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LINEAR HARMONIC EULER SUMS WITH ODD AND EVEN WEIGHT

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Dedicated to Professor Hari Mohan Srivastava on the Occassion of His 85th birthday

Abstract. Euler's groundbreaking work in the 18th century on linear and nonlinear harmonic sums laid a foundation that continues to inspire mathematical research today. Over the years, many important new discoveries have emerged, deepening our understanding of these fascinating structures. In this paper, we explore various representations of linear harmonic Euler sums of both odd and even weights. The findings highlight not only the rich complexity of this field but also the many opportunities that remain, particularly in the study of multiple zeta values and their generalizations. We also present several new examples of Euler sums with even weight, contributing fresh insights to this enduring area of research.

Keywords. Linear harmonic Euler sums, Nonlinear harmonic Euler sums, Riemann zeta function, Dirichlet beta function, Polygamma function, Polylogarithm function.

1. Introduction

The 18th-century mathematician Euler introduced and investigated *linear harmonic sums* of the form

$$\sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n^t}, \tag{1.1}$$

where p and t are positive integers with $t \geqslant 2$, and the sum $p+t$ is referred to as the *weight* of the sum. Building on Eulers foundational work, Flajolet and Salvy [22] made a significant contribution by systematically studying four distinct classes of linear and skew harmonic Euler sums. They provided explicit expressions for sums of the types:

$$\mathbf{S}_{p,t}^{++} := \sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n^t}, \qquad \mathbf{S}_{p,t}^{+-} := \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n^{(p)}}{n^t}, \tag{1.2}$$

$$\mathbf{S}_{p,t}^{-+} := \sum_{\tau=1}^{\infty} \frac{A_n^{(p)}}{n^t}, \qquad \mathbf{S}_{p,t}^{--} := \sum_{n=1}^{\infty} \frac{(-1)^{n+1} A_n^{(p)}}{n^t}, \tag{1.3}$$

where $H_n^{(p)}$ denotes the harmonic numbers of order p (defined in equation (1.13)) and $A_n^{(p)}$ denotes the alternating harmonic numbers (defined in equation (1.15)). In particular, the linear harmonic Euler sums $\mathbf{S}_{p,t}^{++}$, which are also represented as $\mathbf{S}_{p,t}$, were extensively studied by Euler and Goldbach (see Flajolet and Salvy [22]). For sums of odd weight $p+t$, Borwein et al. [10] established the elegant identity:

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$$\begin{aligned}\mathbf{S}_{p,t}^{++} &= \frac{1-(-1)^p}{2}\zeta(p)\zeta(t) + \frac{1-(-1)^p}{2}\binom{p+t}{t}\binom{p+t-1}{p}\zeta(p+t)\\ &+(-1)^p\sum_{j=1}^{\left[\frac{p}{2}\right]}\binom{p+t-2j-1}{t-1}\zeta(2j)\zeta(p+t-2j)\\ &+(-1)^p\sum_{j=1}^{\left[\frac{t}{2}\right]}\binom{p+t-2j-1}{p-1}\zeta(2j)\zeta(p+t-2j),\end{aligned}\tag{1.4}$$

where, in this context, the symbol $[x]$ denotes the greatest integer less than or equal to $x \in \mathbb{R}$. Euler derived this evaluation by calculating numerous examples for weights $p+t \leqslant 13$ and then extrapolating a general formula, albeit without providing a formal proof. In his work [21], Euler stated the result

$$\mathbf{S}_{1,t}^{++} = \left(1+\frac{t}{2}\right)\zeta(t+1) - \frac{1}{2}\sum_{j=1}^{t-2}\zeta(j+1)\zeta(t-j)\tag{1.5}$$

for integer $t \geqslant 2$. For the alternating linear harmonic Euler sums, defined by

$$\mathbf{S}_{p,t}^{+-}(1) := \mathbf{S}_{p,t}^{+-} = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{H_n^{(p)}}{n^t},$$

Flajolet and Salvy [22] provided an explicit expression for the case of odd weight $p+t$:

$$\begin{aligned}2\mathbf{S}_{p,t}^{+-} &= \left(1-(-1)^p\right)\zeta(p)\eta(t) + \eta(p+t)\\ &+2\sum_{j+2k=p}\binom{t+j-1}{t-1}(-1)^{j+1}\eta(t+j)\eta(2k)\\ &+2(-1)^p\sum_{i+2k=t}\binom{p+i-1}{p-1}\zeta(p+i)\eta(2k),\end{aligned}\tag{1.6}$$

where $\eta(\cdot)$ denotes the Dirichlet eta function. For the specific case of odd weight $1+t$, Sitaramachandrarao [39] established the following identity:

$$\mathbf{S}_{1,t}^{+-} = (t+1)\eta(t+1) - \zeta(t+1) - 2\sum_{j=1}^{\frac{t}{2}-1}\zeta(2j)\zeta(t+1-2j).$$

More recently, Alzer and Choi [3] derived the following identity: for $p \in \mathbb{Z}_{\geqslant 2}$,

$$2\mathbf{S}_{p,1}^{+-} = p\zeta(p+1) - \zeta(t+1) - 2\sum_{j=1}^{p}\eta(j)\eta(p+1-j).\tag{1.7}$$

Throughout this paper, let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{C}$ denote the sets of positive integers, integers, real numbers, positive real numbers, and complex numbers, respectively. Also, let $\mathbb{Z}_{\geqslant \ell}$ ($\mathbb{Z}_{\leqslant \ell}$) denote the set of integers greater than or equal to (less than or equal to) some integer ℓ.

Nielsen [34], along with many other researchers (see [1, 3, 6, 8, 9, 10, 14, 22, 26, 43, 47, 48, 49, 50, 51], and [68]), has continued to develop this work. It is now established that $\mathbf{S}_{p,t}^{++}$ can be explicitly evaluated in terms of special functions, such as the Riemann zeta function, in the following cases: when $p = t \in \mathbb{N}$; when $p+t$ is odd; and when $p+t$ is even, but only for the specific pairs $(p,t) = (4,2)$ and $(2,4)$. It is conjectured that $\mathbf{S}_{p,t}^{++}$ with even weight $p+t \geqslant 8$ cannot, in general, be evaluated in terms of special functions.

There are reciprocity (shuffle) relations of the form (consult [22, p. 33]):

$$\mathbf{S}_{p,t}^{-+} + \mathbf{S}_{t,p}^{+-} = \eta(p)\,\zeta(t) + \eta(p+t), \tag{1.8}$$

$$\mathbf{S}_{p,t}^{--} + \mathbf{S}_{t,p}^{--} = \eta(p)\,\eta(t) + \zeta(p+t), \tag{1.9}$$

$$\mathbf{S}_{p,t}^{++} + \mathbf{S}_{t,p}^{++} = \zeta\,(p)\,\zeta\,(t) + \zeta\,(p+t)\,, \tag{1.10}$$

which can be used to evaluate, for example, $\mathbf{S}_{p,t}^{++}$ if $\mathbf{S}_{t,p}^{++}$ is known (or vice versa).

In a similar manner, parametric extensions of these sums are defined (as in [3]) by introducing parameters a, b, and q into the summation:

$$\begin{aligned} \mathbf{S}_{p,t}^{++}(a,b;q) &= \sum_{n=1}^{\infty} \frac{H_{qn}^{(p)}(a)}{(n+b)^t}, \qquad \mathbf{S}_{p,t}^{+-}(a,b;q) = \sum_{n=1}^{\infty} \frac{(-1)^{j+1} H_{qn}^{(p)}(a)}{(n+b)^t}, \\ \mathbf{S}_{p,t}^{-+}(a,b;q) &= \sum_{n=1}^{\infty} \frac{A_{qn}^{(p)}(a)}{(n+b)^t}, \qquad \mathbf{S}_{p,t}^{--}(a,b;q) = \sum_{n=1}^{\infty} \frac{(-1)^{j+1} A_{qn}^{(p)}(a)}{(n+b)^t}, \end{aligned} \tag{1.11}$$

for the case $q = 1$, and further extended by Sofo and Choi [47] to $q \in \mathbb{R}^+$. Here, $a,\, b \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant -1}$, and $p,\, t \in \mathbb{C}$ are chosen so that each of the four series converges. It is easy to see that when $a = 0$ and $b = 0$, these parametric sums reduce to the forms in (1.2) and (1.3):

$$\mathbf{S}_{p,t}^{++}(0,0;1) = \mathbf{S}_{p,t}^{++}, \;\; \mathbf{S}_{p,t}^{+-}(0,0;1) = \mathbf{S}_{p,t}^{+-}, \;\; \mathbf{S}_{p,t}^{-+}(0,0;1) = \mathbf{S}_{p,t}^{-+}, \;\; \mathbf{S}_{p,t}^{--}(0,0;1) = \mathbf{S}_{p,t}^{--}.$$

Harmonic numbers appear in a wide range of research topics. For instance, Flajolet and Sedgewick [23], using the residue theorem from complex analysis, showed that the following finite combinatorial sum can be expressed in terms of generalized harmonic numbers in (1.2) (refer also to Batır [7] and the references therein): For $m,\, n \in \mathbb{N}$, they showed:

$$\begin{aligned} \sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^{k-1}}{k^m} &= \sum_{n \geqslant r_1 \geqslant r_2 \geqslant \cdots \geqslant r_m \geqslant 1} \frac{1}{r_1 r_2 \cdots r_m} \\ &= \sum_{m_1 + 2m_2 + \cdots = m} \frac{1}{m_1!\, m_2! \cdots} \left(\frac{H_n^{(1)}}{1} \right)^{m_1} \left(\frac{H_n^{(2)}}{2} \right)^{m_2} \left(\frac{H_n^{(3)}}{3} \right)^{m_3} \cdots, \end{aligned}$$

where $H_n^{(k)}$ denotes the generalized harmonic numbers of order k.

The paper [51] discusses a notable integral that can be expressed in terms of harmonic numbers and finds applications in statistical plasma physics, particularly in the Sommerfeld temperature expansion of electronic entropy. For a comprehensive overview of research on multiple zeta values and Euler sums of arbitrary degree, readers are referred to the expository paper [11]. Since then, the study of multiple zeta values and Euler sums has remained an active and evolving area of research; see, for example, [5, 18, 20, 24, 25, 32, 33, 35, 36, 37, 38], and [42].

A fundamental and widely used special function in this research is the Riemann zeta function, defined by

$$\zeta\,(t) := \begin{cases} \sum_{j \geqslant 1} \frac{1}{j^t} = \frac{1}{1-2^{-t}} \sum_{j \geqslant 1} \frac{1}{(2j-1)^t} & (\Re(t) > 1), \\ \frac{1}{1-2^{1-t}} \sum_{j \geqslant 1} \frac{(-1)^{j-1}}{j^t} & (\Re(t) > 0,\ t \neq 1). \end{cases} \tag{1.12}$$

The Dirichlet eta function, $\eta(t)$, is defined by

$$\eta(t) := \sum_{n \geqslant 1} \frac{(-1)^{n-1}}{n^t} = \left(1 - 2^{1-t}\right)\, \zeta(t) \quad (\Re(t) > 0).$$

The generalized harmonic numbers $H_n^{(p)}(v)$ of order p are defined by

$$H_n^{(p)}(v) := \sum_{j=1}^{n} \frac{1}{(j+v)^p} \quad \left(p \in \mathbb{C},\ v \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant -1},\ n \in \mathbb{N}\right), \tag{1.13}$$

where $H_n^{(p)} := H_n^{(p)}(0)$ are the harmonic numbers of order p. In particular, the classical harmonic numbers $H_n := H_n^{(1)}$ are given by

$$H_n = \sum_{j=1}^{n} \frac{1}{j} = \gamma + \psi\left(n+1\right) \quad \left(n \in \mathbb{Z}_{\geqslant 0}\right), \tag{1.14}$$

with the convention $H_0 := 0$. The generalized alternating, or skew, harmonic numbers $A_n^{(t)}\left(a\right)$ of order t, as introduced in (1.3), are defined by

$$A_n^{(t)}\left(a\right) := \sum_{j=1}^{n} \frac{(-1)^{j+1}}{\left(j+a\right)^t} \quad \left(t \in \mathbb{C},\ n \in \mathbb{N},\ a \setminus \mathbb{Z}_{\leqslant -1}\right), \tag{1.15}$$

where $A_n := A_n^{(1)}$. An interesting relationship between the harmonic numbers $H_n^{(s)}$ and the alternating harmonic numbers $A_n^{(s)}$, is given by

$$A_n^{(s)} = H_n^{(s)} - 2^{1-s} H_{[n/2]}^{(s)}, \tag{1.16}$$

where $[x]$ represents the greatest integer less than or equal to x. Here, γ denotes the Euler-Mascheroni constant (see, e.g., [53, Section 1.2]), and $\psi\left(t\right)$ is the digamma (or psi) function defined as

$$\psi\left(t\right) := \frac{d}{dz}\left(\log \Gamma\left(t\right)\right) = \frac{\Gamma'\left(t\right)}{\Gamma\left(t\right)} \quad \left(t \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0}\right), \tag{1.17}$$

where $\Gamma\left(t\right)$ is the Gamma function (see, e.g., [53, Section 1.1]). The generalized (or Hurwitz) zeta function $\zeta(t,m)$ is defined by

$$\zeta(t,m) = \sum_{j \geqslant 0} \frac{1}{(j+a)^t} \quad \left(\Re(t) > 1,\ a \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0}\right), \tag{1.18}$$

and admits a meromorphic continuation to the entire complex t-plane (except for a simple pole at $t=1$ with residue 1) (see, e.g., [53, Section 1.3]). Furthermore. the m-th derivative of the digamma function, known as the polygamma function $\psi^{(m)}(t)$, satisfies

$$\psi^{(m)}(t) = (-1)^{m+1} m! \sum_{j \geqslant 0} \frac{1}{(j+t)^{m+1}} = (-1)^{m+1} m! \zeta(m+1,t).$$

Two useful identities for the Hurwitz zeta function $\zeta(t,m)$ are

$$\zeta(t,1) = \zeta(t) \text{ and } \zeta\left(t, \tfrac{1}{2}\right) = 2^t \lambda(t), \tag{1.19}$$

where $\lambda(t)$ denotes the Dirichlet lambda function. The Dirichlet lambda function $\lambda\left(s\right)$ is defined as the term-wise arithmetic mean of the Dirichlet eta function $\eta(s)$ and the Riemann zeta function $\zeta(s)$, given by

$$\lambda\left(m\right) = \frac{\eta(m)+\zeta(m)}{2} = \lim_{n\to\infty} O_n^{(s)} = \sum_{j=1}^{\infty} \frac{1}{(2j-1)^m} \quad (\Re(m) > 1).$$

The Dirichlet beta function $\beta(z)$ is defined by

$$\beta(z) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)^z} \quad (\Re(z) > 0), \tag{1.20}$$

and specifically, $\beta(2) = G$, where G denotes the Catalan constant. Among various properties and identities involving $\beta(z)$, we have

$$\beta(z) = 4^{-z}\left\{\zeta\left(z,\, \tfrac{1}{4}\right) - \zeta\left(z,\, \tfrac{3}{4}\right)\right\} = \frac{\mathbf{i}}{2}\left\{\mathrm{Li}_z(-\mathbf{i}) - \mathrm{Li}_z(\mathbf{i})\right\}, \tag{1.21}$$

where we use the bold symbol, $\mathbf{i} = \sqrt{-1}$, the imaginary unit, in this paper, and $\mathrm{Li}_p(z)$ denotes the polylogarithm function.

The polylogarithm function $\mathrm{Li}_p(z)$ of order $p \geqslant 1$ is defined by (see, e.g., [53, p. 198])

$$\mathrm{Li}_p(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^p} \quad \left(|z| \leqslant 1;\ p \in \mathbb{Z}_{\geqslant 2}\right), \tag{1.22}$$

and, in particular,

$$\mathrm{Li}_1(z) = -\log\left(1 - z\right), \quad \left(|z| \leqslant 1\right).$$

The polylogarithm $\mathrm{Li}_s(-z)$ can be expressed as

$$\mathrm{Li}_s(-z) = -\frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}}{z^{-1}\exp(t) + 1}\, \mathrm{d}t, \tag{1.23}$$

where $\Gamma(s)$ is the gamma function. In this context, $\mathrm{Li}_s(-z)$ is sometimes referred to as a Fermi-Dirac integral (see [2] or [54]). Similarly, the polylogarithm $\mathrm{Li}_s(z)$ is given by

$$\mathrm{Li}_s(z) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}}{z^{-1}\exp(t) - 1}\, \mathrm{d}t \tag{1.24}$$

is sometimes called a Bose-Einstein integral [54]. According to Lewin (see [31, p. 299]; see also [16], [19, pp. 30–31], [29], [53, pp. 197–198]), Jonquière's relation states that for $t \in \mathbb{Z}_{\geqslant 2}$,

$$\mathrm{Li}_t\left(\frac{1}{z}\right) = (-1)^{t+1}\left(\mathrm{Li}_t(z) + \frac{(2\pi\mathbf{i})^t}{t!} B_t\left(\frac{\ln z}{2\pi\mathbf{i}}\right)\right) \tag{1.25}$$

where $\mathrm{Li}_t(z)$ is the polylogarithm function and $B_t(x)$ denotes the Bernoulli polynomial. Expanding $B_t\left(\frac{\ln z}{2\pi\mathbf{i}}\right)$ explicitly, we can also write

$$\mathrm{Li}_t\left(\frac{1}{z}\right) = (-1)^{t+1}\,\mathrm{Li}_t(z) + (-1)^{t+1}\frac{(2\pi\mathbf{i})^t}{t!}\sum_{j=0}^{t}\binom{t}{j}\left(\frac{\ln z}{2\pi\mathbf{i}}\right)^{t-j} B_j,$$

where B_j are the Bernoulli numbers. For a negative argument, Jonquière's relation gives

$$\mathrm{Li}_t\left(-\frac{1}{z}\right) = (-1)^{t+1}\left(\mathrm{Li}_t(-z) + \sum_{j=0}^{\left[\frac{t}{2}\right]} \frac{(\ln z)^{t-2j}}{(t-2j)!}\eta(2j)\right) \ (t \in \mathbb{Z}_{\geqslant 2}), \tag{1.26}$$

where $\eta(2j)$ denotes the Dirichlet eta function evaluated at even integers.

The relations (1.25) and (1.26) thus provide the analytic continuation of the polylogarithm $\mathrm{Li}_s(z)$, originally defined in (1.22), beyond its initial domain of convergence $|z| \leqslant 1$.

The Bernoulli polynomials $B_j(t)$ are defined by the generating function

$$\frac{z\, e^{tz}}{e^z - 1} = \sum_{j \geqslant 0} \frac{z^j}{j!} B_j(t) \quad \left(|z| < 2\pi\right).$$

They have the following explicit representation (see, e.g., [53, Section 1.7]):

$$B_j\left(t\right)=\sum_{k=0}^{j}\binom{j}{k}B_k\,t^{j-k}=\sum_{k=0}^{j}\binom{j}{k}B_{t-k}t^k \tag{1.27}$$

where $B_k=B_k\left(0\right)$ are the Bernoulli numbers. The Bernoulli numbers themselves are defined by the generating function

$$\frac{z}{e^z-1}=\sum_{j\geqslant 0}\frac{z^j}{j!}B_j \quad (|z|<2\pi).$$

The Euler polynomials $E_n\left(x\right)$ and Euler numbers E_n are defined by the generating functions

$$\frac{2\;e^{xt}}{e^t+1}=\sum_{n\geqslant 0}\frac{t^n}{n!}E_n\left(x\right) \quad \left(|t|<\pi\right),$$

and

$$\frac{2\;e^{t}}{e^{2t}+1}=\frac{1}{\cosh t}=\sum_{n\geqslant 0}\frac{t^n}{n!}E_n \quad \left(|t|<\frac{\pi}{2}\right). \tag{1.28}$$

Alternatively, the Euler polynomials can be expressed explicitly as

$$E_n\left(x\right)=\sum_{k=0}^{n}\frac{1}{2^k}\sum_{j=0}^{k}(-1)^j\binom{k}{j}\left(x+j\right)^n.$$

Some of the first few Bernoulli numbers and Euler numbers are:

$$\begin{aligned} B_0 &= 1, B_1=-\frac{1}{2}, B_2=\frac{1}{6}, B_3=0, B_4=-\frac{1}{30}, B_5=0,\; B_6=\frac{1}{42}; \\ E_0 &= 1, E_1=0,\; E_2=-1,\; E_3=0,\; E_4=5,\; E_5=0,\; E_6=-61. \end{aligned} \tag{1.29}$$

Moreover, it holds that

$$B_{2n+1}=0 \quad (n\in\mathbb{N}),$$

and

$$E_{2n-1}=0 \quad (n\in\mathbb{N}).$$

The extended harmonic numbers $H_\alpha^{(m)}$ of order $m\in\mathbb{N}$ with index $\alpha\in\mathbb{C}\setminus\mathbb{Z}_{\leqslant -1}$ are defined by the following expressions (see [53]):

$$H_\alpha^{(m)}:=\begin{cases}\gamma+\psi(\alpha+1) & (m=1),\\ \zeta(m)+\frac{(-1)^{m-1}}{(m-1)!}\,\psi^{(m-1)}(\alpha+1) & (m\in\mathbb{Z}_{\geqslant 2}).\end{cases} \tag{1.30}$$

The case for $m=1$ in (1.30) is given in (1.14). By employing (1.30), we obtain the recurrence relation:

$$H_\alpha^{(m)}=H_{\alpha-1}^{(m)}+\frac{1}{\alpha^m} \quad \left(m\in\mathbb{N},\,\alpha\in\mathbb{C}\setminus\mathbb{Z}_{\leqslant 0}\right). \tag{1.31}$$

Next, applying (1.30) to the multiplication formula for polygamma functions

$$\psi^{(n)}(mz)=\delta_{n,0}\log m+\frac{1}{m^{n+1}}\sum_{j=1}^{m}\psi^{(n)}\left(z+\tfrac{j-1}{m}\right) \quad \left(m\in\mathbb{N},\,n\in\mathbb{Z}_{\geqslant 0}\right), \tag{1.32}$$

where $\delta_{n,j}$ is the Kronecker delta, leads to the following multiplication formula for the extended harmonic numbers:

$$H_{m\alpha}^{(p)} = \frac{1}{m^p} \sum_{j=1}^{m} H_{\alpha+\frac{j}{m}-1}^{(p)} + \left(1 - m^{1-p}\right) \zeta(p) \tag{1.33}$$
$$\left(m \in \mathbb{N},\, p \in \mathbb{Z}_{\geqslant 2};\; m\alpha + 1,\, \alpha + \frac{j}{m} \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0}\right).$$

In this article, we provide a variety of specific evaluations of harmonic Euler sums with both odd and even weights. Evaluations of linear harmonic Euler sums in closed form, whether of odd or even weight, refer to cases where the sums can be expressed as polynomials in zeta values and other special functions. The Euler sums that can be evaluated in this manner are referred to as reducible sums. On the other hand, sums that cannot be expressed in terms of special functions are called irreducible Euler sums. Relevant references regarding reducible and irreducible sums include [12, 13, 15, 17, 27, 28, 30, 34, 42, 44, 45, 46, 52, 56, 55, 57, 65].

2. General Results for Odd Weights

The first general result pertains to a closed-form expression for $\mathbf{S}_{p,t}^{++}(0,0;q) = \mathbf{S}_{p,t}^{++}(q)$ for $q \in \mathbb{R}^+$.

Theorem 2.1. *Let $p \in \mathbb{N},\; t \in \mathbb{Z}_{\geqslant 2}$, and $q \in \mathbb{R}^+$. Then, for odd weight $p+1+t$, we have:*

$$\begin{aligned} 2\,(-1)^p\, p! \mathbb{S}_{p+1,t}^{++}(q) = &\left(\frac{(-1)^p\, p!}{q^{p+1}} + q^t\,(t+1)_p\right) \zeta\,(p+t+1) \\ &+ 2\,(-1)^p\, p! \zeta\,(p+1)\,\zeta\,(t) - 2 \sum_{j=1}^{\left[\frac{t}{2}\right]} (t+1-2j)_p\, q^{t-2j} \zeta(2j) \zeta\,(p+t+1-2j) \\ &- \Re\left(\int_0^\infty \frac{\ln^p(y)\, \mathrm{Li}_t(y^q)}{y\,(1-y)}\, \mathrm{d}y\right), \end{aligned} \tag{2.1}$$

where $\mathbb{S}_{p+1,t}^{++}(q)$ is the linear harmonic Euler sum (1.2), $\mathrm{Li}_t(\cdot)$ is the polylogarithmic function defined by (1.22), $\zeta\,(t)$ is the Riemann zeta function, and $(\cdot)_p$ is the Pochhammer symbol. The symbol $\Re\,(z)$ indicates the real part of z.

Proof. The proof is based on expressing the integral

$$\int_0^\infty \frac{\ln^p(y)\, \mathrm{Li}_t(y^q)}{y\,(1-y)}\, \mathrm{d}y,$$

using the Fermi-Dirac polylogarithm (1.24) and the Jonquière relation (1.25), in Euler sums and special finctions. Further details can be found in reference [40]. □

The next general result provides a closed-form expression for $\mathbf{S}_{p,t}^{+-}(0,0;q) = \mathbf{S}_{p,t}^{+-}(q)$ where $q \in \mathbb{R}^+$.

Theorem 2.2. *Let $p,\, t \in \mathbb{N}$, and $q \in \mathbb{R}^+$. For odd weight $p+1+t$, the following identity holds:*

$$\begin{aligned} 2\,(-1)^p\, p! \mathbb{S}_{p+1,t}^{+-}(q) = &\frac{(-1)^p\, p!}{q^{p+1}} \eta\,(p+t+1) + 2\,(-1)^p\, p! \eta(t) \zeta\,(p+1) \\ &- 2 \sum_{j=0}^{\left[\frac{t}{2}\right]} (t+1-2j)_p\, q^{t-2j} \eta(2j) \zeta\,(p+t+1-2j) \\ &+ \int_0^\infty \frac{\ln^p(y)\, \mathrm{Li}_t(-y^q)}{y\,(1-y)}\, \mathrm{d}y, \end{aligned} \tag{2.2}$$

where $\mathbb{S}^{+-}_{p+1,t}(q)$ *is the linear harmonic Euler sum* (1.2), $\mathrm{Li}_t(\cdot)$ *is the polylogarithmic function defined by* (1.22), $\zeta(t)$ *is the Riemann zeta function, and* $(\cdot)_p$ *is the Pochhammer symbol.*

Proof. The proof is based on expressing the integral

$$\int_0^\infty \frac{\ln^p(y)\,\mathrm{Li}_t(-y^q)}{y(1-y)}\,\mathrm{d}y,$$

using the Fermi-Dirac polylogarithm (1.23) and the Jonquière relation (1.26), in Euler sums and special finctions. Further details can be found in reference [40]. □

Some particular cases of the results in Theorems 2.1 and 2.2 are provided.

Example 2.3. For $t \in \mathbb{Z}_{\geqslant 2}$,

$$\begin{aligned}\mathbb{S}^{++}_{5,t}(2) =& \tfrac{1}{64}\zeta(t+5) + \zeta(5)\zeta(t)\\
&- \tfrac{1}{24}\sum_{j=0}^{\left[\frac{t}{2}\right]}(t+1-2j)_4\, 2^{t-2j}\zeta(2j)\zeta(t+5-2j) - \tfrac{1}{16}t\left(15\cdot 2^t - 7\right)\zeta(4)\zeta(t+1)\\
&- \tfrac{1}{96}(t)_3\left(3\cdot 2^{t+2} - 7\right)\zeta(2)\zeta(t+3) + \tfrac{1}{7680}(t)_5\,\zeta(t+5).\end{aligned}$$

Example 2.4. For $t \in \mathbb{N}$,

$$\begin{aligned}\mathbb{S}^{+-}_{4,t}(2) =& \sum_{n=1}^{\infty}\frac{(-1)^{n+1}H^{(4)}_{2n}}{n^t} = \tfrac{1}{12}\left(\tfrac{3}{8} - \tfrac{1}{64}t(1+t)(2+t)(3+t)\right)\eta(t+4)\\
&+ \tfrac{1}{6}\sum_{j=0}^{\left[\frac{t}{2}\right]}5^{t-2j}(t+1-2j)_3\,\eta(2j)\zeta(t+4-2j) - \tfrac{7}{128}\eta(t)\zeta(4)\\
&- 2^{t-5}\pi^3 t\beta(t+1) - \tfrac{1}{3}2^{t-3}\pi t(t+1)(t+2)\beta(t+3)\\
&- \tfrac{1}{64}t(1+t)\eta(t+2)\zeta(2).\end{aligned}$$

Example 2.5. For $t \in \mathbb{Z}_{\geqslant 2}$,

$$\begin{aligned}\mathbb{S}^{+-}_{2,t}\left(\tfrac{1}{5}\right) =& \sum_{n=1}^{\infty}\frac{(-1)^{n+1}H^{(2)}_{\frac{n}{5}}}{n^t} = \left(\tfrac{25}{2} - \tfrac{t(1+t)}{4\cdot 5^t}\right)\eta(t+2) + \left(1 - \tfrac{27}{2\cdot 5^t}\right)\zeta(2)\eta(t)\\
&+ \sum_{j=0}^{\left[\frac{t}{2}\right]}5^{2j-t}(t+1-2j)_1\,\eta(2j)\zeta(t+2-2j)\\
&- 3\cdot 2^{2-t}\cdot 5^{t-1/2}\zeta(2)\,\phi\left(\zeta\left(t,\tfrac{1}{10}\right) - \zeta\left(t,\tfrac{2}{5}\right) - \zeta\left(t,\tfrac{3}{5}\right) + \zeta\left(t,\tfrac{9}{10}\right)\right)\\
&- 3\cdot 2^{2-t}\cdot 5^{t-1/2}\zeta(2)\,\phi^\star\left(\zeta\left(t,\tfrac{1}{5}\right) - \zeta\left(t,\tfrac{3}{10}\right) - \zeta\left(t,\tfrac{7}{10}\right) + \zeta\left(t,\tfrac{4}{5}\right)\right),\end{aligned}$$

where ϕ and $\phi^\star$ are the two zeros of the Fibonacci recurrence:

$$\phi = \tfrac{1+\sqrt{5}}{2}, \qquad \phi^\star = \tfrac{1-\sqrt{5}}{2}.$$

3. General Results for Even Weights

In this section, we explore linear harmonic Euler sums of even weight. It is generally accepted that Euler determined all the closed-form expressions of the type (1.4). For even weight, the following relations hold:

$$\begin{aligned}
\mathbf{S}_{t,t}^{-+}+\mathbf{S}_{t,t}^{+-} &= \eta(t)\,\zeta(t)+\eta(2t);\\
2\mathbf{S}_{t,t}^{--} &= \eta^2(t)+\zeta(2t);\\
2\mathbf{S}_{t,t}^{++} &= \zeta^2\left(t\right)+\zeta\left(2t\right);\\
2\mathbf{S}_{t,t}^{++}-2\mathbf{S}_{t,t}^{--} &= \zeta^2\left(t\right)-\eta^2(t).
\end{aligned}$$

The only known closed-form expressions for the linear Euler sums $\mathbf{S}_{p,t}^{++}$ with even weight $p+t$ and $p\neq t$ are the exceptional cases:

$$\begin{aligned}
\mathbf{S}_{2,4} &= \zeta^2(3)-\tfrac{1}{3}\zeta(6),\\
\mathbf{S}_{4,2} &= \tfrac{37}{12}\zeta(6)-\zeta^2(3).
\end{aligned}$$

The following identity, presented in [41], states that for $q\in\mathbb{R}^+$ and $t\in\mathbb{N}$:

$$\mathbf{S}_{1,2t+1}^{++}\left(\frac{1}{q}\right)+\frac{1}{q^{2t}}\mathbb{S}_{1,2t+1}^{++}\left(q\right)=\sum_{j=1}^{2t-1}\left(-1\right)^{j+1}q^{-j}\,\zeta(j+1)\,\zeta(2t+1-j). \tag{3.1}$$

In particular, for $q=2$, we have the following relations:

$$\mathbf{S}_{1,2t+1}^{++}\left(\tfrac{1}{2}\right)-\mathbf{S}_{1,2t+1}^{+-}=-\mathbf{S}_{1,2t+1}^{++}+\sum_{j=1}^{2t-1}\left(-1\right)^{j+1}2^{-j}\,\zeta(j+1)\,\zeta(2t+1-j),$$

$$\mathbf{S}_{1,2t+1}^{+-}\left(\tfrac{1}{2}\right)-\mathbf{S}_{1,2t+1}^{+-}=\left(2^{-2t}-1\right)\mathbf{S}_{1,2t+1}^{++}+\sum_{j=1}^{2t-1}\left(-1\right)^{j+1}2^{-j}\,\zeta(j+1)\,\eta(2t+1-j),$$

and

$$\mathbf{S}_{1,2t+1}^{+-}\left(\tfrac{1}{2}\right)-\mathbf{S}_{1,2t+1}^{++}\left(\frac{1}{2}\right)=2^{-2t}\mathbf{S}_{1,2t+1}^{++}-2^{-2t}\sum_{j=1}^{2t-1}\left(-1\right)^{j+1}\,\zeta(j+1)\,\zeta(2t+1-j).$$

Moreover, as noted in [45, p. 10], the following even weight linear harmonic Euler sum identity holds:

$$\begin{aligned}
\left(2^{-2t}-1\right)\mathbf{S}_{2,2t}^{++}+2\mathbf{S}_{2,2t}^{+-}-\mathbf{S}_{2t,2}^{+-}+2t\mathbf{S}_{1,2t+1}^{+-}=&\tfrac{1}{2}\zeta\left(2\right)\eta\left(2t\right)-2^{1-2t}\eta\left(2\right)\zeta\left(2t\right)\\
&+\mathbf{S}_{2t+1,1}^{+-}+\left(2t-1\right)\left(1-2^{-2t}\right)\,\mathbf{S}_{1,2t+1}^{++}\\
&-\sum_{j=2}^{2t-1}\frac{(-1)^j}{2^{2t+1-j}}\left(2t-j\right)\eta\left(j\right)\zeta(2t+2-j).
\end{aligned} \tag{3.2}$$

The Euler sum $\mathbf{S}_{2t+1,1}^{+-}$ is determined by the Alzer-Choi identity (1.7), while $\mathbf{S}_{1,2t+1}^{++}$ corresponds to the classical Euler identity (1.5). The identity (3.2) represents Euler sums of even weight. For $t=6$, we have the specific case:

$$2\mathbf{S}_{2,12}^{+-}-\tfrac{4095}{4096}\mathbf{S}_{2,12}^{++}-\mathbf{S}_{12,2}^{+-}+12\mathbf{S}_{1,13}^{+-}=\tfrac{75989887}{1720320}\zeta(14)-\tfrac{189}{32}\zeta(7)^2-\tfrac{375}{32}\zeta(5)\zeta(9)-\tfrac{5883}{512}\zeta(3)\zeta(11). \tag{3.3}$$

We introduce the following constant, which will be used in the next theorem to represent the sum of even weight linear harmonic Euler sums. Define $T(n,k)$ as follows (see [69]):

$$T(n,k):=\binom{n}{k}\frac{(n-1)!}{(k-1)!}. \tag{3.4}$$

Here, $T(n,k)$ represents the number of partially ordered sets (posets) with n elements that are composed of exactly k chains. These numbers also appear as coefficients in the expansion of the rising factorials $(x)_k=x(x+1)\cdots(x+k-1)$ in terms of the falling factorials $\{x\}_k=x(x-1)\cdots(x-k+1)$. Specifically, the relation

$$(x)_n = \sum_{k=1}^{n} T(n,k)\{x\}_k$$

holds. For additional properties of $T(n,k)$, see [69].

Theorem 3.1. *Let $m \in \mathbb{N}$, $t \in \mathbb{Z}_{\geqslant m+1}$, and $q \in \mathbb{R}^+$. Then the following formula holds:*

$$\sum_{n\geqslant 1} \frac{1}{n^{t+1}} \left(q \frac{\partial^m}{\partial q^m} H_{\frac{n}{q}} + m \frac{\partial^{m-1}}{\partial q^{m-1}} H_{\frac{n}{q}} \right) = \sum_{r=1}^{m} \frac{(-1)^m}{q^{m-1}} T(m,r) \sum_{n\geqslant 1} \frac{1}{n} \int_0^1 y^{qn-1} \ln^r(y) \operatorname{Li}_{t-r}(y) dy, \tag{3.5}$$

where $\operatorname{Li}_t(\cdot)$ is the polylogarithm function, $H_{\frac{n}{q}}$ is the extended harmonic number, and $T(m,r)$ is defined as in (3.4).

Proof. Refer to details in [47]. □

The particular case $m = 2,\ q = 1$ of (3.5) are provided in the following corollary.

Corollary 3.2. *The following identities hold: for $t \in \mathbb{Z}_{\geqslant 3}$,*

$$\begin{aligned}(1-2t)\,\mathbf{S}^{++}_{2,2t} - 2\mathbf{S}^{++}_{3,2t-1} =& \tfrac{1}{2}t\,(2t-1)\,(2t+3)\,\zeta(2t+2) - 2\zeta(3)\zeta(2t-1)\\ &- (2t-1)\,\zeta(2)\zeta(2t) - \tfrac{1}{2}t\,(2t-1)\sum_{j=1}^{2t-1} \zeta(2t+1-j)\,\zeta(j+1)\\ &+ \sum_{j=1}^{2t-2} (-1)^{j+1}\, j\zeta(2t-j)\zeta(j+2)\\ &+ \tfrac{1}{2}\sum_{j=1}^{2t-3} (-1)^j\, j\,(j+1)\,\zeta(2t-1-j)\zeta(j+3),\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}(2t-1)\,\mathbf{S}^{++}_{2t,2} + 2\mathbf{S}^{++}_{2t-1,3} =& \tfrac{1}{2}t\,(2t-1)\,(2t+3)\,\zeta(2t+2) + (2t+1)\,\zeta(2t+2)\\ &- \tfrac{1}{2}t\,(2t-1)\sum_{j=1}^{2t-1} \zeta(2t+1-j)\,\zeta(j+1)\\ &+ \sum_{j=1}^{2t-2} (-1)^{j+1}\, j\zeta(2t-j)\zeta(j+2)\\ &+ \tfrac{1}{2}\sum_{j=1}^{2t-3} (-1)^j\, j\,(j+1)\,\zeta(2t-1-j)\zeta(j+3).\end{aligned} \tag{3.7}$$

Some more specific instances of (3.7) are:

$$5\mathbf{S}_{2,6} + 2\mathbf{S}_{3,5} = 10\,\zeta(3)\zeta(5) - \tfrac{21}{4}\zeta(8) \tag{3.8}$$

and

$$5\mathbf{S}_{6,2} + 2\mathbf{S}_{5,3} = \tfrac{247}{12}\zeta(8) - 8\,\zeta(3)\zeta(5); \tag{3.9}$$

$$7\mathbf{S}_{2,8} + 2\mathbf{S}_{3,7} = 14\zeta(3)\zeta(7) + 8\zeta^2(5) - \tfrac{33}{2}\zeta(10) \tag{3.10}$$

and

$$7\mathbf{S}_{8,2} + 2\mathbf{S}_{7,3} = \tfrac{741}{20}\zeta(10) - 8\zeta^2(5) - 12\,\zeta(3)\zeta(7); \tag{3.11}$$

$$2\mathbf{S}_{3,11}+11\mathbf{S}_{2,12}=18\zeta^2(7)+32\zeta(9)\zeta(5)+22\zeta(11)\zeta(3)-65\zeta(14)$$

and

$$2\mathbf{S}_{11,3}+11\mathbf{S}_{12,2}=11\zeta(12)\zeta(2)+78\zeta(14)-18\zeta^2(7)-32\zeta(9)\zeta(5)-20\zeta(11)\zeta(3).$$

The identities (3.8)and (3.10) also appear in [62, p. 595], together with

$$7\mathbf{S}_{2,8}-2\mathbf{S}_{4,6}=14\zeta(3)\zeta(7)+10\zeta^2(5)-\tfrac{227}{10}\zeta(10). \tag{3.12}$$

By subtracting (3.12) from (3.8), side by side, we obtain:

$$\mathbf{S}_{3,7}+\mathbf{S}_{4,6}=\tfrac{31}{10}\zeta(10)-\zeta^2(5).$$

Also, we have

$$2\mathbf{S}_{2,4}^{+,-}-\mathbf{S}_{4,2}^{+,-}+4\mathbf{S}_{1,5}^{+,-}=\tfrac{359}{64}\zeta(6)-\tfrac{9}{16}\zeta^2(3),$$

and similarly,

$$\mathbf{S}_{3,3}^{+,-}+3\mathbf{S}_{2,4}^{+,-}+6\mathbf{S}_{1,5}^{+,-}=\tfrac{155}{16}\zeta(6)-\tfrac{9}{32}\zeta^2(3).$$

Eliminating the $\mathbf{S}_{1,5}^{+,-}$ term from these two, we obtain:

$$\mathbf{S}_{3,3}^{+,-}+\tfrac{3}{2}\mathbf{S}_{4,2}^{+,-}=\tfrac{163}{128}\zeta(6)+\tfrac{9}{16}\zeta^2(3).$$

For $t=3$, from [62, p. 603], we have:

$$12\mathbf{S}_{1,7}^{+,-}-\tfrac{63}{32}\mathbf{S}_{2,6}^{+,+}-2\mathbf{S}_{6,2}^{+,-}+4\mathbf{S}_{2,6}^{+,-}=\tfrac{4825}{192}\zeta(8)-\tfrac{21}{2}\zeta(3)\zeta(5),$$

and

$$6\mathbf{S}_{1,7}^{+,-}-\tfrac{9}{8}\mathbf{S}_{2,6}^{+,+}+\mathbf{S}_{2,6}^{+,-}=\tfrac{9377}{256}\zeta(8)-6\zeta(3)\zeta(5).$$

By combining these two identities, we find:

$$\mathbf{S}_{2,6}^{+,-}-\mathbf{S}_{6,2}^{+,-}=-\tfrac{9}{64}\mathbf{S}_{2,6}^{+,+}+\tfrac{3}{4}\zeta(3)\zeta(5)-\tfrac{481}{768}\zeta(8). \tag{3.13}$$

Using similar evaluations, we record the following results:

$$\begin{aligned} 5\mathbf{S}_{6,2}^{+,-}+2\mathbf{S}_{5,3}^{+,-} &= \tfrac{45}{8}\zeta(3)\zeta(5)-\tfrac{853}{768}\zeta(8), \\ 5\mathbf{S}_{2,6}^{+,-}+\tfrac{99}{64}\mathbf{S}_{2,6}^{+,+}-\tfrac{3}{2}\mathbf{S}_{4,4}^{+,-} &= \tfrac{399}{32}\zeta(3)\zeta(5)-\tfrac{5337}{512}\zeta(8). \end{aligned} \tag{3.14}$$

Using (3.8) and (3.14), we derive:

$$5\mathbf{S}_{6,2}^{+,+}(2)+\mathbf{S}_{5,3}^{+,+}(2)=\tfrac{16661}{384}\zeta(8)-\tfrac{109}{4}\zeta(3)\zeta(5). \tag{3.15}$$

For $t=4$, using [62, p. 605], we get:

$$\mathbf{S}_{2,8}^{+,-}-\mathbf{S}_{8,2}^{+,-}=-\tfrac{27}{256}\mathbf{S}_{2,8}^{+,+}+\tfrac{45}{64}\zeta(3)\zeta(7)-\tfrac{2367}{2560}\zeta(10)+\tfrac{21}{64}\zeta^2(5). \tag{3.16}$$

Similarly,

$$7\mathbf{S}_{8,2}^{+,-}+2\mathbf{S}_{7,3}^{+,-}=\tfrac{567}{64}\zeta(3)\zeta(7)-\tfrac{54957}{5120}\zeta(10)+\tfrac{225}{32}\zeta^2(5), \tag{3.17}$$

and we notice that, from (3.11) and (3.17), we have:

$$7\mathbf{S}_{8,2}^{+,+}(2)+\mathbf{S}_{7,3}^{+,+}(2)=\tfrac{2446535}{2560}\zeta(10)-\tfrac{1335}{32}\zeta(3)\zeta(7)-\tfrac{481}{16}\zeta^2(5). \tag{3.18}$$

Other identities involving even weight ten are:

$$\begin{aligned} \mathbf{S}_{4,6}^{+,-}+5\mathbf{S}_{2,8}^{+,-}+2\mathbf{S}_{8,2}^{+,-}+4\mathbf{S}_{3,7}^{+,-}+\tfrac{1275}{512}\mathbf{S}_{2,8}^{+,+} &= \tfrac{2175}{256}\zeta^2(5)+\tfrac{4551}{256}\zeta(3)\zeta(7)-\tfrac{84951}{5120}\zeta(10), \\ \mathbf{S}_{4,6}^{+,-}+7\mathbf{S}_{2,8}^{+,-}+4\mathbf{S}_{3,7}^{+,-}+\tfrac{1383}{512}\mathbf{S}_{2,8}^{+,+} &= \tfrac{2353}{256}\zeta^2(5)+\tfrac{4911}{256}\zeta(3)\zeta(7)-\tfrac{94419}{5120}\zeta(10), \\ \mathbf{S}_{6,4}^{+,-}+\tfrac{603}{512}\mathbf{S}_{2,8}^{+,+}+2\mathbf{S}_{3,7}^{+,-} &= \tfrac{1475}{512}\zeta(10)+\tfrac{603}{256}\zeta(3)\zeta(7)-\tfrac{387}{256}\zeta^2(5). \end{aligned}$$

Setting $m=4$ and $q=1$ in Theorem 3.1, we obtain the following corollary.

Corollary 3.3. *The following identity holds: for $t \in \mathbb{Z}_{\geqslant 3}$,*

$$\begin{aligned}
&48\mathbf{S}^{++}_{5,2t-3} + 24\left(2t-3\right)\mathbf{S}^{++}_{4,2t-2} + 24\left(t^2-2t+3\right)\mathbf{S}^{++}_{3,2t-1} + 4\left(2t-1\right)\left(t^2-t+6\right)\mathbf{S}^{++}_{2,2t}\\
&\quad = -2t^2\left(t^2+11\right)\mathbf{S}^{++}_{1,2t+1} + 4\left(2t-1\right)\left(t^2-t+6\right)\zeta\left(2\right)\ \zeta\left(2t\right)\\
&\qquad + 48\zeta\left(5\right)\zeta\left(2t-3\right) + 24\left(2t-3\right)\zeta\left(4\right)\zeta\left(2t-2\right) + 24\left(t^2-2t+3\right)\zeta\left(3\right)\zeta\left(2t-1\right)\\
&\qquad + 2\sum_{j=1}^{t-1}\left(-1\right)^{j+1}\left(j-t\right)^2\left(t^2-2jt+j^2+11\right)\zeta\left(j+1\right)\zeta\left(2t+1-j\right).
\end{aligned} \tag{3.19}$$

Some specific examples of (3.19) are:

$$\begin{aligned}
\tfrac{13}{2}\mathbf{S}^{+,+}_{5,7} + \tfrac{35}{2}\mathbf{S}^{+,+}_{4,8} + 21\mathbf{S}^{+,+}_{3,9} &= \tfrac{286645}{2764}\zeta(12) - 56\zeta(7)\zeta(5),\\
\mathbf{S}^{+,+}_{5,9} + \tfrac{9}{2}\mathbf{S}^{+,+}_{4,10} + \tfrac{15}{2}\mathbf{S}^{+,+}_{3,11} &= \tfrac{481}{8}\zeta(14) - \tfrac{75}{4}\zeta^2(7) - 27\zeta(9)\zeta(5),\\
\mathbf{S}^{+,+}_{5,9} + \tfrac{9}{2}\mathbf{S}^{+,+}_{4,10} &= \tfrac{165}{4}\mathbf{S}^{+,+}_{2,12} + \tfrac{2431}{8}\zeta(14) - \tfrac{345}{4}\zeta^2(7) - 147\zeta(9)\zeta(5) - \tfrac{165}{2}\zeta(11)\zeta(3),\\
6\mathbf{S}^{+,+}_{4,10} - \mathbf{S}^{+,+}_{6,8} &= 66\mathbf{S}^{+,+}_{2,12} + \tfrac{3433}{7}\zeta(14) - 141\zeta^2(7) - 240\zeta(9)\zeta(5) - 132\zeta(11)\zeta(3),\\
\mathbf{S}^{+,+}_{8,6} - 6\mathbf{S}^{+,+}_{10,4} &= 66\mathbf{S}^{+,+}_{2,12} + \tfrac{9599}{20}\zeta(14) - 141\zeta^2(7) - 240\zeta(9)\zeta(5) - 132\zeta(11)\zeta(3).
\end{aligned}$$

In the following theorem, we investigate variant Euler sums with even weight.

Theorem 3.4. *Let $(p,t) \in \mathbb{N}^2$, $a \in [-1,1]$, and assume that $p+t+1$ is an even integer. Then the following variant of a harmonic Euler sum of even weight holds:*

$$\begin{aligned}
\tfrac{1}{2^{p+1}}\mathbf{S}^{+,-}_{t,p+1}\left(0,\tfrac{1}{2};1\right) &= \sum_{n\geqslant 1}\frac{\left(-1\right)^{n+1}H^{(t)}_n}{\left(2n+1\right)^{p+1}}\\
&= \tfrac{(-1)^{p+1}}{2p!}J\left(p,t\right) + \left(-1\right)^{p+1}\sum_{j=0}^{\left[\frac{t}{2}\right]}2^{t-2j}\binom{p+t-2j}{p}\eta\left(2j\right)\beta\left(p+t+1-2j\right),
\end{aligned} \tag{3.20}$$

where

$$J\left(p,t\right) := \int_0^\infty \tfrac{\log^p(x)\mathrm{Li}_t(-x^2)}{1+x^2}dx = \lim_{a\to 0}\tfrac{\partial^p}{\partial a^p}\left(\tfrac{\pi}{2}\sec\left(\tfrac{a\pi}{2}\right)\left(\zeta\left(t\right) - \zeta\left(t,\tfrac{1-a}{2}\right)\right)\right).$$

Also, $[m]$ represents the greatest integer less than or equal to m, $\eta\left(\cdot\right)$ is the Dirichlet eta function, $\beta\left(\cdot\right)$ is the Dirichlet beta function, $\mathrm{Li}_t(-x^2)$ is the polylogarithm function, $\zeta\left(t,\frac{1-a}{2}\right)$ is the generalized zeta function.

Proof. By expanding the polylogarithm inside the following integral, we obtain

$$\begin{aligned}
\int_0^1 f\left(p,t;x\right)\,dx &= \sum_{n\geqslant 1}\left(-1\right)^{n+1}H^{(t)}_n\int_0^1 x^{2n}\ \log^p\left(x\right)dx\\
&= \left(-1\right)^p p!\sum_{n\geqslant 1}\tfrac{(-1)^{n+1}H^{(t)}_n}{(2n+1)^{p+1}},
\end{aligned}$$

where

$$f\left(p,t;x\right) := \frac{\log^p\left(x\right)\mathrm{Li}_t(-x^2)}{1+x^2}.$$

By integrating $f(p,t;x)$ over the positive line $x \geqslant 0$ and decomposing the integral into two parts, we obtain

$$\int_0^\infty f(p,t;x)\,dx = \int_0^1 f(p,t;x)\,dx + \int_1^\infty f(p,t;x)\,dx.$$

In the second integral, we apply the substitution $x = y^{-1}$, use the Jonquière relation (1.26), and after some simplifications, we find:

$$\int_0^\infty f(p,t;x)\,dx = 2(-1)^{p+1} p! \sum_{n\geqslant 1} \frac{(-1)^{n+1} H_n^{(t)}}{(2n+1)^{p+1}} + 2\sum_{j=0}^{\left[\frac{t}{2}\right]} \frac{2^{t-2j}(p+t-2j)!}{(t-2j)!} \eta(2j)\,\beta(p+t+1-2j).$$

Rearranging terms, we can express the series as:

$$\sum_{n\geqslant 1} \frac{(-1)^{n+1} H_n^{(t)}}{(2n+1)^{p+1}} = \tfrac{(-1)^{p+1}}{2p!} J(p,t) + (-1)^{p+1} \sum_{j=0}^{\left[\frac{t}{2}\right]} 2^{t-2j} \binom{p+t-2j}{p} \eta(2j)\,\beta(p+t+1-2j).$$

To evaluate $J(p,t)$, we begin by introducing the auxiliary function:

$$X(a,t) := \int_0^\infty \frac{x^a \mathrm{Li}_t(-x^2)}{1+x^2}\,dx,$$

where the parameter $a \in [-1,1]$. Using the Fermi-Dirac integral (1.23), we arrive at:

$$\begin{aligned} X(a,t) &= -\frac{1}{\Gamma(t)} \int_0^\infty x^{t-1} \left(\int_0^\infty \frac{y^a}{(1+y^2)(y^{-2}\exp(x)+1)}\,dy \right) dx \\ &= \frac{\pi}{2} \sec\left(\frac{a\pi}{2}\right) \left(\zeta(t) - \zeta\left(t, \tfrac{1-a}{2}\right)\right). \end{aligned}$$

Taking the p-th derivative of $X(a,t)$ with respect to a and evaluating the limit as $a \to 0$, we obtain

$$\begin{aligned} J(p,t) &= \int_0^\infty \frac{\log^p(x)\,\mathrm{Li}_t(-x^2)}{1+x^2}\,dx \\ &= \lim_{a\to 0} \frac{\partial^p}{\partial a^p} \left(\frac{\pi}{2} \sec\left(\frac{a\pi}{2}\right) \left(\zeta(t) - \zeta\left(t, \tfrac{1-a}{2}\right)\right) \right). \end{aligned}$$

□

Two specific cases of the result in Theorem 3.4 are given in the following corollaries.

Corollary 3.5. *The following identity holds: for $t \in \mathbb{N}$,*

$$\begin{aligned} \tfrac{1}{2}\mathbf{S}^{+,-}_{2t+1,1}\left(0, \tfrac{1}{2}; 1\right) &= \sum_{n\geqslant 1} \frac{(-1)^{n+1} H_n^{(2t+1)}}{2n+1} \\ &= -\pi 2^{2t} \eta(2t+1) - \sum_{j=0}^{\left[\frac{2t+1}{2}\right]} 2^{2t+1-2j} \eta(2j)\,\beta(2t+2-2j). \end{aligned} \tag{3.21}$$

Proof. Setting $p = 0$, and replacing t by $2t+1$ in (3.20) leads to the desired identity. □

Corollary 3.6. *The following identity holds: for $p \in \mathbb{N}$,*

$$\begin{aligned}\tfrac{1}{2^{2p+1}}\mathbf{S}^{+,-}_{1,2p+1}\left(0,\tfrac{1}{2};1\right) =&\sum_{n\geqslant 1}\frac{(-1)^{n+1}H_n}{(2n+1)^{2p+1}}\\ =&\frac{1}{(2p)!}\left(\frac{\pi}{2}\right)^{2p+1}|E_{2p}|\ln 2-(2p+1)\,\beta\,(2p+2)\\ &+\frac{1}{2\,(2p)!}\left(\frac{\pi}{2}\right)^{2p+1}\sum_{j=1}^{p}(-1)^{j+1}\frac{2^{4j}}{\zeta(2j)}\binom{2p}{2j}|E_{2p-2j}|\,|B_{2j}|\,\lambda\,(2j+1)\,.\end{aligned} \tag{3.22}$$

Here, B_{2j} are the Bernoulli numbers (1.29), E_{2p} are the Euler numbers (1.28), and $\lambda\,(2j+1)$ are the Dirichlet Lambda functions.

Proof. Substituting p into $2p$ and $t=1$ in (3.20) yields the desired identity. □

Two particular cases of (3.20) for weight eight when $p=5$, $t=2$ and $p=1$, $t=6$, respectively, are provided in the following examples.

Example 3.7.

$$\sum_{n\geqslant 1}\frac{(-1)^{n+1}H_n^{(2)}}{(2n+1)^6}=42\beta\,(8)-\tfrac{35}{1536}\pi^5\zeta(3)-\tfrac{31}{64}\pi^3\zeta(5)-\tfrac{381}{64}\pi\zeta(7).$$

Example 3.8.

$$\sum_{n\geqslant 1}\frac{(-1)^{n+1}H_n^{(6)}}{(2n+1)^2}=32\beta\,(8)+80\eta\,(2)\,\beta\,(6)+12\eta\,(4)\,\beta\,(4)+\eta\,(6)\,\beta\,(2)-\tfrac{381}{4}\pi\zeta(7). \tag{3.23}$$

4. Extended Discussion

An illustrative example highlights the difference between two linear harmonic Euler sums of even weight with multiple arguments. Specifically, from equation (3.20), by expansion, we have:

$$\tfrac{1}{4^{p+1}}\mathbf{S}^{+,-}_{t,p+1}\left(0,-\tfrac{1}{4};2\right)-\tfrac{1}{4^{p+1}}\mathbf{S}^{+,-}_{t,p+1}\left(0,\tfrac{1}{4};2\right)=\sum_{n\geqslant 1}\frac{(-1)^{n+1}H_n^{(t)}}{(2n+1)^{p+1}}+\sum_{n=1}^{\infty}\frac{1}{(2n)^t\,(4n-1)^{p+1}}.$$

Setting $t=6$, $p=1$, this simplifies to

$$\begin{aligned}\tfrac{1}{16}\mathbf{S}^{+,-}_{6,2}\left(0,-\tfrac{1}{4};2\right)-\tfrac{1}{16}\mathbf{S}^{+,-}_{6,2}\left(0,\tfrac{1}{4};2\right)=&224\beta\,(8)+80\eta\,(2)\,\beta\,(6)+12\eta\,(4)\,\beta\,(4)+\eta\,(6)\,\beta\,(2)\\ &-\tfrac{381}{4}\pi\zeta(7)+\tfrac{1}{8}\zeta(5)+4\zeta(3)-288\ln 2+\tfrac{1}{64}\zeta(6)\\ &+\tfrac{3}{4}\zeta(4)+44\zeta(2)+48\pi-32G,\end{aligned}$$

where G is the Catalan constant.

Nonlinear harmonic Euler sums, which involve at least the product of two generalized harmonic numbers, are deeply connected to linear harmonic Euler sums. Let $P=(p_1,\,\dots,\,p_k)$ be a partition of an integer p into k summands such that $p=p_1+\cdots+p_k$ and $p_1\leqslant p_2\leqslant\cdots\leqslant p_k$. The Euler sum of index P,t is defined as

$$\mathbf{S}_{P;t}:=\sum_{j=1}^{\infty}\frac{H_j^{(p_1)}H_j^{(p_2)}\cdots H_j^{(p_k)}}{j^t}, \tag{4.1}$$

where $H_j^{(p)}$ represents the p-th generalized harmonic number. The quantity $t+p_1+\cdots+p_k$ is referred to as the weight of the sum, while k is called the degree. For simplicity, repeated summands in the partition are represented using exponents. For example,

$$\mathbf{S}_{1^4,2^3,6;t} = \mathbf{S}_{1,1,1,1,2,2,2,6;t} = \sum_{j=1}^{\infty} \frac{H_j^4 \left\{H_j^{(2)}\right\}^3 H_j^{(6)}}{j^t}.$$

For a comprehensive overview of multiple zeta values and Euler sums of arbitrary degree, readers are referred to the expository article [11]. The study of multiple zeta values and Euler sums remains an active area of research; see, for example, [42, 43, 49, 57, 62, 60, 61, 59, 58, 63, 64, 66, 67].

Several notable results have been obtained regarding nonlinear harmonic Euler sums. Flajolet and Salvy [22] provided numerous examples of quadratic sums, such as:

$$\mathbf{S}_{2^2;4} = \tfrac{457}{18}\zeta(8) - 40\zeta(3)\zeta(5) + 6\zeta(2)\zeta^2(3) + 11\mathbf{S}_{2,6}^{+,+}.$$

Wang and Lyu [57] in Corollary 3.4, explicitly list the Borwein-Borwein-Girgenshon reduction formula, which expresses the quadratic Euler sums $\mathbf{S}_{1^2;2t+2}$ in terms of the irreducible Euler sums $\mathbf{S}_{2,2t+2}^{+,+}$. Moreover, in Theorem 5.5, they showed that all twenty-nine Euler sums of a given weight can be reduced to values of $\mathbf{S}_{2,6}^{+,+}$. For example

$$\mathbf{S}_{3^2;2} = \tfrac{677}{24}\zeta(8) - 35\zeta(3)\zeta(5) + 42\zeta(2)\zeta^2(3) + \tfrac{15}{2}\mathbf{S}_{2,6}^{+,+}.$$

In view of the linear Euler sum (1.5), the following proposition illustrates that evaluating the quadratic nonlinear Euler sum $\mathbb{S}_{1^2,t}$ is equivalent to evaluating the linear Euler sum $\mathbf{S}_{2,t}^{+,+}$.

Proposition 4.1. *For all weights, the quadratic sums $\mathbb{S}_{1^2;t}$ reduce to linear sums and polynomials in zeta values as follows:*

$$\begin{aligned}\mathbb{S}_{1^2;t} &= \mathbf{S}_{2,t}^{+,+} + t\,\mathbf{S}_{1,t+1}^{+,+} - \tfrac{t(t+1)}{6}\,\zeta(t+2) + \zeta(2)\zeta(t) \\ &\quad - \tfrac{1}{3}\sum_{\ell=0}^{t+2}\sum_{j=0}^{\ell} a_{t+2-\ell}\,a_j\,a_{\ell-j} \quad (t\in\mathbb{Z}_{\geqslant 2}),\end{aligned} \tag{4.2}$$

where the a_k are defined by

$$a_0 = 1, \quad a_1 = 0, \quad a_k = -\zeta(k) \quad (k\in\mathbb{Z}_{\geqslant 2}).$$

Alternatively, the double sum can be expressed as

$$\sum_{\ell=0}^{t+2}\sum_{j=0}^{\ell} a_{t+2-\ell}\,a_j\,a_{\ell-j} = -\sum_{i+j+k=t+2}\zeta(i)\zeta(j)\zeta(k),$$

where $\zeta(0) := -1$ and $\zeta(1) := 0$ are used whenever they occur.

Si et al. [38, p. 427] presented the identity:

$$\mathbb{S}_{1^3,5} - \tfrac{11}{4}\mathbf{S}_{2,6}^{+,+} = \tfrac{469}{32}\zeta(8) - 16\,\zeta(3)\zeta(5) + \tfrac{3}{2}\zeta(2)\zeta^2(3).$$

Cubic Euler sums $\mathbb{S}_{1^3,t}$ for $(t = 2,\ 3,\ 4,\ 6)$ can be expressed as polynomials in terms of Riemann zeta values (see [38, p. 432]; see also [4, 22, 66]). Xu [61] showed that certain cubic sums

$$\mathbb{S}_{1^2,p,p} = \sum_{n\geq 1}\frac{H_n^2\,H_n^{(p_k)}}{n^p} \text{ and } \mathbb{S}_{1,2p+1,2p+1} = \sum_{n\geq 1}\frac{H_n\left(H_n^{(2p+1)}\right)^2}{n^{2p+1}}$$

are reducible to combinations of quadratic and linear Euler sums. Moreover, it is known that quadratic Euler sums of the form

$$\mathbb{S}_{p_1,p_2,t} = \sum_{n\geq 1} \frac{H_n^{(p_1)} H_n^{(p_2)}}{n^t},$$

can also be expressed in terms of linear sums, provided the weight $p_1 + p_2 + t$ is even. For example,

$$\begin{aligned}\mathbb{S}_{2,3,5} \quad = \quad & -\tfrac{2227}{32}\zeta(10) + \tfrac{89}{2}\zeta^2(5) + 56\zeta(3)\zeta(7) - 15\zeta(2)\zeta(3)\zeta(5) \\ & -10\mathbf{S}^{+,+}_{2,8} - \tfrac{5}{2}\zeta(2)\mathbf{S}^{+,+}_{2,6}.\end{aligned}$$

5. Concluding Remarks

From the brilliant early work of Euler in the 18th century on linear and nonlinear harmonic number sums, such as those in (1.1) and (4.1), an enduring and vibrant area of study was born, one that remains active today. In the 19th and 20th centuries, researchers such as Nielsen, Borwein, Flajolet, and many others revitalized this field, leading to the regular publication of new and significant results. Harmonic Euler sums, both linear and nonlinear, have found wide-ranging applications across combinatorics, number theory, physics, and even music.

In Section 2, we presented some of the latest developments concerning Euler sums of odd weight. In Section 3, we compiled a collection of current and new results related to Euler sums of even weight. Nonlinear harmonic Euler sums, in particular, are known for their difficulty, and extracting meaningful results remains challenging. Ongoing research is focused on multivariable zeta values and Tornheim-like series. Additionally, in Section 3, we have provided several new examples of Euler sums of even weight.

Statements and Declarations

The authors declare that they have no conflict of interest, and the manuscript has no associated data.

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