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LERAY–SCHAUDER ALTERNATIVES FOR MAPS WITH AN UPPER SEMICONTINUOUS INCLUSION PROPERTY

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ABSTRACT. In this paper, we present a variety of Leray-Schauder alternatives in which the maps considered have an upper semicontinuous or continuous selection.

Keywords. Fixed points, Leray–Schauder alternatives.

1. INTRODUCTION

This paper presents some new Leray–Schauder alternatives for multivalued maps which have either an upper semicontinuous selection [6, 7] or a continuous single valued selection [4, 8, 9, 13]. Our theory is based on a fixed point result of the author [10] and some ideas initiated in [12].

Next we describe the maps considered in this paper i.e. we consider a general class of maps, namely the PK maps of Park (which include Kak and AC maps). Let X and Y be Hausdorff topological spaces. Given a class $\mathbf{X}$ of maps, $\mathbf{X}(X,Y)$ denotes the set of maps $F : X \to 2^Y$ (nonempty subsets of Y) belonging to $\mathbf{X}$, and $\mathbf{X}_c$ the set of finite compositions of maps in $\mathbf{X}$. We let

$$\mathbf{F}(\mathbf{X}) = \{Z : \ Fix\, F \neq \emptyset \ \text{ for all } \ F \in \mathbf{X}(Z,Z)\}$$

where $Fix\, F$ denotes the set of fixed points of F.

The class $\mathbf{U}$ of maps is defined by the following properties:

(i). $\mathbf{U}$ contains the class C of single valued continuous functions;

(ii). each $F \in \mathbf{U}_c$ is upper semicontinuous and compact valued; and

(iii). $B^n \in \mathbf{F}(\mathbf{U}_c)$ for all $n \in \{1, 2, ...\}$; here $B^n = \{x \in \mathbf{R}^n : \ \|x\| \leq 1\}$.

We say $F \in PK(X,Y)$ if for any compact subset K of X there is a $G \in \mathbf{U}_c(K,Y)$ with $G(x) \subseteq F(x)$ for each $x \in K$.

An example of a PK map is a Kakutani map. An upper semicontinuous map $\phi : X \to CK(Y)$ is said to be Kakutani (and we write $\phi \in Kak(X,Y)$); here X is a Hausdorff topological space, Y is a Hausdorff topological vector space and $CK(Y)$ denotes the family of nonempty, convex, compact subsets of Y. Another example is an acyclic map which we now describe. Let X and Z be subsets of Hausdorff topological spaces and let $F : X \to K(Z)$ i.e. F has nonempty compact values. Recall a nonempty topological space is said to be a acyclic if all its reduced Čech homology groups over the rationals are trivial. Now we consider maps $F : X \to Ac(Z)$ i.e. $F : X \to K(Z)$ with acyclic values (i.e. F has nonempty acyclic compact values). We say $F \in AC(X,Z)$ (i.e. F is an acyclic map) if $F : X \to Ac(Z)$ is upper semicontinuous.

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For a subset K of a topological space X, we denote by $Cov_X(K)$ the directed set of all coverings of K by open sets of X (usually we write $Cov(K) = Cov_X(K)$). Given two maps $F, G : X \to 2^Y$ and $\alpha \in Cov(Y)$, F and G are said to be α–close if for any $x \in X$ there exists $U_x \in \alpha$, $y \in F(x) \cap U_x$ and $w \in G(x) \cap U_x$.

Let Q be a class of topological spaces. A space Y is an extension space for Q (written $Y \in ES(Q)$) if for any pair (X, K) in Q with $K \subseteq X$ closed, any continuous function $f_0 : K \to Y$ extends to a continuous function $f : X \to Y$. A space Y is an approximate extension space for Q (written $Y \in AES(Q)$) if for any $\alpha \in Cov(Y)$ and any pair (X, K) in Q with $K \subseteq X$ closed, and any continuous function $f_0 : K \to Y$ there exists a continuous function $f : X \to Y$ such that $f|_K$ is α–close to f_0.

Let V be a subset of a Hausdorff topological vector space E. Then we say V is Schauder admissible if for every compact subset K of V and every covering $\alpha \in Cov_V(K)$ there exists a continuous function $\pi_\alpha : K \to V$ such that

(i). π_α and $i : K \to V$ are α–close;

(ii). $\pi_\alpha(K)$ is contained in a subset $C \subseteq V$ with $C \in AES(\text{compact})$.

The following fixed point result can be found in [10].

Theorem 1.1. *Let X be a Schauder admissible subset of a Hausdorff topological vector space and $T \in PK(X, X)$ a compact upper semicontinuous map with closed values. Then there exists a $x \in X$ with $x \in T(x)$.*

Remark 1.2. (i). Other variations of Theorem 1.1 can be found in [11].

(ii). Every convex subset of a Hausdorff locally convex linear topological space is $AES(\text{compact})$ (see [3]).

Next we describe the maps due to Wu [13]. Let X and Y be subsets lying in Hausdorff topological vector spaces and we say $\Phi \in W(X, Y)$ if $\Phi : X \to 2^Y$ and there exists a lower semicontinuous map $\theta : X \to 2^Y$ with $\overline{co}\,(\theta(x)) \subseteq \Phi(x)$ for $x \in X$. Next we recall a selection theorem [1] (see the proof in Theorem 1.1) for Wu maps.

Theorem 1.3. *Let X be a paracompact subset of a Hausdorff topological vector space and Y a metrizable complete subset of a Hausdorff locally convex linear topological space. Suppose $\Phi \in W(X, Y)$ and let $\theta : X \to 2^Y$ be a lower semicontinuous map with $\overline{co}\,(\theta(x)) \subseteq \Phi(x)$ for $x \in X$. Then there exists an upper semicontinuous map $\Psi : X \to CK(Y)$ (collection of nonempty convex compact subsets of Y) with $\Psi(x) \subseteq \overline{co}\,(\theta(x)) \subseteq \Phi(x)$ for $x \in X$.*

Remark 1.4. Let X be paracompact and Y a metrizable subset of a complete Hausdorff locally convex linear topological space E and $\Phi \in W(X, Y)$ with $\theta : X \to 2^Y$ a lower semicontinuous map and $\overline{co}\,(\theta(x)) \subseteq \Phi(x)$ for $x \in X$. Note [9] that $\overline{co}\,\theta : X \to 2^Y$ (since $\overline{co}\,(\theta(x)) \subseteq \Phi(x) \subseteq Y$ for $x \in X$) is lower semicontinuous, so from Michael's selection theorem there exists a continuous (single valued) map $f : X \to Y$ with $f(x) \in \overline{co}\,(\theta(x))$ for $x \in X$, so consequently $f(x) \in \overline{co}\,(\theta(x)) \subseteq \Phi(x)$ for $x \in X$.

We now note two properties for W maps.

(i). Let $F \in W(X, Y)$ and $Z \subseteq X$. Then $F \in W(Z, Y)$.

To see this note there exists a lower semicontinuous map $\theta : X \to 2^Y$ with $\overline{co}\,(\theta(x)) \subseteq F(x)$ for $x \in X$. Let Ω be a closed subset of Y. Then $\{x \in Z : \ \theta(x) \subseteq \Omega\} = Z \cap \{x \in X : \ \theta(x) \subseteq \Omega\}$ which is closed in Z since $\theta : X \to 2^Y$ is lower semicontinuous. Thus $F \in W(Z, Y)$.

(ii). Let $F \in W(X, Y)$ and $F(X) \subseteq W \subseteq Y$. Then $F \in W(X, W)$.

To see this note there exists a lower semicontinuous map $\theta : X \to 2^Y$ with $\overline{co}\,(\theta(x)) \subseteq F(x)$ for $x \in X$. Let Ω be a closed subset of W. Then $\Omega = W \cap C$ for some closed set C of Y. Now since $F(X) \subseteq W$ then $\{x \in X : \theta(x) \subseteq \Omega\} = \{x \in X : \theta(x) \subseteq C\}$ which is closed in X since $\theta : X \to 2^Y$ is lower semicontinuous. Thus $F \in W(X, W)$.

Let Z be a subset of a Hausdorff topological space Y_1 and W a subset of a Hausdorff topological vector space Y_2 and G a multifunction. We say $F \in HLPY(Z, W)$ [8] if W is convex and there exists a map $S : Z \to W$ (i.e. $S : Z \to P(W)$ (collection of subsets of W)) with $co\,(S(x)) \subseteq F(x)$ for $x \in Z$, $S(x) \neq \emptyset$ for each $x \in Z$ and $Z = \bigcup\{\, int\, S^{-1}(w) : \ w \in W\}$; here $S^{-1}(w) = \{z \in Z : \ w \in S(z)\}$ and note $S(x) \neq \emptyset$ for each $x \in Z$ is redundant since if $z \in Z$ then there exists a $w \in W$ with $z \in int\, S^{-1}(w) \subseteq S^{-1}(w)$ so $w \in S(z)$ i.e. $S(z) \neq \emptyset$. For the selection theorem below see [8].

Theorem 1.5. *Let X be a paracompact subset of a Hausdorff topological space, Y a convex subset of a Hausdorff topological vector space and $F \in HLPY(X, Y)$ (let $S : X \to 2^Y$ with $co\,(S(x)) \subseteq F(x)$ for $x \in X$ and $X = \bigcup\{\, int\, S^{-1}(w) : \ w \in Y\}$). Then there exists a continuous (single–valued) map $f : X \to Y$ with $f(x) \in co\, S(x) \subseteq F(x)$ for all $x \in X$.*

Remark 1.6. These maps are related to the DKT maps in the literature and $F \in DKT(Z, W)$ [4] if W is convex and there exists a map $S : Z \to W$ with $co\,(S(x)) \subseteq F(x)$ for $x \in Z$, $S(x) \neq \emptyset$ for each $x \in Z$ and the fibre $S^{-1}(w)$ is open (in Z) for each $w \in W$. Note these maps were motivated from the Fan maps.

We now note two properties for $HLPY$ maps (here Y is convex).

(i). Let $F \in HLPY(X, Y)$ and $Z \subseteq X$. Then $F \in HLPY(Z, Y)$.

To see this note exists a map $S : X \to Y$ with $co\,(S(x)) \subseteq F(x)$ for $x \in X$ and $X = \bigcup\{\, int\, S^{-1}(y) : y \in Y\}$. Let S also denote the restriction of S to Z. Notice

$$Z = Z \cap X = Z \cap \left(\bigcup\{\, int\, S^{-1}(y) : \ y \in Y\} \right) = \bigcup\{Z \cap \, int\, S^{-1}(y) : \ y \in Y\},$$

so $Z \subseteq \bigcup\{\, int_Z\, S^{-1}(y) : \ y \in Y\}$ since for each $y \in Y$ we have that $Z \cap int\, S^{-1}(y)$ is open in Z. On the other hand clearly $\bigcup\{\, int_Z\, S^{-1}(y) : \ y \in Y\} \subseteq Z$. Thus $Z = \bigcup\{\, int_Z\, S^{-1}(y) : \ y \in Y\}$ so $F \in HLPY(Z, Y)$.

(ii). Let $F \in HLPY(X, Y)$ and $F(X) \subseteq W \subseteq Y$ with W convex. Then $F \in HLPY(X, W)$.

To see this note exists a map $S : X \to Y$ with $co\,(S(x)) \subseteq F(x)$ for $x \in X$ and $X = \bigcup\{\, int\, S^{-1}(y) : y \in Y\}$. Now note for any $x \in X$ there exists a $y \in Y$ with $x \in int\, S^{-1}(y) \subseteq S^{-1}(y)$ so $y \in S(x) \subseteq co\,(S(x)) \subseteq F(x) \subseteq W$. Thus $X = \bigcup\{\, int\, S^{-1}(y) : \ y \in W\}$, so $F \in HLPY(X, W)$.

Let X be a subset of a Hausdorff topological space and Y a subset of a Hausdorff topological vector space. We say $T : X \to 2^Y$ has the strong continuous inclusion property (SCIP) [7] at $x \in X$ if there exists an open set $U(x)$ in X containing x and a $F^x : U(x) \to 2^Y$ such that $F^x(w) \subseteq T(w)$ for all $w \in U(x)$ and $co\, F^x : U(x) \to 2^Y$ is compact valued and upper semicontinuous. We write $T \in KLU(X, Y)$ if T has the SCIP at every $x \in X$.

In this paper our map T will usually be a compact map so T has the SCIP is equivalent (see [2], pp 465) to T has the CIP [6].

We now note two properties for KLU maps which we will use in Section 2.

(i). Let $F \in KLU(X, Y)$ and $Z \subseteq X$. Then $F \in KLU(Z, Y)$.

To see this let $x \in Z$. Then $x \in X$ and since $F \in KLU(X, Y)$ then there exists an open set $U(x)$ in X containing x and a $\Phi^x : U(x) \to 2^Y$ such that $\Phi^x(w) \subseteq F(w)$ for all $w \in U(x)$ and $co\, \Phi^x : U(x) \to 2^Y$ is compact valued and upper semicontinuous. Let $V(x) = Z \cap U(x)$. Note $V(x)$ is open in Z and $co\,\Phi^x : V(x) \to 2^Y$ is upper semicontinuous (consider $co\,\Phi^x \circ i$ where $i : V(x) \to U(x)$ is the inclusion) and compact valued. Thus $F \in KLU(Z, Y)$.

(ii). Let $F \in KLU(X, Y)$ and $F(X) \subseteq W \subseteq Y$. Then $F \in KLU(X, W)$.

Let $x \in X$. Then there exists an open set $U(x)$ in X containing x and a $\Phi^x : U(x) \to 2^Y$ such that $\Phi^x(w) \subseteq F(w)$ for all $w \in U(x)$ and $co\,\Phi^x : U(x) \to 2^Y$ is compact valued and upper semicontinuous. Let $\Psi^x : U(x) \to 2^W$ be obtained by restricing the range of Φ^x and let Ω be open in W. Then $\Omega = W \cap V$ for some open set V of Y. Now since $F(X) \subseteq W$ then $\{y \in U(x) : co\,\Psi^x(y) \subseteq \Omega\} = \{y \in U(x) : co\,\Phi^x(y) \subseteq V\}$ which is open in $U(x)$. Thus $co\,\Psi^x : U(x) \to 2^W$ is upper semicontinuous so $F \in KLU(X, W)$.

Next we recall a selection theorem [7].

Theorem 1.7. *Let X be a paracompact subset of a Hausdorff topological space, Y a subset of a Hausdorff topological vector space and $T \in KLU(X, Y)$. Then there exists an upper semicontinuous map $G : X \to CK(Y)$ with $G(w) \subseteq co\,T(w)$ for all $w \in X$.*

Finall we note a well known result from the literature [14].

Theorem 1.8. *Let X and Y be two topological spaces and A an open subset of X. Suppose $F_1 : X \to 2^Y$, $F_2 : A \to 2^Y$ are upper semicontinuous such that $F_2(x) \subset F_1(x)$ for all $x \in A$. Then the map $F : X \to 2^Y$ defined by*

$$F(x) = \begin{cases} F_1(x), & x \notin A, \\ F_2(x), & x \in A \end{cases}$$

is upper semicontinuous.

2. Leray–Schauder Type Alternatives

In this section E will be a Hausdorff topological vector space and U an open subset of E. We will present a variety of Leray–Schauder alternatives for KLU (and W, $HLPY$) maps, the first is when $\overline{U}$ is paracompact (here $\overline{U}$ denotes the boundary of U in E).

Theorem 2.1. *Let E be a Schauder admissible Hausdorff topological vector space, U an open subset of E, $0 \in U$, $\overline{U}$ paracompact, $F \in KLU(\overline{U}, E)$ with $co\,F$ a compact map and $x \notin t\,co\,F(x)$ for $x \in \partial U$ and $t \in (0, 1]$ (here ∂U denotes the boundary of U in E). Then $co\,F$ has a fixed point in U.*

Proof. Now Theorem 1.7 guarantees that there is a compact map $\Phi \in Kak(\overline{U}, E)$ with $\Phi(x) \subseteq co\,F(x)$ for $x \in \overline{U}$ (note Φ is a compact map since $co\,F$ is a compact map). Let

$$\Omega = \left\{x \in \overline{U} : \ x \in t\,\Phi(x) \ \text{ for some } \ t \in [0, 1]\right\}.$$

Now $\Omega \neq \emptyset$ (take $t = 0$ and note $0 \in U$) and Ω is closed since Φ is upper semicontinuous with compact values. In fact Ω is compact. To see this it is enough to note that $D = [0, 1]\,\overline{\Phi(\overline{U})}$ is compact (note $D = p([0, 1] \times \overline{\Phi(\overline{U})})$ where $p(t, y) = t\,y$ for $(t, y) \in [0, 1] \times E$ and $p \in C([0, 1] \times E, E)$). Also $\Omega \cap \partial U = \emptyset$ since $x \notin t\,co\,F(x)$ for $x \in \partial U$ and $t \in [0, 1]$, so $x \notin t\,\Phi(x)$ for $x \in \partial U$ and $t \in [0, 1]$ (note $\Phi(x) \subseteq co\,F(x)$ for $x \in \overline{U}$). Now since Hausdorff topological vector spaces are completely regular then there exists a continuous map $\mu : \overline{U} \to [0, 1]$ with $\mu(\partial U) = 0$ and $\mu(\Omega) = 1$. Let $\theta : E \to 2^E$ be given by

$$\theta(x) = \begin{cases} \mu(x)\,\Phi(x), & x \in \overline{U} \\ \{0\}, & x \in E \backslash \overline{U}. \end{cases}$$

Now Theorem 1.8 guarantees that $\theta \in Kak(E, E)$ is a compact map. To see the compactness let $G(x) = T(x) \times \Psi(x)$, $x \in E$ where $T(x) = \mu(x)$, $\Psi(x) = \Phi(x)$ if $x \in \overline{U}$ and $T(x) = \Psi(x) = \{0\}$ if $x \in E \backslash \overline{U}$. Now note $G(E) \subseteq [0, 1] \times \left(\overline{\Phi(\overline{U})} \cup \{0\}\right)$. Let $p(t, y) = t\,y$ for $(t, y) \in [0, 1] \times E$ and note $p \in C([0,1] \times E, E)$ so $\theta = p\,G$ is a compact map. Now Theorem 1.1 guarantees that there exists a $x \in E$ with $x \in \theta(x)$. If $x \in E \backslash \overline{U}$ then $x \in \{0\}$, a contradiction since $0 \in U$. Thus $x \in U$ so $x \in \mu(x)\,\Phi(x)$ and so $x \in \Omega$. Thus $\mu(x) = 1$ and $x \in \Phi(x) \subseteq co\,F(x)$. □

The analogue of Theorem 2.1 for W and $HLPY$ maps is immediate. For completeness we state the analogue for $HLPY$ maps (the proof is exactly the same as in Theorem 2.1 if we replace $co\,F$ with F and the Kakutani map Φ with a continuous single valued map ϕ using Theorem 1.5 instead of Theorem 1.8).

Theorem 2.2. *Let E be a Schauder admissible Hausdorff topological vector space, U an open subset of E, $0 \in U$, $\overline{U}$ paracompact, $F \in HLPY(\overline{U}, E)$ be a compact map and $x \notin t\,F(x)$ for $x \in \partial U$ and $t \in (0,1]$. Then F has a fixed point in U.*

One could of course obtain an analogue of Theorem 2.1 for general classes of maps described as follows. We say (i). $F \in HYKC(\overline{U}, E)$ if $F : \overline{U} \to 2^E$ and there exists a compact map $\Phi \in Kak(\overline{U}, E)$ with $\Phi(x) \subseteq co\,F(x)$ for $x \in \overline{U}$, and (ii). $F \in HYACC(\overline{U}, E)$ if $F : \overline{U} \to 2^E$ and there exists a compact map $\Phi \in AC(\overline{U}, E)$ with $\Phi(x) \subseteq co\,F(x)$ for $x \in \overline{U}$.

The argument in Theorem 2.1 immediately guarantees the following result.

Theorem 2.3. *Let E be a Schauder admissible Hausdorff topological vector space, U an open subset of E, $0 \in U$, $F \in HYKC(\overline{U}, E)$ or $F \in HYACC(\overline{U}, E)$ and $x \notin t\,co\,F(x)$ for $x \in \partial U$ and $t \in (0,1]$. Then $co\,F$ has a fixed point in U.*

Similarly there is an analogue of Theorem 2.2. We say (i). $F \in HYK(\overline{U}, E)$ if $F : \overline{U} \to 2^E$ and there exists a compact map $\Phi \in Kak(\overline{U}, E)$ with $\Phi(x) \subseteq F(x)$ for $x \in \overline{U}$, and (ii). $F \in HYAC(\overline{U}, E)$ if $F : \overline{U} \to 2^E$ and there exists a compact map $\Phi \in AC(\overline{U}, E)$ with $\Phi(x) \subseteq F(x)$ for $x \in \overline{U}$.

Theorem 2.4. *Let E be a Schauder admissible Hausdorff topological vector space, U an open subset of E, $0 \in U$, $F \in HYK(\overline{U}, E)$ or $F \in HYAC(\overline{U}, E)$ and $x \notin t\,F(x)$ for $x \in \partial U$ and $t \in (0,1]$. Then F has a fixed point in U.*

Our next result removes the paracompactness assumption in Theorem 2.1. We will consider the case when E is a Hausdorff locally convex linear topological space and then we will remark about a more general case after the proof.

Theorem 2.5. *Let E be a Hausdorff locally convex linear topological space, U an open subset of E, $0 \in U$, $F \in KLU(\overline{U}, E)$ with $co\,F$ a compact map and $x \notin t\,co\,F(x)$ for $x \in \partial U$ and $t \in (0,1]$. Then $co\,F$ has a fixed point in U.*

Proof. Since $co\,F$ is a compact map then there exists a compact set K with $co(F(\overline{U})) \subseteq K$. Let $L(K)$ be the linear span of K (i.e. the smallest linear subspace of E that contains K). Now recall [5] that $L(K)$ is Lindelöf so paracompact. Note $U \cap L(K)$ is an open subset of $L(K)$ and the closure of U in $L(K)$ (i.e. $\overline{U \cap L(K)}^{\,L(K)} = \overline{U \cap L(K)} \cap L(K)$) is paracompact. Also since $F \in KLU(\overline{U}, E)$ then from section 1 we have that $F \in KLU(\overline{U \cap L(K)} \cap L(K), L(K))$. Now Theorem 1.7 guarantees that there exists a compact map $\Phi \in Kak(\overline{U \cap L(K)} \cap L(K), L(K))$ with $\Phi(x) \subseteq co\,F(x)$ for $x \in \overline{U \cap L(K)} \cap L(K)$. Let

$$\Omega = \{x \in \overline{U \cap L(K)} \cap L(K) : \; x \in t\,\Phi(x) \;\text{ for some }\; t \in [0,1]\}.$$

Note $\Omega \neq \emptyset$ (take $t = 0$) and Ω is compact. Next we note that $\partial_{L(K)}(U \cap L(K)) \subseteq \partial U$ since

$$\begin{aligned}\partial_{L(K)}(U \cap L(K)) &= (\overline{U \cap L(K)} \cap L(K)) \setminus (U \cap L(K)) \\ &\subseteq (\overline{U} \cap L(K)) \setminus (U \cap L(K)) \\ &= (\overline{U} \cap L(K)) \setminus U \ \cap \ (\overline{U} \cap L(K))) \setminus L(K) \\ &= (\overline{U} \cap L(K)) \setminus U \subseteq \overline{U} \setminus U = \partial U.\end{aligned}$$

Note $\Omega \cap \partial_{L(K)}(U \cap L(K)) \neq \emptyset$ since $x \notin t\, co\, F(x)$ for $x \in \partial U$ and $t \in [0,1]$, so $x \notin t\,\Phi(x)$ for $x \in \partial_{L(K)}(U \cap L(K))$ and $t \in [0,1]$ since $\Phi(x) \subseteq co\, F(x)$ for $x \in \overline{U \cap L(K)} \cap L(K)$. Thus there exists a continuous map $\mu : \overline{U \cap L(K)}^{\,L(K)} \to [0,1]$ with $\mu(\Omega) = 1$ and $\mu(\partial_{L(K)}(U \cap L(K))) = 0$. Let $\theta : L(K) \to 2^{L(K)}$ be given by

$$\theta(x) = \begin{cases} \mu(x)\,\Phi(x), & x \in \overline{U \cap L(K)}^{\,L(K)} \\ \{0\}, & x \in L(K) \setminus (U \cap L(K)). \end{cases}$$

Now $\theta \in Kak(L(K), L(K))$ is a compact map so Theorem 1.1 guarantees that there exists a $x \in L(K)$ with $x \in \theta(x)$. If $x \in L(K) \setminus (U \cap L(K))$ then $x \in \{0\}$, a contradiction since $0 \in U \cap L(K)$. Thus $x \in U \cap L(K)$ and so $x \in \mu(x)\,\Phi(x)$. Consequently $x \in \Omega$ so $\mu(x) = 1$, and so $x \in \Phi(x) \subseteq co\, F(x)$. □

Remark 2.6. (i). Note in Theorem 2.5 we can replace E locally convex with $L(K)$ locally convex.

(ii). Note in Theorem 2.5 we can replace E is a Hausdorff locally convex linear topological space with E is a Hausdorff topological vector space and $L(K)$ is Schauder admissible.

The analogue of Theorem 2.5 for W and $HLPY$ maps is immediate. For completeness we state the analogue for $HLPY$ maps (the proof is exactly the same as in Theorem 2.5 if we replace $co\, F$ with F and the Kakutani map Φ with a continuous single valued map ϕ).

Theorem 2.7. *Let E be a Hausdorff locally convex linear topological space, U an open subset of E, $0 \in U$, $F \in HLPY(\overline{U}, E)$ be a compact map and $x \notin t\,F(x)$ for $x \in \partial U$ and $t \in (0,1]$. Then F has a fixed point in U.*

Remark 2.8. There is an analogue of Remark 2.6 for Theorem 2.7.

Statements and Declarations

The authors declare no conflicts of interest.

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