APPLICABLE NONLINEAR ANALYSIS
Volume 2 (2025), No. 3, 200–206
https://doi.org/10.69829/apna-025-0203-ta05
Tulipa Opera Scholarum

HOLDER REGULARITY OF WEAK SOLUTIONS TO THE QUASI-LINEAR ELLIPTIC EQUATIONS INVOLVING A GENERALIZED VARIABLE EXPONENT LAPLACIAN

MYKOLA IVANOVICH YAREMENKO[1]

[1] *National Technical University of Ukraine, "Igor Sikorsky Kyiv Polytechnic Institute" Kyiv, Ukraine, 37, Prospect Beresteiskyi (former Peremohy), Kyiv, Ukraine, 03056. math.kiev@gmail.com*

Dedicated to Professor Hari Mohan Srivastava on the Occassion of His 85th birthday

ABSTRACT. We prove the Holder continuity of a weak solution $u \in W_1^{p(\cdot)}(\Omega)$ to quasi-linear elliptic equations in the diverged form $div\,(A\,(x,\ u,\ \nabla u)) + b\,(x,\ u,\ \nabla u) = 0$, in the regular bounded domain Ω in R^l for $l \geq 3$, where functions $A = a_i\,(x,\ u,\ k)$ and $b\,(x,\ u,\ k)$ are correctly defined for all $x \in clos\,(\Omega)$ and every u, k, and A, b are measurable. We establish the conditions on coefficients A and b under which the weak solution $u \in W_1^{p(\cdot)}(\Omega)$ belongs to $C_{0,\ \alpha}(\Omega)$ for certain α depending only on M, $\nu\,(M)$, $\mu\,(M)$, $p\,(\cdot)$, and $|\nabla p|$, where $ess\max_{\Omega} |u| \leq M < \infty$.

Keywords. Weak solution, Minimax, Variational functional, Holder continuity, De Giorgi class, Regularity, Variable exponential Lebesgue space, Variable Laplacian, Minimizer.

1. Introduction

This article is dedicated to the Holder regularity of a weak solution to the elliptic differential equation in the divergent form with the nonstandard growth

$$div\,(A\,(x,\ u,\ \nabla u)) + b\,(x,\ u,\ \nabla u) = 0, \tag{1.1}$$

where Ω is a bounded domain in R^l, $l \geq 3$, and where functions $A = a_i\,(x,\ u,\ k)$ and $b\,(x,\ u,\ k)$ are well-defined for all $x \in clos\,(\Omega)$ and every u, k, are measurable. This type of equation is connected with the minimization problem of variational functionals I of the type

$$I = \int_{\Omega} |\nabla u\,(x)|^{p(x)}\,dx$$

with variable exponential growth conditions. More information can be found in [4, 9], and the Holder minimizer continuity was discussed in [6] for variational problems under the assumptions of nonstandard growth.

For general information on the smoothness of the weak solutions to quasilinear Dirichlet problems with standard growth, we refer to [9]. See [1, 11], for breakthrough results in Holder's regularity of solutions to linear elliptic differential equations. The authors employed the estimation of the square integrals of some convex functions of a weak solution.

We are interested in the regularity of a weak solution in the variable exponent Sobolev spaces $W_1^{p(\cdot)}(\Omega)$ under the assumption that such a solution exists and that e $p \in P^{\log}(\Omega)$, $p_m = \inf_{x\in\Omega} p\,(x)$,

*Corresponding author.
E-mail address: math.kiev@gmail.com (M. I. Yaremenko)
2020 Mathematics Subject Classification: 35J60, 35B20, 35J25, 35J05, 35J08.
Accepted: July 07, 2025.

$p_S = \sup_{x\in\Omega} p(x)$ and $|\nabla p| \le const$. From condition (3.2) follows

$$\sum_{i=1,\,\dots,\,l} |a_i(x,\ u,\ k)| \le \mu(|u|)(1+|k|)^{p(x)-1},$$

and, for fixed space variables, we are applying the Young inequality in the form

$$ab \le \frac{p(x)-1}{p(x)} a^{\frac{p(x)}{p(x)-1}} + \frac{1}{p(x)} b^{p(x)},$$

which is valid for all numbers $a,\ b>0$. In [5], the authors studied the existence of weak solutions to the Dirichlet problem for equations involving variable exponent Laplacian and proved the criterion for the existence of infinitely many pairs of weak solutions for such problems. In [6], the Holder continuity of minimizers of variational functional associated with quasi-linear elliptic equations is investigated in the variational exponent framework; authors used methods similar to that are developed in the present paper, however, the De Giorgio classes introduced as follows: a function $u \in W_1^{p(\cdot)}(\Omega)$ belongs to class $\tilde{B}_{p(\cdot)}(\Omega,\ M) = \tilde{B}_{p(\cdot)}(\Omega,\ M,\ \vartheta,\ \vartheta_1,\ \delta)$ if $ess\max_{\Omega}|u(x)| \le M$ and for every components $u(x)$ and $-u(x)$ in arbitrary ball $B_r \subset \Omega$ the inequality

$$\int_{\Lambda_{n,\ r-\gamma r}} |\nabla u|^{p(x)}\,dx \le \vartheta \int_{\Lambda_{n,\ r}} \left|\frac{u(x)-n}{\gamma r}\right|^{p(x)} dx + \tilde{\vartheta} meas(\Lambda_{n,\ r})$$

holds for each $\gamma \in (0,\ 1)$, and here $ess\max_{\Omega} u(x) \le M$, $n \ge \max_{B(r)} u(x) - \delta$ and a ball $B_{r-\gamma r}$ is concentric with the ball B_r, see [6, 9]. In case $p(x)$ is a constant, definitions and general framework can be found in [9]. The results on the existence of a weak solution can be found in [2, 3, 4, 5, 6, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23]; some applications of equations involving variable exponent Laplacian were presented in [14, 15].

In this article, assuming the existence of a weak solution $u \in W_1^{p(\cdot)}(\Omega)$ to the elliptic equation (1.1), we establish the rather minimal conditions under which this weak solution satisfies a Holder condition with some exponent $\alpha \in (0,\ 1]$ and the Holder constant

$$M_1 = \sup \frac{osc\{u,\ B_r \cap \Omega\}}{r^\alpha}.$$

2. Preliminary Information

Let $\Omega \subset R^n$ be a bounded domain. Assume $p \in P^{\log}(\Omega)$, we denote $p_m = \inf_{x\in\Omega} p(x)$ and $p_S = \sup_{x\in\Omega} p(x)$. We define a modular function by

$$\rho_{p(\cdot)}(u) = \int_\Omega |u(x)|^{p(x)}\,dx. \tag{2.1}$$

The norm of the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ is defined by

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \inf_{\lambda>0}\left\{\rho_{p(\cdot)}\left(\frac{u}{\lambda}\right) \le 1\right\}. \tag{2.2}$$

The Sobolev space $W_k^{p(\cdot)}(\Omega)$ consists of all functions $u \in L^{p(\cdot)}(\Omega)$ such that its all weak partial derivatives $\partial_\alpha u \in L^{p(\cdot)}(\Omega)$ with $|\alpha| \le k$. The norm of the Sobolev space is defined by

$$\|u\|_{W_k^{p(\cdot)}(\Omega)} = \inf_{\lambda>0}\left\{\sum_{|\alpha|\le k} \rho_{p(\cdot)}\left(\frac{\partial_\alpha u}{\lambda}\right) \le 1\right\}. \tag{2.3}$$

We assume $u \in W_1^{p(\cdot)}(\Omega)$, and $B_r(x) \subset \Omega$ is a ball of radius $r > 0$ centered at a point $x \in \Omega$, we denote

$$\Lambda_n = \{x \in \Omega \ : \ u(x) > n\},$$
$$\Lambda_{n,\, r} = \{x \in B_r \ : \ u(x) > n\}.$$

A function p is continuous; therefore, there exists a small radius $\tilde{r} < 1$ such that $clos\,(B(\tilde{x},\ \tilde{r})) \subset \Omega$ and $\frac{1}{2}\left(\max_{\Omega}|u|\right)^{osc\{p,\quad B(\tilde{x},\,\tilde{r})\}} \leq 1.$

Definition 2.1. A functional class $B_{p(\cdot)}(\Omega,\ M) = B_{p(\cdot)}(\Omega,\ M,\ \vartheta,\ \vartheta_1,\ \delta)$ consists of all elements u of $W_1^{p(\cdot)}(\Omega)$ such that $ess\max_{\Omega}|u(x)| \leq M$ and for every component $u(x)$ and $-u(x)$ in an arbitrary ball $B_r \subset \Omega$, the inequality

$$\int_{\Lambda_{n,\, r-\gamma r}} |\nabla u|^{p(x)}\, dx \leq$$
$$\leq \vartheta \frac{1}{\gamma^{p_S} \min\left(r^{p_m\left(1-\frac{l}{m}\right)},\ r^{p_S\left(1-\frac{l}{m}\right)}\right)} \times$$
$$\max_{\Lambda_{n,\, r}}(\max((u(x)-n)^{p_S},\ (u(x)-n)^{p_m})) + \vartheta meas^{1-\frac{p_S}{m}}(\Lambda_{n,\, r})$$

holds for each $\gamma \in (0,\ 1)$, and where $ess\max_{\Omega} u(x) \leq M$, $n \geq \max_{B(r)} u(x) - \delta$ and a ball $B_{r-\gamma r}$is concentric with the ball B_r.

Straightforwardly [9, lemma 6.1, also 6, lemmas 2.1 – 2.3], we have that if the function u satisfies the inequality

$$\begin{aligned}&\textstyle\int_{\Lambda_{n,\, r}} |\nabla u|^{p(x)}\, \zeta^{p(x)} dx \leq \\ &\leq \tilde{\vartheta}\left(\textstyle\int_{\Lambda_{n,\, r}} (u-n)^{p(x)}\, |\nabla \zeta|^{p(x)}\, dx + meas^{1-\frac{p_S}{m}}(\Lambda_{n,\, r})\right)\end{aligned}$$

for $n \geq \max_{B(r)} u(x) - \delta$ and for an arbitrary positive, smooth function ζ, which vanishes on the sphere $S(r) = \partial B(r)$, then the function u satisfies the conditions of the previous definition for B_r and an arbitrary concentric ball $B_{r-\gamma r}$ for all $\gamma \in (0,\ 1)$.

Proposition 2.2. *There exists a positive number $\theta_1 > 0$ such that, for all $u \in B_{p(\cdot)}(\Omega,\ M)$, $B_r \subset \Omega$ and a number $n \geq \max_{B(r)} u(x) - \delta$, we have that from*

$$meas(\Lambda_{n,\, r}) \leq \theta_1 r^l$$

follows

$$meas\left(\Lambda_{n+\frac{K}{2},\, \frac{r}{2}}\right) = 0,$$

where $K = \max_{B(r)} u(x) - n \geq r^{1-\frac{l}{m}}$.

Proposition 2.3. *There exists a positive number $k > 0$ such that, for all $u \in B_{p(\cdot)}(\Omega,\ M)$, $B_r \subset \Omega$ and a concentric ball $B_{4r} \subset \Omega$, we have at least one of the following inequalities*

$$osc\{u,\ B_r\} \leq 2^k r^{1-\frac{l}{m}}$$

or

$$osc\{u,\ B_r\} \leq \left(1 - 2^{-k+1} r^{1-\frac{l}{m}}\right) osc\{u,\ B_{4r}\}$$

holds.

Proposition 2.4. *Let u is a measurable function bounded in a ball B_{r_0}. For fixed $b > 1$, $r \leq \frac{r_0}{b}$, let B_{r_0}, B_r, and B_{br} be concentric balls. Assume that at least one of the inequalities*

$$osc\left\{u,\ B_r \cap \Omega\right\} \leq c_1 r^{\varepsilon},$$

or

$$osc\left\{u,\ B_r \cap \Omega\right\} \leq \sigma\ osc\left\{u,\ B_{br} \cap \Omega\right\}$$

holds for some $c_1, \varepsilon \leq 1$ and $\sigma < 1$. Then, for all $r \leq r_0$, the inequality

$$osc\left\{u,\ B_r \cap \Omega\right\} \leq c\left(\frac{r}{r_0}\right)^{\alpha}$$

holds for

$$\alpha = \min\left(-\ln_b \sigma,\ \varepsilon\right),$$

$$c = b^{\alpha} \max\left(c_1 {r_0}^{\varepsilon},\ osc\left\{u,\ B_{r_0} \cap \Omega\right\}\right).$$

As a result, we have the theorem.

Theorem 2.5. *Let $u \in B_{p(\cdot)}\left(\Omega,\ M\right)$ and $B_{r_0} \subset \Omega$, $r_0 \leq 1$. Then, for each ball $B_r,\ r < r_0$ concentric with B_{r_0}, we have*

$$osc\left\{u,\ B_r\right\} \leq c\left(\frac{r}{r_0}\right)^{\alpha}$$

for some constants α and c.

3. The Regularity for Elliptic Equations Involving the Generalized Variable Exponent Laplacian

We assume that functions $a_i\left(x,\ u,\ k\right)$ and $b\left(x,\ u,\ k\right)$ satisfy the following condition

$$\sum_{i=1,\ \ldots,\ l} a_i\left(x,\ u,\ k\right) k_i \geq \nu\left(|u|\right)|k|^{p(x)} - \mu\left(|u|\right), \tag{3.1}$$

$$\begin{gathered}\textstyle\sum_{i=1,\ \ldots,\ l}\left|a_i\left(x,\ u,\ k\right)\right|\left(1+|k|\right)+\\ +\left|b\left(x,\ u,\ k\right)\right| \leq \mu\left(|u|\right)\left(1+|k|\right)^{p(x)},\end{gathered} \tag{3.2}$$

where ν and μ are positive functions.

Definition 3.1. A function $u \in W_1^{p(\cdot)}\left(\Omega\right)$ is called a weak solution to the elliptic differential equation in the divergent form with the nonstandard growth if the identity

$$\sum_{i=1,\ \ldots,\ l}\int_{\Omega} a_i\left(x,\ u,\ \nabla u\right)\nabla_i \varphi dx - \int_{\Omega} b\left(x,\ u,\ \nabla u\right)\varphi dx = 0 \tag{3.3}$$

holds for all $\varphi \in W_{1,\ 0}^{p(\cdot)}\left(\Omega\right)$.

A weak solution $u \in W_1^{p(\cdot)}\left(\Omega\right)$ to the elliptic differential equation in the divergent form is called bounded if $ess\max\limits_{\Omega}|u| < \infty$.

Remark. Further, we will skip the symbol of the sum, for example

$$\sum_{i=1,\ \ldots,\ l} a_i\left(x,\ u,\ \nabla u\right)\nabla_i \varphi = a_i\left(x,\ u,\ \nabla u\right)\nabla_i \varphi.$$

We prove the following lemma.

Lemma 3.2. *Let the function $u \in W_1^{p(\cdot)}(\Omega)$ be a weak, bounded solution to the elliptic differential equation in the divergent form with the nonstandard growth such that $ess\max_{\Omega}|u| \leq M < \infty$. Then, for any ball $B_r \subset \Omega$, the estimate*

$$\int_{B_r} |\nabla u|^{p(x)}\, dx \leq c_1 r^l \left(1 + \max\left((r_1 - r)^{-p_S},\ (r_1 - r)^{-p_m}\right)\right),$$

where a constant c_1 depends only on M, $\nu(M)$, $\mu(M)$, $p(\cdot)$ and $|\nabla p|$, r_1 is the distance from center of the ball B_r and the boundary $\partial\Omega$.

Proof. We assume $B_r \subset \Omega$ and $\varphi = e^{\lambda u}\zeta^{p(x)}$, where ζ is a smooth finite function such that $0 \leq \zeta(x) \leq 1,\ x \in \Omega$ and $\zeta(x) = 1,\ x \in B_r$, and $\zeta(x) = 0,\ x \in \Omega \backslash B_{r_1},\ r < r_1$ where B_r and B_{r_1} are concentric balls. Taking a test function equal $\varphi = e^{\lambda u}\zeta^{p(x)}$, we obtain

$$\int_\Omega \lambda e^{\lambda u}\zeta^{p(x)} a_i \nabla_i u dx + \int_\Omega e^{\lambda u}\zeta^{p(x)} \ln(\zeta)\, a_i \nabla_i p dx + \int_\Omega p(x)\, e^{\lambda u}\zeta^{p(x)-1} a_i \nabla_i \zeta dx - \int_\Omega b e^{\lambda u}\zeta^{p(x)} dx = 0.$$

We denote $\nu = \nu(M)$, $\mu = \mu(M)$. From assumptions, we deduce

$$\begin{aligned} a_i \nabla_i u \geq& \nu |\nabla u|^{p(x)} - \mu, \\ \left| p(x)\, \zeta^{p(x)-1} a_i \nabla_i \zeta \right| \leq& p(x)\, \mu \left(1 + |\nabla u|\right)^{p(x)-1} \zeta^{p(x)-1} |\nabla \zeta| \\ &\leq (p(x) - 1)\left(1 + |\nabla u|\right)^{p(x)} \zeta^{p(x)} + \mu^{p(x)} |\nabla \zeta|^{p(x)}, \end{aligned}$$

and

$$\begin{aligned} \left| \zeta^{p(x)} \ln(\zeta)\, a_i \nabla_i p \right| \leq& \mu \zeta^{p(x)-1} \left(1 + |\nabla u|\right)^{p(x)-1} |\nabla p| \\ &\leq |\nabla p| \left(\frac{p(x) - 1}{p(x)} \left(1 + |\nabla u|\right)^{p(x)} \zeta^{p(x)} + \frac{1}{p(x)} \mu^{p(x)} 1_{B_r} \right), \end{aligned}$$

where we employed the inequality $|\ln(\zeta)| \leq \frac{1}{\zeta}$ for all $\zeta \in (0,\ 1]$.

By assumption $|\nabla p| \leq const$, therefore, we obtain

$$\begin{aligned} \lambda \nu \int_\Omega e^{\lambda u} |\nabla u|^{p(x)} \zeta^{p(x)} dx \leq& c \int_\Omega e^{\lambda u} \left(\left(1 + |\nabla u|\right)^{p(x)} \zeta^{p(x)} + |\nabla \zeta|^{p(x)} \right) dx \\ &+ c \frac{\max\left(\mu^{p_m},\ \mu^{p_S}\right)}{p_m} \int_\Omega e^{\lambda u} 1_{B_r} dx. \end{aligned}$$

We take $\lambda = \frac{2c}{\nu}$ and obtain

$$\int_\Omega |\nabla u|^{p(x)} \zeta^{p(x)} dx \leq e^{\frac{4c}{\nu}M} \int_\Omega \left(\zeta^{p(x)} + |\nabla \zeta|^{p(x)} \right) dx + \tilde{c} \max\left(\mu^{p_m},\ \mu^{p_S}\right) e^{\frac{2c}{\nu}M} r^l.$$

Now, changing constants and choosing $\max_\Omega |\nabla \zeta| \leq const\left((r_1 - r)^{-1}\right)$, we conclude

$$\int_{B_r} |\nabla u|^{p(x)}\, dx \leq \tilde{c}_1 r^l \left(1 + \max\left((r_1 - r)^{-p_S},\ (r_1 - r)^{-p_m}\right)\right) + \tilde{c}_2 r^l.$$

This concludes the proof of the lemma.

Lemma 3.3. *Let the function $u \in W_1^{p(\cdot)}(\Omega)$ be a weak, bounded solution to the elliptic differential equation in the divergent form with the nonstandard growth such that $ess\max_{\Omega}|u| \leq M < \infty$. Let $u(x)|_{\partial\Omega} = \psi(x)|_{\partial\Omega}$, where ψ is a bounded function from $W_1^{p(\cdot)}(\Omega)$. Then, for any ball $B_r \subset \Omega$, the estimate*

$$\int_{B_r \cap \Omega} |\nabla u|^{p(x)}\, dx \leq c \max\left(r^{l - p_S},\ r^{l - p_m}\right),$$

where a constant c depends only on M, $\nu(M)$, $\mu(M)$, $\max_\Omega |\psi|$, $\|\nabla \psi\|_{L^{p(\cdot)}(\Omega)}$, $p(\cdot)$, and $|\nabla p|$.

Proof. We take the test function as $\varphi = \left(e^{\lambda u} - e^{\lambda \psi}\right)\zeta$, where ζ is the cutoff for an arbitrary ball B_{2r}, equal to the unit in a concentric ball B_r. Applying similar to the previous lemma arguments, we obtain the statement of the lemma.

Theorem 3.4. *Let conditions (5) and (6) be satisfied. Then, any weak, bounded solution $u \in W_1^{p(\cdot)}(\Omega)$ to the elliptic differential equation in the divergent form with the nonstandard growth such that $ess\max_{\Omega}|u| \leq M < \infty$ belongs to the Holder class $C_{0,\ \alpha}(\Omega)$ for certain α depending only on M, $\nu(M)$, $\mu(M)$, $p(\cdot)$, and $|\nabla p|$.*

Proof. We assume that a function $u \in W_1^{p(\cdot)}(\Omega)$ is a weak bounded solution. We take

$$\varphi(x) = \zeta^{p(x)}(x)\max(u(x) - n,\ 0),$$

where ζ is the cutoff for an arbitrary ball B_r, and the number n is arbitrary for all inside balls B_r and larger than the maximum of the solution $u(x)$ on $B_r \cap \partial\Omega$, when $B_r \cap \partial\Omega$ is not empty. We have

$$\begin{aligned}&\textstyle\int_{\Lambda_{n,\ r}} \zeta^{p(x)} a_i \nabla_i u dx + \int_{\Lambda_{n,\ r}} p(x)\zeta^{p(x)-1} a_i (u(x) - n)\nabla_i \zeta dx\\ &+\textstyle\int_{\Lambda_{n,\ r}} \zeta^{p(x)} \ln(\zeta) a_i (u(x) - n)\nabla_i p dx - \int_{\Lambda_{n,\ r}} b(u(x) - n)\zeta^{p(x)} dx = 0,\end{aligned}$$

where $\Lambda_{n,\ r} = \{x \in B_r \cap \Omega\ :\ u(x) > n\}$. Applying conditions (3.1), (3.2), we obtain

$$\begin{aligned}\nu\int_{\Lambda_{n,\ r}} \zeta^{p(x)} |\nabla u|^{p(x)} dx \leq& \mu\int_{\Lambda_{n,\ r}} \left(\zeta^{p(x)} + p(x)(1 + |\nabla u|)^{p(x)-1}\zeta^{p(x)-1}|\nabla\zeta|(u(x) - n)\right)dx\\ &+ \mu\int_{\Lambda_{n,\ r}} |\nabla p|\zeta^{p(x)-1}(1 + |\nabla u|)^{p(x)-1}(u(x) - n)dx\\ &+ \mu\int_{\Lambda_{n,\ r}} (1 + |\nabla u|)^{p(x)}\zeta^{p(x)}(u(x) - n)dx.\end{aligned}$$

By the Young inequality, we obtain

$$\begin{aligned}&p(x)(1 + |\nabla u|)^{p(x)-1}\zeta^{p(x)-1}|\nabla\zeta|(u(x) - n)\\ &\leq (p(x) - 1)\varepsilon(1 + |\nabla u|)^{p(x)}\zeta^{p(x)} + \varepsilon^{1-p(x)}(u(x) - n)^{p(x)}|\nabla\zeta|^{p(x)}\\ &\leq p(x)2^{p(x)}\varepsilon(1 + |\nabla u|)^{p(x)}\zeta^{p(x)} + \varepsilon^{1-p(x)}(u(x) - n)^{p(x)}|\nabla\zeta|^{p(x)},\end{aligned}$$

and, we similarly calculate

$$\begin{aligned}\zeta^{p(x)-1}(1 + |\nabla u|)^{p(x)-1}(u(x) - n) \leq& \varepsilon\frac{p(x) - 1}{p(x)}(1 + |\nabla u|)^{p(x)}\zeta^{p(x)} + \frac{1}{p(x)}\varepsilon^{1-p(x)}(u(x) - n)^{p(x)}\\ \leq& 2^{p(x)}\varepsilon(1 + |\nabla u|)^{p(x)}\zeta^{p(x)} + \frac{1}{p(x)}\varepsilon^{1-p(x)}(u(x) - n)^{p(x)}.\end{aligned}$$

Then, we choose $\varepsilon = \frac{\nu}{2^{p_S+4}\mu p_S}$ and number n such that

$$\max_{B_r\cap\Omega} u(x) - n \leq \delta = \frac{\nu}{2^{p_S+6}\mu}.$$

Then, we have

$$\textstyle\int_{\Lambda_{n,\ r}} \zeta^{p(x)}|\nabla u|^{p(x)} dx \leq \tilde{c}\int_{\Lambda_{n,\ r}} \left((u(x) - n)^{p(x)}|\nabla\zeta|^{p(x)} + \zeta^{p(x)}\right)dx,$$

where a constant $\tilde{c}$ depends only on ν, μ, $p(\cdot)$, and $|\nabla p|$. Let $\gamma \in (0,\ 1)$, we choose the function ζ equal to the unit in a ball $B_{r-\gamma r}$ concentric with the ball B_r, and such that $|\nabla\zeta| \leq \frac{c}{\gamma r}$, and we obtain

$$\begin{aligned}&\textstyle\int_{\Lambda_{n,\ r-\gamma r}} |\nabla u|^{p(x)} dx\\ &\leq \vartheta\left(\frac{1}{\min((\gamma r)^{p_S},\ (\gamma r)^{p_m})}\max_{\Lambda_{n,\ r}}(\max((u(x) - n)^{p_S},\ (u(x) - n)^{p_m})) + 1\right) meas(\Lambda_{n,\ r}).\end{aligned}$$

The same inequality is applicable to the negative component $-u(x)$ of the weak solution $u(x)$, if we take

$$\varphi(x) = \zeta^{p(x)}(x)\max(-u(x) - n,\ 0),$$

where a number n is chosen so that

$$\max_{B_r}(-u(x)) - n \leq \delta = \frac{\nu}{2^{p_S+6}\mu}$$

and

$$\max_{B_r\cap\partial\Omega}(-u(x)) \leq n.$$

So, the weak solution u belongs to a certain De Giorgio class $B\left(clos\left(\Omega\right),\ M,\ \vartheta,\ \delta\right)$,where $\vartheta,\ \delta$ depending only on M, $\nu\left(M\right)$, $\mu\left(M\right)$, $p\left(\cdot\right)$, and $\left|\nabla p\right|$ thus u belongs to $C_{0,\ \alpha}\left(\Omega\right)$.

Statements and Declarations

The author declare that he has no conflict of interest, and the manuscript has no associated data.

References

[1] E. De Giorgi. Sulla differenziabilita e lanaliticita delle estremali degli integrali multipli regolari. *Memorie Della Reale Accademia Delle Scienze di Torino*, 3:25-43, 1957.

[2] M. Ding, C. Zhang, and S. Zhou. Global boundedness and Holder regularity of solutions to general p(x, t)-Laplace parabolic equations. *Mathematical Methods in the Applied Sciences*, 43(9):5809-5831, 2020.

[3] M. Ding, C. Zhang, and S. Zhou. On optimal $C^{1,\alpha}$ estimates for p(x)-Laplace type equations. *Nonlinear Analysis*, 200:Article ID 112030, 2020.

[4] L. Diening, P. Harjulehto, P. Hasto, and M. Ruzicka. *Lebesgue and Sobolev Spaces with Variable Exponents. Lecture Notes in Math.*, vol. 2017, Springer-Verlag, Heidelberg, 2011.

[5] X. L. Fan and Q. H. Zhang. Existence of solutions for p(x)-Laplacian Dirichlet problems. *Nonlinear Analysis*, 52:1843-1852, 2003.

[6] X. Fan and D. Zhao. A class of De Giorgi type and Holder continuity. *Nonlinear Analysis:Theory, Methods & Applications*, 36(3):295-318, 1999.

[7] T. Hara. Global Holder solvability of linear and quasilinear Poisson equations. *ArXiv Preprint*, arXiv:2311.09701, 2023.

[8] Y. C. Kim. Nonlocal Harnack inequalities for nonlocal heat equations. *Journal of Differential Equations*, 267(11):6691-6757, 2019.

[9] O. A. Ladyzhenskaya and N. N. Uraltseva. *Linear and Quasilinear Elliptic Equations*, Elsevier, 1st edition, Nauka, Moscow, 1968.

[10] G. Li, D. Motreanu, H. Wu, and Q. Zhang. Multiple solutions with constant sign for a (p, q)-elliptic system Dirichlet problem with product nonlinear term. *Boundary Value Problems*, 2018:Article ID 67, 2018.

[11] J. Moser. A new proof of De Giorgi's theorem concerning the regularity problem for elliptic differential equations. *Communications on Pure and Applied Mathematics*, 13:457-468, 1960.

[12] F. Yao. Holder regularity for the general parabolic p(x,t)-Laplacian equations. *Nonlinear Differential Equations and Applications NoDEA*, 22:105-119, 2015.

[13] C. Yu and D. Ri. Global L∞-estimates and Holder continuity of weak solutions to elliptic equations with the general nonstandard growth conditions. *Nonlinear Analysis*, 156:144-166, 2017.

[14] X. J. Wang. Neumann problems of semilinear elliptic equations involving critical Sobolev exponents. *Journal of Differential Equations*, 93(2):283-310, 1991.

[15] X. Zhang and W. Huang. Adaptive Neural Network Sliding Mode Control for Nonlinear Singular Fractional Order Systems with Mismatched Uncertainties. *Fractal Fractional*, 4:Article ID 50, 2020.

[16] B. L. Zhang, N. Pan, and J. Cao. Weak solutions for parabolic equations with p(x)-growth. *Electronic Journal of Differential Equations*, 2016:Article ID 209, 2016.

[17] Q. H. Zhang. Existence of solutions for p(x)-Laplacian equations with singular coefficients in RN. *Journal of Mathematical Analysis and Applications*, 348:38-50, 2008.

[18] D. Liu, B. Wang, and P. Zhao. On the trace regularity results of Musielak–Orlicz–Sobolev spaces in a bounded domain. *Communications on Pure & Applied Analysis*, 15:1643-1659, 2016.

[19] D. Liu and P. Zhao. Solutions for a quasilinear elliptic equation in Musielak–Sobolev spaces. *Nonlinear Analysis: Real World Applications*, 26:315-329, 2016.

[20] M. Nikolay Yaremenko. Existence of non-trivial Yang-Mills theory on R∧4. *Journal of Advances in Physics*, 10:2682-2691, 2015.

[21] D. S. Li and L. H. Wang. Boundary differentiability of solutions of elliptic equations on convex domains. *Manuscripta Mathematica*, 121:137–156, 2006.

[22] D. S. Li and L. H. Wang. Elliptic equations on convex domains with nonhomogeneous Dirichlet boundary conditions. *Journal of Differential Equations*, 246:1723-1743, 2009.

[23] Z. Li and Q. S. Zhang. Regularity of weak solutions of elliptic and parabolic equations with some critical or supercritical potentials. *Journal of Differential Equations*, 263(1):57-87, 2017.