

## ABSTRACT CONVEXITY WITH RESPECT TO NORM LINEAR FUNCTIONS

SAMET BILA<sup>1</sup> AND REFAIL KASIMBEYLI<sup>2,\*</sup>

<sup>1</sup>*Department of Mathematics, Faculty of Science, Eskisehir Technical University, 26470, Eskisehir, Turkey*

<sup>2</sup>*Department of Industrial Engineering, Faculty of Engineering, Eskisehir Technical University, 26555, Eskisehir, Turkey*

**ABSTRACT.** In this article, we study abstract convexity, also known as convexity without linearity, for a special class of elementary functions of the form  $\varphi(x) = \langle x^*, x - a \rangle - c\|x - b\| + \alpha$ . Each function  $\varphi(x)$  is characterized by a pair  $(x^*, c)$ , and the class is defined in terms of such pairs. Within the framework of abstract convexity, probably such a characterization arises first. In this work, we define the class of functions that are representable as a pointwise supremum of this family and investigate its properties.

**Keywords.** Weak subdifferential, Nonconvex analysis, Abstract convexity.

© Applicable Nonlinear Analysis

### 1. INTRODUCTION

In recent years, the field of convex analysis has experienced significant expansion. The initial motivation is provided by operations research, where the duality theorem and the simplex method have driven the success of linear programming. This success sparked interest among engineers in such problems. Then, it led to a search for similar results in nonlinear optimization. A deeper understanding of convex functions has yielded the important concepts, such as conjugate convex functions and subdifferentiability.

Nonconvexity may arise in many different forms and investigating nonconvex problems is very challenging. Many nonconvex optimization problems have been approached using a method known as abstract convexity. The idea of abstract convexity dates back to the study in [5]. Later, this concept is employed to prove a generalized Krein-Milman theorem in Banach spaces [6]. Further, the idea of abstract convexity was introduced to extend the notion of duality to the nonconvex functions and provided a scheme for analyzing important classes of non-smooth analysis [15].

Abstract convexity provides powerful global tools for analyzing nonconvex structures and offers a robust framework for addressing global optimization problems. Abstract convexity has found widespread applications in mathematical analysis and optimization [14]. The book by Pallaschke and Rolewics presents a comprehensive selection of numerous results in the area of abstract convexity with particular emphasis on subdifferentials and duality [10].

In this work, we study abstract convexity w.r.t. a special class of functions defined as  $\varphi(x) = \langle x^*, x - a \rangle - c\|x - b\| + \alpha$ . This class of functions is characterized by a pair  $(x^*, c)$ . We study class of functions which are representable as a pointwise supremum of this family, and the abstract convexity concepts related to these functions, such as subdifferentiability, duality and so on. One of the main characteristics of these functions is that they are lower (locally) Lipschitz. Now we recall the definition of lower (locally) Lipschitz functions.

\*Corresponding author.

E-mail address: rkasimbeyli@eskisehir.edu.tr (R. Kasimbeyli), sametbila@eskisehir.edu.tr (S. Bila)

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**Definition 1.1.** A function  $f : X \rightarrow \mathbb{R}$  is called lower locally Lipschitz at  $\bar{x} \in X$ , if there exists a positive number  $L$  and a neighborhood  $\mathcal{N}(\bar{x})$  of  $\bar{x}$  such that

$$f(x) - f(\bar{x}) \geq -L\|x - \bar{x}\| \text{ for all } x \in \mathcal{N}(\bar{x}). \quad (1.1)$$

$f$  is called lower Lipschitz at  $\bar{x}$  with the Lipschitz constant  $L$  if the inequality (1.1) holds for all  $x \in X$ .

This paper is organized as follows.

Section 2 provides main concepts and necessary definitions of abstract convexity. Section 3 presents main results of the article.

## 2. ABSTRACT CONVEXITY

A.M. Rubinov introduced abstract convex functions as upper envelopes of families of simpler functions in his excellent book [12]. This approach enables to analyze some classes of nonconvex problems and extends the concept of convexity beyond linear spaces. This book explores IPH functions, normal sets, and monotonic functions, explains their role in abstract convexity and nonconvex optimization. It also presents duality principles within the context of abstract convexity, offering insights into the relationships between primal and dual problems under certain conditions. Furthermore, Rubinov discussed the use of Lagrange and penalty functions within this framework, providing methods to handle constraints in optimization problems. The concept of convexity is further extended to star-shaped sets and functions, which are essential for modeling and solving specific classes of nonconvex problems.

Now we recall main definitions related to abstract convexity.

**Definition 2.1.** [12, Definition 1.1] Let  $V \subset \overline{\mathbb{R}}$  and  $H$  be a nonempty set of functions  $h : X \rightarrow V$ . A function  $f : X \rightarrow \overline{V}$  is called abstract convex with respect to  $H$  (or  $H$ -convex) if there exists a set  $U \subset H$  such that  $f$  is the upper envelope of this set:

$$f(x) = \sup\{h(x) : h \in U\} \text{ for all } x \in X.$$

It follows directly from Definition 2.1 that  $f$  is  $H$ -convex if and only if

$$f(x) = \sup\{h(x) : h \in H, h \leq f\} \text{ for all } x \in X.$$

**Definition 2.2.** [12, Definition 1.2] Let  $f : X \rightarrow \overline{V}$ . The set

$$\text{Supp}(f, H) = \{h \in H, h \leq f\}$$

of all  $H$ -minorants of  $f$  is called the support set of the function  $f$  with respect to the set of elementary functions  $H$ .

Abstract subgradient is defined for abstract convex functions in the following form.

**Definition 2.3.** [12, Definition 1.7] Let  $L$  be a set of finite elementary functions defined on  $X$ . A function  $l \in L$  is called an abstract subgradient of a proper function  $f : X \rightarrow \mathbb{R}_{+\infty}$  at a point  $y$  if

$$f(x) \geq f(y) + l(x) - l(y) \text{ for all } x \in X.$$

The set  $\partial_L f(y)$  of all abstract subgradients of  $f$  at  $y$  is referred to as the abstract subdifferential of  $f$  at the point  $y$ . If the subdifferential  $\partial_L f(y)$  is not empty then  $f$  is called abstract subdifferentiable at  $y \in \text{dom}(f) = \{x \in X : f(x) < +\infty\}$ .

Let  $X$  and  $L$  be two sets. Consider a function  $\phi : L \times X \rightarrow \mathbb{R}$ . The function  $\phi$  will be referred to as a coupling function and we shall use the notation  $\phi(x, l) = \langle l|x \rangle$ . The Fenchel-Moreau  $L$ -conjugate and The Fenchel-Moreau second  $L$ -conjugate functions are defined as follows:

**Definition 2.4.** [12, Definition 1.5, p. 12] Let  $f : X \rightarrow \overline{\mathbb{R}}$ . The Fenchel-Moreau  $L$ -conjugate,  $f_L^*$ , of a function  $f$  is defined in the following form.

$$f_L^*(l) = \sup_{x \in X} (\langle l|x \rangle - f(x)), \quad l \in L.$$

The Fenchel-Moreau second  $L$ -conjugate,  $f_L^{**}$ , of a function  $f$  is defined as follows:

$$f_L^{**}(l) = \sup_{x \in X} (\langle l|x \rangle - f_L^*(l)), \quad x \in X.$$

*Remark 2.5.* The abstract duality and zero duality gap conditions are investigated in [12] w.r.t. increasing positively homogeneous functions. Gasimov studied duality with zero gap in nonconvex optimization, by constructing augmented dual problems with respect to norm linear functions in [8], and presented an efficient solution method for nonconvex constrained problems without differentiability. Rubinov and Gasimov studied the duality w.r.t. norm linear and increasing positively homogeneous functions in [13]. Duality w.r.t. normlinear functions was further developed by Dinc Yalcin and Kasimbeyli in [3].

### 3. ABSTRACT CONVEXITY WITH RESPECT TO THE NORM LINEAR FUNCTIONS

Let  $X$  be a real normed space and let  $X^*$  be its dual. In this paper, we use a family of elementary functions defined by:

$$H = \{\varphi : \varphi(x) = \langle x^*, x - a \rangle - c\|x - b\| + \alpha, x^* \in X^*, a, b \in X, c \in \mathbb{R}^+, \alpha \in \mathbb{R}\}.$$

The following is the definition of  $H$ -convex functions.

**Definition 3.1.** Let  $H$  be a nonempty set of functions consisting of the functions  $\varphi$ . A function  $f : \mathbb{X} \rightarrow (-\infty, +\infty]$  is called abstract convex with respect to  $H$  (or  $H$  convex) if there exists a set  $U \subset H$  such that  $\varphi(x) \leq f(x)$ ,  $\forall x \in X$  and  $f$  is the pointwise supremum of this set:

$$f(x) = \sup\{\varphi(x) : \varphi \in U\} \text{ for all } x \in X.$$

**Definition 3.2.** Let function  $f : \mathbb{X} \rightarrow (-\infty, +\infty]$  be a given function. The set

$$\text{supp}(f, H) = \sup\{\varphi \in H : \varphi \leq f\}$$

of all  $H$ -minorants of  $f$  is called the support set of  $f$  with respect to the set of elementary functions  $H$ .

Now we present the definition of  $H$ -subgradient. We use the general definition 2.3, where the class of elementary functions  $H$  is used.

**Definition 3.3** ( $H$ -weak subgradient). An element  $(x^*, c) \in H \times \mathbb{R}_+$  is called  $H$ -subgradient of a function  $f : \mathbb{X} \rightarrow (-\infty, +\infty]$  at  $\bar{x} \in \text{dom}(f)$  if

$$f(x) - f(\bar{x}) \geq \langle x^*, x - \bar{x} \rangle - c\|x - \bar{x}\|, \quad \forall x \in \mathbb{X}$$

holds.

The set of all  $H$ -subgradients of  $f$  at  $\bar{x}$  is denoted as  $\partial_H(\bar{x})$ . Note that this subgradients is the same as the weak subgradient notion earlier introduced by Kasimbeyli in his Ph.D. Dissertation [7]. Note that, this concept is an extension of classical subgradient concept of the convex analysis, see e.g. [4, 9, 11]

We will denote the class of functions defined on a set  $V \subset X$ , which are representable as a pointwise supremum of some set of elementary functions from  $H$ , by  $H(V)$ . Despite the fact that all the functions

in  $H$  are lower Lipschitz, the functions from  $H(V)$  are not necessarily lower Lipschitz. By  $H_0(V)$  we will denote the subset of  $H(V)$  which consists of lower Lipschitz functions.

The following theorem shows that  $H$ -convex functions are lower semicontinuous.

**Theorem 3.4.** *Assume that  $f \in H(V)$ . Then  $f$  is a lower semicontinuous.*

*Proof.* Let  $f(x) = \sup\{g_{(x_i^*, c_i, x_i, \alpha_i)}(x) : g_{(x_i^*, c_i, x_i, \alpha_i)} \in H : i \in I\}$ . Let  $x, y \in \text{dom}(g_i), i \in I$ . By the definition of the supremum,

$$\forall \varepsilon > 0, \text{ there exists } j_0 \in I : g_{(x_{j_0}^*, x_{j_0}, c_{j_0}, \alpha_{j_0})}(x) > \sup_i g_{(x_i^*, c_i, x_i, \alpha_i)}(x) - \varepsilon \quad (3.1)$$

Multiplying (3.1) by -1 yields to,

$$-\sup_i g_{(x_i^*, c_i, x_i, \alpha_i)}(x) + \varepsilon > -g_{(x_{j_0}^*, x_{j_0}, c_{j_0}, \alpha_{j_0})}(x) \quad (3.2)$$

and we have

$$\sup_{i \in I} g_{(x_i^*, c_i, x_i, \alpha_i)}(y) \geq g_{(x_{j_0}^*, x_{j_0}, c_{j_0}, \alpha_{j_0})}(y), \quad \forall i \quad (3.3)$$

From (3.2) and (3.3) and considering  $g_{j_0}$  is a lower Lipschitz with the Lipschitz constant  $L_{j_0}$  we obtain that

$$\begin{aligned} f(y) - f(x) &= \sup_{i \in I} g_{(x_i^*, c_i, x_i, \alpha_i)}(y) - \sup_{i \in I} g_{(x_i^*, c_i, x_i, \alpha_i)}(x) \\ &\geq g_{(x_{j_0}^*, x_{j_0}, c_{j_0}, \alpha_{j_0})}(y) - g_{(x_{j_0}^*, x_{j_0}, c_{j_0}, \alpha_{j_0})}(x) - \varepsilon \\ &\geq -L_{j_0} \|y - x\| - \varepsilon \end{aligned}$$

Passing to the liminf as  $y \rightarrow x$  leads to

$$\liminf_{y \rightarrow x} f(y) \geq f(x) - \varepsilon.$$

Since  $\varepsilon$  is arbitrary; this implies that  $f$  is lower semicontinuous.  $\square$

We present the following  $H$ -conjugate and the second  $H$ -conjugate functions, by using our elementary functions in Definition 2.4. Note that these functions are the weak conjugate functions, earlier presented by Kasimbeyli in [7] (see also [1, 2]).

**Definition 3.5** ( $H$ -weak conjugate). Let  $f$  be a proper function from  $X$  into  $\overline{\mathbb{R}}$ . The function

$$f_H^w(\bar{x}, x^*, \alpha) = \sup_{x \in X} \{-c\|x - \bar{x}\| + c\|x\| + \langle x, x^* \rangle - f(x)\}$$

is called the  $H$ -weak conjugate of  $f$ . Accordingly, the second  $H$ -weak conjugate is defined as

$$f_H^{ww}(x) = \sup_{(\bar{x}, x^*, \alpha) \in X \times X^* \times \mathbb{R}_+} \{-c\|x - \bar{x}\| + c\|x\| + \langle x, x^* \rangle - f_H^w(\bar{x}, x^*, \alpha)\}$$

The following theorem establishes a relation between the given function and its weak conjugate, via the weak subgradient. Note that this relation extends the well-known Fenchel-Moreau relation of convex analysis.

**Theorem 3.6** (Theorem 3.4, [1]). *Let  $f$  be a proper function from  $X$  into  $\overline{\mathbb{R}}$ . The following conditions are equivalent:*

- (i)  $f(x) + f_H^w(x, x^*, c) = c\|x\| + \langle x, x^* \rangle$
- (ii)  $(x^*, c) \in \partial_H^w f(x)$ .

*Remark 3.7.* Note that A.M. Rubinov established optimality conditions in [12] by using the so-called increasing positively homogeneous functions as a set of elementary functions. It is remarkable that the functions used in our paper as a set of elementary functions defined as  $g(x) = \langle x^*, x \rangle - c\|x\|$  satisfy the positive homogeneity condition. However, our elementary functions may or may not be increasing positively homogeneous, in general.

The following proposition gives a criterion for equality between a given function and its second weak conjugate.

**Proposition 3.8.** [1, Theorem 3.3] *Let  $f$  be a proper function from  $X$  into  $\overline{\mathbb{R}}$ . If  $\partial_H^w f(x_0) \neq \emptyset$  then  $f(x_0) = f_H^{ww}(x_0)$ .*

Now we present an illustrative example for Theorem 3.4. This example demonstrates that the pointwise supremum of elementary functions used in this paper, is lower semicontinuous. On the other hand, this example demonstrates that the pointwise supremum of these functions may not be necessarily lower Lipschitz at all.

**Example 3.9.** Consider the following functions :

$$\begin{aligned} g_1(x) &= \langle -\frac{1}{4}, x + 2 \rangle - \frac{5}{4}|x + 2| + 6, \\ g_2(x) &= \langle -\frac{3}{2}, x + 1 \rangle - \frac{5}{2}|x + 1| + 7, \\ g_3(x) &= \langle -5, x + \frac{1}{12} \rangle - 6|x + \frac{1}{12}| + \frac{91}{12}. \\ g_4(x) &= \langle -24, x + \frac{1}{10} \rangle - 25|x + \frac{1}{10}| + \frac{79}{10}. \\ g_5(x) &= \langle -249, x + \frac{1}{100} \rangle - 250|x + \frac{1}{100}| + \frac{799}{100}. \end{aligned}$$

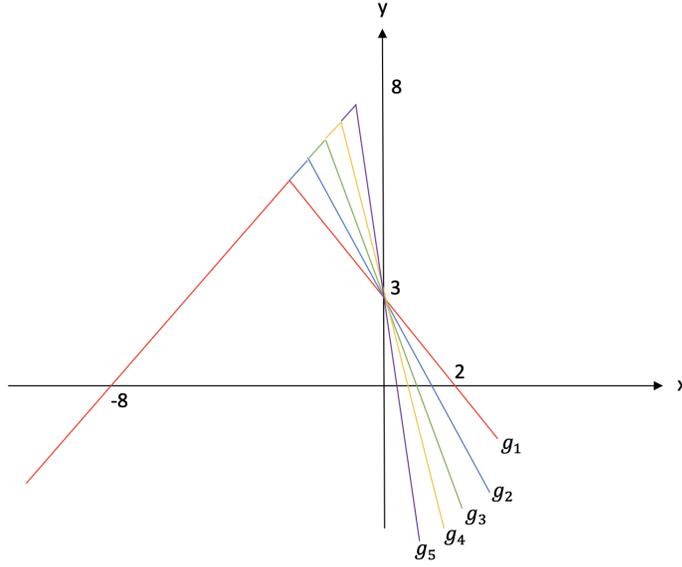
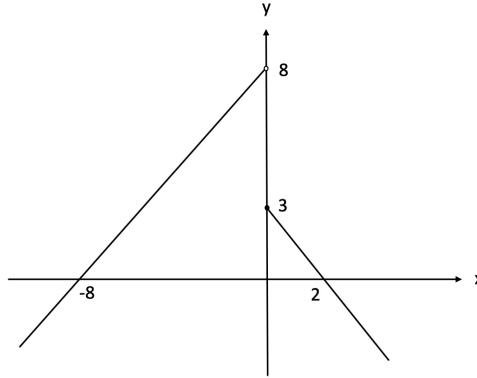
The graph of these functions is provided in Figure 1. It is obvious that  $g_1, g_2, g_3, g_4$  and  $g_5$  belong to  $H$ . Let us construct the family of functions in such a way that all the functions pass through the points  $(-8, 0)$  and  $(0, 3)$ . We construct these functions in such a way that all the function  $f$  attains its maximum at the point  $x_i$ , and define them as follows:

$$g_i(x) = \langle x_i^*, x - x_i \rangle - c_i|x - x_i| + \alpha_i.$$

If  $x < x_i$  then  $|x - x_i| = -(x - x_i)$ . Substituting this into the definition of  $g_i$ , we get:  $g_i(x) = \langle x_i^*, x - x_i \rangle + c_i(x - x_i) + \alpha_i = (x_i^* + c_i)x + (-x_i^*x_i - c_ix_i + \alpha_i)$ . Considering the graph of these family of functions, when  $x < 0$ , the graph of  $g(x) = x + 8$  lays above all these functions and thus implies that if  $x \leq x_i$  then  $x_i^* + c_i = 1$  and  $-x_i^*x_i - c_ix_i + \alpha_i = 8$ . If  $x \geq x_0$  the line must pass through the point  $(0, 3)$ . By considering  $x \geq x_i$  we obtain  $g_i(x) = \langle x_i^*, x - x_i \rangle - c_i(x - x_i) + \alpha_i = (x_i^* - c_i)x + (-x_i^*x_i + c_ix_i + \alpha_i)$ . Finally, we have  $-x_i^*x_i + c_ix_i + \alpha_i = 3$ . We can write the conditions in the following way,  $x_i^* + c_i = 1$ ,  $-x_i^*x_i - c_ix_i + \alpha_i = 8$ , and  $-x_i^*x_i + c_ix_i + \alpha_i = 3$ . If we consider the equations  $-x_i^*x_i - c_ix_i + \alpha_i = 8$  and  $-x_i^*x_i + c_ix_i + \alpha_i = 3$ , we obtain that  $c_ix_i = -\frac{5}{2}$ . Thus, the family, named I, consists of the  $(x_i^*, x_i, c_i, \alpha_i)'$  s such that

$$x_i^* + c_i = 1, -x_i^*x_i - c_ix_i + \alpha_i = 8 \text{ and } c_ix_i = -\frac{5}{2}$$

Let us verify this for the functions  $g_1$  and  $g_5$ . For  $g_1$ , we have  $x_1^* = -\frac{1}{4}, x_1 = -2, c_1 = \frac{5}{4}$  and  $\alpha_1 = 6$ . Clearly,  $x_1^* + c_1 = -\frac{1}{4} + \frac{5}{4} = 1$ ,  $-x_1^*x_1 - c_1x_1 + \alpha_1 = -\frac{1}{4}(-2) - (\frac{5}{4})(-2) + 6 = 8$  and  $c_1x_1 =$

FIGURE 1. The graphs of  $g_i, i = 1, \dots, 5$ FIGURE 2. The pointwise supremum of a family of given functions  $g_i(x), i \in I$ 

$\frac{5}{4}(-2) = -\frac{5}{2}$ . In the same way for  $g_5$ , we have  $x_5^* = -249$ ,  $x_5 = -\frac{1}{100}$ ,  $c_5 = 250$  and  $\alpha_5 = \frac{799}{100}$ . Clearly,  $x_5^* + c_5 = -249 + 250 = 1$ ,  $-x_5^*x_5 - c_5x_5 + \alpha_5 = -(-249)(-\frac{1}{100}) - (250)(-\frac{1}{100}) + \frac{799}{100} = 8$  and  $c_4x_4 = 250(-\frac{1}{100}) = -\frac{5}{2}$ .

If we consider the pointwise supremum of this family, we obtain the function whose graph is depicted in Figure 2.

Clearly this function is lower semicontinuous. However, it is not lower Lipschitz at  $x = 0$ .

Another example of a function that can be represented as a pointwise supremum of a family of functions that is not lower Lipschitz, is the following. Consider the function  $f(x) = -\sqrt{|x|}$ . The function  $f$  is lower semicontinuous and it can be represented as a pointwise supremum of the family of functions  $g_{(x_i^*, c_i, x_i, \alpha_i)}(x) = g_{(0, c_i, 0, 0)}(x) = -c_i|x|$ , which means that,  $-\sqrt{|x|} = \sup_{i \in I} \{-c_i|x|\}$ . However, the function  $f(x) = -\sqrt{|x|}$  is not lower Lipschitz at  $x = 0$ . To see this, suppose the contrary that that  $f$  is lower Lipschitz at  $x = 0$ . Then there exists an  $L > 0$  such that  $f(x) - f(0) \geq -L|x - 0|$ ,

equivalently  $-\sqrt{|x|} \geq -L|x|$ . Multiplying the last inequality yields that  $\sqrt{|x|} \leq L|x|$ . Then it yields that  $|x| \leq L^2x^2$ . This inequality does not hold for every  $L > 0$ . Thus,  $f$  is not lower Lipschitz at 0. This example shows that the pointwise supremum of lower Lipschitz functions may fail to be lower Lipschitz.

#### 4. CONCLUSION

In this paper, we study a possibility of the use of a special class of homogeneous concave elementary functions which are characterized by a pair  $(x^*, c) \in X^* \times R_+$ . We demonstrated that these functions can be used to define abstract convex functions and investigated their properties, such as subdifferentiability and lower semicontinuity. Although the duality and optimality conditions in nonconvex optimization was well investigated in terms of abstract convexity in recent years, there are still open problems in this area, waiting their solutions. For example zero duality gap conditions for inequality constrained problems, is still unsolved problem. We hope that this problem will be considered by using the abstract convexity tools, soon.

#### STATEMENTS AND DECLARATIONS

The authors declare that they have no conflict of interest, and the manuscript has no associated data.

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