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A STUDY OF A NEW SUBCLASS OF ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS

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Dedicated to Professor Hari Mohan Srivastava on the Occasion of His 85th Birthday

ABSTRACT. In this paper,we introduce the subclasses $A_j(n,m,\lambda,\alpha)$ and $T_j(n,m,\lambda,\alpha)$ defined by generalized Salagean operator D_λ^n. Coefficients estimates for the function belong to class $T_j(n,m,\lambda,\alpha)$ and distortion theorems are given. By making use of the familiar concept of neighborhoods of analytic functions is defined the neighborhoods of functions in the class $T_j(n,m,\lambda,\alpha)$. Furthermore, distortion theorems for fractional calculius of functions in the class $T_j(n,m,\lambda,\alpha)$ has been studied. Additionally, Upper bound of the Hankel determinants $H_{2,1}(f)$ for the coefficients of the functions belonging to class $A(n,m,\lambda,\alpha)$ determined.

Keywords. Analytic function with Negative Coefficients, Coefficient estimates, Salagean Operator, Neighborhoods, Fractional Calculus, Hankel determinant.

1. Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which in the open unit disk $\Delta = \{z \in C : |z| < 1\}$.

Let A_j be the family of functions of the form

$$f(z) = z + \sum_{k=j+1}^{\infty} a_k z^k, \quad (j \in \mathbb{N} = \{1,2,3,\ldots\}) \tag{1.2}$$

which are analytic in the open unit disk $\Delta = \{z \in C : |z| < 1\}$.

For a function $f(z) \in A_j$, we define

$$\begin{aligned} D^0 f(z) &= f(z) \\ D^1 f(z) &= Df(z) = zf'(z), \ldots \end{aligned}$$

and

$$D^n f(z) = D\left(D^{n-1} f(z)\right) \quad (n \in \mathbb{N})$$

The differential operator D^n was introduced by Salagean [12].

With the help of the differential operator D^n, we say that a function $f(z) \in A_j$ is in the class $A_j(n,m,\alpha)$ if and only if

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$$Re\left\{\frac{D^{n+m}f(z)}{D^{n}f(z)}\right\} > \alpha \quad (n\in\mathbb{N}^*=\mathbb{N}\cup\{0\}, m\in\mathbb{N}, \lambda\geq 0) \tag{1.3}$$

for some $\alpha(0\leq\alpha<1)$, and for all $z\in\Delta$.
The operator D^{n+m} was introduced by Sekine [13], Aouf et al. [2].
In [1], AL-Oboudi defined the generalized Salagean operator as following:
Let $n\in\mathbb{N}^*$ and $\lambda\geq 0$. We let D_λ^n denote the operator defined by

$$D_\lambda^n : A\to A$$

$$D_\lambda^0 f(z) = f(z)$$

$$D_\lambda^1 f(z) = (1-\lambda)D_\lambda^0 f(z) + \lambda z\left(D_\lambda^0 f(z)\right)' = (1-\lambda)f(z)+\lambda z f'(z)$$

$$\ldots$$

$$D_\lambda^{n+1} f(z) = (1-\lambda)D_\lambda^n f(z) + \lambda z\left(D_\lambda^n f(z)\right)'$$

When $\lambda=1$, we get Salagean's differential operator [12].
For $f\in A_j(n,m,\alpha)$ using the definition of the D_λ^n operator, we write [5]

$$D_\lambda^n f(z) = z + \sum_{k=j+1}^{\infty}[1+\lambda(k-1)]^n a_k z^k \tag{1.4}$$

Let T_j denote the subclass of A_j consisting of functions of the form:

$$f(z) = z - \sum_{k=j+1}^{\infty} a_k z^k, \quad (a_k\geq 0, j\in\mathbb{N}) \tag{1.5}$$

which are analytic in the open unit disk Δ.
With the above operator D_λ^n, we say that a function $f(z)$ belonging to A_j is in the class $\mathrm{A}_j(n,m,\lambda,\alpha)$ if and only if

$$Re\left\{\frac{D^{n+m}f(z)}{D^{n}f(z)}\right\} > \alpha \quad (n,m\in\mathbb{N}^*, \lambda\geq 0) \tag{1.6}$$

for some α, $(0\leq\alpha<1)$, and for all $z\in\Delta$.
We note that $A_1(0,1,1,\alpha)=S^*(\alpha)$ is the class of starlike functions of order α, $A_1(1,1,1,\alpha)=K(\alpha)$ is the class of convex functions of order α, and that $A_1(n,1,1,\alpha)=S_n(\alpha)$ is the class of functions defined by Salagean [12].
Further, we define the class $T_j(n,m,\lambda,\alpha)$ by

$$T_j(n,m,\lambda,\alpha) = A_j(n,m,\lambda,\alpha)\cap T_j$$

Then we observe that $T_1(0,1,1,\alpha)=T^*(\alpha)$ is the subclass of starlike functions of order α [14], $T_1(1,1,1,\alpha)=C(\alpha)$ is the subclass of convex functions of order α[6], and that $T_j(0,1,1,\alpha)$ and $T_j(1,1,1,\alpha)$ are the classes defined by Chatterjea [4]

2. Coefficients Estimates And Distortion Theorems

We begin with the statement and the proof of the following result.

Theorem 2.1. *Let the function $f(z)$ be defined by*

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad (a_k\geq 0)$$

Then $f(z)\in T_1(n,m,\lambda,\alpha)$ if and only if

$$\sum_{k=2}^{\infty}[1+\lambda(k-1)]^n\left\{[1+\lambda(k-1)]^m-\alpha\right\}a_k \le 1-\alpha \tag{2.1}$$

for $n \in \mathbb{N}^*, m \in \mathbb{N}^*$ and $0 \le \alpha < 1$. The result is sharp.

Proof. Let $|z| = 1$. Assume that the inequality (2.1) holds. Then we have

$$\begin{aligned}
\left|\frac{D_\lambda^{n+m}f(z)}{D_\lambda^n f(z)}-1\right| &= \left|\frac{z-\sum_{k=2}^{\infty}[1+\lambda(k-1)]^{n+m}a_k z^k}{z-\sum_{k=2}^{\infty}[1+\lambda(k-1)]^n a_k z^k}-1\right| \\
&= \left|\frac{\sum_{k=2}^{\infty}[1+\lambda(k-1)]^n\left\{[1+\lambda(k-1)]^m-1\right\}a_k z^{k-1}}{1+\sum_{k=2}^{\infty}[1+\lambda(k-1)]^n a_k z^{k-1}}\right| \\
&\le \frac{\sum_{k=2}^{\infty}[1+\lambda(k-1)]^n\left\{[1+\lambda(k-1)]^m-1\right\}a_k|z|^{k-1}}{1-\sum_{k=2}^{\infty}[1+\lambda(k-1)]^n a_k|z|^{k-1}} \\
&= \frac{\sum_{k=2}^{\infty}[1+\lambda(k-1)]^n\left\{[1+\lambda(k-1)]^m-1\right\}a_k}{1-\sum_{k=2}^{\infty}[1+\lambda(k-1)]^n a_k} \\
&= \frac{\sum_{k=2}^{\infty}[1+\lambda(k-1)]^n\left\{[1+\lambda(k-1)]^m-\alpha+\alpha-1\right\}a_k}{1-\sum_{k=2}^{\infty}[1+\lambda(k-1)]^n a_k} \\
&= \frac{\sum_{k=2}^{\infty}[1+\lambda(k-1)]^n\left\{[1+\lambda(k-1)]^m-\alpha\right\}a_k-(1-\alpha)\sum_{k=2}^{\infty}[1+\lambda(k-1)]^n a_k}{1-\sum_{k=2}^{\infty}[1+\lambda(k-1)]^n a_k} \\
&\le \frac{(1-\alpha)-(1-\alpha)\sum_{k=2}^{\infty}[1+\lambda(k-1)]^n a_k}{1-\sum_{k=2}^{\infty}[1+\lambda(k-1)]^n a_k} \\
&= (1-\alpha).
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
\left|\frac{D_\lambda^{n+m}f(z)}{D_\lambda^n f(z)}-1\right| < 1-\alpha \Rightarrow & \left|Re\left\{\frac{D_\lambda^{n+m}f(z)}{D_\lambda^n f(z)}-1\right\}\right| < 1-\alpha \\
&\Rightarrow -(1-\alpha) < Re\left\{\frac{D_\lambda^{n+m}f(z)}{D_\lambda^n f(z)}-1\right\} < 1-\alpha \\
&\Rightarrow -(1-\alpha) < Re\left\{\frac{D_\lambda^{n+m}f(z)}{D_\lambda^n f(z)}-1\right\} < 1-\alpha \\
&\Rightarrow Re\left\{\frac{D_\lambda^{n+m}f(z)}{D_\lambda^n f(z)}\right\} > \alpha
\end{aligned}$$

So $f(z) \in T_1(n,m,\lambda,\alpha)$.Conversely, assume that the function $f(z)$ is in the class defined $T_1(n,m,\lambda,\alpha)$. Then

$$Re\left\{\frac{D_\lambda^{n+m}f(z)}{D_\lambda^n f(z)}\right\} = \left\{\frac{1-\sum_{k=2}^{\infty}[1+\lambda(k-1)]^{n+m}a_k z^{k-1}}{1-\sum_{k=2}^{\infty}[1+\lambda(k-1)]^n a_k z^{k-1}}\right\} > \alpha \tag{2.2}$$

for $z \in \Delta$. Choose values of z on the real axis so that $\frac{D_\lambda^{n+m}f(z)}{D_\lambda^n f(z)}$ is real. Upon clearing the denominator in (2.2) and letting $z \to 1^-$through real values, we obtain

$$\begin{aligned}
&\frac{1-\sum_{k=2}^{\infty}[1+\lambda(k-1)]^{n+m}a_k z^{k-1}}{1-\sum_{k=2}^{\infty}[1+\lambda(k-1)]^n a_k z^{k-1}} > \alpha \\
&\Rightarrow 1-\sum_{k=2}^{\infty}[1+\lambda(k-1)]^{n+m}a_k z^{k-1} > \alpha\left\{1-\sum_{k=2}^{\infty}[1+\lambda(k-1)]^n a_k z^{k-1}\right\} \\
&\Rightarrow 1-\sum_{k=2}^{\infty}[1+\lambda(k-1)]^{n+m}a_k \ge \alpha\left\{1-\sum_{k=2}^{\infty}[1+\lambda(k-1)]^n a_k\right\} \\
&\Rightarrow \sum_{k=2}^{\infty}[1+\lambda(k-1)]^n\left\{[1+\lambda(k-1)]^m-\alpha\right\}a_k \le 1-\alpha
\end{aligned}$$

So (2.1) is correct. The result is sharp with the extremal function $f(z)$ defined by

$$f(z) = z - \frac{2(1-\alpha)}{[1+\lambda(k-1)]^n \{[1+\lambda(k-1)]^m - \alpha\} k(k+1)} z^k \quad (k \geq 2) \tag{2.3}$$

□

Theorem 2.2. *Let the function $f(z)$ be defined by*

$$f(z) = z - \sum_{k=j+1}^{\infty} a_k z^k, \quad (a_k \geq 0) \tag{2.4}$$

Then $f(z) \in T_j(n, m, \lambda, \alpha)$ if and only if

$$\sum_{k=j+1}^{\infty} [1+\lambda(k-1)]^n \{[1+\lambda(k-1)]^m - \alpha\} a_k \leq 1 - \alpha \tag{2.5}$$

for $n \in \mathbb{N}^*, m \in \mathbb{N}^*$ and $0 \leq \alpha < 1$. The result is sharp for the function

$$f(z) = z - \frac{2(1-\alpha)}{[1+\lambda(k-1)]^n \{[1+\lambda(k-1)]^m - \alpha\} k(k+1)} z^k \quad (k \geq j+1) \tag{2.6}$$

Proof. Putting $a_k = 0 (k = 2, 3, \ldots, j)$ in Theorem 2.1, we can prove the assertion of Theorem 2.2. □

Corollary 2.3. *Let the function $f(z)$ defined by (2.4) be in the class function $T_j(n, m, \lambda, \alpha)$. Then*

$$a_k \leq \frac{1-\alpha}{[1+\lambda(k-1)]^n \{[1+\lambda(k-1)]^m - \alpha\}} z^k \quad (k \geq j+1) \tag{2.7}$$

The equality in (2.7) is attained for the function $f(z)$ given by (2.6).

Remark 2.4. Taking $(n, m, \lambda, \alpha) = (n, m, 1, \alpha)$ in Theorem 2.1, Theorem 2.2 and Corollary 2.3, we have the corresponding results by Sekine [13]. Taking $(j, n, m, \lambda, \alpha) = (1, 1, 0, 1, \alpha)$ and $(j, n, m, \lambda, \alpha) = (1, 1, 1, 1, \alpha)$ in Theorem 2.2 , we have the corresponding results by Silverman [14]. Additionally,taking $(j, n, m, \lambda, \alpha) = (j, 0, 1, 1, \alpha)$ and $(j, n, m, \lambda, \alpha) = (1, 1, 1, 1, \alpha)$ in Theorem 2.2, we have the corresponding results by Chatterjea [4].

Theorem 2.5. *Let the function $f(z)$ defined by (1.5) be in the class function $T_j(n, m, \lambda, \alpha)$. Then*

$$\left|D_\lambda^\Omega f(z)\right| \geq |z| - \frac{1-\alpha}{[1+\lambda j]^{n-\Omega} \{(1+\lambda j)^m - \alpha\}} |z|^{j+1} \tag{2.8}$$

and

$$\left|D_\lambda^\Omega f(z)\right| \leq |z| + \frac{1-\alpha}{[1+\lambda j]^{n-\Omega} \{(1+\lambda j)^m - \alpha\}} |z|^{j+1} \tag{2.9}$$

for $z \in \Delta$, where $0 \leq \Omega \leq n$. The equalities in (2.8) and (2.9) are attained for the function $f(z)$ given by

$$f(z) = z - \frac{1-\alpha}{[1+\lambda j]^n \{[1+\lambda j]^m - \alpha\}} z^{j+1} \tag{2.10}$$

Proof. Note that $f(z) \in T_j(n, m, \lambda, \alpha)$ if and only if $D_\lambda^\Omega f(z) \in T_j(n - \Omega, m, \lambda, \alpha)$, and that

$$D_\lambda^\Omega f(z) = z - \sum_{k=j+1}^{\infty} [1+\lambda(k-1)]^\Omega a_k z^k \tag{2.11}$$

Using Theorem 2.2, we know that

$$\begin{aligned}
& \sum_{k=j+1}^{\infty} [1+\lambda(k-1)]^{n-\Omega+\Omega} \{[1+\lambda(k-1)]^m - \alpha\} a_k \leq 1-\alpha \\
\Rightarrow \quad & \sum_{k=j+1}^{\infty} [1+\lambda(k-1)]^{n-\Omega}[1+\lambda(k-1)]^{\Omega} \{[1+\lambda(k-1)]^m - \alpha\} a_k \leq 1-\alpha \\
\Rightarrow \quad & [1+\lambda j]^{n-\Omega} \{(1+\lambda j)^m - \alpha\} \sum_{k=j+1}^{\infty} [1+\lambda(k-1)]^{\Omega} a_k \leq 1-\alpha
\end{aligned} \tag{2.12}$$

that is, that

$$\sum_{k=j+1}^{\infty} [1+\lambda(k-1)]^{\Omega} a_k \leq \frac{1-\alpha}{[1+\lambda j]^{n-\Omega} \{(1+\lambda j)^m - \alpha\}} \tag{2.13}$$

It follow from (2.11) and (2.12) that

$$\left|D_\lambda^\Omega f(z)\right| \geq |z| - \frac{1-\alpha}{[1+\lambda j]^{n-\Omega} \{(1+\lambda j)^m - \alpha\}} |z|^{j+1} \tag{2.14}$$

and

$$\left|D_\lambda^\Omega f(z)\right| \leq |z| + \frac{1-\alpha}{[1+\lambda j]^{n-\Omega} \{(1+\lambda j)^m - \alpha\}} |z|^{j+1} \tag{2.15}$$

Finally, we note that the equalities in (2.8) and (2.9) are attained for the function $f(z)$ defined by

$$D_\lambda^\Omega f(z) = z - \frac{1-\alpha}{[1+\lambda j]^{n-\Omega} \{[1+\lambda j]^m - \alpha\}} z^{j+1} \tag{2.16}$$

This completes the proof of Theorem 2.5. □

Taking $\lambda = 1$ in Theorem 2.5, the following Corollary 2.6 given by Sekine [13].

Corollary 2.6. *Let the function $f(z)$ defined by (1.6) be in the class function $T_j(n,m,1,\alpha) = T_j(n,m,\alpha)$. Then*

$$\left|D^\Omega f(z)\right| \geq |z| - \frac{1-\alpha}{[1+j]^{n-\Omega} \{(1+j)^m - \alpha\}} |z|^{j+1} \tag{2.17}$$

and

$$\left|D^\Omega f(z)\right| \leq |z| + \frac{1-\alpha}{[1+j]^{n-\Omega} \{(1+j)^m - \alpha\}} |z|^{j+1} \tag{2.18}$$

for $z \in \Delta$, where of $0 \leq \Omega \leq n$. The equality in (2.17) and (2.18) are attained for the function $f(z)$ given by

$$f(z) = z - \frac{1-\alpha}{[1+j]^{n} \{[1+j]^m - \alpha\}} z^{j+1} \tag{2.19}$$

Taking $\Omega = 0$ in Theorem 2.5, we have the following Corollary 2.7.

Corollary 2.7. *Let the function $f(z)$ defined by (1.5) be in the class function $T_j(n,m,\lambda,\alpha)$. Then*

$$|f(z)| \geq |z| - \frac{1-\alpha}{[1+\lambda j]^{n} \{(1+\lambda j)^m - \alpha\}} |z|^{j+1} \tag{2.20}$$

and

$$|f(z)| \leq |z| + \frac{1-\alpha}{[1+\lambda j]^{n} \{(1+\lambda j)^m - \alpha\}} |z|^{j+1} \tag{2.21}$$

for $z \in \Delta$. The equalities in (2.20) and (2.21) are attained for the function $f(z)$ defined by (2.10).

Taking $\Omega = 1$ in Theorem 2.5 , we can write following Corollary 2.8 .

Corollary 2.8. *Let the function $f(z)$ defined by (1.5) be in the class function $T_j(n,m,\lambda,\alpha)$. Then*

$$\left|D^1 f(z)\right| \geq 1 - \frac{1-\alpha}{[1+\lambda j]^{n-1}\left\{(1+\lambda j)^m - \alpha\right\}}|z|^j \tag{2.22}$$

and

$$\left|D^1 f(z)\right| \leq 1 + \frac{1-\alpha}{[1+\lambda j]^{n}\left\{(1+\lambda j)^m - \alpha\right\}}|z|^j \tag{2.23}$$

for $z \in \Delta$.
The equalities in (2.22) and (2.23) are attained for the function $f(z)$ defined by (2.10).

Taking $\Omega = 1$ and $\lambda = 1$ in Theorem 2.5, we can write following Corollary 2.9.

Corollary 2.9. *Let the function $f(z)$ defined by (1.5) be in the class function $T_j(n,m,\lambda,\alpha)$. Then*

$$\left|f'(z)\right| \geq 1 - \frac{1-\alpha}{[1+j]^{n-1}\left\{(1+j)^m - \alpha\right\}}|z|^j \tag{2.24}$$

and

$$\left|f'(z)\right| \leq 1 + \frac{1-\alpha}{[1+j]^{n-1}\left\{(1+j)^m - \alpha\right\}}|z|^j \tag{2.25}$$

for $z \in \Delta$. The equalities in (2.22) and (2.23) are attained for the function $f(z)$ define by (2.10).

3. Neighborhoods of Functions In The Class $T_j(n,m,\lambda,\alpha)$ With Negative Coefficients

For any $f(z) \in T_j$ and $\delta \geq 0$, we define

$$N_{j,\delta}(f) = \left\{ g \in T_j : g(z) = z - \sum_{k=j+1}^{\infty} b_k z^k, \sum_{k=j+1}^{\infty} k\left|a_k - b_k\right| \leq \delta \right\}, \tag{3.1}$$

which was called (j,δ) - neighborhood of $f(z)$. So, for $e(z) = z$, we see that

$$N_{j,\delta}(e) = \left\{ g \in T_j : g(z) = z - \sum_{k=j+1}^{\infty} b_k z^k, \sum_{k=j+1}^{\infty} k\left|b_k\right| \leq \delta \right\}. \tag{3.2}$$

The concept of neighborhoods was first introduced by A. W. Goodman [6] and then generalized by Ruscheweyh [15].

Theorem 3.1. *Let*

$$\delta = \frac{(1-\alpha)[1+\lambda j]^n + \left|(1+\lambda j)^n - \lambda(nj+1)\right|(1-\alpha)}{\lambda\left[(1+\lambda j)^m - \alpha\right](1+\lambda j)^n}$$

Then $T_j(n,m,\lambda,\alpha) \subset N_{j,\delta}(e)$.

Proof. For $f(z) \in T_j(n,m,\lambda,\alpha)$, Theorem 2.1 immediately yields

$$\sum_{k=j+1}^{\infty} [1+\lambda(k-1)]^n \left\{[1+\lambda(k-1)]^m - \alpha\right\} a_k \leq 1-\alpha$$

$$\Rightarrow [1+\lambda j]^n \left\{[1+\lambda j]^m - \alpha\right\} \sum_{k=j+1}^{\infty} a_k \leq 1-\alpha$$

$$\Rightarrow \quad \sum_{k=j+1}^{\infty} a_k \leq \frac{1-\alpha}{[1+\lambda j]^n \left\{[1+\lambda j]^m - \alpha\right\}}$$

On the other hand, we also find

$$
\begin{aligned}
&[(1+\lambda j)^m-\alpha]\sum_{k=j+1}^{\infty}[1+\lambda(k-1)]^n a_k \\
=&[(1+\lambda j)^m-\alpha]\sum_{k=j+1}^{\infty}\left[\begin{array}{c}\binom{n}{0}\lambda^n(k-1)^n+\binom{n}{1}\lambda^{n-1}(k-1)^{n-1}\\ +\ldots+\binom{n}{n-1}\lambda(k-1)+1\end{array}\right]a_k \\
=&[(1+\lambda j)^m-\alpha]\sum_{k=j+1}^{\infty}\left[\begin{array}{c}\lambda^n(k-1)^n+\binom{n}{1}\lambda^{n-1}(k-1)^{n-1}\\ +\ldots+\lambda k-\lambda+1\end{array}\right]a_k \\
=&[(1+\lambda j)^m-\alpha]\sum_{k=j+1}^{\infty}\left[\begin{array}{c}\lambda^n(k-1)^n+\binom{n}{1}\lambda^{n-1}(k-1)^{n-1}\\ +\ldots-\lambda+1\end{array}\right]a_k \\
&+[(1+\lambda j)^m-\alpha]\,\lambda\sum_{k=j+1}^{\infty}ka_k\le 1-\alpha \\
\Rightarrow&[(1+\lambda j)^m-\alpha]\,\lambda\sum_{k=j+1}^{\infty}ka_k \\
\le&1-\alpha-[(1+\lambda j)^m-\alpha]\sum_{k=j+1}^{\infty}\left[\begin{array}{c}\lambda^n(k-1)^n+\binom{n}{1}\lambda^{n-1}(k-1)^{n-1}\\ +\ldots-\lambda+1\end{array}\right]a_k \\
\Rightarrow&[(1+\lambda j)^m-\alpha]\,\lambda\sum_{k=j+1}^{\infty}ka_k \\
\le&1-\alpha-[(1+\lambda j)^m-\alpha]\sum_{k=j+1}^{\infty}\left[\begin{array}{c}(\lambda j)^n+\binom{n}{1}(\lambda j)^{n-1}\\ +\ldots+\binom{n}{n-1}(\lambda j)-\binom{n}{n-1}(\lambda j)-\lambda+1\end{array}\right]a_k \\
\Rightarrow&[(1+\lambda j)^m-\alpha]\,\lambda\sum_{k=j+1}^{\infty}ka_k \\
\le&1-\alpha-[(1+\lambda j)^m-\alpha]\sum_{k=j+1}^{\infty}[(1+\lambda j)^n-\lambda(nj+1)]\,a_k \\
\Rightarrow&[(1+\lambda j)^m-\alpha]\,\lambda\sum_{k=j+1}^{\infty}ka_k \\
\le&1-\alpha-[(1+\lambda j)^m-\alpha]\,[(1+\lambda j)^n-\lambda(nj+1)]\,\frac{1-\alpha}{[1+\lambda j]^n\left\{[1+\lambda j]^m-\alpha\right\}} \\
\Rightarrow&[(1+\lambda j)^m-\alpha]\,\lambda\sum_{k=j+1}^{\infty}ka_k \\
\le&1-\alpha-[(1+\lambda j)^n-\lambda(nj+1)]\,\frac{1-\alpha}{[1+\lambda j]^n} \\
\Rightarrow&[(1+\lambda j)^m-\alpha]\,\lambda\sum_{k=j+1}^{\infty}ka_k \\
\le&\frac{(1-\alpha)[1+\lambda j]^n+|(1+\lambda j)^n-\lambda(nj+1)|\,(1-\alpha)}{(1+\lambda j)^n}
\end{aligned}
$$

Thus,

$$\begin{aligned} &\sum_{k=j+1}^{\infty} k\,|a_k| \\ \leq\ & \frac{(1-\alpha)[1+\lambda j]^n + |(1+\lambda j)^n - \lambda(nj+1)|\,(1-\alpha)}{\lambda\left[(1+\lambda j)^m - \alpha\right](1+\lambda j)^n} \\ =\ & \delta \end{aligned}$$

As result, Theorem 3.1 is proven according to the definition of (3.2).

□

4. Distortion Theorems for Fractional Calculus of Functions in The Class $T_j(n, m, \lambda, \alpha)$.

Let us first give the following definitions of fractional calculus by Owa [9], which we will use in the theorems we will give in this section.

Definition 4.1. The fractional integral of order β is defined by

$$D_z^{-\beta} f(z) = \frac{1}{\Gamma(\beta)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\beta}} d\zeta \tag{4.1}$$

where $\beta > 0$, $f(z)$ is an analytic function in a simply connected region of the z - plane containing the origin and the multiplicity of $(z-\zeta)^{\beta-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

Definition 4.2. The fractional derivative of order β is defined by

$$D_z^{\beta} f(z) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\beta}} d\zeta \tag{4.2}$$

where $0 \leq \beta < 1$, $f(z)$ is an analytic function in a simply connected region of the z - plane containing the origin and the multiplicity of $(z-\zeta)^{-\beta}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

Definition 4.3. Under the hypotheses of Definition 4.2 , the fractional derivative of order $(n+\beta)$ is defined by

$$D_z^{\beta+n} f(z) = \frac{d^n}{dz^n} D_n^{\beta} f(z) \tag{4.3}$$

where $0 \leq \beta < 1$ and analytic $n \in \mathbb{N}^* = \mathbb{N} \cup \{0\} = \{0, 1, 2, 3, \ldots\}$.

Theorem 4.4. *Let the function $f(z)$ defined by (1.5) be in the class $T_j(n, m, \lambda, \alpha)$. Then*

$$\left|D_z^{-\beta}\left(D_\lambda^{\Omega} f(z)\right)\right| \geq \frac{|z|^{\beta+1}}{\Gamma(2+\beta)} \left\{1 - \frac{\Gamma(j+2)\Gamma(2+\beta)(1-\alpha)}{\Gamma(j+2+\beta)[1+\lambda j]^{n-\Omega}\left\{(1+\lambda j)^m - \alpha\right\}} |z|^j\right\} \tag{4.4}$$

and

$$\left|D_z^{-\beta}\left(D_\lambda^{\Omega} f(z)\right)\right| \leq \frac{|z|^{\beta+1}}{\Gamma(2+\beta)} \left\{1 + \frac{\Gamma(j+2)\Gamma(2+\beta)(1-\alpha)}{\Gamma(j+2+\beta)[1+\lambda j]^{n-\Omega}\left\{(1+\lambda j)^m - \alpha\right\}} |z|^j\right\} \tag{4.5}$$

for $\beta > 0$, $0 \leq \Omega \leq n$, and $z \in \Delta$. The equalities in (4.4) and (4.5) are attained for the function $f(z)$ define by (2.10).

Proof. It is easy to see that

$$\begin{aligned}\Gamma(2+\beta)z^{-\beta}D_z^{-\beta}\left(D_\lambda^\Omega f(z)\right) &= z-\sum_{k=j+1}^{\infty}\frac{\Gamma(k+1)\Gamma(2+\beta)}{\Gamma(k+1+\beta)}[1+\lambda(k-1)]^\Omega a_k z^k\\ &= z-\sum_{k=j+1}^{\infty}\varphi(k)[1+\lambda(k-1)]^\Omega a_k z^k \end{aligned} \tag{4.6}$$

Where

$$\varphi(k)=\frac{\Gamma(k+1)\Gamma(2+\beta)}{\Gamma(k+1+\beta)}\quad (k\geq j+1) \tag{4.7}$$

Noting that $\varphi(k)$ is a decreasing of k, we have

$$0<\varphi(k)\leq\varphi(j+1)=\frac{\Gamma(j+2)\Gamma(2+\beta)}{\Gamma(j+2+\beta)} \tag{4.8}$$

Therefore, by using (2.13) and (4.8), we can see that

$$\begin{aligned}\left|\Gamma(2+\beta)z^{-\beta}D_z^{-\beta}\left(D_\lambda^\Omega f(z)\right)\right| &\geq |z|-\varphi(j+1)|z|^{j+1}\sum_{k=j+1}^{\infty}[1+\lambda(k-1)]^\Omega a_k\\ &\geq |z|-\frac{\Gamma(j+2)\Gamma(2+\beta)(1-\alpha)}{\Gamma(j+2+\beta)[1+\lambda j]^{n-\Omega}\left\{(1+\lambda j)^m-\alpha\right\}}|z|^{j+1}\end{aligned} \tag{4.9}$$

which implies (4.4), and that

$$\begin{aligned}\left|\Gamma(2+\beta)z^{-\beta}D_z^{-\beta}\left(D_\lambda^\Omega f(z)\right)\right| &\leq |z|+\varphi(j+1)|z|^{j+1}\sum_{k=j+1}^{\infty}[1+\lambda(k-1)]^\Omega a_k\\ &\leq |z|+\frac{\Gamma(j+2)\Gamma(2+\beta)(1-\alpha)}{\Gamma(j+2+\beta)[1+\lambda j]^{n-\Omega}\left\{(1+\lambda j)^m-\alpha\right\}}|z|^{j+1}\end{aligned} \tag{4.10}$$

which shows (4.5). Furthermore, note that the equalities in (4.4) and (4.5) are attained for the function $f(z)$ defined by

$$D_z^{-\beta}\left(D_\lambda^\Omega f(z)\right)=\frac{z^{\beta+1}}{\Gamma(2+\beta)}\left\{1-\frac{\Gamma(j+2)\Gamma(2+\beta)(1-\alpha)}{\Gamma(j+2+\beta)[1+\lambda j]^{n-\Omega}\left\{(1+\lambda j)^m-\alpha\right\}}z^j\right\} \tag{4.11}$$

Thus we complete the assertion of Theorem 4.4. □

Taking $\lambda=1$ in Theorem 4.4, the following Corollary 4.5 given by Sekine is obtained [13].

Corollary 4.5. *Let the function $f(z)$ defined by (1.5) be in the class function $T_j(n,m,1,\alpha)=T_j(n,m,\alpha)$. Then*

$$\left|D_z^{-\beta}\left(D_1^\Omega f(z)\right)\right|\geq\frac{|z|^{\beta+1}}{\Gamma(2+\beta)}\left\{1-\frac{\Gamma(j+2)\Gamma(2+\beta)(1-\alpha)}{\Gamma(j+2+\beta)[1+j]^{n-\Omega}\left\{(1+j)^m-\alpha\right\}}|z|^j\right\} \tag{4.12}$$

and

$$\left|D_z^{-\beta}\left(D_1^\Omega f(z)\right)\right|\leq\frac{|z|^{\beta+1}}{\Gamma(2+\beta)}\left\{1+\frac{\Gamma(j+2)\Gamma(2+\beta)(1-\alpha)}{\Gamma(j+2+\beta)[1+j]^{n-\Omega}\left\{(1+j)^m-\alpha\right\}}|z|^j\right\} \tag{4.13}$$

for $\beta>0, 0\leq\Omega\leq n$, and $z\in\Delta$. The equalities in (4.12) and (4.13) are attained for the function $f(z)$ define by

$$D_z^{-\beta}\left(D_1^\Omega f(z)\right)=\frac{z^{\beta+1}}{\Gamma(2+\beta)}\left\{1-\frac{\Gamma(j+2)\Gamma(2+\beta)(1-\alpha)}{\Gamma(j+2+\beta)[1+j]^{n-\Omega}\left\{(1+j)^m-\alpha\right\}}z^j\right\} \tag{4.14}$$

Taking $\lambda = 1$ and $\Omega = 0$ Theorem 4.4, we have the following Corollary 4.6.

Corollary 4.6. *Let the function $f(z)$ defined by (1.5) be in the class function $T_j(n,m,1,\alpha) = T_j(n,m,\alpha)$. Then*

$$\left|D_z^{-\beta} f(z)\right| \geq \frac{|z|^{\beta+1}}{\Gamma(2+\beta)} \left\{1 - \frac{\Gamma(j+2)\Gamma(2+\beta)(1-\alpha)}{\Gamma(j+2+\beta)[1+j]^n \left\{(1+j)^m - \alpha\right\}} |z|^j\right\}$$

and

$$\left|D_z^{-\beta} f(z)\right| \leq \frac{|z|^{\beta+1}}{\Gamma(2+\beta)} \left\{1 + \frac{\Gamma(j+2)\Gamma(2+\beta)(1-\alpha)}{\Gamma(j+2+\beta)[1+j]^n \left\{(1+j)^m - \alpha\right\}} |z|^j\right\}$$

for $\beta > 0$ and $z \in \Delta$. The equality cases in the inequalities here are valid for the function $f(z)$ define by (2.10).

Theorem 4.7. *Let the function $f(z)$ defined by (1.5) be in the class function $T_j(n,m,\lambda,\alpha)$. Then*

$$\left|D_z^{\beta}\left(D_\lambda^\Omega f(z)\right)\right| \geq \frac{|z|^{1-\beta}}{\Gamma(2-\beta)} \left\{1 - \frac{\Gamma(j+1)\Gamma(2-\beta)(1-\alpha)}{\Gamma(j+2-\beta)[1+\lambda j]^{n-\Omega-1} \left\{(1+\lambda j)^m - \alpha\right\}} |z|^j\right\}$$

and

$$\left|D_z^{\beta}\left(D_\lambda^\Omega f(z)\right)\right| \leq \frac{|z|^{1-\beta}}{\Gamma(2-\beta)} \left\{1 + \frac{\Gamma(j+1)\Gamma(2-\beta)(1-\alpha)}{\Gamma(j+2-\beta)[1+\lambda j]^{n-\Omega-1} \left\{(1+\lambda j)^m - \alpha\right\}} |z|^j\right\}$$

for $0 \leq \beta < 1, \quad 0 \leq \Omega \leq n-1$, and $z \in \Delta$. The equalities in (2.22) and (2.23) are attained for the function $f(z)$ define by (2.10).

Proof. It is easy to see that

$$\Gamma(2-\beta) z^\beta D_z^\beta \left(D_\lambda^\Omega f(z)\right) = z - \sum_{k=j+1}^{\infty} \frac{\Gamma(k+1)\Gamma(2-\beta)}{\Gamma(k+1-\beta)} [1+\lambda(k-1)]^\Omega a_k z^k$$

Since the function

$$\sigma(k) = \frac{\Gamma(k)\Gamma(2-\beta)}{\Gamma(k+1-\beta)} \quad (k \geq j+1)$$

is decreasing in k, we have

$$0 < \sigma(k) \leq \sigma(j+1) = \frac{\Gamma(j+1)\Gamma(2-\beta)}{\Gamma(j+2-\beta)} \tag{4.15}$$

Therefore, by using (2.13) and (4.8), we can see that

$$\begin{aligned}\left|\Gamma(2-\beta) z^{-\beta} D_z^{-\beta}\left(D_\lambda^\Omega f(z)\right)\right| &\geq |z| - \varphi(j+1)|z|^{j+1} \sum_{k=j+1}^{\infty} [1+\lambda(k-1)]^\Omega a_k \\ &\geq |z| - \frac{\Gamma(j+2)\Gamma(2+\beta)(1-\alpha)}{\Gamma(j+2+\beta)[1+\lambda j]^{n-\Omega} \left\{(1+\lambda j)^m - \alpha\right\}} |z|^{j+1}\end{aligned} \tag{4.16}$$

which implies (4.4), and that

$$\begin{aligned}\left|\Gamma(2-\beta) z^{-\beta} D_z^{-\beta}\left(D_\lambda^\Omega f(z)\right)\right| &\leq |z| + \varphi(j+1)|z|^{j+1} \sum_{k=j+1}^{\infty} [1+\lambda(k-1)]^\Omega a_k \\ &\leq |z| + \frac{\Gamma(j+2)\Gamma(2+\beta)(1-\alpha)}{\Gamma(j+2+\beta)[1+\lambda j]^{n-\Omega} \left\{(1+\lambda j)^m - \alpha\right\}} |z|^{j+1}\end{aligned} \tag{4.17}$$

which shows (1.4) Furthermore, note that the equalities in (4.16) and (4.17) are attained for the function $f(z)$defined by

$$D_z^\beta\left(D_\lambda^\Omega f(z)\right)=\frac{z^{1-\beta}}{\Gamma(2-\beta)}\left\{1-\frac{\Gamma(j+1)\Gamma(2-\beta)(1-\alpha)}{\Gamma(j+2-\beta)[1+\lambda j]^{n-\Omega-1}\left\{(1+\lambda j)^m-\alpha\right\}}|z|^j\right\}$$

or (2.16). Thus we complete the assertion of Theorem 4.7. □

Taking $\lambda=1$ in Theorem 4.7, we have the corresponding results by Sekine [13].

Corollary 4.8. *Let the function $f(z)$ defined by (1.4) be in the class function $T_j(n,m,1,\alpha)=T_j(n,m,\alpha)$. Then*

$$\left|D_z^\beta\left(D_1^\Omega f(z)\right)\right|\geq\frac{|z|^{1-\beta}}{\Gamma(2-\beta)}\left\{1-\frac{\Gamma(j+1)\Gamma(2-\beta)(1-\alpha)}{\Gamma(j+2-\beta)[1+j]^{n-\Omega}\left\{(1+j)^m-\alpha\right\}}|z|^j\right\}\tag{4.18}$$

and

$$\left|D_z^\beta\left(D_1^\Omega f(z)\right)\right|\leq\frac{|z|^{1-\beta}}{\Gamma(2-\beta)}\left\{1+\frac{\Gamma(j+1)\Gamma(2-\beta)(1-\alpha)}{\Gamma(j+2-\beta)[1+j]^{n-\Omega-1}\left\{(1+j)^m-\alpha\right\}}|z|^j\right\}\tag{4.19}$$

for $\beta>0$ and $z\in\Delta$. The equalities in (4.18) and (4.19) are attained for the function $f(z)$ define by (2.10).

Taking $\lambda=1$ and $\Omega=0$ Theorem 4.7, we have

Corollary 4.9. *Let the function $f(z)$ defined by (1.4) be in the class function $T_j(n,m,1,\alpha)=T_j(n,m,\alpha)$. Then*

$$\left|D_z^\beta f(z)\right|\geq\frac{|z|^{1-\beta}}{\Gamma(2-\beta)}\left\{1-\frac{\Gamma(j+1)\Gamma(2-\beta)(1-\alpha)}{\Gamma(j+2-\beta)[1+j]^{n}\left\{(1+j)^m-\alpha\right\}}|z|^j\right\}\tag{4.20}$$

and

$$\left|D_z^\beta f(z)\right|\leq\frac{|z|^{1-\beta}}{\Gamma(2-\beta)}\left\{1+\frac{\Gamma(j+1)\Gamma(2-\beta)(1-\alpha)}{\Gamma(j+2-\beta)[1+j]^{n}\left\{(1+j)^m-\alpha\right\}}|z|^j\right\}\tag{4.21}$$

for $\beta>0$ and $z\in\Delta$. The equalities in (4.20) and (4.21) are attained for the function $f(z)$ define by (2.10).

5. Upper Bounds Of The Hankel Determinants $H_{2,1}(f)$ For The Coefficients Of The Functions Belonging To The Class A (n,m,λ,α).

In the 1960s Pommerenke [10, 11] defined the Hankel determinant $H_{q,n}(f)$ for a given f of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n$$

as follows

$$H_{q,n}(f)=\begin{vmatrix}a_n & a_{n+1} & \cdots & a_{n+q-1}\\ a_{n+1} & a_{n+2} & \cdots & a_{n+q}\\ \vdots & \vdots & \cdots & \vdots\\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2}\end{vmatrix}$$

where $n,q\in\mathbb{N}=\{1,2,3,\ldots\}$. In particular,

$$H_{2,1}(f) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2 \quad \text{and} \quad H_{2,2}(f) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2a_4 - a_3^2$$

The studies on Hankel determinants are concentrated on estimating $H_{2,2}(f)$ and $H_{3,1}(f)$ for different subclasses of S. The absolute sharp bounds of the functional $H_{2,2}(f)$ were found in 2.7, 2.8 for each of the families S^*, C. In [8], Janteng et al. proved that $|H_{2,2}(f)| \leq 1$ for S^* and $|H_{2,2}(f)| \leq \frac{1}{8}$ for K, where S^* and K are very well known classes of starlike and convex functions. The estimation of the determinant $|H_{3,1}(f)|$ is very hard as compared to deriving the bound of $|H_{2,2}(f)|$. The paper on $|H_{3,1}(f)|$ was given in 2010 by Babalola[3], in which he obtained the upper bound of $H_{3,1}(f)$ for the families of S^*, C.

For the function $\psi(z) = c_1z + c_2z^2 + c_3z^3 + \dots$ (with $|\psi(z)| < 1, z \in \Delta$) the next relations is valid

$$|c_1| \leq 1, \tag{5.1}$$

$$|c_2| \leq 1 - |c_1|^2 \tag{5.2}$$

$$\left|c_3\left(1 - |c_1|^2\right) + \bar{c}_1 c_2^2\right| \leq \left(1 - |c_1|^2\right)^2 - |c_2|^2. \tag{5.3}$$

From (5.3), we can write

$$\begin{aligned}
&\left|c_3\left(1 - |c_1|^2\right) + \overline{c_1}c_2^2\right| \leq \left(1 - |c_1|^2\right)^2 - |c_2|^2 \\
&\Rightarrow \left|c_3\left(1 - |c_1|^2\right)\right| - \left|\overline{c_1}c_2^2\right| \leq \left|c_3\left(1 - |c_1|^2\right) + \overline{c_1}c_2^2\right|^2 \left(1 - |c_1|^2\right)^2 - |c_2|^2 \\
&\Rightarrow \left|c_3\left(1 - |c_1|^2\right)\right| - \left|\overline{c_1}c_2^2\right| \leq \left(1 - |c_1|^2\right)^2 - |c_2|^2 \\
&\Rightarrow \left|c_3\left(1 - |c_1|^2\right)\right| \leq \left(1 - |c_1|^2\right)^2 - |c_2|^2 + |\overline{c_1}|\left|c_2^2\right| \\
&\Rightarrow |c_3| \leq \frac{\left(1 - |c_1|^2\right)^2 - |c_2|^2 + |\overline{c_1}|\left|c_2^2\right|}{\left|\left(1 - |c_1|^2\right)\right|} \\
&\Rightarrow |c_3| \leq 1 - |c_1|^2 - \frac{|c_2|^2}{(1 + |c_1|)}
\end{aligned} \tag{5.4}$$

Let $0 \leq c_1 \leq 1$. In this case, since $0 \leq c_2 \leq 1$, from (5.4) is written

$$|c_3| \leq 1 - c_1^2 - \frac{c_2^2}{1 + c_1} \tag{5.5}$$

Let's take $c_1 = x$ and $c_2 = y$ to get the maksimum value of the right side of (5.5). Let's take

$$\varphi(x, y) = 1 - x^2 - \frac{y^2}{1 + x}.$$

Let's calculate the maximum value of the bivariate function

$$\left.\begin{array}{r} \varphi_x(x, y) = -2x + \frac{y^2}{(1+x)^2} \\ \varphi_y(x, y) = -\frac{2y}{(1+x)} \end{array}\right\} \Rightarrow x = 0, y = 0$$

$$\varphi_{xx}(x, y) = -2 - \frac{2y^2}{(1 + x)^3}, \quad \varphi_{yy}(x, y) = -\frac{2}{(1 + x)}, \quad \varphi_{xy}(x, y) = \frac{2y}{(1 + x)^2}$$

According to the values of the second-order partial derivatives at the point $(0,0)$, the following inequalities are written as

$$[\varphi_{xy}(0,0)]^2 - \varphi_{xx}(0,0)\varphi_{yy}(0,0) = -4 < 0 \quad \text{and} \quad \varphi_{xx}(0,0) = -2 < 0$$

then the point $(0,0)$ is a maximum of $\varphi(x,y)$ and

$$\max \varphi(x,y) = 1$$

So,

$$|c_3| \leq 1 \tag{5.6}$$

is obtained.

Theorem 5.1. *Let $f(z) = z + a_2z^2 + a_3z^3 + \ldots$ belongs to the class $\mathrm{A}(n,m,\lambda,\alpha), 0 \leq \alpha < 1, \lambda > 0$. Then we have the coefficients estimation:*

$$\begin{aligned} |a_2| &\leq \frac{2(1-\alpha)}{(1+\lambda)^n\left[(1+\lambda)^m - 1\right]}, \\ |a_3| &\leq \frac{2(1-\alpha)}{(1+2\lambda)^n\left[(1+2\lambda)^m - 1\right]}\left\{\frac{|1-2\alpha| + (1+\lambda)^m}{(1+\lambda)^m - 1}\right\}. \end{aligned}$$

Proof. From the definition of the class $A(n,m,\lambda,\alpha)$ we have

$$\frac{D_\lambda^{n+m}f(z)}{D_\lambda^n f(z)} = \alpha + (1-\alpha)\frac{1+\mho(z)}{1-\mho(z)} = 1 + 2(1-\alpha)\left\{\mho(z) + \mho^2(z) + \mho^3(z) + \ldots\right\} \tag{5.7}$$

where $\mho$ is analytic in Δ with $\mho(0) = 0$ and $|\mho(z)| < 1, z \in \Delta$. Let $\mho(z) = c_1z + c_2z^2 + c_3z^3 + \ldots$. From (5.7), we have

$$\begin{aligned} &z + (1+\lambda)^{n+m}a_2z^2 + (1+2\lambda)^{n+m}a_3z^3 + (1+3\lambda)^{n+m}a_4z^4 + \ldots \\ &= \left\{z + (1+\lambda)^n a_2z^2 + (1+2\lambda)^n a_3z^3 + (1+3\lambda)^n a_4z^4 + \ldots\right\}\left\{1 + 2(1-\alpha)\left\{\mho(z) + \mho^2(z) + \ldots\right\}\right\} \end{aligned} \tag{5.8}$$

and

$$\begin{aligned} &1 + 2(1-\alpha)\{\mho(z) + \mho^2(z) + \mho^3(z) + \ldots\} \\ &\qquad = 1 + 2(1-\alpha)\left\{c_1z + \left(c_2 + c_1^2\right)z^2 + \left(c_3 + 2c_1c_2 + c_1^3\right)z^3 + \ldots\right\}. \end{aligned} \tag{5.9}$$

From (5.8) and (5.9), we write

$$\begin{aligned} &z + (1+\lambda)^{n+m}a_2z^2 + (1+2\lambda)^{n+m}a_3z^3 + (1+3\lambda)^{n+m}a_4z^4 + \ldots \\ &= \left\{z + (1+\lambda)^n a_2z^2 + \ldots\right\}\left[1 + 2(1-\alpha)\left\{c_1z + \left(c_2 + c_1^2\right)z^2 + \left(c_3 + 2c_1c_2 + c_1^3\right)z^3 + \ldots\right\}\right] \end{aligned} \tag{5.10}$$

From (5.10), we obtain

$$\begin{aligned} (1+\lambda)^{n+m}a_2 =& (1+\lambda)^n a_2 + 2(1-\alpha)c_1 \\ \Rightarrow \quad a_2 =& \frac{2(1-\alpha)c_1}{(1+\lambda)^n\left[(1+\lambda)^m - 1\right]} \end{aligned} \tag{5.11}$$

and

$$\begin{aligned} (1+2\lambda)^{n+m}a_3 =& 2(1-\alpha)\left(c_2 + c_1^2\right) + 2(1-\alpha)(1+\lambda)^n c_1a_2 + (1+2\lambda)^n a_3 \\ \Rightarrow a_3 & \frac{2(1-\alpha)\left\{c_2 + \frac{(1-2\alpha)+(1+\lambda)^m}{(1+\lambda)^m-1}c_1^2\right\}}{(1+2\lambda)^n\left[(1+2\lambda)^m - 1\right]}. \end{aligned} \tag{5.12}$$

If inequality (5.1) is used (5.11),we write

$$|a_2| \leq \frac{2(1-\alpha)\,|c_1|}{(1+\lambda)^n\left[(1+\lambda)^m - 1\right]} = \frac{2(1-\alpha)}{(1+\lambda)^n\left[(1+\lambda)^m - 1\right]}$$

Similarly, If inequality (5.2) is used (5.12),we obtain

$$\begin{aligned}
|a_3| =& \frac{2(1-\alpha)}{(1+2\lambda)^n\left[(1+2\lambda)^m-1\right]}\left|c_2+\frac{(1-2\alpha)+(1+\lambda)^m}{(1+\lambda)^m-1}c_1^2\right| \\
\leq& \frac{2(1-\alpha)}{(1+2\lambda)^n\left[(1+2\lambda)^m-1\right]}\left\{|c_2|+\frac{|1-2\alpha|+(1+\lambda)^m}{(1+\lambda)^m-1}\left|c_1^2\right|\right\} \\
=& \frac{2(1-\alpha)}{(1+2\lambda)^n\left[(1+2\lambda)^m-1\right]}\left\{1+\frac{|1-2\alpha|+1}{(1+\lambda)^m-1}|c_1|^2\right\} \\
\leq& \frac{2(1-\alpha)}{(1+2\lambda)^n\left[(1+2\lambda)^m-1\right]}\left\{\frac{|1-2\alpha|+(1+\lambda)^m}{(1+\lambda)^m-1}\right\}.
\end{aligned}$$

□

Theorem 5.2. *Let* $f(z)=z+a_2z^2+a_3z^3+\ldots$ *belongs to the class* $\mathrm{A}(n,m,\lambda,\alpha), 0\leq\alpha<1$. *Then we have next sharp estimation:*

$$|H_{2,1}(f)|\leq\frac{2(1-\alpha)\left\{\left[|1-2\alpha|+(1+\lambda)^m\right](1+\lambda)^{2n}\left[(1+\lambda)^m-1\right]+4(1-\alpha)^2\right\}}{(1+\lambda)^{2n}(1+2\lambda)^n\left[(1+\lambda)^m-1\right]^2\left[(1+2\lambda)^m-1\right]}.$$

Proof. From the definition of Hankel determinant has the form of $H_{2,1}(f)=a_3-a_2^2$. In this definition by using (5.11) and (5.12) and taking module $H_{2,1}(f)$ is written as

$$H_{2,1}(f)=a_3-a_2^2=\frac{2(1-\alpha)\left\{c_2+\frac{(1-2\alpha)+(1+\lambda)^m}{(1+\lambda)^m-1}c_1^2\right\}}{(1+2\lambda)^n\left[(1+2\lambda)^m-1\right]}-\left\{\frac{2(1-\alpha)c_1}{(1+\lambda)^n\left[(1+\lambda)^m-1\right]}\right\}^2.$$

If the modulus of both sides of this last equation is taken and necessary operations are made,we obtain

$$\begin{aligned}
|H_{2,1}(f)|=\left|a_3-a_2^2\right|=&\left|\frac{2(1-\alpha)\left\{c_2+\frac{(1-2\alpha)+(1+\lambda)^m}{(1+\lambda)^m-1}c_1^2\right\}}{(1+2\lambda)^n\left[(1+2\lambda)^m-1\right]}-\left\{\frac{2(1-\alpha)c_1}{(1+\lambda)^n\left[(1+\lambda)^m-1\right]}\right\}^2\right| \\
\leq& \frac{2(1-\alpha)}{(1+2\lambda)^n\left[(1+2\lambda)^m-1\right]}\left\{|c_2|+\frac{(1-2\alpha)+(1+\lambda)^m}{(1+\lambda)^m-1}\left|c_1^2\right|\right\} \\
&+\left\{\frac{2(1-\alpha)}{(1+\lambda)^n\left[(1+\lambda)^m-1\right]}\right\}^2|c_1|^2 \\
=& \frac{2(1-\alpha)}{(1+2\lambda)^n\left[(1+2\lambda)^m-1\right]}\left\{1+\begin{bmatrix}-1+\frac{(1-2\alpha)+(1+\lambda)^m}{(1+\lambda)^m-1}\\+\frac{4(1-\alpha)^2}{(1+\lambda)^{2n}\left[(1+\lambda)^m-1\right]^2}\end{bmatrix}|c_1|^2\right\} \\
\Rightarrow|H_{2,1}(f)|\leq& \frac{2(1-\alpha)}{(1+2\lambda)^n\left[(1+2\lambda)^m-1\right]}\left\{1+\begin{bmatrix}-1+\frac{(1-2\alpha)+(1+\lambda)^m}{(1+\lambda)^m-1}\\+\frac{4(1-\alpha)^2}{(1+\lambda)^{2n}\left[(1+\lambda)^m-1\right]^2}\end{bmatrix}|c_1|^2\right\} \\
\Rightarrow|H_{2,1}(f)|\leq& \frac{2(1-\alpha)}{(1+2\lambda)^n\left[(1+2\lambda)^m-1\right]}\left\{\frac{(1-2\alpha)+(1+\lambda)^m}{(1+\lambda)^m-1}+\frac{4(1-\alpha)^2}{(1+\lambda)^{2n}\left[(1+\lambda)^m-1\right]^2}\right\} \\
\Rightarrow|H_{2,1}(f)|\leq& \frac{2(1-\alpha)\left\{\left[|1-2\alpha|+(1+\lambda)^m\right](1+\lambda)^{2n}\left[(1+\lambda)^m-1\right]+4(1-\alpha)^2\right\}}{(1+\lambda)^{2n}(1+2\lambda)^n\left[(1+\lambda)^m-1\right]^2\left[(1+2\lambda)^m-1\right]}
\end{aligned}$$

□

6. Conclusion

Defining subclasses in the theory of analytic functions and giving coefficient estimates of these subclasses is one of the main problems of this theory. In this study, a new subclass of analytic functions is defined using the generalized Salagean operator, coefficient estimates are given for the functions of this subclass, and distortion theorems are proven. Distortion theorems have been studied for fractional calculus belonging to defined subclass of analytic functions with negative coefficients. A relative relation has been obtained for neighborhoods of defined subclass of analytical functions with negative coefficients. In addition, the upper bounds of the Hankel determinant, which is one of the important problems of this theory, was examined for a special case.

In this study, the results of previously studied subclasses can be compared by giving special values to some parameters used to define the new subclass.

Statements and Declarations

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References

[1] F. M. AL-Oboudi. On Univalent Functions defined by a Generalized Salagean Operator. *International Journal of Mathematics and Mathematical Sciences*, 27:1429-1436, 2004.
[2] M. K. Aouf, H. E. Darwish, and A.A. Attiya. Generalization of certain subclasses of analytic functions with negative coefficients. *Studia University Babes-Bolyai Mathematica*, 45(1):11-22, 2000.
[3] K. O. Babalola. On $H_{3,1}(f)$ Hankel determinants for some classes of univalent functions, Inequality theory and applications. *Arxiv*, 6:1-7, 2007.
[4] S. K. Chatterjee. On starlike functions. *Journal of Pure Mathematics*, 1:23-26, 1981.
[5] H. E. Darwish. Certain Subclasses of Analytic Functions with Negative Coefficients Defined by Generalized Salagean Operator. *General Mathematics*, 15(4):69-82, 2007.
[6] A. W. Goodman. Univalent Functions and Nonanalytic Curves. *Proceedings of the American Mathematical Society*, 8:598-601, 1957.
[7] A. Janteng, S. A. Halim, and M. Darus. Coefficient inequality for a function whose derivative has a positive real part. *Journal of Inequalities in Pure and Applied Mathematics*, 7:Article ID 50, 2006.
[8] A. Janteng, S. A. Halim, and M. Darus. Hankel determinant for starlike and convex functions. *International Journal of Mathematical Analysis*, 1:619-625, 2007.
[9] S. Owa. On the distortion theorems. *Kyungpook Mathematical Journal*, 18:53-59, 1978.
[10] C. Pommerenke. On the coefficients and Hankel determinants of univalent functions. *Journal of London Mathematical Society*, 41:111–122, 1966.
[11] C. Pommerenke. On the Hankel determinants of univalent functions. *Mathematika*, 14:108–112, 1967.
[12] G. S. Salagean. Subclasses of univalent functions. In C. A. Cazacu, N. Boboc, M. Jurchescu, I. Suciu, editors, *Complex Analysis Fifth Romanian-Finnish Seminar. Lecture Notes in Mathematics*, vol 1013, pages 362–372, Springer, Berlin, Heidelberg, 1983.
[13] T. Sekine. Generalization of certain subclasses of analytic functions. *International Journal Mathematics and Mathematical Sciences*. 10(4):725-732, 1987.
[14] H. Silverman. Univalent functions with negative coefficients. *Proceedings of the American Mathematical Society*, 51:109-116, 1975.
[15] S. Ruscheweyh. Neighborhoods of univalent functions. *Proceedings of the American Mathematical Society*, 81:521-527, 1981.