For T in the above example, the Picard iteration generated by sequence $(z_n)_{n\in\mathbb{N}}$ connected with T, specifically, the sequence whose recurrence relation is given by $z_{n+1} = 1 - z_n,\ n \ge 0$, fails to converge for any initial guess z_0 which is not equal to $\frac{u}{2}$, which is the sole element of $\mathcal{F}(T)$.

(5) In the preceding example, it is noteworthy to observe that the map T despite not qualifying the test of a contraction map, $\mathcal{F}(T)$ consists of a single element.

Theorem 3.6. *Let $(\mathcal{Z}, \|\cdot\|, q)$ be a QBS, where $q \geq 1$, and $T\colon \mathcal{Z} \to \mathcal{Z}$ be a (τ, g) enriched contraction. Then, we have the following results:*

- $\mathcal{F}(T) = \{\vartheta\}$;
- *With $\lambda = \frac{1}{q(1+g)} \in (0,1)$, the sequence $(z_n)_{n=0}^{\infty}$ generated by the iterative method*

$$z_{n+1} = (1-\lambda q)z_n + \lambda q T z_n,\ n \in \mathbb{N}_0, \tag{3.3}$$

converges to ϑ, for any $z_0 \in \mathcal{Z}$.

Proof. Our proof consists of two cases.

Case 1. g is positive. Denote $\lambda = \frac{1}{q(1+g)} \in (0,1)$. The enriched contraction rule (3.2) becomes

$$\|(\frac{1}{\lambda q} - 1)(y-z) + Ty - Tz\| \leq \tau\|y-z\| \text{ for all } y,\ z \in \mathcal{Z},$$

which is rewritten as

$$\|\mathcal{S}_\lambda y - \mathcal{S}_\lambda z\| \leq f\|y-z\| \ \text{ for all } y,\ z \in \mathcal{Z}. \tag{3.4}$$

Here, $f = \lambda q \tau = \frac{q\tau}{q(g+1)} \in [0,1)$. Therefore, the quasi-averaged mapping is an f-contraction. Following the definition of the averaged mapping $\mathcal{S}_\lambda$, the Krasnoselskij iterative scheme $(z_n)_{n\in\mathbb{N}}$ defined by (3.3) precisely represents the Picard iteration associated with $\mathcal{S}_\lambda$, that is,

$$z_{n+1} = \mathcal{S}_\lambda z_n,\ n \in \mathbb{N}_0. \tag{3.5}$$

Substituting $y = z_n$ and $z = z_{n-1}$ in (3.4) gives

$$\|z_{n+1} - z_n\| \leq f\|z_n - z_{n-1}\|,\ n \geq 1. \tag{3.6}$$

From Lemma 2.5, $(z_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Consequently, it exhibits convergence within the space $\mathcal{Z}$ with a limit denoted as ϑ. In order to show that ϑ is in $\mathcal{F}(\mathcal{S}_\lambda)$, utilizing relaxed triangle inequality and (3.4) yields:

$$\begin{aligned}\|\vartheta - \mathcal{S}_\lambda\vartheta\| &\leq q[\|\vartheta - z_{n+1}\| + \|z_{n+1} - \mathcal{S}_\lambda\vartheta\|]\\ &\leq q[\|\vartheta - z_{n+1}\| + f\|z_n - \vartheta\|].\end{aligned}$$

As $n \to \infty$, we get $\|\vartheta - \mathcal{S}_\lambda\vartheta\| = 0$, i.e., $\vartheta \in \mathcal{F}(\mathcal{S}_\lambda)$.

Next, we prove that ϑ is the only element of $\mathcal{F}(\mathcal{S}_\lambda)$. On the contrary, assume that $\vartheta \neq \tilde{\vartheta} \in \mathcal{F}(\mathcal{S}_\lambda)$. Then, by (3.4), we have

$$0 < \|\vartheta - \tilde{\vartheta}\| \leq f\|\vartheta - \tilde{\vartheta}\| < \|\vartheta - \tilde{\vartheta}\|, \tag{3.7}$$

which is untrue. Hence, $\mathcal{F}(\mathcal{S}_\lambda) = \{\vartheta\}$.

Since $\mathcal{F}(\mathcal{S}_\lambda) = \mathcal{F}(T),\ \mathcal{F}(T) = \{\vartheta\}$.

Case 2. g is zero. Under this scenario, the map boils down to a usual Banach contraction. The proof is immediate.

□

3.2. **Enriched Chatterjea mapping.**

Definition 3.7 (Enriched Chatterjea mapping)**.** Consider $(\mathcal{Z}, \|\cdot\|, q)$ as a QNLS. A mapping $T\colon \mathcal{Z} \to \mathcal{Z}$ earns the designation of an enriched Chatterjea map if we can find a nonnegative $\overline{k}$ and $\tau \in [0, \frac{1}{2})$ for which the following condition is true:

$$\begin{aligned}\|\overline{k}(y-z) + Ty - Tz\| \leq{}& \tau\, [\|(\overline{k}+1)(y-z) + z - Tz\|\\ &+ \|(\overline{k}+1)(z-y) + y - Ty\|],\end{aligned} \tag{3.8}$$

for all $y,\ z \in \mathcal{Z}$. To emphasise two real scalars in (3.8), we term T as an enriched Chatterjea map with parameters $\overline{k}$ and τ.

Remark 3.8. For $\overline{k} = 0$, the above map becomes a Chatterjea mapping in a QBS.

Example 3.9. Consider the space $\mathcal{Z} = l^p$, where $p = \frac{1}{2}$. This space is equipped with the quasi-norm $\|x\| = \left(\sum_{j=1}^{\infty} |x_j|^{\frac{1}{2}}\right)^2$, and it has a quasi-index denoted as $q = 2^{\frac{1}{p}} = 4$.
Let $T\colon \mathcal{Z} \to \mathcal{Z}$ be defined by $Tz = u - z$, where $u = (1, \frac{1}{2^4}, \frac{1}{3^4}, \dots) \in l^{\frac{1}{2}}$, for all $z \in \mathcal{Z}$.
The simplified expression for the LHS expression of (3.8) is

$$\|\overline{k}(y-z) + Ty - Tz\| = |(\overline{k}-1)|\|y-z\|.$$

The RHS of (3.8) is given by

$$\begin{aligned}\tau\,[\|(\overline{k}+1)(y-z) + z - Tz\| &+ \|(\overline{k}+1)(z-y) + y - Ty\|] \\ &= \tau[\|(\overline{k}+1)y - (\overline{k}-1)z - u\| + \|(\overline{k}+1)z - (\overline{k}-1)y - u\|].\end{aligned}$$

For T to qualify as a quasi-enriched Chatterjea mapping, it is necessary for there to exist values $0 \le \tau < \frac{1}{2}$ and $0 \le k < \infty$ such that the condition expressed in (3.8) is satisfied, which is identical to:

$$|(\overline{k}-1)|\|y-z\| \le \tau[\|(\overline{k}+1)y - (\overline{k}-1)z - u\| + \|(\overline{k}+1)z - (\overline{k}-1)y - u\|]. \tag{3.9}$$

In light of the following inequality,

$$\begin{aligned}2\overline{k}\|y-z\| =& \|[(\overline{k}+1)y - (\overline{k}-1)z - u] - [(\overline{k}+1)z - (\overline{k}-1)y - u]\| \\ \le& q\|(\overline{k}+1)y - (\overline{k}-1)z - u\| + \|(\overline{k}+1)z - (\overline{k}-1)y - u\|,\end{aligned}$$

(3.9) to be satisfied for all $y,\ z \in l^{\frac{1}{2}}$, it is essential that $\frac{|\overline{k}-1|}{2\overline{k}} \le \frac{\tau}{q}$ holds for a specific $\tau \in [0, \frac{1}{2})$.
One possibility is to have $\frac{1}{1+\frac{1}{q}} < \overline{k} \le 1$ and $\frac{(1-\overline{k})q}{2\overline{k}} \le \tau < \frac{1}{2}$. By selecting $\frac{(1-\overline{k})q}{2\overline{k}} = \tau$, we can take $\overline{k} = \frac{1}{\frac{2\tau}{q}+1}$. Consequently, for any $\tau \in [0, \frac{1}{2})$, T qualifies as a $(\frac{1}{\frac{2\tau}{q}+1}, \tau)$-quasi-enriched Chatterjea mapping, and $\mathcal{F}(T) = \frac{u}{2}$.
In the alternative scenario, it is required that $1 \le \overline{k} < \frac{1}{1-\frac{1}{q}}$. By choosing $\overline{k} = \frac{1}{1-\frac{2\tau}{q}}$, for any $\tau \in [0, \frac{1}{2})$, T becomes a $(\frac{1}{1-\frac{2\tau}{q}}, \tau)$-quasi-enriched Chatterjea contraction, and it possesses a unique fixed point $\frac{u}{2}$.

Theorem 3.10. *Consider $(\mathcal{Z}, \|\cdot\|, q)$ as a QBS, where $q \ge 1$. Let $T\colon \mathcal{Z} \to \mathcal{Z}$ be an enriched Chatterjea mapping with parameters $\overline{k}$ and τ. If $\tau < \min\{\frac{1}{q^3}, \frac{1}{2q^2}\}$, then*

- *$\mathcal{F}(T) = \{\vartheta\}$;*
- *With $\lambda = \frac{1}{q(\overline{k}+1)}$, the sequence $(z_n)_{n=0}^{\infty}$ generated by the iterative method*

$$z_{n+1} = (1-\lambda q)z_n + \lambda q T z_n,\ n \ge 0, \tag{3.10}$$

converges to ϑ, for any $z_0 \in \mathcal{Z}$.

Proof. With Remark 3.8, the map is a Chatterjea mapping for $\overline{k} = 0$. Therefore, we only discuss the case for $\overline{k} > 0$. If $\overline{k}$ is positive in (3.8), then set $\lambda = \frac{1}{q(\overline{k}+1)}$. Evidently, $0 < \lambda \le 1$ and (3.8) becomes

$$\begin{aligned}\|(\frac{1}{\lambda q} - 1)(y-z) + Ty - Tz\| \le& \tau[\|\frac{1}{\lambda q}(y-z) + z - Tz\| \\ &+ \|\frac{1}{\lambda q}(z-y) + y - Ty\|]\ \text{ for all }\ y,\ z \in \mathcal{Z},\end{aligned}$$

which is equivalent to

$$\begin{aligned}\|(1-\lambda q)(y-z) + \lambda q(Ty - Tz)\| \le \tau[&\|y - z + \lambda q(z - Tz)\| \\ &+ \|z - y + \lambda q(y - Ty)\|]\ \text{for all}\ y,\ z \in \mathcal{Z}.\end{aligned} \tag{3.11}$$

The last inequality can be expressed more succinctly as

$$\|\mathcal{S}_\lambda y - \mathcal{S}_\lambda z\| \le \tau[\|y - \mathcal{S}_\lambda z\| + \|z - \mathcal{S}_\lambda y\|], \tag{3.12}$$

for all $y,\ z \in \mathcal{Z}$. The iterative scheme produced by $(z_n)_{n\in\mathbb{N}}$ according to (3.10) corresponds to the Picard iteration associated with the quasi-averaged operator, specifically,

$$z_{n+1} = \mathcal{S}_\lambda z_n,\ n \in \mathbb{N}_0.$$

Select $y = z_n$ and $z = z_{n-1}$ and substitute in (3.12) so that

$$\begin{aligned}\|z_{n+1} - z_n\| &\le q\tau\left(\|z_n - z_n\| + \|z_{n-1} - z_{n+1}\|\right)\\ &\le q^2\tau\left[\|z_{n-1} - z_n\| + \|z_n - z_{n+1}\|\right].\end{aligned}$$

After simplification, it reduces to

$$\|z_{n+1} - z_n\| \le \frac{q^2\tau}{1 - q^2\tau}\|z_{n-1} - z_n\|,\ n \in \mathbb{N}.$$

Denoting $\delta = \frac{q^2\tau}{1-q^2\tau}$, we have $0 \le \delta < 1$. It follows from the following.

Case 1. $1 < q < 2$. *Then,* $\tau < \frac{1}{2q^2}$.
Now,

$$\frac{1}{\delta} = \frac{1 - q^2\tau}{q^2\tau} = \frac{1}{q^2\tau} - 1 > 1.$$

Therefore, $0 < \delta < 1$.

Case 2. $q > 2$. *Then,* $\tau < \frac{1}{q^3} \implies \tau q^2 < \frac{1}{q} < \frac{1}{2}$.
Now,

$$\frac{1}{\delta} = \frac{1}{q^2\tau} - 1 > 1.$$

Thus, $(z_n)_{n\in\mathbb{N}}$ follows the following inequality:

$$\|z_{n+1} - z_n\| \le \delta\|z_n - z_{n-1}\|,\ n \in \mathbb{N}.$$

Thus, $(z_n)_{n\in\mathbb{N}}$ is Cauchy in $\mathcal{Z}$. Since $(\mathcal{Z}, \|\cdot\|, q)$ is a QBS, $(z_n)_{n\in\mathbb{N}}$ converges to some element (say ϑ) in $(\mathcal{Z}, \|\cdot||, q)$.

We first prove that $\vartheta \in \mathcal{F}(\mathcal{S}_\lambda)$. We have

$$\begin{aligned}\|\vartheta - \mathcal{S}_\lambda\vartheta\| \le& q\left[\|\vartheta - z_{n+1}\| + \|z_{n+1} - \mathcal{S}_\lambda\vartheta\|\right]\\ =& q\left[\|\vartheta - z_{n+1}\| + \|\mathcal{S}_\lambda z_n - \mathcal{S}_\lambda\vartheta\|\right].\end{aligned} \tag{3.13}$$

From (3.12), we have that

$$\|\mathcal{S}_\lambda z_n - \mathcal{S}_\lambda\vartheta\| \le q\tau[\|z_n - \mathcal{S}_\lambda\vartheta\| + \|\vartheta - \mathcal{S}_\lambda z_n\|].$$

Again arriving at (3.13),

$$\begin{aligned}\|\vartheta - \mathcal{S}_\lambda\vartheta\| &\le q\ \|\vartheta - z_{n+1}\| + q^2\tau\ [\|z_n - \mathcal{S}_\lambda\vartheta\| + \|\vartheta - \mathcal{S}_\lambda z_n\|]\\ &= (q + q^2\tau)(\|\vartheta - z_{n+1}\|)\ + q^2\tau\|z_n - \mathcal{S}_\lambda\vartheta||\\ &\le (q + q^2\tau)\|\vartheta - z_{n+1}\|\ + q^3\tau\ [\|z_n - \vartheta\| + \|\vartheta - \mathcal{S}_\lambda\vartheta\|]\\ \implies (1 - q^3\tau)\|\vartheta - \mathcal{S}_\lambda\vartheta\| &\le (q + q^2\tau)\|z_{n+1} - \vartheta\| + q^3\tau\|z_n - \vartheta\|.\end{aligned}$$

With $\tau < \min\{\frac{1}{q^3}, \frac{1}{2q^2}\}$, it follows that $(1 - q^3\tau) > 0$. Thus, we get $||\vartheta - \mathcal{S}_\lambda\vartheta|| = 0$ as $n \to \infty$. This implies $\mathcal{S}_\lambda\vartheta = \vartheta$.

We now aim to prove that $\mathcal{F}(\mathcal{S}_\lambda)$ contains only ϑ. If not, let $\hat{\vartheta}(\neq \vartheta)$ be another element of $\mathcal{F}(\mathcal{S}_\lambda)$. Then, by (3.12),

$$\|\vartheta - \hat{\vartheta}\| \leq 2\tau\|\vartheta - \hat{\vartheta}\| \implies \|\vartheta - \hat{\vartheta}\| = 0.$$

Hence, $\mathcal{F}(\mathcal{S}_\lambda) = \{\vartheta\}$ and since $\mathcal{F}(T) = \mathcal{F}(\mathcal{S}_\lambda)$, our claim is proven. □

3.3. **Enriched Kannan mapping.**

Definition 3.11 (Enriched Kannan mapping)**.** Consider a Quasi-Normed Linear Space $(\mathcal{Z}, \|\cdot\|, q)$. A mapping $T\colon \mathcal{Z} \to \mathcal{Z}$ earns the title of an enriched Kannan mapping if there exist nonnegative $\overline{k}$ and $\mathfrak{a} \in [0, \frac{1}{2})$ such that the inequality

$$\| \overline{k}(y - z) + Ty - Tz\| \leq \mathfrak{a}[\|y - Ty\| + \|z - Tz\|], \tag{3.14}$$

holds for all $y,\ z \in \mathcal{Z}$. To highlight the scalars in the inequality, we designate T as an enriched Kannan mapping with parameters $\overline{k}$ and $\mathfrak{a}$.

Example 3.12. Consider the space $\mathcal{Z} = l^p$, where $p = \frac{1}{2}$. It is equipped with the quasi-norm $\|x\| = \left(\sum_{j=1}^{\infty} |x_j|^{\frac{1}{2}}\right)^2$, and it has a quasi-index $q = 4$.
Define $T\colon \mathcal{Z} \to \mathcal{Z}$ as $Tx = u - x$, where $u = (1, \frac{1}{2^4}, \frac{1}{3^4}, \dots) \in l^{\frac{1}{2}}$, for all $x \in \mathcal{Z}$.
The simplified expression for the LHS of (3.14) is

$$\|\overline{k}(y - z) + Ty - Tz\| = |(\overline{k} - 1)|\|y - z\|.$$

The RHS expression of (3.14) is given by

$$\mathfrak{a}(\|y - Ty\| + \|z - Tz\|) = \mathfrak{a}[\|2y - u\| + \|2z - u\|].$$

To establish T as an enriched Kannan mapping, it is imperative to find values for $\mathfrak{a} \in [0, \frac{1}{2})$ and $\overline{k} \in [0, \infty)$ such that inequality expressed in (3.14) is satisfied for the above map. This can be reformulated as

$$|(\overline{k} - 1)|\|y - z\| \leq \mathfrak{a}[\|2y - u\| + \|2z - u\|]. \tag{3.15}$$

As

$$2\|y - z\| \leq q[\|2y - u\| + \|2z - u\|]$$

holds for all $y,\ z \in \mathcal{Z}$, it is worth noting that (3.15) holds for any $\mathfrak{a} \in [0, \frac{1}{2})$ if we take $\overline{k} = 1 - \frac{2\mathfrak{a}}{q} \in (1 - \frac{1}{q}, 1]$. Hence, for any $\mathfrak{a} \in [0, \frac{1}{2})$, the mapping T qualifies as an enriched Kannan mapping with parameters $1 - \frac{2\mathfrak{a}}{q}$ and $\mathfrak{a}$, and $\mathcal{F}(T) = \{\frac{u}{2}\}$.
If we take $\overline{k} = 1 + \frac{2\mathfrak{a}}{q} \in [1, 1 + \frac{1}{q})$, (3.15) holds for any $\mathfrak{a} \in [0, \frac{1}{2})$. Therefore, T is characterized as an enriched Kannan mapping with parameters $1 + \frac{2\mathfrak{a}}{q}$, $\mathfrak{a}$ and $\mathcal{F}(T) = \frac{u}{2}$.

Example 3.13. Let $\mathcal{Z} = \mathbb{R}^2$ be endowed with the quasi-norm given by $\|(a, b)\| = (\sqrt{|a|} + \sqrt{|b|})^2$, with the quasi-index $q = 2$.
Let $T\colon \mathcal{Z} \to \mathcal{Z}$ be defined by $Tx = u - x$, where $u = (\zeta, \zeta)$ is a fixed element of $\mathbb{R}^2$. Here T satisfies (14) for certain values of $\bar{k}$ and $\mathfrak{a}$. In particular, if take $\mathfrak{a} = 0.2$, $u = (3, 3)$, $y = (1, 2)$ and $z = (a_1, a_2)$, the LHS of ((3.14)) reduces to $0.2(\sqrt{a_1 - 1} + \sqrt{a_2 - 2})^2$ and the RHS of ((3.14)) reduces to $0.2[4 + (\sqrt{|2a_1 - 3|} + \sqrt{|2a_2 - 3|})^2$, and they are plotted in Figure 1. Here, in general T is an enriched Kannan mapping with parameters $\bar{k} = 1 - \frac{2\mathfrak{a}}{q}$ and $\mathfrak{a}$.

Remark 3.14. We have observed two interesting instances. In Example 3.12, the space is infinite dimensional, while in Example 3.13, T acts on a finite-dimensional space.

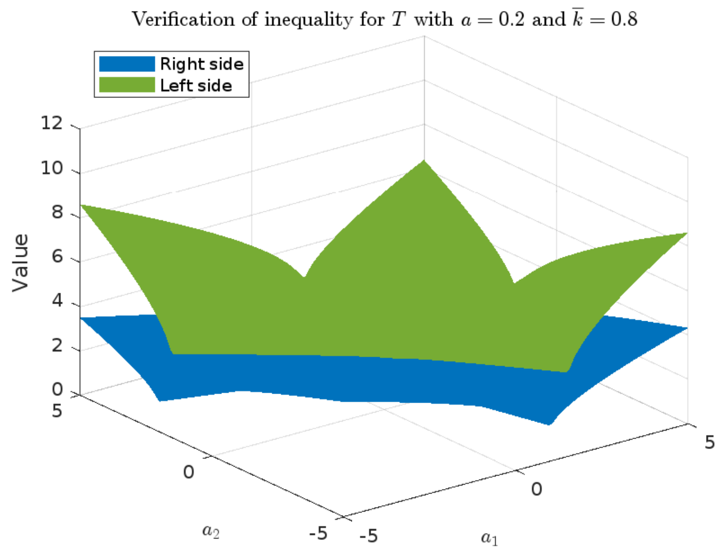

FIGURE 1. Verification of the inequality associated with Kannan enriched mapping

Theorem 3.15. *Let $(\mathcal{Z}, \|\cdot\|, q)$ be a QBS. Suppose that $T\colon \mathcal{Z} \to \mathcal{Z}$ is an enriched Kannan mapping with parameters $\overline{k}$ and $\mathfrak{a}$. If $\mathfrak{a} < \frac{1}{q}$, then*

(1) $\mathcal{F}(T) = \{\tilde{p}\}$;
(2) For any initial value $z_0 \in \mathcal{Z}$, there exists a $\lambda \in (0, 1]$ such that the iterative process $(z_n)_{n\in\mathbb{N}}$ defined through

$$z_{n+1} = (1 - \lambda q)z_n + \lambda q z_n, \; n \in \mathbb{N}_0, \tag{3.16}$$

converges to $\tilde{p}$.

Proof. We will partition our proof into two cases: one for positive values of $\overline{k}$ and other when $\overline{k}$ is zero.

Case 1. $\overline{k} > 0$.

Let λ equal $\frac{1}{q(\overline{k}+1)}$. Clearly, $\lambda \in (0, 1]$ and (3.14) becomes

$$\|(\frac{1}{\lambda q} - 1)(y - z) + Ty - Tz\| \leq \mathfrak{a}[\|\frac{y - \mathcal{S}_\lambda y}{\lambda q}\| + \|\frac{z - \mathcal{S}_\lambda z}{\lambda q}\|]$$

which is

$$\|(1 - \lambda q)(y - z) + \lambda q(Ty - Tz)\| \leq \mathfrak{a}[\|y - \mathcal{S}_\lambda y\| + \|z - \mathcal{S}_\lambda z\|]. \tag{3.17}$$

The aforementioned inequality can be succinctly expressed as

$$\|\mathcal{S}_\lambda y - \mathcal{S}_\lambda z\| \leq \mathfrak{a}[\|y - \mathcal{S}_\lambda y\| + \|z - \mathcal{S}_\lambda z\|], \quad (y, \; z \in \mathcal{Z}). \tag{3.18}$$

In consideration of (3.1), the iterative scheme generated by $(z_n)_{n\in\mathbb{N}}$ given by (3.16) is the Picard scheme related to $\mathcal{S}_\lambda$, equivalently,

$$z_{n+1} = \mathcal{S}_\lambda z_n, \; n \in \mathbb{N}_0.$$

Selecting $y = z_n$ and $z = z_{n-1}$ and substituting in (3.18) yields

$$\|z_{n+1} - z_n\| \leq \mathfrak{a}[\|z_n - z_{n+1}\| + \|z_n - z_{n-1}\|]$$
$$\implies \|z_{n+1} - z_n\| \leq \frac{\mathfrak{a}}{1-\mathfrak{a}}\|z_n - z_{n-1}\|.$$

By defining $\delta = \frac{\mathfrak{a}}{1-\mathfrak{a}}$, where $0 \leq \delta < 1$, $(z_n)_{n\in\mathbb{N}}$ complies with the following

$$\|z_{n+1} - z_n\| \leq \delta\|z_n - z_{n-1}\|,\ n \in \mathbb{N}. \tag{3.19}$$

Consequently, $(z_n)_{n\in\mathbb{N}}$ is Cauchy, implying its convergence in $(\mathcal{Z}, \|\cdot\|, q)$. Let us designate

$$\tilde{p} = \lim_{n\to\infty} z_n.$$

We first establish the inclusion of $\tilde{p}$ in $\mathcal{F}(\mathcal{S}_\lambda)$. One has

$$\begin{aligned}\|\tilde{p} - \mathcal{S}_\lambda\tilde{p}\| &\leq q[\|\tilde{p} - z_{n+1}\| + \|z_{n+1} - \mathcal{S}_\lambda\tilde{p}\|] \\ &= q[\|\tilde{p} - z_{n+1}\| + \|\mathcal{S}_\lambda z_n - \mathcal{S}_\lambda\tilde{p}\|]\end{aligned} \tag{3.20}$$

By replacing $y = z_n$ and $z = \tilde{p}$ into (3.18), we obtain

$$\|\mathcal{S}_\lambda z_n - \mathcal{S}_\lambda\tilde{p}\| \leq \mathfrak{a}\,[\|z_n - z_{n+1}\| + \|\tilde{p} - \mathcal{S}_\lambda\tilde{p}\|].$$

Attention to the last inequality in view of (3.18), one has

$$\|\tilde{p} - \mathcal{S}_\lambda\tilde{p}\| \leq q\|\tilde{p} - z_{n+1}\| + q\mathfrak{a}[\|z_n - z_{n+1}\| + \|\tilde{p} - \mathcal{S}_\lambda\tilde{p}\|]$$
$$\implies (1 - q\mathfrak{a})\|\tilde{p} - \mathcal{S}_\lambda\tilde{p}\| \leq q\|\tilde{p} - z_{n+1}\| + q\mathfrak{a}\|z_n - z_{n+1}\|.$$

Since $(1 - q\mathfrak{a}) > 0$, we obtain

$$\|\tilde{p} - \mathcal{S}_\lambda\tilde{p}\| \leq \frac{q}{1-q\mathfrak{a}}\|\tilde{p} - z_{n+1}\| + \frac{q\mathfrak{a}}{1-q\mathfrak{a}}\|z_n - z_{n+1}\|, \quad n \in \mathbb{N}_0.$$

Now, by tending n to ∞, $||\tilde{p} - \mathcal{S}_\lambda\tilde{p}|| = 0$. Thus, $\tilde{p} \in \mathcal{F}(\mathcal{S}_\lambda)$. We will now present a proof establishing that $\tilde{p}$ is the unique fixed point of $\mathcal{S}_\lambda$. Let's assume the existence of another fixed point $\hat{p}$ (distinct from $\tilde{p}$). Then, by (3.18),

$$||\hat{p} - \tilde{p}|| \leq \mathfrak{a}[||\tilde{p} - \tilde{p}|| + ||\hat{p} - \hat{p}||] = 0,$$

which is not valid unless both coincide. Thus, $\mathcal{F}(\mathcal{S}_\lambda)$ contains only $\tilde{p}$.

Since $\mathcal{F}(\mathcal{S}_\lambda) = \mathcal{F}(T)$, our T has only one fixed point.

Case 2. $\overline{k} = 0$. In this case, the map simplifies to a Kannan mapping. The fixed-point theorem for a Kannan mapping is available in the literature.

3.4. **Enriched Bianchini mapping.**

Definition 3.16 (Enriched Bianchini mapping)**.** Let $(\mathcal{Z}, \|\cdot\|, q)$ be a QNLS. A mapping $T\colon \mathcal{Z} \to \mathcal{Z}$ is referred to as an enriched Bianchini mapping with parameters $\overline{k}$ and h if we can find nonnegative $\overline{k}$ and $h \in [0, 1)$ satisfying the following inequality for all $y,\ z \in \mathcal{Z}$:

$$\|\overline{k}(y - z) + Ty - Tz\| \ \leq h\max\{\|y - Ty\|,\ ||z - Tz\|\}. \tag{3.21}$$

Remark 3.17.

(1) Any enriched Kannan mapping with parameters $\overline{k}$ and a is an enriched Bianchini mapping (parameters are $\overline{k}$ and h) , with $h = 2a$, in light of the result $\frac{s+t}{2} \leq \max\{s, t\}$.

(2) If $s, t \geq 0$, then $\max\{s, t\} \leq s + t$.

(3) An enriched Bianchini mapping is essentially a form of enriched Kannan mapping, provided $h \in [0, \frac{1}{2})$, mindful of the following fact:
If $s, t \geq 0$, then $\max\{s, t\} \leq s + t$.

Theorem 3.18. *Let $(\mathcal{Z}, \|\cdot\|, q)$ be a Quasi-Banach Space and $T\colon \mathcal{Z} \to \mathcal{Z}$ an enriched Bianchini mapping with parameters $\overline{k}$ and h. If $h < \frac{1}{q}$, then the following hold:*

(1) $\mathcal{F}(T) = \{\vartheta\}$*;*

(2) There exists a positive λ*, not larger than one, such that the recursive scheme generated by* $(z_n)_{n\in\mathbb{N}}$*, defined through*

$$z_{n+1} = (1-\lambda q)z_n + \lambda q z_n,\ n \in \mathbb{N}_0, \tag{3.22}$$

converges to ϑ*, for any initial* $z_0 \in \mathcal{Z}$*.*

Proof. Likewise in preceding theorems, we only analyse situation for positive values of $\overline{k}$. We focus on the averaged mapping $\mathcal{S}_\lambda$ for $\lambda = \frac{1}{(\overline{k}+1)q}$ as $0 < \lambda < 1$. Inequality (3.21) becomes

$$||(\frac{1}{\lambda q} - 1)(y-z) + Ty - Tz|| \leq h \max\{||\frac{y - \mathcal{S}_\lambda y}{\lambda q}||, ||\frac{\mathcal{Z} - \mathcal{S}_\lambda \mathcal{Z}}{\lambda q}||\},$$

for all $y,\ z \in \mathcal{Z}$. Rewriting the above equivalently as

$$\|\mathcal{S}_\lambda y - \mathcal{S}_\lambda z\| \leq h \max\{\|y - \mathcal{S}_\lambda y\|, \|z - \mathcal{S}_\lambda z\|\}, \quad (y,\ z \in \mathcal{Z}). \tag{3.23}$$

In accordance with (3.1), (3.22) reduces to $z_{n+1} = \mathcal{S}_\lambda z_n, \quad n \in \mathbb{N}_0$.

Selecting $y = z_n$ and $z = z_{n-1}$ and substituting in (3.23) yields

$$\|z_{n+1} - z_n\| \leq h \max\{\|z_n - z_{n+1}\|, \|z_{n-1} - z_n\|\}.$$

If

$$\max\{\|z_n - z_{n+1}\|, \|z_{n-1} - z_n\|\} = \|z_n - z_{n+1}\|,$$

then

$$\|z_{n+1} - z_n\| \leq h\|z_n - z_{n+1}\| < \|z_n - z_{n+1}\|.$$

This statement presents a contradictory assertion. If

$$\max\{\|z_n - z_{n+1}\|, \|z_{n-1} - z_n\|\} = \|z_n - z_{n-1}\|,$$

then

$$\|z_{n+1} - z_n\| \leq h\|z_n - z_{n-1}\|. \tag{3.24}$$

Given that $0 \leq h < 1$, the sequence $(z_n)n \in \mathbb{N}$ is Cauchy. With the assumption that $\mathcal{Z}$ is a QBS, there exists a $\vartheta \in \mathcal{Z}$ such that $\lim\limits_{n\to\infty} z_n = \vartheta$. Let us demonstrate that ϑ is an element of $\mathcal{F}(\mathcal{S}_\lambda)$.

We have

$$\|\vartheta - \mathcal{S}_\lambda \vartheta\| \leq q\ [\|z_{n+1} - \vartheta\| + \|\mathcal{S}_\lambda \vartheta - \mathcal{S}_\lambda z_n\|]. \tag{3.25}$$

By (3.23),

$$\|\mathcal{S}_\lambda z_n - \mathcal{S}_\lambda \vartheta\| \leq h \max\{\|z_n - \mathcal{S}_\lambda z_n\|, \|\vartheta - \mathcal{S}_\lambda \vartheta\|\}$$

Now, if

$$\max\{\|z_n - \mathcal{S}_\lambda z_n\|, \|\vartheta - \mathcal{S}_\lambda \vartheta\|\} = \|z_n - \mathcal{S}_\lambda z_n\|,$$

then one has

$$\begin{aligned}\|\mathcal{S}_\lambda z_n - \mathcal{S}_\lambda \vartheta\| &\leq h\ \|z_n - \mathcal{S}_\lambda z_n\| \\ &= h\ \|z_n - z_{n+1}\|.\end{aligned}$$

By (3.25), one obtains

$$\|\vartheta - \mathcal{S}_\lambda \vartheta\| \leq q\ \|z_{n+1} - \vartheta\| + qh\ \|z_{n+1} - z_n\|,\ n \in \mathbb{N}_0.$$

Tending n to infinity in the above inequality, we have $||\vartheta - \mathcal{S}_\lambda \vartheta|| = 0$, that is, $\vartheta \in \mathcal{F}(\mathcal{S}_\lambda)$.

If

$$\max\{\|z_n - \mathcal{S}_\lambda z_n\|, \|\vartheta - \mathcal{S}_\lambda \vartheta\|\} = \|\vartheta - \mathcal{S}_\lambda \vartheta\|,$$

Then, by (3.25), we obtain

$$\begin{aligned} &\|\vartheta - \mathcal{S}_\lambda \vartheta\| \leq q\|z_{n+1} - \vartheta\| + qh\|\vartheta - \mathcal{S}_\lambda \vartheta\| \\ \implies &(1 - qh)\|\vartheta - \mathcal{S}_\lambda \vartheta\| \leq q\|z_{n+1} - \vartheta\| \\ \implies &\|\vartheta - \mathcal{S}_\lambda \vartheta\| \leq \frac{q}{(1-qh)}\|z_{n+1} - \vartheta\|,\ n \geq 0, \end{aligned}$$

As $n \to \infty$, we get

$$\|\vartheta - \mathcal{S}_\lambda \vartheta\| = 0 \implies \vartheta \in \mathcal{F}(\mathcal{S}_\lambda).$$

To prove the unique invariant point of $\mathcal{S}_\lambda$ as ϑ, if not let $\hat{\vartheta} \neq \vartheta$ be a different element of $\mathcal{F}(\mathcal{S}_\lambda)$. Then, by (3.23),

$$||\vartheta - \hat{\vartheta}|| \leq h[\ ||\vartheta - \vartheta|| + ||\hat{\vartheta} - \hat{\vartheta}||\ ] = 0,$$

which can not be true. Hence, $\mathcal{F}(\mathcal{S}_\lambda) = \{\vartheta\}$.
Since $\mathcal{F}(\mathcal{S}_\lambda) = \mathcal{F}(T)$, our T has only one fixed point. □

3.5. **Enriched Hardy-Rogers contraction.**

Definition 3.19 (Enriched Hardy-Rogers contraction)**.** Let $(\mathcal{Z}, \|\cdot\|, q)$ be a QNLS, $a, b, c, d, e, \overline{k} \in \mathbb{R}^+$. If $T\colon \mathcal{Z} \to \mathcal{Z}$ satisfies

$$\begin{aligned} \|\overline{k}(y-z) + Ty - Tz\| \leq &a\|y - z\| + b\|y - Ty\| + c\|z - Tz\| + d\|(\overline{k}+1)(y-z) \\ &+ z - Tz\| + e\|(\overline{k}+1)(z-y) + y - Ty\|\ \forall\, y,\ z \in \mathcal{Z}, \end{aligned} \tag{3.26}$$

then T is called an enriched Hardy-Rogers contraction.

Example 3.20. Let $\mathcal{Z} = l^p$, where $p = \frac{1}{2}$, be equipped with the quasi-norm $\|x\| = (\sum_j |x_j|^{\frac{1}{2}})^2$ with the quasi-index $q = 4$. Define $T\colon \mathcal{Z} \to \mathcal{Z}$ as $Tx = -x$, for all $x \in \mathcal{Z}$.
The simplified expression for the LHS of inequality (3.26) is $|(\overline{k}-1)|\|y - z\|$. The RHS of (3.26) is given by

$$\begin{aligned} &a\|y - z\| + b\|y - Ty\| + c\|z - Tz\| + d\|(\overline{k}+1)(y-z) + z - Tz\| + e\|(\overline{k}+1)(z-y) + y - Ty\| \\ =&a\|y - z\| + b\|2y\| + c\|2z\| + d\|(\overline{k}+1)(y-z) + 2z\| + e\|(\overline{k}+1)(z-y) + 2y\|. \end{aligned}$$

For T to be classified as an enriched Hardy-Rogers mapping, it is necessary for there to exist $\overline{k},\ a,\ b,\ c,\ d,\ e \in \mathbb{R}^+$ such that (3.26) holds, which is equivalent to

$$\begin{aligned} |(\overline{k}-1)|\|y - z\| \leq a\|y - z\| + b\|2y\| + c\|2z\| + d\|(\overline{k}+1)(y-z) + 2z\| \\ + e\|(\overline{k}+1)(z-y) + 2y\|. \end{aligned} \tag{3.27}$$

It is very true that (3.27) holds if $|(\overline{k}-1)| \leq a$.
Now, taking $\overline{k} = (1+a) \in [1, \infty)$, our mapping T qualifies as a quasi-enriched Hardy-Rogers contraction with parameters $1 + a$ and a.

Theorem 3.21. *Let $(\mathcal{Z}, \|\cdot\|, q)$ be a Quasi-Banach space, and $T\colon \mathcal{Z} \to \mathcal{Z}$ be an enriched Hardy-Rogers contraction. If $b + eq < 1$, $aq + d + e < 1$, $(1 - cq - dq^2) > 0$ and $aq + b + c + 2eq < 1$, then*

(1) $\mathcal{F}(T) = \{p\}$;
(2) we can find a $\lambda \in (0, 1]$ for which the iterative scheme $(z_n)_{n\in\mathbb{N}}$ prescribed by

$$z_{n+1} = (1 - \lambda q)z_n + \lambda q T z_n,\ n \in \mathbb{N}_0, \tag{3.28}$$

is convergent to p, for any $z_0 \in \mathcal{Z}$.

Proof. If $\overline{k}$ is positive in (3.26), then choose $\overline{k} = \frac{1}{\lambda q} - 1$. Obviously, $0 < \lambda < 1$. Thus equation (3.26) becomes

$$\|(\frac{1}{\lambda q} - 1)(y - z) + Ty - Tz\| \leq a\|y - z\| + b\|y - Ty\| + c\|z - Tz\| \\ + d\|\frac{1}{\lambda q}(y - z) + z - Tz\| + e\|\frac{1}{\lambda q}(\mathcal{Z} - y) + y - Ty\|,$$

for all $y,\ z \in \mathcal{Z}$. Whence

$$\|\mathcal{S}_\lambda y - \mathcal{S}_\lambda z\| \leq a\lambda q\|y - z\| + b\|y - \mathcal{S}_\lambda y\| + c\|z - \mathcal{S}_\lambda z\| + d\|y - \mathcal{S}_\lambda z\| + e\|z - \mathcal{S}_\lambda y\|.$$

Since $a\lambda \leq a$, we have

$$\|\mathcal{S}_\lambda y - \mathcal{S}_\lambda z\| \leq aq\|y - z\| + b\|y - \mathcal{S}_\lambda y\| + c\|z - \mathcal{S}_\lambda z\| + d\|y - \mathcal{S}_\lambda z\| + e\|z - \mathcal{S}_\lambda y\|, \qquad (3.29)$$

for all $y,\ z \in \mathcal{Z}$. According to the definition of the averaged mapping, the iterative process of the sequence $(z_n)_{n\in\mathbb{N}}$ defined by (3.28) corresponds to the Picard iteration associated with $\mathcal{S}_\lambda$, equivalently,

$$z_{n+1} = \mathcal{S}_\lambda z_n,\ n \in \mathbb{N}_0.$$

Plugging $y = z_n$ and $z = z_{n-1}$ into (3.29), we can deduce that

$$\begin{aligned} \|z_{n+1} - z_n\| \leq& aq\|z_n - z_{n-1}\| + b\|z_n - z_{n+1}\| + c\|z_{n-1} - z_n\| \\ &+ d\|z_n - z_n\| + e\|z_{n+1} - z_{n-1}\| \\ \implies (1-b)\|z_{n+1} - z_n\| \leq& (aq + c)\|z_n - z_{n-1}\| + e\|z_{n+1} - z_{n-1}\| \\ \implies (1-b-qe)\|z_{n+1} - z_n\| \leq& (aq + c)\|z_n - z_{n-1}\| + eq\|z_n - z_{n-1}\|) \\ \implies (1-b-qe)\|z_{n+1} - z_n\| \leq& (aq + c + eq)\|z_n - z_{n-1}\|) \\ \implies \|z_{n+1} - z_n\| \leq& \delta\|z_n - z_{n-1}\|), \end{aligned}$$

where $\delta = \frac{(aq+c+eq)}{(1-b-eq)} < 1$. Hence, the sequence $(z_n)_{n\in\mathbb{N}}$ is Cauchy in $\mathcal{Z}$. As $\mathcal{Z}$ is a QBS, it converges in $\mathcal{Z}$, and let the limit be p. To demonstrate that p is an element of $\mathcal{F}(\mathcal{S}_\lambda)$, let us analyse the expression:

$$\begin{aligned} \|p - \mathcal{S}_\lambda p\| &\leq q[\|p - z_{n+1}\| + \|z_{n+1} - \mathcal{S}_\lambda p\|] \\ &= q\|p - z_{n+1}\| + q\|\mathcal{S}_\lambda z_n - \mathcal{S}_\lambda p\|. \end{aligned} \qquad (3.30)$$

We substitute $y = z_n$ and $z = p$ in (3.29), we have

$$\begin{aligned} \|\mathcal{S}_\lambda z_n - \mathcal{S}_\lambda p\| \leq& aq\|z_n - p\| + b\|z_n - z_{n+1}\| + c\|p - \mathcal{S}_\lambda p\| + d\|z_n - \mathcal{S}_\lambda p\| \\ &+ e\|p - z_{n+1}\|. \end{aligned}$$

$$\begin{aligned} \|\mathcal{S}_\lambda z_n - \mathcal{S}_\lambda p\| \leq& aq\|z_n - p\| + b\|z_n - z_{n+1}\| + c\|p - \mathcal{S}_\lambda p\| + dq(\|z_n - p\| + \|p - \mathcal{S}_\lambda p\|) \\ &+ e\|p - z_{n+1}\|. \end{aligned} \qquad (3.31)$$

Again, employing (3.31) in (3.30), we get,

$$\begin{aligned} \|p - \mathcal{S}_\lambda p\| \leq& q\|p - z_{n+1}\| + q[aq\|z_n - p\| + b\|z_n - z_{n+1}\| + c\|p - \mathcal{S}_\lambda p\| + dq\|z_n - p\| \\ &+ dq\|p - \mathcal{S}_\lambda p\| + e\|p - z_{n+1}\|]. \end{aligned}$$

Taking the limit as $n \to \infty$ on both sides, we obtain

$$\begin{aligned} \|p - \mathcal{S}_\lambda p\| &\leq cq\|p - \mathcal{S}_\lambda p\| + dq^2\|p - \mathcal{S}_\lambda p\| \\ \implies (1 - cq)\|p - \mathcal{S}_\lambda p\| &\leq dq^2(\|p - \mathcal{S}_\lambda p\| \\ \implies (1 - cq - dq^2)\|p - \mathcal{S}_\lambda p\| &\leq 0. \end{aligned}$$

Since $(1-cq-dq^2)>0,\ p$ is an element of $\mathcal{F}(\mathcal{S}_\lambda)$.
We now establish that p is the only point that is invariant under $\mathcal{S}_\lambda$. Assume that $\hat{p}\neq p$ is another element of $\mathcal{F}(\mathcal{S}_\lambda)$. Then, owing to (3.29),

$$\begin{aligned}\|p-\hat{p}\| &= \|\mathcal{S}_\lambda p-\mathcal{S}_\lambda\hat{p}\|\\ &\leq qa\|p-\hat{p}\|+b\|p-\mathcal{S}_\lambda p\|+c\|\hat{p}-\mathcal{S}_\lambda\hat{p}\|+d\|p-\mathcal{S}_\lambda\hat{p}\|+e\|\hat{p}-\mathcal{S}_\lambda p\|\\ &= (a\cdot q+d+e)\|p-\hat{p}\|.\end{aligned}$$

Since $q\geq 1$ and a,c,e are nonnegative,

$$\|p-\hat{p}\|<\|p-\hat{p}\|,$$

which can not be true. Hence, $\mathcal{F}(\mathcal{S}_\lambda)=\{p\}$. As $\mathcal{F}(T)=\mathcal{F}(\mathcal{S}_\lambda)$, it follows that T has a unique fixed point. □

4. Weakly Enriched Mapping

In this section, we explore the concept of points that are fixed under a weakly enriched mapping, which serves as a generalization encompassing both contraction mappings in a Banach space and enriched contraction mappings in a QNLS. This is done with quasi-double averaged mappings defined on convex subsets.

Definition 4.1 (Weakly enriched mapping)**.** Consider a convex subset $\mathfrak{C}$ of a Quasi-normed linear space denoted by $(\mathcal{Z},\|\cdot\|,q)$. Let T be a self-mapping on $\mathfrak{C}$. If there exist nonnegative real numbers $\mathcal{A},\ \mathcal{B}$, and $\mu\in[0,1+\mathcal{A}+\mathcal{B})$ such that T adheres to the inequality

$$\|\mathcal{A}(y-z)+Ty-Tz+\mathcal{B}(T^2y-T^2z)\|\leq\mu\|y-z\|,\forall\, y,\ z\in\mathfrak{C}, \tag{4.1}$$

then T is termed as a weakly enriched mapping with parameters $\mathcal{A},\mathcal{B},$ and μ.

Remark 4.2. On setting $\mathcal{B}=0$ in (4.1), the inequality boils down to an enriched contraction mapping.

Now, let us see an example of a weakly enriched contraction mapping in a QNLS.

Example 4.3. Let $\mathcal{Z}=\mathbb{R}^2$ be equipped with the quasi-norm given by $\|(a_1,a_2)\|=(\sqrt{|a_1|}+\sqrt{|a_2|})^2$, with the quasi-index $q=2$.
Consider a convex subset of $\mathbb{R}^2$ as $C=[-1,1]\times[-1,1]$. Define $T\colon\mathcal{Z}\to\mathcal{Z}$ as

$$T(a_1,a_2)=\begin{cases}(0,a_2^2), & \text{if } (a_1,a_2)\in[-1,1]\times[-1,0),\\ (0,1-a_2), & \text{if } (a_1,a_2)\in[-1,1]\times[0,1].\end{cases}$$

We claim that T functions as a weakly enriched mapping for $\mathcal{A}=1,\ \mathcal{B}=1$ and any $\mu\in[1,\mathcal{A}+\mathcal{B}+1]$.
Now, we consider three cases for $y=(a_1,a_2)$ and $z=(c_1,c_2)$.
Case 1. Take $y,\ z\in[-1,1]\times[-1,0]$. We have

$$\begin{aligned}\|\mathcal{A}(y-z)+Ty-Tz+\mathcal{B}(T^2y-T^2z\| &= \|(y-z)+((0,a_2^2)-(0,c_2^2))+((0,1-a_2^2)-(0,1-c_2^2))\|\\ &\leq\mu\|y-z\|.\end{aligned}$$

Case 2. Take $y,\ z\in[-1,1]\times[0,1]$. One has

$$\begin{aligned}\|\mathcal{A}(y-z)+Ty-Tz+\mathcal{B}(T^2y-T^2z\| &= \|(y-z)+((0,1-a_2)-(0,1-c_2))+((0,a_2)-(0,c_2))\|\\ &\leq\mu\|y-z\|.\end{aligned}$$

Case 3. Take $y\in[-1,1]\times[-1,0)$ and $z\in[-1,1]\times[0,1]$.
We have

$$\begin{aligned}\|\mathcal{A}(y-z)+Ty-Tz+\mathcal{B}(T^2y-T^2z\| &= \|(y-z)+((0,a_2^2)-(0,1-c_2))+((0,1-a_2^2)-(0,c_2))\|\\ &\leq\mu\|y-z\|.\end{aligned}$$

Case 4. Take $z \in [-1,1] \times [-1,0)$ and $y \in [-1,1] \times [0,1]$.
We follow the same steps as we did in Case 3.

Fixing the values of $\mathcal{A}$, $\mathcal{B}$ and μ, inequality (4.1) is verified numerically in the following two figures:

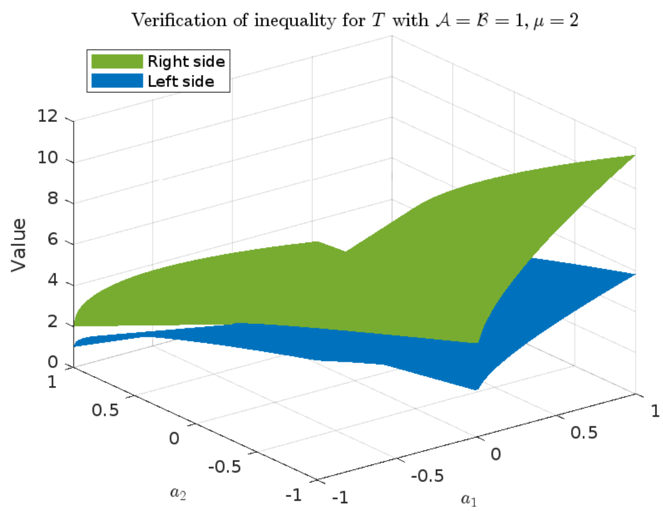

FIGURE 2. 3D plot of LHS and RHS of (4.1) with $y = (0,1)$ and $z = (a_1, a_2)$

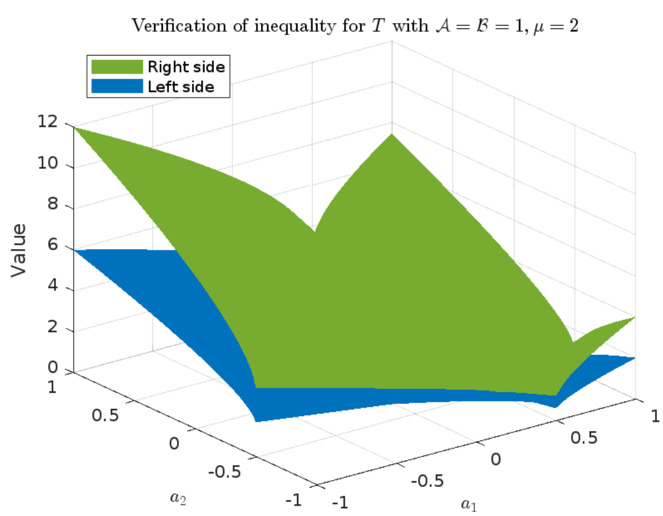

FIGURE 3. 3D plot of LHS and RHS of (4.1) with $y = (1/2, -1/2)$ and $z = (a_1, a_2)$

Ergo, it follows that T can be regarded as a gracefully weakly enriched contraction mapping, where the parameters $\mathcal{A}$ and $\mathcal{B}$ are both assigned the value of 1. This holds true for any μ falling within the captivating range of $[0, \mathcal{A} + \mathcal{B} + 1)$.

4.1. **Quasi-double averaged mapping of type -I.**

Definition 4.4. Suppose that $\mathfrak{C}$ is a convex subset of $\mathcal{Z}$, where $(\mathcal{Z}, \|\cdot\|, q)$ is a QNLS. We define a new self-mapping on $\mathfrak{C}$ as

$$\mathcal{T}_{q\beta_1,\beta_2}(x) := (1 - q\beta_1 - \beta_2)x + q\beta_1 Tx + \beta_2 T^2 x,$$

where $\beta_1 > 0,\ \beta_2 \geq 0$ and $\beta_1 + \beta_2 \in (0,1]$.

Remark 4.5. For $q = 1,\ \mathcal{T}_{q\beta_1,\beta_2}$ becomes a double-averaged mapping T_{β_1,β_2} in a normed linear space [17].

Remark 4.6. For $\beta_1 = \lambda,\ q = 1, \beta_2 = 0,\ \mathcal{T}_{q\beta_1,\beta_2}$ reduces to $\mathcal{S}_\lambda$ in a normed linear space, that is, $\mathcal{T}_{q\beta_1,\beta_2}$ is a generalization of $\mathcal{S}_\lambda$.

Remark 4.7. If $\beta_2 = 0,\ \mathcal{T}_{q\beta_1,\beta_2}$ becomes a quasi-averaged mapping $\mathcal{S}_\lambda$ in a QNLS.

In the upcoming theorem, we establish the existence and uniqueness of an invariant point for $\mathcal{T}_{q\beta 1,\beta_2}$.

Theorem 4.8. *Consider a closed convex subset $\mathfrak{C}$ of a QBS $(\mathcal{Z}, \|\cdot\|, q)$. Suppose that T is a weakly enriched contraction mapping on $\mathfrak{C}$. Then, we can find constants $\beta_1 > 0$ and $\beta_2 \geq 0$ with $\beta_1 q + \beta_2 \in (0,1]$ for which the following results are valid:*

$\mathfrak{A}$. *$\mathcal{F}(\mathcal{T}_{q\beta_1,\beta_2})$ contains only one element.*

$\mathfrak{B}$. *For any given $z_0 \in \mathfrak{C}, (z_n) \subset \mathfrak{C}$ defined by the iteration:*

$$z_n = (1 - q\beta_1 - \beta_2)z_{n-1} + q\beta_1 T z_{n-1} + \beta_2 T^2 z_{n-1},\ n \in \mathbb{N}$$

converges to the only member of $\mathcal{F}(\mathcal{T}_{q\beta_1,\beta_2})$.

Proof. In the context of T being a weakly enriched contraction, there exist nonnegative constants $\mathcal{A},\ \mathcal{B}$ satisfying equation (4.1). In the case $\mathcal{B} = 0$, T transforms into an enriched contraction mapping in a QNLS. The objective is to prove this for nonnegative values of $\mathcal{A}$ and positive values of $\mathcal{B}$. Set $\beta_1 := \frac{1}{q(\mathcal{A}+\mathcal{B}+1)} > 0$ and $\beta_2 := \frac{\mathcal{B}}{\mathcal{A}+\mathcal{B}+1} \geq 0$. Then, (4.1) becomes

$$\|\frac{(1 - \beta_1 q_1 - \beta_2)(y - z) + \beta_1 q(Ty - Tz) + \beta_2(T^2 y - T^2 z)}{q\beta_1}\| \leq \mu\|y - z\|.$$

Since $q\beta_1 > 0$, the above inequality becomes

$$\|(1 - \beta_1 q_1 - \beta_2)(y - z) + \beta_1 q(Ty - Tz) + \beta_2(T^2 y - T^2 z)\| \leq q\beta_1\mu\|y - z\|.$$

Expressing the above inequality in terms of double averaged mapping, we get

$$\|\mathcal{T}_{q\beta_1,\beta_2} y - \mathcal{T}_{q\beta_1,\beta_2} z\| \leq \zeta\|y - z\| \quad (y,\ z \in \mathfrak{C}), \tag{4.2}$$

where $0 \leq \zeta = q\beta_1\mu < 1$.
Given an initial point z_0 in the closed convex subset $\mathfrak{C}$, we define a sequence $(z_n)_{n\in\mathbb{N}} \subseteq \mathfrak{C}$ as follows

$$z_n = \mathcal{T}_{q\beta_1,\beta_2} z_{n-1},\ n \in \mathbb{N}.$$

For each natural number n, we have

$$\|z_{n+1} - z_n\| = \|\mathcal{T}_{q\beta_1,\beta_2} z_n - \mathcal{T}_{q\beta_1,\beta_2} z_{n-1}\| \leq \zeta\|z_n - z_{n-1}\|.$$

Iteratively, we obtain

$$\|z_{n+1} - z_n\| \leq \zeta^n\|z_1 - z_0\| \quad (n \in \mathbb{N}).$$

This inequality indicates that $(z_n)_{n\in\mathbb{N}}$ forms a Cauchy sequence in $\mathfrak{C}$. Considering $\mathfrak{C}$ to be complete, there exists a point $x^* \in \mathfrak{C}$ such that z_n converges to x^* as $n \to \infty$.

Let us establish that x^* is an element of $\mathcal{F}(\mathcal{T}_{q\beta_1,\beta_2})$. Implementing (4.2), we have

$$\begin{aligned}\|x^* - \mathcal{T}_{q\beta_1,\beta_2} x^*\| &\leq q[\|x^* - z_{n+1}\| + \|\mathcal{T}_{q\beta_1,\beta_2} z_n - \mathcal{T}_{q\beta_1,\beta_2} x^*\| \\ &\leq q[\|x^* - z_{n+1}\| + \zeta\|z_n - x^*\|].\end{aligned}$$

As we take the limit as n approaches infinity on both sides of the given inequality, the expression $|x^* - \mathcal{T}_{q\beta_1,\beta_2} x^*| = 0$.

Now, we aim to show that the quasi-double averaged mapping $\mathcal{F}(\mathcal{T}_{q\beta_1,\beta_2})$ contains only one element. We assume that $\mathcal{F}(\mathcal{T}_{q\beta_1,\beta_2})$ contains not less than two distinct elements, represented x^* and z^*. By (4.2), we obtain

$$\|x^* - z^*\| = \|\mathcal{T}_{q\beta_1,\beta_2}x^* - \mathcal{T}_{q\beta_1,\beta_2}z^*\| \leq \zeta\|x^* - z^*\| < \|x^* - z^*\|,$$

which is not true. Thus, $\mathcal{F}(\mathcal{T}_{q\beta_1,\beta_2})$ contains only one element. □

Remark 4.9. In Example 4.3, set $\beta_1 = \frac{1}{q(1+\mathcal{A}+\mathcal{B})} = \frac{1}{6}$ and $\beta_2 = \frac{1}{1+\mathcal{A}+\mathcal{B}} = \frac{1}{3}$. With these choices, it is obvious that $\beta_1 q + \beta_2 < 1$, therefore $|\mathcal{F}(\mathcal{T}_{q\beta_1,\beta_2})| = 1$. Moreover, consider the following iterative sequence

$$z_n = (1 - \beta_1 q - \beta_2)z_{n-1} + \beta_1 qTz_{n-1} + \beta_2 T^2 z_{n-1},\ n \in \mathbb{N}.$$

This iteration starts from the initial point $z_0 \in [-1, 1]$. This sequence converges to the unique invariant element of $\mathcal{T}_{q\frac{1}{6},\frac{1}{3}}$.

4.1.1. *Adequate Conditions for* $\mathcal{F}(T) = \mathcal{F}(\mathcal{T}_{q\beta_1,\beta_2})$. The preceding section unveils the presence and uniqueness of an invariant point of $\mathcal{T}_{q\beta_1,\beta_2}$ (type -I) linked with a weakly enriched mapping. This section is dedicated to giving adequate conditions for the equivalence of $\mathcal{F}(\mathcal{T}_{q\beta_1,\beta_2})$ and $\mathcal{F}(T)$.

Theorem 4.10. *Consider a closed convex subset* $\mathfrak{C}$ *of a QBS* $(\mathcal{Z}, \|\cdot\|, q)$. *Let* T *be a self-map defined on* $\mathfrak{C}$. *Suppose that we can find* $\beta_1 > 0$ *and* $\beta_2 \geq 0$ *with* $q\beta_1 + \beta_2 \in (0, 1]$ *for which the following result holds:*
(I.1) If for each $j \in [0, 1)$ *and* $u \in \mathcal{F}(\mathcal{T}_{q\beta_1,\beta_2})$, *one has*

$$\|u - Tu\| \leq \|u - (1 - j)Tu - jT^2u\|. \tag{4.3}$$

Then, $\mathcal{F}(\mathcal{T}_{q\beta_1,\beta_2}) = \mathcal{F}(T)$.

Proof. By observation, $\mathcal{F}(T) \subseteq \mathcal{F}(\mathcal{T}_{q\beta_1,\beta_2})$. Also, this inclusion can be strict. If $\mathcal{F}(\mathcal{T}_{q\beta_1,\beta_2}) = \emptyset$, then $\mathcal{F}(T) = \emptyset$ and therefore $\mathcal{F}(T) \subseteq \mathcal{F}(\mathcal{T}_{q\beta_1,\beta_2})$. In the remaining proof, we assume that $\mathcal{F}(\mathcal{T}_{q\beta_1,\beta_2}) \neq \emptyset$. Let $u \in \mathcal{F}(\mathcal{T}_{q\beta_1,\beta_2})$. Putting $j := \frac{\beta_2}{q\beta_1+\beta_2} \in (0, 1)$ in (4.3), we have

$$\begin{aligned}
\|u - Tu\| &\leq \|u - \frac{q\beta_1}{q\beta_1 + \beta_2}Tu - \frac{\beta_2}{q\beta_1 + \beta_2}T^2u\| \\
&= \frac{1}{q\beta_1 + \beta_2}\|u - (1 - q\beta_1 - \beta_2)u - q\beta_1 Tu - \beta_2 T^2 u\| \\
&= \frac{1}{q\beta_1 + \beta_2}\|u - \mathcal{T}_{q\beta_1,\beta_2}u\| \\
\implies \|u - Tu\| &= 0.
\end{aligned}$$

Hence, $\mathcal{F}(\mathcal{T}_{q\beta_1,\beta_2}) \subseteq \mathcal{F}(T)$. Therefore, $\mathcal{F}(T) = \mathcal{F}(\mathcal{T}_{q\beta_1,\beta_2})$. □

Theorem 4.11. *Consider a closed convex subset* $\mathfrak{C}$ *of a QBS* $(\mathcal{Z}, \|\cdot\|, q)$. *Let* T *be a self-mapping with positive* β_1 *and nonnegative* β_2 *satisfying* $\beta_1 + \beta_2 \in (0, 1]$. *The assertion*
(I.2)

$$\|\mathcal{T}_{q\beta_1,\beta_2}x - Tx\| \leq \overline{k}\|x - Tx\|,\ \forall x \in \mathfrak{C}, \tag{4.4}$$

holds for a nonnegative $\overline{k} < 1$. *Then,* $\mathcal{F}(T) = \mathcal{F}(\mathcal{T}_{q\beta_1,\beta_2})$.

Proof. If $\mathcal{F}(\mathcal{T}_{q\beta_1,\beta_2}) = \emptyset$, then $\mathcal{F}(T) = \emptyset$ and we get $\mathcal{F}(\mathcal{T}_{q\beta_1,\beta_2}) = \mathcal{F}(T)$. Now, suppose that $\mathcal{F}(T_{\beta_1,\beta_2}) \neq \emptyset$. Now, for each $z \in \mathcal{F}(T_{\beta_1,\beta_2})$, we have

$$\|\mathcal{T}_{q\beta_1,\beta_2}z - Tz\| \leq \overline{k}\|z - Tz\| \implies z \in \mathcal{F}(T).$$

Therefore, $\mathcal{F}(\mathcal{T}_{q\beta_1,\beta_2}) = \mathcal{F}(T)$. □

Theorem 4.12. *Consider a closed convex subset $\mathfrak{C}$ of a QBS $(\mathcal{Z}, \|\cdot\|, q)$. Suppose that T is a self-mapping defined on $\mathfrak{C}$, $\beta_1 > 0$ and $\beta_2 \geq 0$ and $\beta_1 + \beta_2 \in (0, 1]$ for which $\mathcal{F}(\mathcal{T}_{\beta_1,\beta_2}) \neq \emptyset$ and the following assertion holds:*
(I.3) for every $x \in \mathcal{F}(\mathcal{T}_{q\beta_1,\beta_2})$, we can find a closed convex subset $\mathfrak{B} \subseteq \mathfrak{C}$ containing x so that $T(\mathfrak{B}) \subseteq \mathfrak{B}$ and T exhibits inequality (4.4) only on set $\mathfrak{B}$.
Then, the sets $\mathcal{F}(T|_{\mathfrak{B}})$ and $\mathcal{F}(\mathcal{T}_{q\beta_1,\beta_2}|_{\mathfrak{B}})$ are equal.

Proof. The result can readily be obtained when T acts solely on set $\mathfrak{B}$. □

In the forthcoming theorem, we will utilize the aforementioned results to establish the fixed-point theorem for weakly enriched contraction mappings.

Theorem 4.13. *Consider a closed convex subset $\mathfrak{C}$ of a QBS $(\mathcal{Z}, \|\cdot\|, q)$. Suppose that T is a self-mapping defined on $\mathfrak{C}$ satisfying (4.1). Then the associated quasi-double averaged mapping of type-I contains only one element in its set of fixed points. Moreover, if T, β_1, β_2 obey (I.1) or (I.2) or (I.3), then*

(1) $\mathcal{F}(T)$ contains only one element.
(2) For any given $z_0 \in \mathfrak{C}$, $(z_n) \subseteq \mathfrak{C}$ whose iterative scheme is given by

$$z_n = (1 - q\beta_1 - \beta_2)z_{n-1} + q\beta_1 T z_{n-1} + \beta_2 T^2 z_{n-1},\ n \in \mathbb{N} \tag{4.5}$$

converges to the member of set $\mathcal{F}(T)$.

Proof. With reference to Theorem 4.8, there exist values for $\beta_1, \beta_2 \in (0, 1]$ such that $\mathcal{A}$, $\mathcal{B}$ hold. That is, $\mathcal{F}(\mathcal{T}_{q\beta_1,\beta_2})$ has only one element and the iterative scheme in (4.5) converges to the element of $\mathcal{F}(\mathcal{T}_{q\beta_1,\beta_2})$. Since β_1, β_2 satisfy (I.1) or (I.2) or (I.3), the results are guaranteed by the adequate condition theorems which are Theorem 4.10, Theorem 4.11, Theorem 4.12 respectively. □

4.2. **Quasi-double averaged mapping of type -II.**

Definition 4.14. Let $(\mathcal{Z}, \|\cdot\|, q)$ be a QNLS. Suppose that $\mathfrak{C}$ is a convex subset of $\mathcal{Z}$. We define a new mapping

$$\mathcal{T}_{\beta_1,q\beta_2}(x) := (1 - \beta_1 - q\beta_2)x + \beta_1 Tx + q\beta_2 T^2 x,$$

where $\beta_1 > 0,\ \beta_2 \geq 0$ and $\beta_1 + \beta_2 \in (0, 1]$.

Remark 4.15. For $q = 1,\ \mathcal{T}_{\beta_1,q\beta_2}$ becomes a double-averaged mapping in an NLS [17].

Remark 4.16. For $\beta_1 = \lambda,\ \beta_2 = 0,\ \mathcal{T}_{\beta_1,q\beta_2}$ reduces to $\mathcal{S}_\lambda$, that is, $\mathcal{T}_{q\beta_1,\beta_2}$ is a generalization of $\mathcal{T}_\lambda$.

In the subsequent theorem, we show that $\mathcal{F}(\mathcal{T}_{\beta_1,q\beta_2})$ is nonempty, and we further establish that it does not have more than one element.

Theorem 4.17. *Consider a closed convex subset $\mathfrak{C}$ of a Quasi-Banach space $(\mathcal{Z}, \|\cdot\|, q)$. Suppose that $T\colon \mathfrak{C} \to \mathfrak{C}$ satisfies (4.1). Then, there exist $\beta_1 > 0$ and $\beta_2 \geq 0$ with $\beta_1 + \beta_2 q \in (0, 1]$ such that the following statements are valid:*

$\mathcal{C}$. $\mathcal{F}(T_{\beta_1,\beta_2})$ contains only one element.
$\mathcal{D}$. For any given $z_0 \in \mathfrak{C}$, the iteration $(z_n) \subset \mathfrak{C}$ generated from

$$z_n = (1 - \beta_1 - q\beta_2)z_{n-1} + \beta_1 T z_{n-1} + q\beta_2 T^2 z_{n-1},\ n \in \mathbb{N}$$

converges to the only element of $\mathcal{F}(\mathcal{T}_{\beta_1,q\beta_2})$.

Proof. In (4.1), for $\mathcal{B} = 0$, T is an enriched contraction mapping. We need to verify the result for nonegative $\mathcal{A}$ and positive $\mathcal{B}$. Denote $\beta_1 := \frac{1}{(\mathcal{A}+\mathcal{B}+1)} > 0$ and $\beta_2 := \frac{\mathcal{B}}{q(\mathcal{A}+\mathcal{B}+1)} \geq 0$. Then, (4.1) becomes

$$\left\|\frac{(1 - \beta_1 - q\beta_2)(y - z) + \beta_1(Ty - Tz) + q\beta_2(T^2y - T^2z)}{\beta_1}\right\| \leq \mu\|y - z\|.$$

Since $\beta_1 > 0$, the above inequality becomes

$$\big\|(1-\beta_1 q_1 - \beta_2)(y-z) + \beta_1 q(Ty - Tz) + \beta_2(T^2 y - T^2 z)\big\| \le \beta_1 \mu \|y - z\|.$$

Expressing the above inequality in terms of double averaged mapping, we get

$$\|\mathcal{T}_{\beta_1, q\beta_2} y - \mathcal{T}_{\beta_1, q\beta_2} z\| \le \zeta \|y - z\|, \tag{4.6}$$

where $0 \le \zeta = \mu\beta_1 < 1$, for every x, y. For a pre-assigned $z_0 \in \mathfrak{C}$, construct a sequence $(z_n) \subseteq \mathfrak{C}$ designed as $z_n = T_{\beta_1,\beta_2} z_{n-1}$ for $n \in \mathbb{N}$. Consequently, one has

$$\begin{aligned}\|z_{n+1} - z_n\| &= \|\mathcal{T}_{\beta_1, q\beta_2} z_n - \mathcal{T}_{\beta_1, q\beta_2} z_{n-1}\| \\ &\le \zeta \|z_n - z_{n-1}\|.\end{aligned}$$

Iteratively, we obtain

$$\|z_{n+1} - z_n\| \le \zeta^n \|z_1 - z_0\| \quad (n \in \mathbb{N})$$

entailing that $(z_n)_{n\in\mathbb{N}}$ is eligible to be Cauchy in $\mathfrak{C}$. Furthermore, the completeness of $\mathfrak{C}$ guarantees the existence of $x^* \in \mathfrak{C}$, which serves as the converging limit of $z_n(\to x^*)$.

Let us show that $x^* \in \mathcal{F}(\mathcal{T}_{\beta_1, q\beta_2})$. Making use of (4.6), we have

$$\begin{aligned}\|x^* - \mathcal{T}_{\beta_1, q\beta_2} x^*\| &\le q[\|x^* - z_{n+1}\| + \|\mathcal{T}_{\beta_1, q\beta_2} z_n - \mathcal{T}_{\beta_1, q\beta_2} x^*\| \\ &\le q[\|x^* - z_{n+1}\| + \zeta\|z_n - x^*\|].\end{aligned}$$

Tending n to ∞ , we obtain

$$\|x^* - \mathcal{T}_{\beta_1, q\beta_2} x^*\| = 0.$$

Next, we demonstrate the uniqueness of the fixed point for the double-averaged mapping $\mathcal{T}_{\beta_1, q\beta_2}$. Let's assume the contrary, that $\mathcal{F}(\mathcal{T}_{\beta_1, q\beta_2})$ is not a singleton set and contains two distinct elements x^* and z^*. Indeed, using (4.6), we get

$$\|x^* - z^*\| \le \zeta \|x^* - z^*\| < \|x^* - z^*\|,$$

which is not true. Thus, $\mathcal{F}(\mathcal{T}_{\beta_1, q\beta_2}) = 1$. □

Remark 4.18. Again referring to Example 4.3, we can define $\beta_1 = \frac{1}{(1+\mathcal{A}+\mathcal{B})} = \frac{1}{3}$ and $\beta_2 = \frac{B}{q(1+\mathcal{A}+\mathcal{B})} = \frac{1}{6}$. Then, $\beta_1 + \beta_2 q < 1$ which implies that $\mathcal{T}_{\beta_1, q\beta_2}$ has a unique fixed point.

4.2.1. *Adequate Conditions for Equality of $\mathcal{F}(T)$ and $\mathcal{F}(\mathcal{T}_{\beta_1, q\beta_2})$.* The preceding segment establishes the presence and sole existence of an invariant point of $\mathcal{T}_{\beta_1, q\beta_2}$ (of type -II) linked to a weakly enriched mapping. This section devotes to giving adequate conditions for the equivalence between $\mathcal{F}(T)$ and $\mathcal{F}(\mathcal{T}_{\beta_1, q\beta_2})$.

Theorem 4.19. *Consider a closed convex subset $\mathfrak{C}$ of a QBS $(\mathcal{Z}, \|\cdot\|, q)$, and assume that T is a self-map on $\mathfrak{C}$. If for $\beta_1 > 0$, $\beta_2 \ge 0$, $\beta_1 + q\beta_2 \in (0, 1]$, the following condition is valid:*
(II.1) for every nonnegative j (not larger than one) and $u \in \mathcal{F}(\mathcal{T}_{\beta_1, q\beta_2})$, one has

$$\|u - Tu\| \le \|u - (1-j)Tu - jT^2 u\|. \tag{4.7}$$

Then, $\mathcal{F}(\mathcal{T}_{\beta_1, q\beta_2}) = \mathcal{F}(T)$.

Proof. From the definition of the double-averaged mapping of the above type , $\mathcal{F}(T) \subseteq \mathcal{F}(\mathcal{T}_{\beta_1, q\beta_2})$. If $\mathcal{F}(T_{\beta_1,\beta_2}) = \emptyset$, then $\mathcal{F}(T) = \emptyset$, and therefore $\mathcal{F}(T) \subseteq \mathcal{F}(\mathcal{T}_{\beta_1, q\beta_2})$. Navigating through the remaining proof, we assume that $\mathcal{F}(\mathcal{T}_{\beta_1,q\beta_2}) \neq \emptyset$. Let $u \in \mathcal{F}(\mathcal{T}_{\beta_1,q\beta_2})$. Putting $j := \frac{q\beta_2}{\beta_1+q\beta_2} \in [0,1)$ in (4.7), we have

$$\begin{aligned}\|u - Tu\| &\leq \left\|u - \frac{\beta_1}{\beta_1+q\beta_2}Tu - \frac{q\beta_2}{\beta_1+q\beta_2}\right\| \\ &= \frac{1}{\beta_1+q\beta_2}\|u-(1-\beta_1-q\beta_2)u - \beta_1 Tu - q\beta_2 T^2 z\| \\ &= \frac{1}{\beta_1+q\beta_2}\|u - \mathcal{T}_{\beta_1,q\beta_2}u\| \\ &= 0 \\ \implies u \in \mathcal{F}(T) &\implies \mathcal{F}(\mathcal{T}_{\beta_1,q\beta_2}) \subseteq \mathcal{F}(T).\end{aligned}$$

Therefore, $\mathcal{F}(\mathcal{T}_{\beta_1,q\beta_2})$ and $\mathcal{F}(T)$ coincide. □

Theorem 4.20. *Consider a closed convex subset $\mathfrak{C}$ of a QBS $(\mathcal{Z}, \|\cdot\|, q)$. Assume that T is a self-map on $\mathfrak{C}$, there are $\beta_1 > 0$ and $\beta_2 \geq 0$ with $\beta_1 + q\beta_2 \in (0,1]$ for which the succeeding condition holds:*
(II.2) We can find $\overline{k}$ in $[0,1)$ such that

$$\|\mathcal{T}_{\beta_1,q\beta_2}x - Tx\| \leq \overline{k}\|x - Tx\| \quad \textit{for all } x \in \mathfrak{C}. \tag{4.8}$$

Then, the sets $\mathcal{F}(\mathcal{T}_{\beta_1,q\beta_2})$ and $\mathcal{F}(T)$ coincide.

Proof. If $\mathcal{F}(\mathcal{T}_{\beta_1,q\beta_2}) = \emptyset$, then $\mathcal{F}(T) = \emptyset$ and we get $\mathcal{F}(T_{\beta_1,\beta_2}) = \mathcal{F}(T)$. Now, suppose that $\mathcal{F}(T_{\beta_1,\beta_2}) \neq \emptyset$. Now, for each $z \in \mathcal{F}(\mathcal{T}_{\beta_1,q\beta_2})$, we have

$$\|z - Tz\| \leq \overline{k}\|z - Tz\| \implies (1-\overline{k})\|z - Tz\| \leq 0.$$

As a consequence, $z \in \mathcal{F}(T)$. Thus, $\mathcal{F}(T) = \mathcal{F}(\mathcal{T}_{\beta_1,q\beta_2})$. □

Theorem 4.21. *Consider a closed convex subset $\mathfrak{C}$ of a Quasi-Banach space $(\mathcal{Z}, \|\cdot\|, q)$. Assume that T is a self-map on $\mathfrak{C}$, there are $\beta_1 > 0$ and $\beta_2 \geq 0$, $\beta_1 + q\beta_2 \in (0,1]$ for which $\mathcal{F}(\mathcal{T}_{\beta_1,q\beta_2}) \neq \emptyset$ and the succeeding assertion holds:*
(II.3) for every $x \in \mathcal{F}(\mathcal{T}_{\beta_1,q\beta_2})$, there exists a closed convex subset $\mathfrak{B} \subseteq \mathfrak{C}$ containing x such that $T(\mathfrak{B}) \subseteq \mathfrak{B}$ and T adheres to inequality (4.8) only on set $\mathfrak{B}$.
Then, $\mathcal{F}(T|_{\mathfrak{B}}) = \mathcal{F}(\mathcal{T}_{\beta_1,q\beta_2}|_{\mathfrak{B}})$.

Proof. The result can readily be obtained when T acts solely on set $\mathfrak{B}$. □

Theorem 4.22. *Consider a closed convex subset $\mathfrak{C}$ of a Quasi-Banach space $(\mathcal{Z}, \|\cdot\|, q)$. Assume that T is a self-map on $\mathfrak{C}$ such that it is a weakly enriched contraction mapping. Then, the corresponding quasi-double averged mapping of type-III namely $\mathcal{T}_{\beta_1,q\beta_2}$ has a unique fixed point with the convergence of the sequence $(z_n)_{n\in\mathbb{N}}$ (given by $\mathcal{D}$). Moreover, if T and the coreesponding scalars in the quasi-double averaged mapping of type-II namely β_1, β_2 satisfy (II.1) or (II.2) or (II.3), then*

(1) $\mathcal{F}(T)$ contains a single element.
(2) For any initial $z_0 \in \mathfrak{C}$, $(z_n) \subseteq \mathfrak{C}$ whose iterative scheme is given by

$$z_n = (1-\beta_1-q\beta_2)z_{n-1} + \beta_1 T z_{n-1} + q\beta_2 T^2 z_{n-1}, \quad n \in \mathbb{N}, \tag{4.9}$$

converges to an element of $\mathcal{F}(T)$.

Proof. In accordance with Theorem 4.17, there exist $\beta_1, \beta_2 \in (0,1]$ such that $\mathcal{C}, \mathcal{D}$ hold. That is, $\mathcal{T}_{\beta_1,q\beta_2}$ possesses a unique point that is fixed, and the iterative scheme prescribed by $(z_n)_{n\in\mathbb{N}}$ in (4.9) converges to the single point of $\mathcal{F}(\mathcal{T}_{\beta_1,q\beta_2})$. Since β_1, β_2 satisfy (II.1) or (II.2) or (II.3), the results follow from Theorem 4.19, Theorem 4.20, Theorem 4.21 respectively. □

4.3. **Quasi-double averaged mapping of type -III.**

Definition 4.23. Consider a closed convex subset $\mathfrak{C}$ of a QNLS $(\mathcal{Z}, \|\cdot\|, q)$. We define a new mapping

$$\mathcal{T}_{q\beta_1,q\beta_2} := (1 - q\beta_1 - q\beta_2)x + q\beta_1 Tx + q\beta_2 T^2 x,$$

where $\beta_1 > 0,\ \beta_2 \geq 0$ and $\beta_1 + \beta_2 \in (0, 1]$.

Remark 4.24. For $q = 1,\ \mathcal{T}_{q\beta_1,q\beta_2}$ becomes a double-averaged mapping [17].

Remark 4.25. For $\beta_1 = \lambda,\ q = 1,\ \beta_2 = 0, \mathcal{T}_{q\beta_1,q\beta_2}$ reduces to $\mathcal{S}_\lambda$, that is, $\mathcal{T}_{q\beta_1,q\beta_2}$ is a generalization of $\mathcal{S}_\lambda$.

In the following theorem, we illustrate that $\mathcal{F}(\mathcal{T}_{q\beta_1,q\beta_2})$ isa singleton set.

Theorem 4.26. *Consider a closed convex subset $\mathfrak{C}$ of a QBS $(\mathcal{Z}, \|\cdot\|, q)$. Assume that T is a self-map on $\mathfrak{C}$, there are $\beta_1 > 0$ and $\beta_2 \geq 0, q\beta_1 + q\beta_2 \in (0, 1]$ for which the following statements hold:*

$\mathcal{E}$. *$\mathcal{F}(\mathcal{T}_{q\beta_1,q\beta_2})$ has a single element.*

$\mathcal{F}$. *For any initial $z_0 \in \mathfrak{C}$, $(z_n) \subset \mathfrak{C}$ whose iterative scheme is given by*

$$z_n = (1 - q\beta_1 - q\beta_2)z_{n-1} + q\beta_1 T z_{n-1} + q\beta_2 T^2 z_{n-1},\ n \in \mathbb{N}$$

converges to the element of $\mathcal{F}(T_{\beta_1,\beta_2})$.

Proof. Given T satisfies (4.1), there exist constants $\mathcal{A},\ \mathcal{B}$ satisfying (4.1). Likewise preceding theorems, we need to prove for nonnegative $\mathcal{A}$ and positive $\mathcal{B}$. Define $\beta_1 := \frac{1}{q(\mathcal{A}+\mathcal{B}+1)} > 0$ and $\beta_2 := \frac{B}{q(\mathcal{A}+\mathcal{B}+1)} \geq 0$. Then, (4.1) becomes

$$\|\frac{(1 - \beta_1 q_1 - \beta_2)(y - z) + \beta_1 q(Ty - Tz) + \beta_2(T^2 y - T^2 z)}{q\beta_1}\| \leq \mu\|y - z\|.$$

Since $q\beta_1 > 0$, the above inequality becomes

$$\|(1 - \beta_1 q - \beta_2 q)(y - z) + \beta_1 q(Ty - Tz) + \beta_2 q(T^2 y - T^2 z)\| \leq q\beta_1\mu\|y - z\|.$$

Expressing the above inequality in terms of double averaged mapping, we get

$$\|\mathcal{T}_{q\beta_1,q\beta_2} y - \mathcal{T}_{q\beta_1,q\beta_2} z\| \leq \zeta\|y - z\|, \tag{4.10}$$

where $0 \leq \zeta = q\mu\beta_1 < 1$, for all $x, y \in \mathfrak{C}$. Initialising from a $z_0 \in \mathfrak{C}$, define a sequence (z_n) (whose components are in $\mathfrak{C}$) is given by $z_n = \mathcal{T}_{q\beta_1,q\beta_2} z_{n-1}$, $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, we have

$$\begin{aligned}\|z_{n+1} - z_n\| &= \|\mathcal{T}_{q\beta_1,q\beta_2} z_n - \mathcal{T}_{q\beta_1,q\beta_2} z_{n-1}\| \\ &\leq \zeta\|z_n - z_{n-1}\|\end{aligned}$$

Iterating n-times yields

$$\|z_{n+1} - z_n\| \leq \zeta^n\|z_1 - z_0\|.$$

According to Lemma 2.4), $(z_n)_{n\in\mathbb{N}}$ fulfills to be Cauchy in $\mathfrak{C}$, and the existence of $x^* = \lim\limits_{n\to\infty} z_n \in \mathfrak{C}$ is confirmed by the the completeness of $\mathfrak{C}$.

We aim to prove that $x^* \in \mathcal{F}(\mathcal{T}_{q\beta_1,q\beta_2})$ using (4.10). We have

$$\begin{aligned}\|x^* - \mathcal{T}_{q\beta_1,q\beta_2} x^*\| &\leq q[\|x^* - z_{n+1}\| + \|\mathcal{T}_{q\beta_1,q\beta_2} z_n - \mathcal{T}_{q\beta_1,q\beta_2} x^*\|] \\ &\leq q[\|x^* - z_{n+1}\| + \zeta\|z_n - x^*\|].\end{aligned}$$

Tending n to infinity, we obtain

$$\|x^* - \mathcal{T}_{q\beta_1,q\beta_2} x^*\| = 0.$$

Now, let us move on proving the distinctiveness of the fixed point of $\mathcal{T}_{q\beta_1,q\beta_2}$. Assuming that $\mathcal{T}_{q\beta_1,q\beta_2}$ has at least two fixed points, denoted x^* and z^* as elements of $\mathcal{F}(\mathcal{T}_{q\beta_1,q\beta_2})$ such that $x^* \neq z^*$. In view of inequality (4.10), we obtain

$$\|x^* - z^*\| \leq \zeta\|x^* - z^*\| < \|x^* - z^*\|,$$

which is untrue. Thus, $|\mathcal{F}(\mathcal{T}_{q\beta_1,q\beta_2})| = 1$. □

Remark 4.27. We can define $\beta_1 = \frac{1}{q(1+a+b)} = \frac{1}{6}$ and $\beta_2 = \frac{b}{q(1+a+b)} = \frac{1}{6}$. Then, T_{β_1,β_2} has a unique invariant point.

4.3.1. *Adequate Conditions for Equality of $\mathcal{F}(T)$ and $\mathcal{F}(\mathcal{T}_{q\beta_1,q\beta_2})$.* The prior segment shows that the set $\mathcal{F}(\mathcal{T}_{q\beta_1,q\beta_2})$ (of type -III) associated with a weakly enriched mapping contains a single element. The subsequent division is devoted to deriving adequate conditions for the Equality of $\mathcal{F}(\mathcal{T}_{q\beta_1,q\beta_2})$ (of type -III) and the collection of invariant points of the corresponding weakly enriched mapping. Note that $\mathcal{F}(T) \subseteq \mathcal{F}(\mathcal{T}_{q\beta_1,q\beta_2})$, but this inclusion can be strict.

Theorem 4.28. *Suppose a closed convex subset $\mathfrak{C}$ of a QNLS $(\mathcal{Z}, \|\cdot\|, q)$ is given. Assume that T is a self-map on $\mathfrak{C}$, there are $\beta_1 > 0$ and $\beta_2 \geq 0$, $q\beta_1 + q\beta_2 \in (0, 1]$ for which the following holds:*
(III.1) for each $j \in [0, 1)$ and $u \in \mathcal{F}(\mathcal{T}_{q\beta_1,q\beta_2})$, one has

$$\|u - Tu\| \leq q\|u - (1 - j)Tu - jT^2u\|. \tag{4.11}$$

Then, the sets $\mathcal{F}(T)$ and $\mathcal{F}(\mathcal{T}_{q\beta_1,q\beta_2})$ coincide.

Proof. It is straight forward to observe that $\mathcal{F}(T) \subseteq \mathcal{F}(\mathcal{T}_{q\beta_1,q\beta_2})$. If $\mathcal{F}(\mathcal{T}_{q\beta_1,q\beta_2}) = \emptyset$, then $\mathcal{F}(T) = \emptyset$, and thus $\mathcal{F}(T) \subseteq \mathcal{F}(\mathcal{T}_{q\beta_1,q\beta_2})$. In the subsequent proof, we assume that $\mathcal{F}(\mathcal{T}_{q\beta_1,q\beta_2}) \neq \emptyset$. Let $u \in \mathcal{F}(\mathcal{T}_{q\beta_1,q\beta_2})$. Putting $j := \frac{\beta_2}{\beta_1+\beta_2} \in [0, 1)$ in (4.11), we have

$$\begin{aligned}\|u - Tu\| &\leq q\|u - \frac{\beta_1}{\beta_1 + \beta_2}Tu - \frac{\beta_2}{\beta_1 + \beta_2}\| \\ &= \frac{1}{\beta_1 + \beta_2}\|u - (1 - q\beta_1 - q\beta_2)u - q\beta_1Tu - q\beta_2T^2z\| \\ &= \frac{1}{\beta_1 + \beta_2}\|u - \mathcal{T}_{q\beta_1,q\beta_2}u\| \\ &= 0.\end{aligned}$$

Therefore, $\mathcal{F}(T) = \mathcal{F}(\mathcal{T}_{q\beta_1,q\beta_2})$. □

Theorem 4.29. *Consider a closed convex subset $\mathfrak{C}$ of a QNLS $(\mathcal{Z}, \|\cdot\|, q)$. Assume that T is a self-map on $\mathfrak{C}$, there are $\beta_1 > 0$ and $\beta_2 \geq 0$, $q\beta_1 + q\beta_2 \in (0, 1]$ for which the following assertion is valid:*

$$\|\mathcal{T}_{q\beta_1,q\beta_2}x - Tx\| \leq \overline{k}\|x - Tx\| \qquad (x \in \mathfrak{C}). \tag{4.12}$$

Then, $\mathcal{F}(\mathcal{T}_{q\beta_1,q\beta_2})$ and $\mathcal{F}(T)$ contain same elements.

Proof. If $\mathcal{F}(T_{\beta_1,\beta_2}) = \emptyset$, then $\mathcal{F}(T) = \emptyset$ and we get $\mathcal{F}(T_{\beta_1,\beta_2}) = \mathcal{F}(T)$. Now, suppose that $\mathcal{F}(T_{\beta_1,\beta_2}) \neq \emptyset$. Now, for each $z \in \mathcal{F}(T_{\beta_1,\beta_2})$, we have

$$\|z - Tz\| = \|T_{\beta_1,\beta_2}z - Tz\| \leq \overline{k}\|z - Tz\| \implies (1 - \overline{k})\|z - Tz\| \leq 0.$$

Therefore, $\mathcal{F}(T) = \mathcal{F}(T_{\beta_1,\beta_2})$. □

Theorem 4.30. *Consider a closed convex subset $\mathfrak{C}$ of a QNLS $(\mathcal{Z}, \|\cdot\|, q)$. Assume that T is a self-map on $\mathfrak{C}$, there are $\beta_1 > 0$ and $\beta_2 \geq 0$, $q\beta_1 + q\beta_2 \in (0, 1]$ for which $\mathcal{F}(\mathcal{T}_{q\beta_1,q\beta_2}) \neq \emptyset$ and the following inequality verifies:*

(III.3) For each $x \in \mathcal{F}(\mathcal{T}_{q\beta_1,q\beta_2})$, we can get a closed convex subset $\mathfrak{B} \subseteq \mathfrak{C}$ which possesses x such that $T(\mathfrak{B}) \subseteq \mathfrak{B}$ and T obeys (4.12) only on set $\mathfrak{B}$.
Then, $\mathcal{F}(\mathcal{T}_{q\beta_1,q\beta_2}|_{\mathfrak{B}}) = \mathcal{F}(T|_{\mathfrak{B}})$.

Proof. The result can readily be obtained when T acts solely on set $\mathfrak{B}$. □

Theorem 4.31. *Consider a closed convex subset $\mathfrak{C}$ of a QNLS $(\mathcal{Z}, \|\cdot\|, q)$. Assume that T is a self-map on $\mathfrak{C}$, such that it satisfies the condition of a weakly enriched contraction mapping. Then, there exist a positive β_1 and a nonnegative β_2 with $q\beta_1 + q\beta_2 \leq 1$ for which $\mathcal{E}$ and $\mathcal{F}$ in Theorem 4.26 hold. Moreover, if T, β_1, β_2 satisfy (III.1) (Theorem 4.28) or (III.2) (Theorem 4.29) or (III.3) (Theorem 4.30) , then*

(1) $|\mathcal{F}(T)|$ is one;
(2) for any initial $z_0 \in \mathfrak{C}$, $(z_n) \subseteq \mathfrak{C}$ given by

$$z_n = (1 - q\beta_1 - q\beta_2)z_{n-1} + q\beta_1 T z_{n-1} + q\beta_2 T^2 z_{n-1},\ n \in \mathbb{N}, \tag{4.13}$$

converges to the only element of $\mathcal{F}(T)$.

Proof. In view of Theorem 4.26, there exist $\beta_1, \beta_2 \in (0, 1]$ such that $\mathcal{E}, \mathcal{F}$ hold. That is, the quasi-double averaged mapping of type -III has a single fixed point and the iterative scheme in (4.13) converges to the invariant point of $\mathcal{T}_{q\beta_1,q\beta_2}$. Since β_1, β_2 satisfy (III.1) or (III.2) or (III.3), the results follow from Theorem 4.28, Theorem 4.29, Theorem 4.30 respectively. □

5. Conclusion

In this manuscript, we have coined the averaged mapping and double-averaged mapping in the frame work of QBS, referred to as quasi-averaged mapping and quasi-double averaged mapping (of three types) respectively. With these mappings, we discuss the fixed point theorems of various contractions - enriched contraction, enriched Chatterjea mapping, enriched Kannan mapping, enriched Bianchini mapping, enriched Ćirić mapping, enriched Hardy-Rogers mapping, enriched almost contractions. Some of these results are well-known in b-metric space settings but we know that it is possible to add any two points and multiply a vector by a scalar in a QNLS. This structure allows us for the creation of generalised contraction conditions (here we call them enriched versions) and the construction of more iterative schemes in a QNLS. The fixed point theorems are meticulously crafted, incorporating refined conditions on the parameters of various enriched maps, interwined with the quasi-index of the space. In particular, we work with Krasnoselskij iterative scheme for quasi-averaged mapping and the Kirk's iterative scheme of order two for quasi-double averaged mapping in QBSs. The limited applicability of enriched contractions and weakly enriched contractions in a normed linear space is addressed in this paper by generalizing them to a QNLS (for example in $\mathcal{L}^{\frac{1}{2}}$).

Our findings present various opportunities to explore the feasibility of identifying less stringent conditions. These conditions can aim to establish that invariant points of a quasi-double averaged mapping coincide with those of its initial mapping. Additionally, we are delving into the application of quasi-double averaged mappings to develop fresh contractive conditions, eliminating the necessity for relying on sufficient conditions for the existence and uniqueness of fixed points. Extensions of this nature can offer new perspectives on the characteristics and dynamics of contractive mappings within these spaces, shedding light on their potential applications across diverse mathematical domains.

Statements and Declarations

The authors declare that they have no conflict of interest, and the manuscript has no associated data.

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