APPLICABLE NONLINEAR ANALYSIS
Volume 2 (2025), No. 3, 146–158
https://doi.org/10.69829/apna-025-0203-ta01
*Tulipa Opera Scholarum*

# LAGUERRE-TYPE FRACTIONAL PARAMETRIC POPULATION DYNAMICS MODELS

DIEGO CARATELLI[1] AND PAOLO EMILIO RICCI[2,*]

[1] *The Antenna Company, High Tech Campus 29, 5656 AE - Eindhoven, The Netherlands & Eindhoven University of Technology, PO Box 513, 5600 MB - Eindhoven, The Netherlands*
[2] *Mathematics Section, International Telematic University UniNettuno, Corso Vittorio Emanuele II, 39, 00186 - Roma, Italia*

Dedicated to Professor Hari Mohan Srivastava on the Occasion of His 85th Birthday

Abstract. In a previous article, we used a parametric Laguerre-type operator, which behaves like an exponential with respect to its eigenfunction, to introduce parametric models of population dynamics. In this work, starting from the results obtained there, we compare them with those in which the ordinary derivative is replaced by the fractional one, represented by means of the classical Euler definition. The major difficulty of this extension consists in the fact that the eigenfunctions of the fractional case are not obtained simply by replacing the factorials with the corresponding values expressed through the Gamma function, but required a more in-depth analysis. This fact resulted in a greater analytical complexity of the coefficients through which the coefficients of the different fractional models considered are obtained, by recurrence. Some examples are shown, obtained by the first author, using the computer algebra system Mathematica©.

**Keywords.** Fractional exponential function, Generalized Mittag-Leffler functions, Laguerre-type exponentials, Parametric population dynamics models, Fractional Laguerre-type models.



## 1. Introduction

Fractional calculus is becoming increasingly important both in the field of applied sciences (see e.g. [29]–[33]) and in that of special functions.
In a recent article [7], we have used a fractional version of the exponential function, defined as

$$\mathrm{Exp}_\alpha(t) = 1 + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots + \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} + \ldots\,, \tag{1.1}$$

to derive a generalized form of the Bernoulli and Euler numbers and polynomials.
It is worth to recall that the fractional exponential function (1.1) is a special case of the Mittag-Leffler function $\mathrm{E}_\alpha(x)$ [19], since it results

$$\mathrm{Exp}_\alpha(x) = \mathrm{E}_\alpha(x^\alpha).$$

Note that the fractional exponential function satisfies the eigenvalue property

$$D_x^\alpha \mathrm{Exp}_\alpha(xt) = t^\alpha\, \mathrm{Exp}_\alpha(xt)\,,$$

*Corresponding author.
E-mail address: paoloemilio.ricci@uninettunouniversity.net (P. E. Ricci), d.caratelli@tue.nl (D. Caratelli)
2020 Mathematics Subject Classification:26A33, 33B10, 33E12, 34A08, 92D25.
Accepted: January 10, 2025.

with respect the fractional derivative $D_x^\alpha$, defined by the Euler equation

$$D_x^\alpha x^n = \begin{cases} \dfrac{\Gamma(n+1)}{\Gamma(n+1-\alpha)}\, x^{n-\alpha}\,, & \text{if} \quad n > \lceil\alpha\rceil - 1\,, \\ 0\,, \quad \text{if} \quad n = 0, 1, \dots, \lceil\alpha\rceil - 1\,, \end{cases} \tag{1.2}$$

where $n \geq 0$ and $\lceil\alpha\rceil$ denotes the ceiling function (the smallest integer greater than or equal to $\alpha$). If $c$ is a constant then $D_x^\alpha c = 0$.
This definition falls as a special case, of the fractional derivative introduced by Caputo [4], defined as follows

$$D_{a+}^\alpha f(x) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_a^x \frac{f^{(m)}(\tau)}{(x-\tau)^{\alpha-m+1}}\, d\tau\,, \quad \text{where} \quad m = \lceil\alpha\rceil\,, \quad \text{if} \quad \alpha \notin \mathbb{N} \\ f^{(\alpha)}(x), \quad \text{if} \quad \alpha \in \mathbb{N}\,, \end{cases}$$

and reduces to the above equation (1.2) when $a = 0$ and $f(x) = x^n$.

Studies on population dynamics have a long tradition, which has seen the contribution of many important scholars ([1, 2, 18, 26, 34], and the references therein). See in this regard the text by Nicolas Bacaër [3], which recalls the relevant history.

Many models have been proposed, starting with Malthus's model dating back to 1798, based on exponential growth. In 1838, the Belgian mathematician and statistician Pierre F. Verhulst proposed a modification that took into account the capabilities of the environment.

Subsequently, many other scholars have proposed variants and generalizations of these models, which took into account specific problems, especially from the point of view of verifications on real data.

The problem of two species - predators and prey - fighting for survival in the same environment was studied independently by Alfred J. Lotka and Vito Volterra. The model they proposed was validated by Patrick H. Leslie in 1945, who was able to analyze the population cycles of hares and their predator, the lynx, using the archives of the Hudson's Bay Company in Canada.

The mathematical models studied in the literature are based on the exponential function that intervenes in the evolution studied.

More recently, we proposed to substitute the exponential function with the Laguerre-type operator [5, 12, 13, 28], which is a particular case of the Ditkin and Prudnikov [16]) hyper-Bessel type operators. The *Bessel-type differential operators* were introduced by Dimovski, in 1966 [15] and later called by Kiryakova *hyper-Bessel operators* [23, 24], because their eigenfunctions were called hyper-Bessel by Delerue [14], in 1953.

In previous articles we have studied the fractional case of both the classical and Laguerre-type population dynamics models [6] proving results similar to the ordinary Malthus, Verhulst, minimum thereshold logistic and Volterra-Lotka models.

Since the parametric Laguerre-type operator $D_t\, t\, D_t + m\, D_t$ ($m$ a positive integer) admits the eigenfunction $e_{1,m}(t) = \sum_{k=0}^\infty t^k/k!(k+m)!$, as was proven in [12], in [10] we have introduced parametric population dynamics models extending the classical ones. The parameter $m$ is useful to adapt numerical and graphical values to real cases.

In this article we summarize the results obtained in [10] and show their extension to the fractional case using the operator $D_t^\alpha\, t^\alpha\, D_t^\alpha + m\, D_t^\alpha$ which has an eigenfunction of the type

$$e_{(\alpha|1,m)}(t) = \sum_{k=0}^\infty t^{k\alpha}/[\Gamma(k\alpha+1)\lambda_k(m,\alpha)]\,,$$

where $\lambda_k(m,\alpha)$ is a function defined in what follows.

In the course of the article, numerical and graphical results obtained by the first author using the computer algebra program Mathematica© are derived.

This paper is organized as follows. After presenting in Section 2 the pseudo and true exponential functions, we introduce the parametric Laguerre-type operator both in the ordinary and fractional case. In the following Sections we present the applications to population dynamics models: the Malthus case in Section 3, the Verhulst case in Section 4, the minimum threshold logistic case in Section 5 and the Volterra-Lotka case in Section 6. The technique used is summarized in the Conclusion section.

Some numerical and graphical results are shown, obtained by the first author using the computer algebra system Mathematica©.

## 2. Pseudo and True Exponentials

Abhishek Mishra [25] introduced a set of pseudo-exponential functions defined as

$$e(z,m) = \sum_{k=0}^{\infty} \frac{z^k}{k!\,(k+m)!}\,.$$

These functions satisfy the equation

$$D_z e(z,m) = e(z,m+1)\,,$$

which is similar to that of the exponential but with a shift of the parameter.

Actually these functions are a special case of the generalized Mittag-Leffler function studied by T.R. Prabhakar [27]:

$$\mathrm{E}^{\gamma}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k\, z^k}{k!\,\Gamma(\alpha k+\beta)}\,,$$

where $(\gamma)_k$ denotes the Pochhammer symbol

$$(\gamma)_0 = 1\,, \quad (\gamma)_k = \gamma(\gamma+1)(\gamma+2)\cdots(\gamma+k-1),$$

since

$$e(z,m) = \mathrm{E}^{0}_{1,m}(z)\,.$$

In [12] we have proven the result, which shows a *true exponential function*

**Theorem 2.1.** *The function*

$$e_{1,m}(t) = e(t,m) = \sum_{k=0}^{\infty} \frac{t^k}{k!\,(k+m)!}\,, \tag{2.1}$$

*$m$ an integer number, is an eigenfunction of the operator*

$$D_t\, tD_t + mD_t\,, \qquad \left(D_t := \frac{d}{dt}\right)\,, \tag{2.2}$$

*since, for every complex constant $a$, it results*

$$(D_t\, tD_t + mD_t)\, a\, e_{1,m}(t) = a\, e_{1,m}(t)\,. \tag{2.3}$$

*Remark* 2.2. Introducing the generalized circular functions

$$\cos_{1,m}(x) = \frac{e_{1,m}(ix) + e_{1,m}(-ix)}{2}\,,$$

$$\sin_{1,m}(x) = \frac{e_{1,m}(ix) - e_{1,m}(-ix)}{2\,i}\,,$$

it results that they are solutions of the differential equation

$$(DxD + mD)^2\, y(x) = -y(x)\,.$$

In fact we have, for example, for the cosine:

$$(DxD + mD)^2 \cos_{1,m}(x) = (DxD + mD) \left[\frac{i\, e_{1,m}(ix) - i\, e_{1,m}(-ix)}{2}\right] =$$

$$= \frac{-\, e_{1,m}(ix) - e_{1,m}(-ix)}{2} = -\cos_{1,m}(x)\,.$$

An analogous result, changing signs, holds for the hyperbolic case.

2.1. **The fractional case.** Introducing the function

$$\lambda_k(m,\alpha) = \begin{cases} \dfrac{1}{m+\Gamma(\alpha+1)}, & k=0 \\ 1, & k=1 \\ \displaystyle\prod_{h=1}^{k-1}\left(m + \frac{\Gamma(1+(h+1)\alpha)}{\Gamma(1+h\alpha)}\right), & k \geq 2\,, \end{cases} \tag{2.4}$$

we can prove the theorem

**Theorem 2.3.** *The function*

$$e_{(\alpha|1,m)}(t) = \sum_{k=0}^{\infty} \frac{t^{k\alpha}}{\Gamma(k\alpha+1)\lambda_k(m,\alpha)}\,, \tag{2.5}$$

$m$ *a real number, is an eigenfunction of the operator*

$$D_t^\alpha\, t^\alpha\, D_t^\alpha + m\, D_t^\alpha\,, \qquad \left(D_t := \frac{d}{dt}\right)\,, \tag{2.6}$$

*since, for every complex constant* $a$*, it results*

$$(D_t^\alpha\, t^\alpha\, D_t^\alpha + m\, D_t^\alpha)\, a\, e_{(\alpha|1,m)}(t) = a\, e_{(\alpha|1,m)}(t)\,. \tag{2.7}$$

Proof - In fact, we have

$$(D_t^\alpha\, t^\alpha\, D_t^\alpha + m\, D_t^\alpha) \sum_{k=0}^{\infty} \frac{t^{k\alpha}}{\Gamma(1+k\alpha)\,\lambda_k(m,\alpha)}$$

$$= D_t^\alpha\, t^\alpha \sum_{k=1}^{\infty} \frac{t^{(k-1)\alpha}}{\Gamma(1+(k-1)\alpha)\,\lambda_k(m,\alpha)} + m \sum_{k=1}^{\infty} a_k\, \frac{t^{(k-1)\alpha}}{\Gamma(1+(k-1)\alpha)\,\lambda_k(m,\alpha)}$$

$$= D_t^\alpha \sum_{k=0}^{\infty} \frac{t^{(k+1)\alpha}}{\Gamma(1+k\alpha)\,\lambda_{k+1}(m,\alpha)} + m \sum_{k=0}^{\infty} \frac{t^{k\alpha}}{\Gamma(1+k\alpha)\,\lambda_{k+1}(m,\alpha)}$$

$$= \sum_{k=0}^{\infty} \frac{t^{k\alpha}\,\Gamma(1+(k+1)\alpha)}{\Gamma(1+k\alpha)^2\,\lambda_{k+1}(m,\alpha)} + m \sum_{k=0}^{\infty} \frac{t^{k\alpha}}{\Gamma(1+k\alpha)\,\lambda_{k+1}(m,\alpha)}$$

$$= \sum_{k=0}^{\infty} \frac{t^{k\alpha}}{\Gamma(1+k\alpha)\,\lambda_k(m,\alpha)} \left[m + \frac{\Gamma(1+(k+1)\alpha)}{\Gamma(1+k\alpha)}\right] \frac{\lambda_k(m,\alpha)}{\lambda_{k+1}(m,\alpha)}$$

$$= \sum_{k=0}^{\infty} \frac{t^{k\alpha}}{\Gamma(1+k\alpha)\,\lambda_k(m,\alpha)}\,,$$

since

$$\left[m + \frac{\Gamma(1+(k+1)\alpha)}{\Gamma(1+k\alpha)}\right] \frac{\lambda_k(m,\alpha)}{\lambda_{k+1}(m,\alpha)} = 1, \tag{2.8}$$

for $k = 0, 1, 2, \ldots$ as one can readily infer from the definition of $\lambda_k(m,\alpha)$ in (2.4).

*Remark* 2.4. It is worth noting that the function $\lambda_k$ in equation *(2.4)* appears as an extension of the Pochhammer symbol. Indeed, for $\alpha = 1$, it returns the Pochhammer symbol $(m+2)_{k-1}$, since it results, for $k = 0, 1, 2$, $\lambda_k(m,1) = (m+2)_{k-1}$.

Since the function (2.5) behaves as a true exponential with respect to the operator (2.7), in this paper we extend the population dynamics models using this operator, which depends on the real parameter $m$.

## 3. The Malthus Case

**• The parametric Malthus model**

Using the operator in equation (2.2), the parametric Malthus model writes

$$(D_t\, t\, D_t + m\, D_t) P(t) = r\, P(t)\,, \tag{3.1}$$

where $r$ is a positive constant. Assuming the initial conditions

$$\begin{cases} P(0) = P_0\,, \\ P'(0) = \dfrac{1}{m+1}\, r\, P_0\,, \end{cases} \tag{3.2}$$

the solution is derived using a suitable expansion in power series, since in [10] have proven the result

**Theorem 3.1.** *Putting*

$$P(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!\,(k+m)!} \tag{3.3}$$

*the solution of the problem* (3.1)-(3.2) *is given by*

$$P(t) = m!\, P_0 \sum_{k=0}^{\infty} \frac{(r\,t)^k}{k!\,(k+m)!} = m!\, P_0\, e_{1,m}(r\,t)\,. \tag{3.4}$$

*Remark* 3.2. The result of Theorem 3.1 can also be obtained by observing that the correspondence between the ordinary derivative $D_t$ and the operator $D_t^\alpha\, t^\alpha\, D_t^\alpha + m\, D_t^\alpha$, taking into account the inverse operator $D_t^{-1}$, can be framed in the so-called monomial principle [11, 28] of the second kind together with the related differential isomorphism. In [10] more information on this can be found.

**• The fractional parametric Malthus model**

The fractional parametric Malthus model writes

$$(D_t^\alpha\, t^\alpha\, D_t^\alpha + m\, D_t^\alpha) P(t) = r\, P(t)\,, \tag{3.5}$$

where $r$ is a positive constant. Assuming the initial conditions

$$\begin{cases} P(0) = P_0 \,, \\ D_t^\alpha \, P(0) = \dfrac{1}{m+\Gamma(\alpha+1)} \, r \, P_0 \,, \end{cases} \tag{3.6}$$

we find the result

**Theorem 3.3.** *Putting*

$$P(t) = \sum_{k=0}^{\infty} a_k \, \frac{t^{k\alpha}}{\Gamma(k\alpha+1)\,\lambda_k(m,\alpha)} \tag{3.7}$$

*the solution of the problem* (3.5)-(3.6) *is given by*

$$\begin{aligned} P(t) = \lambda_0(m,\alpha)\,P_0 \sum_{k=0}^{\infty} \frac{(r^{1/\alpha}\,t)^{k\alpha}}{\Gamma(k\alpha+1)\,\lambda_k(m,\alpha)} &= \lambda_0(m,\alpha)\,P_0\,e_{(\alpha|1,m)}(r^{1/\alpha}\,t) = \\ &= \frac{P_0\,e_{(\alpha|1,m)}(r^{1/\alpha}\,t)}{m+\Gamma(\alpha+1)} \,, \end{aligned} \tag{3.8}$$

*thus extending the classical case.*

## 4. The Verhulst Case

### • The parametric Verhulst model

Pierre Verhulst considered the so called *logistic model*, considering that the growth rate depends on the environmental resources, and cannot be constant.
Using the operator in equation (2.2), the parametric Verhulst model writes

$$(D_t\,t\,D_t + m\,D_t)P(t) = r\,P(t)\left[1 - \frac{1}{K}P(t)\right], \tag{4.1}$$

where $r$ is a positive constant. Assuming the initial conditions

$$\begin{cases} P(0) = P_0 \,, \\ P'(0) = \dfrac{1}{m+1}\, r\, P_0 \left(1 - \dfrac{P_0}{K}\right) \,, \end{cases} \tag{4.2}$$

and considering the expansion (3.3) for the solution, in [10] we have proven the result

**Theorem 4.1.** *The coefficients of the expansion* (3.3) *for the solution of the parametric Verhulst model* (4.1)-(4.2) *satisfy the recursion*

$$\begin{cases} a_0 = m!\,P_0 \,, \\ a_{k+1} = r\left[a_k - \dfrac{1}{K}\displaystyle\sum_{n=0}^{k} \binom{k}{n}\binom{k+m}{n+m}\frac{a_n\,a_{k-n}}{(k-n)_m}\right] \,, \end{cases} \tag{4.3}$$

*where* $(k-n)_m$ *denotes the Pochhammer symbol.*

### • The fractional parametric Verhulst model

The fractional parametric Verhulst model writes

$$(D_t^\alpha\,t^\alpha\,D_t^\alpha + m\,D_t^\alpha)P(t) = r\,P(t)\left[1 - \frac{1}{K}P(t)\right], \tag{4.4}$$

where $r$ is a positive constant. Assuming the initial conditions

$$\begin{cases} P(0) = P_0 \,, \\ D_t^\alpha \, P(0) = \dfrac{1}{m+\Gamma(\alpha+1)} \, r \, P_0 \left(1 - \dfrac{P_0}{K}\right) \,, \end{cases} \tag{4.5}$$

and considering the following expansion for the solution

$$P(t) = \sum_{k=0}^{\infty} a_k \, \frac{t^{k\alpha}}{\Gamma(1+k\alpha)\,\lambda_k(m,\alpha)} \,, \tag{4.6}$$

where the function $\lambda_k(m,\alpha)$ is defined in equation (2.4), we find the result

**Theorem 4.2.** *The coefficients of the expansion* (4.6) *for the solution of the problem* (4.4)-(4.5) *satisfy the recursion*

$$\begin{cases} a_0 = \lambda_0(m) P_0 = \dfrac{P_0}{m+\Gamma(\alpha+1)} \\ a_{k+1} = r \left[ a_k - \dfrac{1}{K} \displaystyle\sum_{n=0}^{k} a_n \, a_{k-n} \frac{\Gamma(1+k\alpha)\,\lambda_k(m,\alpha)}{\Gamma(1+n\alpha)\,\lambda_n(m,\alpha)\,\Gamma(1+(k-n)\alpha)\,\lambda_{k-n}(m,\alpha)} \right] \end{cases} \tag{4.7}$$

Proof - In fact, recalling equation (2.8), we have

$$\begin{aligned}
&(D_t^\alpha \, t^\alpha \, D_t^\alpha + m \, D_t^\alpha) \sum_{k=0}^{\infty} a_k \, \frac{t^{k\alpha}}{\Gamma(1+k\alpha)\,\lambda_k(m,\alpha)} \\
&= D_t^\alpha \, t^\alpha \sum_{k=1}^{\infty} a_k \, \frac{t^{(k-1)\alpha}}{\Gamma(1+(k-1)\alpha)\,\lambda_k(m,\alpha)} + m \sum_{k=1}^{\infty} a_k \, \frac{t^{(k-1)\alpha}}{\Gamma(1+(k-1)\alpha)\,\lambda_k(m,\alpha)} \\
&= D_t^\alpha \sum_{k=0}^{\infty} a_{k+1} \, \frac{t^{(k+1)\alpha}}{\Gamma(1+k\alpha)\,\lambda_{k+1}(m,\alpha)} + m \sum_{k=0}^{\infty} a_{k+1} \, \frac{t^{k\alpha}}{\Gamma(1+k\alpha)\,\lambda_{k+1}(m,\alpha)} \\
&= \sum_{k=0}^{\infty} a_{k+1} \, \frac{t^{k\alpha}\,\Gamma(1+(k+1)\alpha)}{\Gamma(1+k\alpha)^2\,\lambda_{k+1}(m,\alpha)} + m \sum_{k=0}^{\infty} a_{k+1} \, \frac{t^{k\alpha}}{\Gamma(1+k\alpha)\,\lambda_{k+1}(m,\alpha)} \\
&= \sum_{k=0}^{\infty} a_{k+1} \, \frac{t^{k\alpha}}{\Gamma(1+k\alpha)\,\lambda_k(m,\alpha)} \left[ m + \frac{\Gamma(1+(k+1)\alpha)}{\Gamma(1+k\alpha)} \right] \frac{\lambda_k(m,\alpha)}{\lambda_{k+1}(m,\alpha)} \\
&= \sum_{k=0}^{\infty} a_{k+1} \, \frac{t^{k\alpha}}{\Gamma(1+k\alpha)\,\lambda_k(m,\alpha)} \,,
\end{aligned} \tag{4.8}$$

so that, comparing the first and second member of equation (4.4), the recursion (4.7) follows.

**Example 1.** Assuming $m = 2$, $r = 1.0$, $K = 2.0$, and putting $P_0 = 1.0$, the solution of the fractional parametric Verhulst model (4.4)-(4.5) is given by the expansion (4.6) whose coefficients are reported in Figure 1.

The graph of the solution using the $a_k$ coefficients is reported in Figure 2.

0.346473, 0.173237, $2.77556\times10^{-17}$, -0.0597694, $-3.46945\times10^{-17}$, 0.0616018, $6.245\times10^{-17}$, -0.111164, $-1.80411\times10^{-16}$, 0.299138, $7.77156\times10^{-16}$, -1.10507, $-3.55271\times10^{-15}$, 5.31933, $2.4869\times10^{-14}$, -32.1673, $-1.91847\times10^{-13}$, 237.812, $1.7053\times10^{-12}$, -2104.33, $-1.77351\times10^{-11}$

FIGURE 1. Table of the $a_k$ coefficients for $0 \le n \le 20$

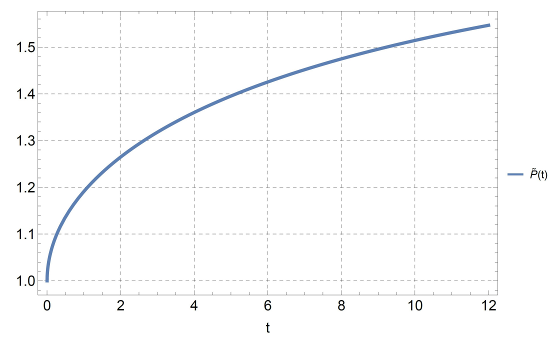


FIGURE 2. Graph of the solution using the $a_k$ coefficients reported in Figure 1

## 5. THE MINIMUM THRESHOLD LOGISTIC CASE

**• The minimum threshold logistic case**

Using the operator in equation (2.2), the minimum threshold parametric logistic model writes

$$(D_t\, t\, D_t + m\, D_t)P(t) = \gamma\, P\left(1-\frac{P}{K}\right)\left(1-\frac{M}{P}\right), \tag{5.1}$$

where $\gamma$ is a positive constant, and $M$ denotes the population minimum threshold. Assuming the initial conditions

$$\begin{cases} P(0) = P_0\,, \\ P'(0) = \dfrac{1}{m+1}\,\gamma\, P_0\left(1-\dfrac{P_0}{K}\right)\left(1-\dfrac{M}{P_0}\right). \end{cases} \tag{5.2}$$

In [10] we have proven the result

**Theorem 5.1.** *The coefficients of the expansion (4.6) for the solution of the parametric population logistic minimum threshold model (5.1)-(5.2) satisfy the recursion*

$$\begin{cases} a_0 = m!\, P_0\,, \\ a_1 = \gamma\left[\left(1+\dfrac{M}{K}\right)a_0 - \dfrac{1}{m!\,K}\,a_0^2 - m!\,M\right], \\ a_{k+1} = \gamma\left[\left(1+\dfrac{M}{K}\right)a_k - \dfrac{1}{K}\displaystyle\sum_{n=0}^{k}\binom{k}{n}\binom{k+m}{n+m}\frac{a_n\, a_{k-n}}{(k-n)_m}\right], \quad (k>0)\,, \end{cases} \tag{5.3}$$

*where $(k-n)_m$ denotes the Pochhammer symbol.*

### • The fractional parametric minimum threshold logistic case

Using the operator in equation (2.6), the fractional minimum threshold parametric logistic model writes

$$(D_t^\alpha \, t^\alpha \, D_t^\alpha + m \, D_t^\alpha) P(t) = \gamma \, P \left(1 - \frac{P}{K}\right)\left(1 - \frac{M}{P}\right) , \tag{5.4}$$

where $\gamma$ is a positive constant, and $M$ denotes the population minimum threshold. Assuming the initial conditions

$$\begin{cases} P(0) = P_0 \, , \\ D_t^\alpha \, P(0) = \dfrac{1}{m + \Gamma(\alpha+1)} \, \gamma \, P_0 \left(1 - \dfrac{P_0}{K}\right)\left(1 - \dfrac{M}{P_0}\right) , \end{cases} \tag{5.5}$$

and considering the expansion (4.6) for the solution, we find the result

**Theorem 5.2.** *The coefficients of the expansion* (4.6) *for the solution of the fractional parametric population logistic minimum threshold model* (5.4)-(5.5) *satisfy the recursion*

$$\begin{cases} a_0 = \dfrac{P_0}{m + \Gamma(\alpha+1)} \\ a_1 = \gamma \left[\left(1 + \dfrac{M}{K}\right) a_0 - \dfrac{m + \Gamma(\alpha+1)}{K} \, a_0^2 - \dfrac{M}{m + \Gamma(\alpha+1)}\right] , \\ a_{k+1} = \gamma \left[\left(1 + \dfrac{M}{K}\right) a_k - \dfrac{1}{K} \displaystyle\sum_{n=0}^{k} a_n \, a_{k-n} \frac{\Gamma(k\alpha+1) \, \lambda_k(m,\alpha)}{\Gamma(n\alpha+1) \, \lambda_n(m,\alpha) \, \Gamma((k-n)\alpha+1) \, \lambda_{k-n}(m,\alpha)}\right] . \end{cases} \tag{5.6}$$

Proof - Using the expansion (4.6) in equation (5.4) and performing calculations, the result follows comparing the coefficients of the two members.

**Example 2.** Assuming $m = 2$, $r = 1.0$, $K = 2.0$, $M = 2.0$, and putting $P_0 = 1.0$, the solution of the parametric fractional logistic minimum threshold model (5.4)-(5.5) for $\alpha = 1/2$ is given by the expansion (4.6) whose coefficients are reported in the following Figure 3.

0.346473, -0.173237, -0.173237, -0.233006, -0.382881, -0.73496, -1.60429, -3.90976, -10.4959, -30.7171, -97.1952, -330.275, -1198.46, -4621.84, -18 865.4, -81 212.9, -367 565., $-1.74417 \times 10^6$, $-8.65603 \times 10^6$, $-4.48289 \times 10^7$, $-2.4179 \times 10^8$

FIGURE 3. Table of the $a_k$ coefficients for $0 \leq n \leq 20$

The graph of the solution using the $a_k$ coefficients is reported in Figure 4.

## 6. The Volterra-Lotka Case

### • The parametric Volterra-Lotka model

Using the operator in equation (2.2),), the parametric Volterra-Lotka model writes

$$\begin{cases} (D_t \, t \, D_t + m \, D_t) \, X(t) = sX(t) - \gamma X(t) Y(t) \, , \\ (D_t \, t \, D_t + m \, D_t) \, Y(t) = -rY(t) + \beta X(t) Y(t) \, , \end{cases} \tag{6.1}$$

where the constants $\gamma, \beta$ have the same meaning as before.

Assuming for the solution $[X(t), Y(t)]$ the expansions

$$X(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k! \, (k+m)!} \, , \quad Y(t) = \sum_{k=0}^{\infty} b_k \frac{t^k}{k! \, (k+m)!} \, , \tag{6.2}$$

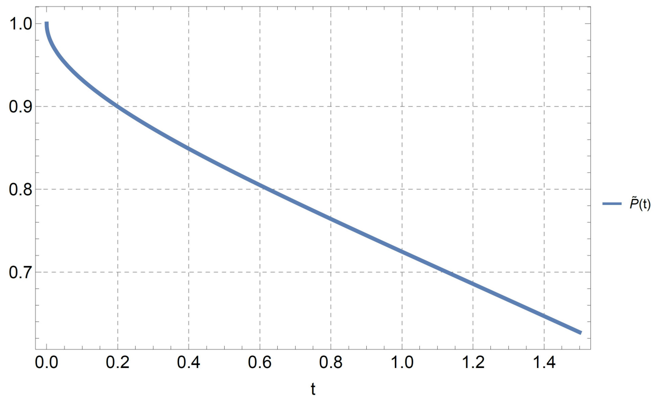


FIGURE 4. Graph of the solution using the $a_k$ coefficients reported in Figure 3

putting $X(0) = X_0\,, Y(0) = Y_0$, and after performing the relevant calculations in equations (6.1), in [10] we have proven the result

**Theorem 6.1.** *The solution of the parametric Volterra-Lotka model (6.1) is obtained by using the following recursions for the $a_k$ and $b_k$ coefficients*

$$\begin{cases} a_0 = m!\, X_0\,, \\ b_0 = m!\, Y_0\,, \\ a_{k+1} = s\, a_k - \gamma \displaystyle\sum_{n=0}^{k} \binom{k}{n} \binom{k+m}{n+m} \frac{a_n\, b_{k-n}}{(k-n)_m}\,, \\ b_{k+1} = -r\, b_k + \beta \displaystyle\sum_{n=0}^{k} \binom{k}{n} \binom{k+m}{n+m} \frac{a_n\, b_{k-n}}{(k-n)_m}\,, \end{cases} \tag{6.3}$$

*where $(k-n)_m$ denotes the Pochhammer symbol.*

- **The fractional parametric Volterra-Lotka model**

Let us consider the fractional parametric Volterra-Lotka model

$$\begin{cases} (D_t^\alpha\, t^\alpha\, D_t^\alpha + m\, D_t^\alpha) X(t) = X(t)[s - \gamma Y(t)] \\ (D_t^\alpha\, t^\alpha\, D_t^\alpha + m\, D_t^\alpha) Y(t) = Y(t)[\beta X(t) - r] \end{cases} \tag{6.4}$$

The orbits are given by the parametric equations $X = X(t), Y = Y(t)$ with $X(0) = x_0$, $Y(0) = y_0$, respectively. The equilibrium point is the same as the classical mode, that is:

$$\begin{cases} X_0 = \dfrac{r}{\beta} \\ Y_0 = \dfrac{s}{\gamma} \end{cases} \tag{6.5}$$

The following theorem holds true.

**Theorem 6.2.** *Upon putting*

$$X(t) = \sum_{n=0}^{\infty} a_n \frac{t^{\alpha n}}{\Gamma(\alpha\, n+1)\, \lambda_n(m,\alpha)}\,, \tag{6.6}$$

*and*

$$Y(t) = \sum_{n=0}^{\infty} b_n \, \frac{t^{\alpha n}}{\Gamma(\alpha\, n+1)\, \lambda_n(m,\alpha)}\,, \tag{6.7}$$

*the solution of the fractional parametric Volterra-Lotka model* (6.4) *under the initial conditions* $X(0) = X_0$ *and* $Y(0) = Y_0$ *is obtained by computing the coefficients* $a_n$ *and* $b_n$ *through the following recursive formulas*

$$\begin{cases} a_0 = \dfrac{X_0}{m+\Gamma(\alpha+1)} \\[2ex] b_0 = \dfrac{Y_0}{m+\Gamma(\alpha+1)} \\[2ex] a_{k+1} = s\, a_k - \gamma c_k \\[2ex] b_{k+1} = -r\, b_k + \beta\, c_k\,, \end{cases} \tag{6.8}$$

*where*

$$c_k = \sum_{n=0}^{k} a_n\, a_{k-n} \, \frac{\Gamma(k\alpha+1)\,\lambda_k(m,\alpha)}{\Gamma(n\alpha+1)\,\lambda_n(m,\alpha)\,\Gamma((k-n)\alpha+1)\,\lambda_{k-n}(m,\alpha)}\,.$$

Proof - Using the expansions (6.6)-(6.7) in equation (6.4) and performing calculations, the result follows comparing the coefficients in the members of the resulting system.

**Example 3.** Assuming $m = 1, r = 1.0, s = 1.0, \beta = 2.0, \gamma = 2.0$, and putting $X_0 = 0.5, Y_0 = 0.75$, we find the $a = a_n$ and $b = b_n$ coefficients reported in Figure 5.

**a**

0.397619, -0.19881, 0.397619, -1.00946, 3.12373, -11.394, 47.7568,
-225.691, 1185.43, -6842.96, 43 023.4, -292 443., $2.13582\times 10^{6}$, $-1.66719\times 10^{7}$, $1.38465\times 10^{8}$,
$-1.21875\times 10^{9}$, $1.13294\times 10^{10}$, $-1.10889\times 10^{11}$, $1.13965\times 10^{12}$, $-1.22685\times 10^{13}$, $1.38034\times 10^{14}$

**b**

0.265079, 0.331349, -0.927778, 2.33486, -6.46805, 20.9858, -80.1367,
353.585, -1764.71, 9793.09, -59 659.4, 395 126., $-2.82339\times 10^{6}$, $2.16311\times 10^{7}$, $-1.76768\times 10^{8}$,
$1.53398\times 10^{9}$, $-1.40822\times 10^{10}$, $1.36301\times 10^{11}$, $-1.38684\times 10^{12}$, $1.4795\times 10^{13}$, $-1.65098\times 10^{14}$

FIGURE 5. The $a_n$ and $b_n$ coefficients for $0 \le n \le 50$

The graph of the solution using the $a_n$ and $b_n$ coefficients is reported in Figure 6.

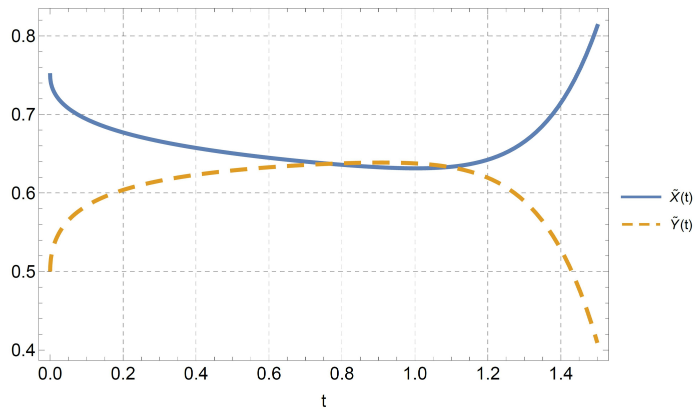


FIGURE 6. Graph of the solution, using the $a_n$ and $b_n$ coefficients reported in Figure 5

## 7. Conclusion

A Laguerre-type operator containing a positive real parameter, which behaves as an exponential with respect to its eigenfunction, has been used to extend some classical models of population dynamics to the fractional parametric case.

The eigenfunction of the fractional operator $D_t^\alpha\, t^\alpha\, D_t^\alpha + m\, D_t^\alpha$ cannot be obtained by simply replacing the factorials with the Gamma function, in the eigenfunction of the ordinary operator $D_t\, t\, D_t + m\, D_t$, but required a more in-depth analysis. As a result, a greater complexity of the recurrence formulas for the calculation of the coefficients of the related solutions has been detected.

The analytical and graphical results shown, with the aid of the computer algebra system Mathematica©, have confirmed the possibility of using such models, which can allow a better fit to real data.

## Statements and Declarations

The authors declare that they have not received funds from any institution and that they have no conflict of interest.

## References

[1] W. C. Alee. *Animal Aggregations*. University Chicago Press, 1931.

[2] R. Almeida, Nuno R. O. Bastos, and M. T. T. Monteiro. A fractional Malthusian growth model with variable order using an optimization approach. *Statistics, Optimization & Information Computing*, 6(1):4–11, 2018.

[3] N. Bacaër. *A short History on Mathematical Population Dynamics*. Springer Verlag, Berlin-Heidelberg-New York, 2010.

[4] L. Beghin and M. Caputo. Commutative and associative properties of the Caputo fractional derivative and its generalizing convolution operator. *Communications in Nonlinear Science and Numerical Simulation*, 89:Article ID 105338, 2020.

[5] G. Bretti and P. E. Ricci. Laguerre-type special functions and population dynamics. *Applied Mathematics and Computation*, 187:89–100, 2007.

[6] D. Caratelli and P. E. Ricci. A note on fractional-type models of population dynamics. *Mathematical Modeling & Analysis*, 29(3):480-492, 2024.

[7] D. Caratelli, P. Natalini, and P. E. Ricci. Fractional Bernoulli and Euler Numbers and related fractional polynomials. A Symmetry in Number Theory. *Symmetry*, 15:Article ID 1900, 2023.

[8] D. Caratelli, P. Natalini, and P. E. Ricci. Fractional differential equations and expansions in fractional powers. *Symmetry*, 15:Article ID 1842, 2023.

[9] D. Caratelli, P. Natalini, and P. E. Ricci. Examples of expansions in fractional powers and applications. *Symmetry*, 15:Article ID 1702, 2023.

[10] D. Caratelli, P. Natalini, and P. E. Ricci. Laguerre-type parametric population dynamics models. *Journal of Nonlinear Sciences and Applications*. 18:239-249, 2025.

[11] G. Dattoli. Hermite-Bessel and Laguerre-Bessel functions: A by-product ot the monomiality principle. In D. Cocolicchio, G. Dattoli and H.M. Srivastava, Editors, *Proceedings of the Melfi School on Advanced Topics in Mathematics and Physics, Melfi, 9-12 May 1999, Advanced Special Functions and Applications*, Pages 147–164, Aracne Editrice, Rome, 2000.

[12] G. Dattoli and P. E. Ricci. Laguerre-type exponentials, and the relevant L-circular and L-hyperbolic functions. *Georgian Mathematical Journal*, 10:481-494, 2003.

[13] S. De Andreis and P. E. Ricci. Modelling population growth via Laguerre-type exponentials. *Mathematical and Computer Modelling*, 42:1421-1428, 2025.

[14] P. Delerue. Sur le calcul symbolique à $n$ variables et fonctions hyper-besséliennes (II). *Annales De La Société Scientifique De Bruxelles*, Ser. 1, 3:229-274, 1953.

[15] I. Dimovski. Operational calculus for a class of differential operators. *Comptes Rendus de L'Academie Bulgare des Sciences*, 19:1111–1114, 1966.

[16] A. P. Ditkin and V. A. Prudnikov. *Transforms and Operational Calculus, Pergamon Press.* Oxford, U.K., 1965.

[17] A. El-Ajou, O. A. Arqub, Z. A. Zhour, and S. Momani. New Results on Fractional Power Series. Theories and Applications, *Entropy*, 15:5305-5323, 2013.

[18] G. F. Gause. *Verifications Sperimentales De La Théorie Mathématique De La Lutte Pour La Vie.* Actualités Scientifiques et Industrielles, Hermann & Cie, Paris, 1935.

[19] R. Gorenflo, A. A. Kilbas, F. Mainardi, and S. V. Rogosin. *Mittag-Leffler Functions, Related Topics and Applications.* Springer, New York, 2014.

[20] G. Groza and M. Jianu. Functions represented into fractional Taylor series. *ITM Web of Conferences*, 29:Article ID 01017, 2019.

[21] G. Groza, M. Jianu, and I. Mierluş-Mazilu. Some properties of the functions representable as fractional power series. *Mathematics*, 12:Article ID 961, 2024.

[22] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo. *Theory and Applications of Fractional Differential Equations, North-Holland Mathematical Studies*, Vol. 204, Elsevier (North-Holland) Science Publishers, Amsterdam, London and New York, 2006.

[23] V. Kiryakova. Multiple Dzrbashjan-Gelfond-Leontiev fractional differintegrals. In *Interational Workshop on the Recent Advances in Applid Mathematics*, pages 281-294, May 4-7, 1996, Kuwait, 1996.

[24] V. Kiryakova. *Generalized Fractional Calculus and Applications.* Pitman Research Notes in Mathematics, Ser. 301, Longman, Harlow, United Kingdom, 1994.

[25] A. Mishra. Birla Institute of Technology & Science, Pilani, Jhunjhunu, Rajasthan, India. Private communication, 2022.

[26] J. Murray. *Mathematical Biology*, Springer Verlag, Berlin–Heidelberg–New York, 1989.

[27] T. R. Prabhakar. A singular integral equation with a generalized Mittag-Leffler function in the kernel. *Yokohama Mathematical Journal*, 19:7-15, 1971.

[28] P. E. Ricci and I. Tavkhelidze. An introduction to operational techniques and special polynomials. *Journal of Mathematical Sciences*, 157:161–189, 2009.

[29] K. M. Saad, H. M. Srivastava, and J. F. Gomez-Aguilar. A fractional quadratic autocatalysis associated with chemical clock reactions involving linear inhibition. *Chaos, Solitons & Fractals*, 132:1-9, 2020.

[30] S. Samko, A. A. Kilbas, and O. Marichev. Fractional Integrals and Derivatives. Taylor & Francis, Abingdon, Oxfordshire U.K., 1993.

[31] H. M. Srivastava, R. S. Dubey, and M. Jain. A study of the fractional-order mathematical model of diabetes and its resulting complications. *Mathematical Methods in the Applied Sciences*, 42:4570–4583, 2019.

[32] H. M. Srivastava, F. A. Shah, and R. Abass. An application of the Gegenbauer wavelet method for the numerical solution of the fractional Bagley-Torvik equation. *Russian Journal of Mathematical Physics*, 26:77-93, 2019.

[33] H. M. Srivastava and H. Gunerhan. Analytical and approximate solutions of fractional-order susceptible-infected-recovered epidemic model of childhood disease. *Mathematical Methods in the Applied Sciences*, 42:935–941, 2019.

[34] H. R. Thieme. *Mathematics in Population Biology*, Princeton University Press, Princeton, New Jersey, 2003.