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COEFFICIENT BOUNDS FOR A CLASS OF SYMMETRIC-STARLIKE FUNCTION DEFINED BY THE SALAGEAN OPERATOR

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Dedicated to Professor Hari Mohan Srivastava on the Occasion of His 85th Birthday

ABSTRACT. Complex analysis provides the foundation for understanding analytic functions while Geometric Function Theory (GFT), a branch of complex analysis investigates the geometric aspects of analytic functions exploring their properties and behavior. In GFT, coefficient problems focus on finding bounds and relationship among the coefficients of analytic functions which plays an important role in studying their geometric properties. Symmetric starlike function is a class of analytic functions that exhibit a specific type of symmetry and is starlike with respect to a symmetric point. Researchers have explored various subclasses of symmetric starlike functions, including those related to q-calculus, Janowski type functions, and conic domains. Previous studies investigated coefficient inequalities, sufficient conditions, convolution properties, and other geometric characteristics for classes of symmetric starlike functions. Many operators are being used in the study of analytic functions. Salagean introduced differential and integral operators which are the most interesting of all. These operators are very inspiring and many researchers in GFT are establishing results using the operators. The focus of this study is on symmetric-starlike function which was defined using Salagean differential operator and subordinate to the Gegenbauer polynomial. The initial coefficient bounds, Fekete-Szego and second Hankel determinant were obtained. Furthermore, Zalcman functional and the Krushkal inequality for the class of the Salagean type symmetric-starlike function were established.

Keywords. Symmetric-starlike function, Hankel determinant, Fekete-Szego functional, Zalcman functional, Krushkal inequality, Gegenbauer polynomial.

1. INTRODUCTION

In complex analysis, the concept "symmetry" meaning objects remaining invariant under some transformation is recently used in geometric function theory to modify the earlier works on starlike functions. Motivated by the works of Faisal et al [3] and Nak [8], this study presented some results on symmetric-starlike function defined using the Salagean differential operator for $n \geq 0$ given as $\mathcal{S}s^*$

$$\mathcal{S}s^* = \left\{ \psi \in S : \Re \frac{2D^{n+1}\psi(\xi)}{D^n(\psi(\xi) - \psi(-\xi))} > 0, (\xi \in \mathbb{U}) \right\}.$$

This paper is organized as follows. In Section 2, the preliminaries comprises the background of the study, definition of terms and literature review while in section 3, the main results of the study which contains the coefficient bounds, the second Hankel determinant and the Zalcman functional for the

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generalized symmetric-starlike function were given. The conclusion of the work is given in Section 4 while the statement and declarations, the acknowledgements and the references for the study follows respectively.

2. Preliminaries

Denote by $\mathfrak{A}$ the class of functions of the form

$$\psi(\xi) = \xi + \sum_{k=2}^{\infty} l_k \xi^k \tag{2.1}$$

analytic and normalized in the unit disc $\mathbb{U} := \{\xi \in \mathbb{C} : |\xi| < 1\}$ satisfying $\psi(0) = \psi'(0) - 1 = 0$. Denote by $\mathbb{S}$ a subclass of $\mathfrak{A}$, the class of normalised univalent functions. Two subclasses of $\mathbb{S} \in \mathfrak{A}$ among others, denoted by $\mathcal{S}^*$ and $\mathcal{C}^*$ known as starlike function and convex function respectively are defined below. The definitions are given in [15].

Definition 2.1. A function $\psi(\xi) \in \mathfrak{A}$ is called starlike if it satisfies

$$\Re\left(\frac{\xi\psi'(\xi)}{\psi(\xi)}\right) > 0 \ (\xi \in \mathbb{U}).$$

Denote this class by $\mathcal{S}^*$.

Definition 2.2. A function $\psi(\xi) \in \mathfrak{A}$ is called convex if it satisfies

$$\Re\left(1 + \frac{\xi\psi''(\xi)}{\psi'(\xi)}\right) > 0 \ (\xi \in \mathbb{U}).$$

Denote this class by $\mathcal{C}$.

Definition 2.3. p analytic in $\mathbb{U}$ with $p(0) = 1$ is called functions with positive real part or Caratheodory functions and is denoted by P. It has Taylor series expansion of the form

$$p(\xi) = 1 + \sum_{n=1}^{\infty} c_n \xi^n \tag{2.2}$$

satisfying

$$\Re p(\xi) > 0 \ (\xi \in \mathbb{U}).$$

Definition 2.4. ψ and ϕ are two functions which are analytic in $\mathbb{U}$, then $\psi(\xi)$ is subordinate to $\phi(\xi)$ in $\mathbb{U}$ and write

$$\psi(\xi) \prec \phi(\xi) \ (\xi \in \mathbb{U}) \tag{2.3}$$

if there exists a Schwartz function $w(\xi)$ analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(\xi)| < 1$ with $(\xi \in \mathbb{U})$ such that

$$\psi(\xi) = \phi(w(\xi))$$

In particular, if the function ϕ is univalent in $\mathbb{U}$, the above subordination is equivalent to $\psi(0) = \phi(0)$ and $\psi(\mathbb{U}) \subset \phi(\mathbb{U})$.

Definition 2.5. Let $\psi(\xi) \in \mathfrak{A}$ with $\psi(\xi)$ given as (2.1) . Then for $n \geq 0$, the Salagean differential operator D^n are defined by

$$D^n[\psi](\xi) = D(D^{n-1}[\psi](\xi))$$

where $D^1\psi(\xi) = \xi\psi'(\xi)$ and $D^0\psi(\xi) = \psi(\xi)$, so that $D^n[\psi](\xi) = \xi + \sum_{k=2}^{\infty} k^n l_k \xi^k$ for $n \in N_0$

Symmetric-starlike and symmetric-convex functions denoted by $\mathcal{S}s^*$ and $\mathcal{C}s^*$ respectively have attracted the attention of many researchers. For example, [3, 8] and [14] worked on symmetric starlike functions of different kind. Analytic formulation of the class investigated by some authors were provided as follows:

$$\mathcal{S}s^* = \left\{ \psi \in \mathbb{S} : \Re \frac{2\xi\psi'(\xi)}{\psi(\xi) - \psi(-\xi)} > 0, (\xi \in U) \right\} \tag{2.4}$$

$$\mathcal{C}s^* = \left\{ \psi \in \mathbb{S} : \Re \frac{(2\xi\psi'(\xi))'}{(\psi(\xi) - \psi(-\xi))'} > 0, (\xi \in U) \right\} \tag{2.5}$$

The qth Hankel determinant for $q \geq 1, n \geq 0$ expressed as

$$H_q(n) = \begin{pmatrix} l_n \; l_{n+1} \dots l_{n+q-1} \\ l_{n+1} \dots \dots l_{n+q} \\ l_{n+2} \dots \dots l_{n+q+1} \\ \dots \dots \dots \dots \\ l_{n+q-1} \; l_{n+q} \dots l_{n+2q-2} \end{pmatrix}, q \geq 1, n \geq 0$$

was introduced by Pommerenke [12]. For $n = 1$ and $q = 2$

$$H_2(1) = \begin{vmatrix} l_1 & l_2 \\ l_2 & l_3 \end{vmatrix} = l_3 - l_2^2$$

where $l_1 = 1$ is known as the Fekete-Szego functional which has been investigated for several subclasses of analytic functions. For $n = 2$ and $q = 2$

$$H_2(2) = \begin{vmatrix} l_2 & l_3 \\ l_3 & l_4 \end{vmatrix} = l_2 l_4 - l_3^2$$

is the second Hankel determinant. Authors in [11] and [13] and many more, have examined and established results for second Hankel determinant . Zalcman propounded a theory in the study of GFT which states that the coefficients of class $\mathbb{S}$ satisfy the inequality

$$|l_k^2 - l_{2k-1}| \leq (k-1)^2 \text{ with } k \geq 2 \text{ for } \psi \in \mathbb{S}.$$

Equality holds for Koebe function $\psi(\xi) = \frac{\xi}{(1-\xi)^2}$ and its rotations. The generalised form of the Zalcman functional for different subclasses of $\mathbb{S}$ is given as

$$|\lambda l_k^2 - l_{2k-1}| \leq \lambda k^2 - 2k + 1 (\lambda \geq 0).$$

A broader Zalcman hypothesis for $\psi \in S$ was given as

$$|l_k l_m - l_{k+s-1}| \leq (k-1)(s-1) \text{ for } k \geq 2, s \geq 2.$$

For details, see [3] page 2, [5, 6] page 4, and [15, 16] page 2. Krushkal [7] introduced and proved the inequality for $n \geq 4$ and $p = 1$

$$|l_n^p - l_2^{p(n-1)}| \leq 2^{p(n-1)} - n^p$$

for the entire class $\mathbb{S}$. Further results were obtained in [11].

Gegenbauer polynomial is very useful in the definition of several subclasses of analytic functions which in turn promotes the advancement of GFT. See [1, 2, 4, 10] and [18] .

Gegenbauer polynomials has the generating function given by

$$\mathcal{G}_\tau(x, \xi) = \frac{1}{(1 - 2x\xi + \xi^2)^\tau}, \tag{2.6}$$

for non-zero real constant τ, where x $\in$ [-1,1] and $\xi \in \mathbb{U}$. For fixed x the function $\mathcal{G}_\tau(x,\xi)$ is analytic in $\mathbb{U}$, so it can be expanded on a Taylor series as

$$\mathcal{G}_\tau(x,\xi) = \xi + \sum_{n=2}^{\infty} C_{n-1}^{\tau}(x)\xi^n, \tag{2.7}$$

which is the normalised Gegenbauer polynomial with $\tau > -\frac{1}{2}$. and initial values

$$C_0^\tau(x) = 1,$$
$$C_1^\tau(x) = 2\tau x$$

given

$$C_n^\tau(x) = \frac{2x(n+\tau-1)C_{n-1}^\tau(x) - (n+2\tau-2)C_{n-2}^\tau(x)}{n} \tag{2.8}$$

representing the Gegenbauer polynomial of degree n. Then for $n = 2, 3$ and 4, we have

$$C_2^\tau(x) = 2\tau(1+\tau)x^2 - \tau \tag{2.9}$$

$$C_3^\tau(x) = \frac{4\tau(\tau+1)(\tau+2)}{3}x^3 - 2\tau(\tau+1)x \tag{2.10}$$

$$C_4^\tau(x) = \frac{2\tau(\tau+1)(\tau+2)(\tau+3)}{3}x^4 - 2\tau(\tau+1)(\tau+2)x^2 + \frac{\tau(\tau+1)}{2}. \tag{2.11}$$

Lemma 2.6. [15] *Let $p \in P$, with $p(\xi)$ given by (2.2). Then,*

$$|c_n| \leq 2, \ n \in N.$$

Definition for the new subclass $\mathcal{G}(\tau, x, n)$ is as follows.

Definition 2.7. Let $\psi \in \mathfrak{A}$. If

$$\frac{2D^{n+1}\psi(\xi)}{D^n(\psi(\xi) - \psi(-\xi))} \prec \mathcal{G}_\tau(x,\xi). \tag{2.12}$$

Then $\psi \in \mathcal{G}(\tau, x, n)$, where $x \in [-1,1]$, $\tau \geq 0$, $n \geq 0$.

3. Main Results

Theorem 3.1. *Let $\psi \in \mathfrak{A}$ of the form (2.1). Let $\psi \in \mathcal{G}(\tau, x, n)$, with $x \in [-1,1]$, $\tau \geq 0$, $n \geq 0$. Then,*

$$|l_2| \leq \frac{2\tau x}{2^n},$$

$$|l_3| \leq \frac{1}{3^n}\left(2\tau x + 4\tau^2x^2 + 4\tau x^2 - 2\tau\right),$$

$$|l_4| \leq \frac{1}{4^n}\left(\tau x + 6x^2\tau^2 + 4x^2\tau - 2\tau + 2\left(\frac{10x^3\tau^3}{3} - 3x\tau^2 - 2x\tau + 6x^3\tau^2 + \frac{8x^3\tau}{3}\right)\right),$$

$$\begin{aligned} |l_5| \leq{}& \frac{1}{5^n}\left(\tau x + 2\tau(\tau+1)x^2 - \tau + 2 + 2(4\tau(\tau+1)(\tau+2)x^3 - 6\tau(\tau+1)x)\right) \\ &+ \frac{4}{5^n}\left(\frac{\tau(\tau+1)}{2} - 2\tau(\tau+1)(\tau+2)x^2 + 2\tau(\tau+1)(\tau+2)(\tau+3)3x^4\right) \\ &+ \frac{1}{5^n}\left(2\tau^2x^2 + 4\tau x(2x^2\tau^2 + 2x^2\tau - \tau) + 2(2x^2\tau^2 - 2x^2\tau - \tau)^2\right). \end{aligned}$$

Proof:
From (2.12),

$$\frac{2D^{n+1}\psi(\xi)}{D^n(\psi(\xi) - \psi(-\xi))} = \mathcal{G}_\tau(x, (w(\xi)) \tag{3.1}$$

From the L.H.S. of (3.1), we have

$$\frac{2D^{n+1}\psi(\xi)}{D^n(\psi(\xi)-\psi(-\xi))} = 1+(2^{n+1}l_2)\xi+2.3^n l_3\xi^2+(4^{n+1}l_4-2^{n+1}3^n l_2 l_3+)\xi^3+(4.5^n l_5+2.3^{2n}l_3^2)\xi^4, \tag{3.2}$$

where

$$\begin{aligned}
\psi(\xi) =&\xi+\sum_{k=2}^{\infty} l_k\xi^k\\
\psi(-\xi) =&-\xi+\sum_{k=2}^{\infty} l_k(-\xi)^k\\
D^n\psi(\xi) =&\xi+\sum_{k=2}^{\infty} k^n l_k\xi^k\\
D^n\psi(-\xi) =&-\xi+\sum_{k=2}^{\infty} k^n l_k(-\xi)^k\\
D^{n+1}\psi(\xi) =&\xi+\sum_{k=2}^{\infty} k^{n+1} l_k\xi^k\\
2D^{n+1}\psi(\xi) =&\xi+\sum_{k=2}^{\infty} k^{n+1} l_k\xi^k
\end{aligned}$$

so that by simple computation, the L.H.S. of (3.1) yeilds (3.2).
Similarly from the R.H.S. of (3.1) using (2.6),

$$\begin{aligned}
\mathcal{G}(x,\xi) =&\frac{1}{(1-2x\xi+\xi^2)^\tau}\\
=&1+(2\tau x)\xi+(2\tau(\tau+1)x^2-\tau)\xi^2+\left(\frac{4\tau(\tau+1)(\tau+2)x^3}{3}-2\tau(\tau+1)x\right)\xi^3 \\
&+\left(\frac{2\tau(\tau+1)(\tau+2)(\tau+3)x^4}{3}-2\tau(\tau+1)(\tau+2)x^2+\frac{\tau(\tau+1)}{2}\right)\xi^4+\cdots
\end{aligned} \tag{3.3}$$

By the principle of subordination we have $f(\xi)=g(w(\xi))$ with

$$w(\xi)=c_1\xi+c_2\xi^2+c_3\xi^3+c_4\xi^4+c_5\xi^5+\cdots$$

Therefore,

$$\begin{aligned}
\mathcal{G}_\tau(x,(w(\xi)) =&1+(2\tau x c_1)\xi+\left(2\tau x c_2+(2\tau^2x^2+2\tau x^2-\tau)c_1^2\right)\xi^2\\
&+\left(2\tau x c_3+\left(\frac{4\tau^3x^3}{3}-2x\tau^2-2x\tau+4x^3\tau^2+\frac{8\tau x^3}{3}\right)c_1^3+(4x^2\tau^2+4x^2\tau-2\tau)c_1c_2\right)\xi^3\\
&+\Bigg(2\tau x c_4+(2\tau(\tau+1)x^2-\tau)c_2^2+2c_1c_3+(4\tau(\tau+1)(\tau+2)x^3-6\tau(\tau+1)x)c_1^2c_2\\
&+(\frac{\tau(\tau+1)}{2}-2\tau(\tau+1)(\tau+2)x^2+\frac{2\tau(\tau+1)(\tau+2)(\tau+3)x^4}{3})c_1^4\Bigg)\xi^4+\cdots
\end{aligned} \tag{3.4}$$

Equating coefficients of powers of ξ in (3.2) with (3.4) and applying Lemma 2.6 completes the proof.

Theorem 3.2. Fekete-Szego functional
Let $\psi\in\mathfrak{A}$ of the form (2.1). If $\psi\in\mathcal{G}(\tau,x,n)$, with $x\in[-1,1]$, $\tau\geq 0$, $n\geq 0$ and μ any real number then,

$$|l_3-\mu l_2^2| \leq \frac{2^{2n+1}\tau x+2^{2n+2}\tau^2x^2+2^{2n+2}\tau x^2-2^{2n+1}\tau-2^23^n\mu\tau^2x^2}{2^{2n}.3^n}$$

Proof: The proof follows from Theorem 3.1

Theorem 3.3. Second order Hankel determinant
Let $\psi \in \mathfrak{A}$ of the form (2.1). If $\psi \in \mathcal{G}(\tau,x,n)$, with $x \in [-1,1]$, $\tau \geq 0$, $n \geq 0$, then

$$\begin{aligned}
&|l_2l_4-l_3^2| \\
&\leq \Bigg|\frac{1}{2^n4^{n+1}}\Bigg(2\tau^2x^2c_1c_3+(6x^3\tau^3+4x^3\tau^2-2x\tau^2)c_1^2c_2+\left(\frac{10x^4\tau^4}{3}-3x^2\tau^3-2x^2\tau^2+6x^4\tau^3+\frac{8x^4\tau^2}{3}\right)c_1^4\Bigg) \\
&-\left(\frac{1}{2^23^{2n}}\left(4\tau^2x^2c_2^2+(8x^3\tau^3+8x^3\tau^2-4x\tau^2)c_1c_2+(2x^2\tau^2+2x^2\tau-\tau)^2c_1^4\right)\right)\Bigg|
\end{aligned}$$

Proof: The second order Hankel determinant is given as

$$|H_2(2)| = |l_2l_4-\mu l_3^2|$$

and the proof follows from Theorem 3.1 .

Theorem 3.4. *Let $\psi \in \mathfrak{A}$ of the form (2.1). If $\psi \in \mathcal{G}(\tau,x,n)$ with $x \in [-1,1]$, $\tau \geq 0$, $n \geq 0$ and $k \geq 2$, then*

$$|l_2^2-l_3| \leq \frac{4\tau^2x^2}{2^n}+\frac{2\tau+2\tau x-4\tau(\tau+1)x^2}{3^n}$$

Proof: Zalcman's inequality is given as

$$|l_k^2-l_{2k-1}| \leq (k-1)^2 \text{ with } k \geq 2 \text{ for } \psi \in \mathbb{S}$$

Now, for $k=2$,

$$|l_2^2-l_3| = \left|(\frac{2\tau x}{2^n})^2-\frac{2\tau+2\tau x-4\tau(\tau+1)x^2}{3^n}\right|$$

Theorem 3.5. *Let $\psi \in \mathfrak{A}$ of the form (2.1). If $\psi \in \mathcal{G}(\tau,x,n)$, with $x \in [1,-1]$, $\tau \geq 0$, $n \geq 0$ and $k \geq 2$, then*

$$\begin{aligned}
|l_3^2-l_5| \leq &\frac{1}{5^n}\left((4\tau^2+4\tau)x^2+(8\tau^2+8\tau)(\tau+2)x^2-(2\tau^2+2\tau)x^2-\tau-\tau x\right) \\
&+\frac{1}{5^n}\left(12\tau(\tau+1)x-8\tau(\tau+1)(\tau+2)x^3-2\tau(\tau+1)-2(\tau x+2\tau(\tau+1)x^2-\tau)^2\right) \\
&+\frac{1}{3^{2n}}\left(4\tau^2x^2+4((2\tau^2+2\tau)x^2-\tau)^2+(16\tau^2+16\tau)x^3\right) \\
&-\frac{1}{3(5^n)}\left(8\tau^2x-(8\tau^2+8\tau)(\tau+2)(\tau+3)x^4\right)
\end{aligned}$$

Proof: Zalcman's inequality is given as

$$|l_k^2-l_{2k-1}| \leq (k-1)^2 \text{ with } k \geq 2 \text{ for } \psi \in \mathbb{S}$$

Now, for $k=3$,

$$\begin{aligned}
|l_3^2-l_5| =&\Bigg|\left(\frac{1}{3^n}\left(2\tau x+4\tau^2x^2+4\tau x^2-2\tau\right)\right)^2 \\
&-\Bigg[\frac{1}{5^n}\left(\tau x+2\tau(\tau+1)x^2-\tau+2+2(4\tau(\tau+1)(\tau+2)x^3-6\tau(\tau+1)x\right) \\
&+\frac{4}{5^n}\left(\frac{\tau(\tau+1)}{2}-2\tau(\tau+1)(\tau+2)x^2+2\tau(\tau+1)(\tau+2)(\tau+3)3x^4\right) \\
&+\frac{1}{5^n}\left(2\tau^2x^2+4\tau x(2x^2\tau^2+2x^2\tau-\tau)+2(2x^2\tau^2-2x^2\tau-\tau)^2\right)\Bigg]\Bigg|
\end{aligned}$$

Theorem 3.6. *Let $\psi \in \mathfrak{A}$ of the form (2.1). If $\psi \in \mathcal{G}(\tau, x, n)$, with $x \in [1, -1]$, $\tau \geq 0$, $n \geq 0$ and $k \geq 2$, then*

$$|l_4 - l_2^3| \leq \frac{\tau x + 6x^2\tau^2 + 4x^2\tau - 2\tau + 2\left(\frac{10x^3\tau^3}{3} - 3x\tau^2 - 2x\tau + 6x^3\tau^2 + \frac{8x^3\tau}{3}\right)}{4^n} + \frac{8\tau^3 x^3}{2^{3n}}$$

Proof: The Krushkal inequality

$$|l_n^p - l_2^{p(n-1)}| \leq 2^{p(n-1)} - n^p$$

is proven for $n = 4, p = 1$, as follows

$$|l_4 - l_2^3| \leq 2^3 - 4$$

and the proof is completed by Theorem 3.1

Theorem 3.7. *Let $\psi \in \mathfrak{A}$ of the form (2.1). If $\psi \in \mathcal{G}(\tau, x, n)$, with $x \in [1, -1]$, $\tau \geq 0$, $n \geq 0$ and $k \geq 2$, then*

$$\begin{aligned} |l_5 - l_2^4| \leq & \frac{1}{5^n}\left(\tau x + 2\tau(\tau+1)x^2 - \tau + 2 + 2(4\tau(\tau+1)(\tau+2)x^3 - 6\tau(\tau+1)x)\right) \\ & + \frac{4}{5^n}\left(\frac{\tau(\tau+1)}{2} - 2\tau(\tau+1)(\tau+2)x^2 + 2\tau(\tau+1)(\tau+2)(\tau+3)3x^4\right) \\ & + \frac{1}{5^n}\left(2\tau^2x^2 + 4\tau x(2x^2\tau^2 + 2x^2\tau - \tau) + 2(2x^2\tau^2 - 2x^2\tau - \tau)^2\right) + \frac{16\tau^4 x^4}{2^{4n}} \end{aligned}$$

Proof: The Krushkal inequality is given as

$$|l_n^p - l_2^{p(n-1)}| \leq 2^{p(n-1)} - n^p$$

so that for $n = 5, p = 1$, we have

$$|l_5 - l_2^4| \leq 2^4 - 5.$$

Thus, the proof follows from Theorem 3.1

4. Conclusion

In this investigation, the class of symmetric-starlike function was defined using the Salagean differential operator. Coefficient bounds and some functionals were established. In further study, higher order Hankel and Toeplitz determinants can be examined for symmetric-convex function.

Statements and Declarations

The authors declare that they have no conflict of interest, and the manuscript has no associated data.

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