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RECENT PROGRESS ON HARTOGS-BOCHNER EXTENSION PHENOMENA

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Abstract. We survey recent progress on Hartogs-Bochner extension phenomena for functions of several quaternionic variables and functions of several vector variables.

Keywords. Hartogs-Bochner extension, CR functions, Quaternionic k-regular functions, Monogenic functions of several vector variables, The boundary operator.

1. The Hartogs-Bochner Extension for Holomorphic Functions

For a point $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, write $z_l := x_l + \mathbf{i}y_l$, $l = 1, \ldots, n$. For a domain $\Omega \subset \mathbb{C}^n$, the Cauchy-Riemann operator $\overline{\partial} : C^1(\Omega, \mathbb{C}) \to C(\Omega, \mathbb{C}^n)$ is given by

$$\overline{\partial} f = \begin{pmatrix} \partial_{\bar{z}_1} f \\ \vdots \\ \partial_{\bar{z}_n} f \end{pmatrix}, \tag{1.1}$$

for $f \in C^1(\Omega, \mathbb{C})$, where

$$\partial_{\bar{z}_l} = \frac{1}{2}(\partial_{x_l} + \mathbf{i}\partial_{y_l}), \tag{1.2}$$

$l = 1, \ldots, n$. A function $f : \Omega \to \mathbb{C}$ is called *holomorphic on a domain* Ω if $\overline{\partial} f(z) = 0$ for any $z \in \Omega$.

In 1906, the German mathematician Friedrich Hartogs [15] first proved Hartogs' theorem, by using the Cauchy integral formula for several complex variables.

Theorem 1.1. [15] *For the complex Euclidean space $\mathbb{C}^n$ with $n \geq 2$, if K is a compact subset of $\mathbb{C}^n$ and $\mathbb{C}^n \setminus K$ is connected, then any holomorphic function on $\mathbb{C}^n \setminus K$ can be holomorphically extended to the entire $\mathbb{C}^n$.*

This result reveals the property that the singular set of a holomorphic function of several complex variables cannot be a compact set, and it is an important milestone in the theory of several complex variables.

Without loss of generality, we can assume the defining function ρ of the domain satisfying $\overline{\partial}_{\bar{z}_1} \rho \neq 0$ in a neighborhood U of a point $z \in \partial\Omega$. Then

$$\overline{Z}_l = \partial_{\bar{z}_l} - \partial_{\bar{z}_l}\rho\, (\partial_{\bar{z}_1}\rho)^{-1}\, \partial_{\bar{z}_1}, \quad l = 2, \ldots, n. \tag{1.3}$$

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are complex tangential vector fields, because $\overline{Z}_l\rho = 0$. The boundary version of the Cauchy-Riemann operator is called the *tangential Cauchy-Riemann operator* on the boundary:

$$\overline{\partial}_b : \Gamma\left(U \cap \partial\Omega, \mathbb{C}\right) \to \Gamma\left(U \cap \partial\Omega, \mathbb{C}^{n-1}\right),$$

given by

$$\overline{\partial}_b f = \begin{pmatrix} \overline{Z}_2 f \\ \vdots \\ \overline{Z}_n f \end{pmatrix}, \tag{1.4}$$

for $f \in C^1(U \cap \partial\Omega, \mathbb{C})$. A function f on $\partial\Omega$ is called *CR* if $\overline{\partial}_b f = 0$ in sense of distributions, i.e. $\overline{Z}_l f = 0, l = 2, \dots, n$. The definition of $\overline{\partial}_b f = 0$ is independent of the choice of local coordinates.

In the 1940s, Salomon Bochner [8] made an important generalization of Hartogs' theorem. Bochner proved

Theorem 1.2. [8] *For a smooth bounded domain $\Omega \subset \mathbb{C}^n$ such that $\mathbb{C}^n \setminus \overline{\Omega}$ is connected, if a continuous function f satisfies the tangential Cauchy-Riemann equation $\bar{\partial}_b f = 0$ on the boundary $\partial\Omega$, then f can be extended to a holomorphic function on Ω and is continuous on $\overline{\Omega}$.*

Henkin [16] generalized the Hartogs-Bochner extension theorem to CR functions on arbitrary 1-concave CR manifolds. A manifold M is said to be *generic CR manifold* if it can be locally represented in the form

$$M = \{z \in \Omega : \rho_1(z) = \cdots = \rho_k(z) = 0\}, \tag{1.5}$$

where $\rho = \{\rho_1, \dots, \rho_k\}$ is a collection of real smooth functions in the domain Ω with the property $\bar{\partial}\rho_1 \wedge \cdots \wedge \bar{\partial}\rho_k \neq 0$ on M . Then the *complex tangential subspace* at z is

$$T_z^{\mathbb{C}}(M) = \left\{\zeta \in \mathbb{C}^n : \sum_{j=1}^{n} \frac{\partial \rho_\nu}{\partial z_j}(z)\zeta_j = 0, \quad \nu = 1, \dots, k\right\}.$$

A smooth function f on a CR manifold M in (1.5) is called a *CR function* if it satisfies the tangential Cauchy–Riemann equations $\bar{\partial}_b f = 0$ in sense of distributions, i.e. for any complex vector field $\zeta = (\zeta_1, \dots, \zeta_n) \in T_z^C(M), z \in M$, we have

$$\sum_{j=1}^{n} \bar{\zeta}_j \frac{\partial f}{\partial \bar{z}_j}(z) = 0.$$

It is equivalent to the equation

$$\bar{\partial}\tilde{f} \wedge \bar{\partial}\rho_1 \wedge \cdots \wedge \bar{\partial}\rho_k = 0 \text{ on } M,$$

where $\tilde{f}$ is any smooth extension of the function f to the domain $\Omega \subset M$. In this general case, $\overline{\partial}_b$ can also be written as in (1.3)-(1.4).

By the *Levi form* of the CR-manifold M at the point $z \in M$ in the direction $t = (t^1, \dots, t^k) \in S^{k-1}$, the unit sphere of $\mathbb{R}^k$, we shall mean the scalar quadratic form

$$L_z^t(w) = \sum_{j,l=1}^{n} \frac{\partial^2 \langle t, \rho\rangle}{\partial z_j \partial \overline{z}_l} w_j \overline{w}_l$$

for $w \in T_z^{\mathbb{C}}(M)$, where $\langle t, \rho\rangle = \sum_{j=1}^{k} t_j \rho_j$. The manifold M is said to be q*-concave* (respectively *weakly* q*-concave*) at the point $z \in M$ in the direction $t \in S^{k-1}$ if the form L_z^t has at least q negative (respectively q non-positive) eigenvalues on $T_z^{\mathbb{C}}(M)$. We shall say that the manifold M is q-concave (respectively weakly q -concave) at the point $z \in M$ if it is q-concave (respectively weakly q-concave) in each direction $t \in S^{k-1}$.

Theorem 1.3. [16] *Suppose that $\Omega \subset \mathbb{C}^n$ be a pseudoconvex domain, $\rho = \{\rho_1, \dots, \rho_k\}$ is a collection of real functions of class $C^2(\Omega)$ such that $\bar{\partial}\rho_1 \wedge \cdots \wedge \bar{\partial}\rho_k \neq 0$ in Ω, and each CR manifold M_ε given by $M_\varepsilon = \{z \in \Omega : \rho(z) = \varepsilon\}$, where $\varepsilon = (\varepsilon_1, \dots, \varepsilon_k)$, is 1-concave. Let $\Omega_0 = \{z \in \Omega : \rho_0(z) < 0\}$ be a relatively compact subdomain of Ω and let $\varepsilon^0 = \{\varepsilon_1^0, \dots, \varepsilon_k^0\}$ be a collection such that the manifold $M_0 = M_{\varepsilon^0} \cap \Omega_0$ has a connected complement in M_{ε^0} and $\bar{\partial}\rho_1 \wedge \cdots \wedge \bar{\partial}\rho_k \wedge \bar{\partial}\rho_0 \neq 0$ almost everywhere on ∂M_0. Then, any continuous CR function f on ∂M_0 can be extended to a continuous CR function on M_0.*

Note that in general, the manifold ∂M_0 is only an almost CR manifold. We say that a function f on ∂M_0 is a *CR function* if

$$\int_{\partial M_0} f \wedge \bar{\partial}\varphi = 0$$

for each smooth $(n, n-k-2)$-form φ. This definition is satisfied, for example, by the restriction to ∂M_0 of any CR-function defined in a neighbourhood of ∂M_0 on M_{ε^0} .

The Hartogs-Bochner extension can also be generalized to pluriharmonic functions. A function f on a domain Ω in $\mathbb{C}^n$ is called *pluriharmonic* if it is harmonic on every complex line. It is equivalent to require

$$\partial\bar{\partial} f = 0.$$

In 1883, Poincaré [24] set a problem to find necessary and sufficient conditions under which a (real) function u given on the boundary of a bounded domain $\Omega \in \mathbb{C}^n$ is the boundary value on $\partial\Omega$ of a pluriharmonic function in Ω. To solve the Poincaré problem is just to establish the Hartogs-Bochner extension theorem. The history and some known results about this problem can be found in [14]. This problem was solved by Bedford [6] on the unit ball in $\mathbb{C}^n$ and then by Bedford and Federbush [7] for general domain with C^3 boundary. Later, Andreotti and Nacinovich [4] constructed the boundary complex of $\partial\bar{\partial}$-complex on complex manifolds, in particular, the boundary operator $(\partial\bar{\partial})_b$ of $\partial\bar{\partial}$ operator. A function (u_0, u_1) on $\partial\Omega$ is called *tangential pluriharmonic* if $(\partial\bar{\partial})_b\,(u_0 + \rho u_1) = 0$, where ρ is the defining function of the domain.

Theorem 1.4. [4, Theorem 3] *Let $H^0\,(X^-, \mathscr{H})$ denote the space of C^∞ functions on $X^- = \{z \in X : \rho(z) \leq 0\}$ which are pluriharmonic in X^- for some smooth complex manifold X and let $H^0\,(X^-, \mathscr{H}_S)$ denote the space of couples of functions $u_0 + \rho u_1$ satisfying the equations*

$$(\partial\bar{\partial})_b\,(u_0 + \rho u_1) = 0 \qquad \text{on} \quad \partial\Omega.$$

If the manifold X is $(n-2)$-complete $(n > 2)$ and $H_c^2(X, \mathbb{C}) = 0$, then the natural map $r : H^0\,(X^-, \mathscr{H}) \to H^0\,(X^-, \mathscr{H}_S)$ is an isomorphism.

In Section 2, we will discuss Hartogs-Bochner extension phenomena for quaternionic k-regular functions, while the Hartogs-Bochner extension phenomena for monogenic functions of several vector variables will be discussed in Section 3.

2. The Hartogs-Bochner Extension for Quaternionic k-Regular Functions

2.1. The k-Cauchy-Fueter complex.

For a point $q = (q_0, \dots, q_n) \in \mathbb{H}^{n+1}$, write

$$\mathbf{q}_l := x_{4l} + x_{4l+1}\mathbf{i} + x_{4l+2}\mathbf{j} + x_{4l+3}\mathbf{k}, \qquad l = 0, \dots, n. \tag{2.1}$$

For a domain $\Omega \subset \mathbb{H}^{n+1}$, the quaternionic generalization of the Cauchy-Riemann operator is the *Cauchy-Fueter operator* $\mathcal{D}: C^1(\Omega, \mathbb{H}) \to C(\Omega, \mathbb{H}^{n+1})$ given by

$$\mathcal{D}f = \begin{pmatrix} \overline{\partial}_{\mathbf{q}_0} f \\ \vdots \\ \overline{\partial}_{\mathbf{q}_n} f \end{pmatrix}, \tag{2.2}$$

for $f \in C^1(\Omega, \mathbb{H})$, where

$$\overline{\partial}_{\mathbf{q}_l} = \partial_{x_{4l}} + \mathbf{i}\partial_{x_{4l+1}} + \mathbf{j}\partial_{x_{4l+2}} + \mathbf{k}\partial_{x_{4l+3}}, \qquad l = 0, 1, \ldots, n \tag{2.3}$$

A function $f : \Omega \to \mathbb{H}$ is called *(left) regular in a domain* Ω if $\mathcal{D}f(\mathbf{q}) = 0$ for any $\mathbf{q} \in \Omega$.

For an $\mathbb{H}$-valued function $f = f_0 + f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k}$ on $\mathbb{H}^{n+1}$, set $\phi_{0'} = f_2 - \mathbf{i}f_3$ and $\phi_{1'} = -f_0 - \mathbf{i}f_1$. Then f is regular if and only if the $\mathbb{C}^2$-valued function $(\phi_{0'}, \phi_{1'})$ satisfies

$$\nabla_{\dot{A}}^{A'} \phi_{A'} = 0, \qquad \dot{A} = 0, \ldots, 2n+1, \tag{2.4}$$

where complex vector fields

$$(\nabla_{\dot{A}A'}) := \begin{pmatrix} \partial_1 + \mathbf{i}\partial_2 & -\partial_3 - \mathbf{i}\partial_4 \\ \partial_3 - \mathbf{i}\partial_4 & \partial_1 - \mathbf{i}\partial_2 \\ \vdots & \vdots \\ \partial_{4l+1} + \mathbf{i}\partial_{4l+2} & -\partial_{4l+3} - \mathbf{i}\partial_{4l+4} \\ \partial_{4l+3} - \mathbf{i}\partial_{4l+4} & \partial_{4l+1} - \mathbf{i}\partial_{4l+2} \\ \vdots & \vdots \end{pmatrix}, \tag{2.5}$$

$\partial_j = \frac{\partial}{\partial x_j}$, and use

$$(\varepsilon_{A'B'}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \text{and} \qquad \left(\varepsilon^{A'B'}\right) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{2.6}$$

to raise or lower primed indices, where $(\varepsilon^{A'B'})$ is the inverse of $(\varepsilon_{A'B'})$. For example,

$$f_{\cdots}{}^{A'}{}_{\cdots} = f_{\cdots B' \cdots}\varepsilon^{B'A'}, \qquad f_{\cdots}{}^{A'}{}_{\cdots}\varepsilon_{A'C'} = f_{\cdots C' \cdots}. \tag{2.7}$$

Here and in the sequel, we adopt the following index notations: $\dot{A}, \dot{B}, \dot{C}, \cdots \in \{0, 1, \cdots, 2n+1\}$, $A, B, C, \cdots \in \{0, 1, \cdots, 2n-1\}$, $A', B', C', D', \cdots \in \{0', 1'\}$, and use the Einstein convention of taking summation for repeated indices.

These complex vector fields [34] are motivated by the embedding τ of quaternionic algebra $\mathbb{H}$ into $\mathfrak{gl}(2, \mathbb{C})$:

$$\tau(x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k}) = \begin{pmatrix} x_1 + \mathbf{i}x_2 & -x_3 - \mathbf{i}x_4 \\ x_3 - \mathbf{i}x_4 & x_1 - \mathbf{i}x_2 \end{pmatrix}. \tag{2.8}$$

The quaternionic structure of $\mathbb{H}^{n+1}$ is encoded in these vector fields. In the sequel, we identify $\mathbb{H}^{n+1}$ with the underlying space $\mathbb{R}^{4(n+1)}$.

The Cauchy-Fueter operator (2.4) can be generalized to the k*-Cauchy-Fueter operator* $\mathcal{D}_0 : \Gamma(\Omega, \odot^k\mathbb{C}^2) \to \Gamma(\Omega, \odot^{k-1}\mathbb{C}^2 \otimes \mathbb{C}^{2n})$ given by

$$(\mathcal{D}_0 f)_{AA'_2\cdots A'_k} = \nabla_A^{A'_1} f_{A'_1\cdots A'_k}, \tag{2.9}$$

for $f = (f_{A_1'\cdots A_k'}) \in \Gamma(\Omega, \odot^k\mathbb{C}^2)$. Here the *symmetric power* $\odot^p\mathbb{C}^2$ is a subspace of $\otimes^p\mathbb{C}^2$, and an element of $\odot^p\mathbb{C}^2$ is given by a 2^p-tuple $(f_{A_1'\ldots A_p'}) \in \otimes^p\mathbb{C}^2$ with $A_1', \ldots, A_p' = 0', 1'$ such that $f_{A_1'\ldots A_p'}$ is invariant under permutations of subscripts, i.e.

$$f_{A_1'\ldots A_p'} = f_{A'_{\sigma(1)}\ldots A'_{\sigma(p)}}$$

for any $\sigma \in S_p$, the group of permutations of p letters. As the quaternionic counterpart of the Cauchy-Riemann operator, the k-Cauchy-Fueter operators are a family of operators acting on $\odot^k\mathbb{C}^2$-valued functions. This is because the group $\mathrm{SU}(2)$ of unit quaternionic numbers has a family of irreducible representations $\odot^k\mathbb{C}^2$, while the group of unit complex numbers has only one irreducible representation space $\mathbb{C}$. On a domain $\Omega \subset \mathbb{H}^{n+1}$, a function $f \in \Gamma(\Omega, \odot^k\mathbb{C}^2)$ is called *k-regular* if

$$\mathcal{D}_0 f(\mathbf{q}) = 0 \qquad \text{for any } \mathbf{q} \in \Omega.$$

The space of all k-regular functions on a domain Ω is denoted by $\mathscr{O}_k(\Omega)$.

As the Dolbeault complex plays the central role in the theory of several complex variables, the corresponding complexes, the k-Cauchy-Fueter complexes, are already known explicitly (cf. [5, 34, 35] and [1, 9, 10, 11] for $k = 1$, and references therein) and used to show several interesting properties of k-regular functions. Denote

$$\mathcal{V}^{\sigma,\tau} := \odot^\sigma\, \mathbb{C}^2 \otimes \wedge^\tau\mathbb{C}^{2(n+1)},$$

where $\odot^\sigma\mathbb{C}^2$ is the σ-th symmetric product of $\mathbb{C}^2$, and $\wedge^\tau\mathbb{C}^{2(n+1)}$ is the τ-th exterior product of $\mathbb{C}^{2(n+1)}$. For fixed $k = 0, 1, \cdots$, the *k-Cauchy-Fueter complex* on a domain Ω in $\mathbb{H}^{n+1}$ is given by

$$0 \to C^\infty(\Omega, \mathcal{V}_0) \xrightarrow{\mathcal{D}_0} \cdots \to C^\infty(\Omega, \mathcal{V}_j) \xrightarrow{\mathcal{D}_j} C^\infty(\Omega, \mathcal{V}_{j+1}) \to \cdots \xrightarrow{\mathcal{D}_{2n}} C^\infty(\Omega, \mathcal{V}_{2n+1}) \to 0, \tag{2.10}$$

where $\mathcal{V}_j := \mathcal{V}^{\sigma_j,\tau_j}$ with

$$\begin{aligned} \sigma_j &:= \begin{cases} k-j, & \text{if } \quad j = 0, \ldots, k, \\ j-k-1, & \text{if } \quad j = k+1, \ldots, 2n+1, \end{cases} \\ \tau_j &:= \begin{cases} j, & \text{if } \quad j = 0, \ldots, k, \\ j+1, & \text{if } \quad j = k+1, \ldots, 2n+1. \end{cases} \end{aligned} \tag{2.11}$$

It is convenient to write operators in the k-Cauchy-Fueter complex (2.10) in terms of a pair of anti-commutative differential operators $d_{A'} : C^\infty(\Omega, \wedge^\tau\mathbb{C}^{2(n+1)}) \to C^\infty(\Omega, \wedge^{\tau+1}\mathbb{C}^{2(n+1)})$, which was introduced by Wan-Wang in [33]. Fix a basis $\{\omega^0, \ldots, \omega^{2n+1}\}$ of $\mathbb{C}^{2(n+1)}$, they are

$$d_{A'}F := \nabla_{\dot{A}A'} f_{\dot{\mathbf{A}}}\, \omega^{\dot{A}} \wedge \omega^{\dot{\mathbf{A}}}, \tag{2.12}$$

$A' = 0', 1'$, for $F = f_{\dot{\mathbf{A}}}\, \omega^{\dot{\mathbf{A}}} \in C^\infty(\Omega, \wedge^\tau\mathbb{C}^{2(n+1)})$, where $\omega^{\dot{\mathbf{A}}} := \omega^{\dot{A}_1} \wedge \ldots \wedge \omega^{\dot{A}_\tau}$ for the multi-index $\dot{\mathbf{A}} = \dot{A}_1 \cdots \dot{A}_\tau$. As ∂ and $\overline{\partial}$ in complex analysis, $d_{0'}$ and $d_{1'}$ are a pair of anti-commutative operators behaving like exterior differentials:

$$d_{0'}^2 = d_{1'}^2 = 0, \qquad d_{0'}d_{1'} = -d_{1'}d_{0'}. \tag{2.13}$$

They give a very useful expression of the quaternionic Monge-Ampère operator, and allow us to prove many important results in quaternionic pluripotential theory (cf. [33, 20] and references therein).

By raising primed indices, we get operator $d^{A'}$. It is convenient to identify $\odot^\sigma\mathbb{C}^2$ with the space $\mathcal{P}_\sigma(\mathbb{C}^2)$ of homogeneous polynomials of degree σ on $\mathbb{C}^2$. A $\mathcal{V}_j$-valued function f can be viewed as a function in real variables $x \in \mathbb{R}^{4(n+1)}$, $s^{A'} \in \mathbb{C}^2$ and Grassmannian variables ω^A:

$$f = f_{\mathbf{A}'\dot{\mathbf{A}}}(x) s^{\mathbf{A}'} \omega^{\dot{\mathbf{A}}},$$

where $s^{\mathbf{A}'} := s^{A'_1} \cdot \ldots \cdot s^{A'_\sigma}$ for the multi-index $\mathbf{A}' = A'_1 \cdots A'_\sigma$. Let $\partial_{A'} = \frac{\partial}{\partial s^{A'}}$. Under this identification, differential operators in the k-Cauchy-Fueter complex (2.10) have a very simple form:

$$\mathcal{D}_j = \begin{cases} \partial_{A'} d^{A'}, & \text{if} \quad j = 0, \ldots, k-1, \\ d^{0'} d^{1'}, & \text{if} \quad j = k, \\ s_{A'} d^{A'}, & \text{if} \quad j = k+1, \ldots, 2n. \end{cases} \tag{2.14}$$

Here for $j \geq k+1$, we use $s_{A'} \in \mathbb{C}^2$ instead of $s^{A'} \in \mathbb{C}^2$. Under this realization, we can easily see (2.10) is a complex, i.e. $\mathcal{D}_{j+1} \circ \mathcal{D}_j = 0$, by

$$\partial_{A'} \partial_{B'} d^{A'} d^{B'} = 0, \qquad s_{A'} s_{B'} d^{A'} d^{B'} = 0, \tag{2.15}$$

since $d^{A'} d^{B'}$ is skew-symmetric in A', B' by (2.13), while $\partial_{A'} \partial_{B'}$ and $s_{A'} s_{B'}$ are both symmetric in A', B', and also

$$d^{0'} d^{1'} \partial_{A'} d^{A'} = 0, \qquad s_{A'} d^{A'} d^{0'} d^{1'} = 0. \tag{2.16}$$

For a differential complex, a fundamental problem is to characterize domains on which the complex is exact, i.e. the Poincaré Lemma holds or its cohomology groups vanish. The Neumann problem associated to the k-Cauchy-Fueter complex on k-pseudoconvex domains was investigated in [36]. It is expected that the nonhomogeneous k-Cauchy-Fueter equation is solvable if and only if the domain is k-pseudoconvex.

2.2. **The boundary operator.** In the complex case, when the Dolbeault complex is restricted to a CR submanifold, one obtain the tangential Cauchy-Riemann complex, whose first operator is the tangential Cauchy-Riemann operator. It is a powerful tool to investigate holomorphic functions and the Dolbeault complex. Consider a domain

$$\Omega = \{\mathbf{q} = (\mathbf{q}', q_{n+1}) \in \mathbb{H}^n \times \mathbb{H}; \varrho(\mathbf{q}) > 0\}. \tag{2.17}$$

By rotation if necessary, we can assume the defining function ϱ has the following form near the origin:

$$\varrho(\mathbf{q}) = \operatorname{Re} q_{n+1} - \phi(\mathbf{q}', \operatorname{Im} q_{n+1}), \tag{2.18}$$

where $\phi(\mathbf{q}', \operatorname{Im} q_{n+1}) = O(|\mathbf{q}', \operatorname{Im} q_{n+1}|^2)$. We have *quaternionic tangential vector fields* on the boundary:

$$\overline{Q}_l = \overline{\partial}_{q_l} - \overline{\partial}_{q_l} \varrho \cdot (\overline{\partial}_{q_{n+1}} \varrho)^{-1} \cdot \overline{\partial}_{q_{n+1}} \tag{2.19}$$

$l = 1, \ldots, n$, since $\overline{Q}_l \varrho = 0$ by definition. The definition of the map τ in (2.8) can extended to a mapping from quaternionic $l \times m$-matrices to complex $2l \times 2m$-matrices. For a quaternionic $(p \times m)$-matrix $\mathbf{a}$ and a quaternionic $(m \times l)$-matrix $\mathbf{b}$, we have $\tau(\mathbf{ab}) = \tau(\mathbf{a})\tau(\mathbf{b})$ [33, Lemma 2.1]. Applying it to (2.19), we get complex vector fields tangential to the boundary $\partial\Omega$:

$$Z_{AC'} = \nabla_{AC'} - n_{AB'} \mathbf{N}_{B'C'}, \tag{2.20}$$

for $A = 0, \ldots, 2n-1$, $A' = 0', 1'$, with

$$n_{AB'} = \nabla_{AD'} \varrho \cdot (\mathbf{N}\varrho)^{-1}_{D'B'}. \tag{2.21}$$

Namely, $Z_{AA'} \varrho = 0$. Here we denote $\mathbf{N}_{B'C'} := \nabla_{(2n+o(B'))C'}$, where $o(0') = 0, o(1') = 1$. We have

$$(\mathbf{N}_{B'C'}) = \begin{pmatrix} \nabla_{(2n)0'} & \nabla_{(2n)1'} \\ \nabla_{(2n+1)0'} & \nabla_{(2n+1)1'} \end{pmatrix},$$

and $\mathbf{N}\varrho$ is the 2×2-matrix $\tau\left(\overline{\partial}_{q_{n+1}} \varrho\right)$.

Then by raising primed indices (2.7), we get

$$Z_A^{A'} = \nabla_A^{A'} - n_{AB'} \mathbf{N}_{B'}^{A'}. \tag{2.22}$$

In the quaternionic case, we have extra 3 vectors also tangential to the boundary,

$$\mathbf{T}_{B'}^{A'} := R_{B'C'}\mathbf{N}_{C'}^{A'} - \delta_{B'}^{A'}\partial_{4n+1},$$

with $R_{B'C'} := \varepsilon_{B'D'}(\mathbf{N}\varrho)^{-1}_{D'C'}$, which belong to the quaternionic line containing the normal vector.

As the boundary version of operators $d^{A'}$, we introduce operators $\mathfrak{d}^{A'} : C^\infty(b\Omega, \wedge^\tau\mathbb{C}^{2n}) \to C^\infty(b\Omega, \wedge^{\tau+1}\mathbb{C}^{2n})$,

$$\mathfrak{d}^{A'} f = Z_A^{A'} f_{\mathbf{A}} \omega^A \wedge \omega^{\mathbf{A}}, \tag{2.23}$$

for $f = f_{\mathbf{A}}\omega^{\mathbf{A}}$. Denote

$$\mathscr{V}^{\sigma,\tau} := \odot^\sigma \mathbb{C}^2 \otimes \wedge^\tau \mathbb{C}^{2n}.$$

Theorem 2.1. [37] *The boundary complex of the k-Cauchy-Fueter complex is the differential complex*

$$0 \to C^\infty(b\Omega, \mathscr{V}_0) \xrightarrow{\mathscr{D}_0} \cdots \longrightarrow C^\infty(b\Omega, \mathscr{V}_j) \xrightarrow{\mathscr{D}_j} C^\infty(b\Omega, \mathscr{V}_{j+1}) \to \cdots \xrightarrow{\mathscr{D}_{2n-2}} C^\infty(b\Omega, \mathscr{V}_{2n-1}) \to 0, \tag{2.24}$$

where $\mathscr{V}_j := \mathscr{V}_j^{(1)} \oplus \mathscr{V}_j^{(2)}$ with

$$\begin{array}{llll} \mathscr{V}_j^{(1)} := \mathscr{V}^{\sigma_j,\tau_j}, & \mathscr{V}_j^{(2)} := \mathscr{V}^{\sigma_{j+1},\tau_j-1}, & \text{if} \quad j \neq k, \\ \mathscr{V}_k^{(1)} := \mathscr{V}^{0,k}, & \mathscr{V}_k^{(2)} := \mathscr{V}^{0,k}, & \end{array} \tag{2.25}$$

where σ_j, τ_j are given by (2.11).

Operators in the boundary complex (2.24) were explicitly given by Wang [37, Theorem 4.1]. In particular, for $k = 1, 2, \ldots$ and $\mathbb{F} \in C^\infty(b\Omega, \mathscr{V}_0)$,

$$\mathscr{D}_0\mathbb{F} = \left(\partial_{A'}\mathfrak{d}^{A'}\mathbb{F}_1, -\partial_{A'}\partial_{B'}\mathbf{T}^{A'B'}\mathbb{F}_1\right) \in C^\infty(b\Omega, \mathscr{V}_1^{(1)} \oplus \mathscr{V}_1^{(2)}); \tag{2.26}$$

for $k = 0$ and $\mathbb{F} = (\mathbb{F}_1, \mathbb{F}_2) \in C^\infty(b\Omega, \mathscr{V}_0^{(1)} \oplus \mathscr{V}_0^{(2)})$, $\mathscr{D}_0\mathbb{F} \in C^\infty(bD, \mathscr{V}_1^{(1)} \oplus \mathscr{V}_1^{(2)})$ is given by

$$\mathscr{D}_0\mathbb{F} = \left(\mathfrak{d}^{0'}\mathfrak{d}^{1'}\mathbb{F}_1 - \mathcal{E}_0 \wedge \left(\mathbf{T}^{1'0'}\mathbb{F}_1 + \mathbb{F}_2\right), -\mathfrak{d}_{A'}\mathbb{F}_2 - \mathcal{E}_{A'} \wedge \left(\mathbf{T}^{1'0'}\mathbb{F}_1 + \mathbb{F}_2\right) + 2\mathbf{T}_{A'}^{[0'}\mathfrak{d}^{1']}\mathbb{F}_1\right),$$

where $\mathcal{E}_0 := -\mathfrak{d}^{0'}\mathfrak{d}^{1'}\varrho$.

On a domain D in $b\Omega$, a function $F \in C(D, \odot^k\mathbb{C}^2)$ is called k-*CF* if $\mathscr{D}_0 F = 0$ in the sense of distributions, $k = 0, 1, 2, \ldots$. Wang [37] gave the following Hartogs-Bochner extension for k-regular functions.

Theorem 2.2. [31, Theorem 1.2] *Let Ω be a bounded domain in $\mathbb{H}^{n+1}$ ($n \geq 1$) with smooth boundary such that $\mathbb{H}^{n+1} \setminus \overline{\Omega}$ is connected. If f is a smooth k-CF function on $b\Omega$, then there exists $\widetilde{f} \in \mathscr{O}_k(\Omega)$ smooth up to the boundary such that $\widetilde{f} = f$ on $b\Omega$.*

This theorem for $k = 1$ was proved by Maggesi-Pertici-Tomassini in [21]. They introduced the notion of admissible functions on the boundary, which coincides with the notion of 1-CF functions here, although it is written in a different form.

For a k-CF function f on $b\Omega$, the boundary complex allows us to construct a representative $\widehat{f} \in C^\infty(\Omega, \odot^k\mathbb{C}^2)$ such that $\widehat{f}|_{b\Omega} = f$ and $\mathcal{D}_0\widehat{f}$ is flat on $b\Omega$.

Proposition 2.3. *For fixed k, if $f \in C^\infty(b\Omega, \odot^k\mathbb{C}^2)$ is k-CF, then there exists a representative $\widehat{f} \in C^\infty(\overline{\Omega}, \odot^k\mathbb{C}^2)$ such that $\widehat{f}|_{b\Omega} = f$ and $\mathcal{D}_0\widehat{f}$ is flat on $b\Omega$.*

Such extension for CR functions was constructed by Andreotti-Hill [2], while extension for functions satisfying $(\partial\overline{\partial})_b$-equation was constructed by Andreotti-Nacinovich in [4].

To show the Hartogs-Bochner extension, we need to solve the nonhomogeneous k-Cauchy-Fueter equation

$$\mathcal{D}_0 u = f, \tag{2.27}$$

for f satisfying the compatibility condition

$$\mathcal{D}_1 f = 0. \tag{2.28}$$

Theorem 2.4. [34, Theorem 5.3] *For $f \in C_c(\mathbb{R}^{4n+4}, \mathcal{V}_1)$ such that (2.28) holds in the sense of distributions, then there exists a function $u \in C_c(\mathbb{R}^{4n+4}, \mathcal{V}_0) \cap W^{1,2}(\mathbb{R}^{4n+4}, \mathcal{V}_0)$ ($u \in C_c(\mathbb{R}^{4n+4}, \mathbb{C}) \cap W^{2,2}(\mathbb{R}^{4n+4}, \mathbb{C})$ if $k = 0$) satisfying (2.27) and vanishing on the unbounded connected component of $\mathbb{R}^{4n+4} \setminus \mathrm{supp} f$.*

Then by using the compactly supported solution given by Theorem 2.4, we can construct the k-regular function $\widetilde{f}$ in Theorem 2.2.

To prove Theorem 2.4 (e.g. for $k = 1$), consider the associated Hodge Laplacians of fourth order as

$$\begin{aligned} \Box_0 &:= \left(\mathcal{D}_0^* \mathcal{D}_0\right)^2, \\ \Box_1 &:= \left(\mathcal{D}_0 \mathcal{D}_0^*\right)^2 + \mathcal{D}_1^* \mathcal{D}_1, \\ \Box_2 &:= \mathcal{D}_1 \mathcal{D}_1^* + \left(\mathcal{D}_2{}^* \mathcal{D}_2\right)^2. \end{aligned} \tag{2.29}$$

Let $\mathbf{G}_1$ be the inverse operator of $\Box_1$ on $L^2\left(\mathbb{R}^{kn}, \mathcal{V}_1\right)$. For $f \in C_c^\infty\left(\mathbb{R}^{kn}, \mathcal{V}_1\right)$, noting that $\mathcal{D}_1 \Box_1 f = \Box_2 \mathcal{D}_1 f$ automatically, we have

$$\mathbf{G}_2 \mathcal{D}_1 = \mathcal{D}_1 \mathbf{G}_1, \tag{2.30}$$

and so

$$\mathcal{D}_1 \mathbf{G}_1 f = \mathbf{G}_2 \mathcal{D}_1 f = 0, \tag{2.31}$$

if (2.28) is satisfied. Thus,

$$\mathcal{D}_0 u = \mathcal{D}_0 \mathcal{D}_0^* \mathcal{D}_0 \mathcal{D}_0^* \mathbf{G}_1 f = \left(\left(\mathcal{D}_0 \mathcal{D}_0^*\right)^2 + \mathcal{D}_1^* \mathcal{D}_1\right) \mathbf{G}_1 f = \Box_1 \mathbf{G}_1 f = f, \tag{2.32}$$

i.e. $u = \mathcal{D}_0^* \mathcal{D}_0 \mathcal{D}_0^* \mathbf{G}_1 f$ satisfies (2.27). In this proof of Theorem 2.4, first three operators $\mathcal{D}_0$, $\mathcal{D}_1$ and $\mathcal{D}_2$ in the k-Cauchy-Fueter complex (2.10) are used.

3. The Hartogs-Bochner Extension for Monogenic Functions of Several Vector Variables

3.1. **The Dirac complex.** The notion of a holomorphic function in several complex variables was generalized further to the notion of a monogenic function of several vector variables, which is annihilated by several Dirac operators on k copies of the Euclidean space $\mathbb{R}^n$.

Write $\mathbf{x}_A = (x_{A1}, \ldots, x_{An})$ as the vector variable in the A-th copy of $\mathbb{R}^n$, and $\partial_{Aj} := \frac{\partial}{\partial x_{Aj}}$, where $A = 0, \ldots, k-1$, $j = 1, \ldots, n$. The Dirac operator in the A-th copy is $\nabla_A : C^1\left(\mathbb{R}^n, \mathbb{S}^\pm\right) \to C\left(\mathbb{R}^n, \mathbb{S}^\mp\right)$ with

$$\nabla_A := \sum_{j=1}^n \gamma_j \partial_{Aj}, \qquad A = 0, \ldots, k-1,$$

where $\mathbb{S}^{\pm}$ denote the two spinor modules for n even. The same symbols are used for n odd with the convention that $\mathbb{S}^{+}$ and $\mathbb{S}^{-}$ are isomorphic, and $\gamma_j : \mathbb{S}^{\pm} \to \mathbb{S}^{\mp}$ are Dirac matrices. On a domain Ω in $\mathbb{R}^{kn}$, the *several Dirac operators* $\mathcal{D}_0 : C^1(\Omega, \mathbb{S}^+) \to C(\Omega, \mathbb{C}^k \otimes \mathbb{S}^-)$ are given by

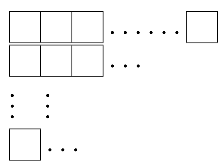
$$\mathcal{D}_0 f = \begin{pmatrix} \nabla_0 f \\ \vdots \\ \nabla_{k-1} f \end{pmatrix}, \quad \text{for } f \in C^{\infty}(\Omega, \mathbb{S}^+).$$

f is called a *monogenic function* on Ω if $\mathcal{D}_0 f(\mathbf{x}) = 0$ for any $\mathbf{x} \in \Omega$. The space of all monogenic functions on Ω is denoted by $\mathcal{O}(\Omega)$. Since 90's [32], one have been interested in generalizing the theory of holomorphic functions to spinor-valued monogenic function on $\mathbb{R}^{kn}$. For this purpose, a fundamental tool again is to solve the non-homogeneous several Dirac equations

$$\mathcal{D}_0 f = g, \tag{3.1}$$

for $g \in C^{\infty}(\Omega, \mathbb{C}^k \otimes \mathbb{S}^-)$. Since it is overdetermined, (3.1) can only be solved under a compatibility condition. Thus the solution of this problem can be obtained if one can provide a description of the so-called the Dirac complex.

Let V_λ be an irreducible $\mathrm{GL}(k)$-module labeled by λ, where $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a partition of n, i.e. $\lambda_1 \geq \lambda_2 \geq \ldots \lambda_k \geq 0$, and $\lambda_1 + \cdots + \lambda_k = n$. The *Young diagram* associated to a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ is



with λ_i boxes in the i-th row. For 2 vector variables with the dimension n even, the Dirac complex was found by Krump-Souček [17], by using parabolic geometry associated to $\mathrm{SO}(2, 2n+2)$. For $k = 2$, the Dirac complex in terms of the Young diagrams is

$$0 \to \bullet \otimes \mathbb{S}^+ \longrightarrow \square \otimes \mathbb{S}^- \Rightarrow \begin{array}{|c|c|}\hline & \\ \hline \\ \cline{1-1}\end{array} \otimes \mathbb{S}^- \longrightarrow \bullet \otimes \mathbb{S}^+ \to 0, \tag{3.2}$$

which was given by Damiano-Sabadini-Souček [13], where $\bullet = \mathbb{C}$.

In [30], Shi-Wang-Wu wrote down operators explicitly in the Dirac complex of two variables

$$0 \to C^{\infty}(\Omega, S^+) \xrightarrow{\mathcal{D}_0} C^{\infty}(\Omega, \mathbb{C}^2 \otimes \mathbb{S}^-) \xrightarrow{\mathcal{D}_1} C^{\infty}(\Omega, \mathbb{C}^2 \otimes \mathbb{S}^-) \xrightarrow{\mathcal{D}_2} C^{\infty}(\Omega, \mathbb{S}^+) \to 0, \tag{3.3}$$

without the restriction of even dimensions, where Ω is a domain in $\mathbb{R}^{2n}$. The explicit forms of operators are

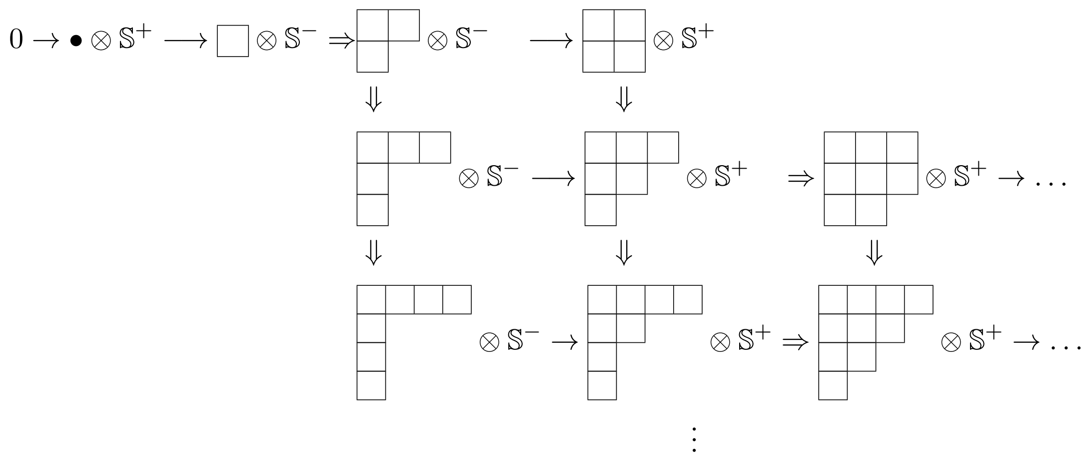
$$\begin{aligned} (\mathcal{D}_0 f)_\alpha :=& \nabla_\alpha f, && \text{for } f \in C^{\infty}(\Omega, \mathbb{S}^+), \\ (\mathcal{D}_1 g)_\alpha :=& \nabla_0 \nabla_\alpha g_1 - \nabla_1 \nabla_\alpha g_0, && \text{for } g \in C^{\infty}(\Omega, \mathbb{C}^2 \otimes \mathbb{S}^-), \\ \mathcal{D}_2 h :=& 2\nabla_{[0} h_{1]} := \nabla_0 h_1 - \nabla_1 h_0, && \text{for } h \in C^{\infty}(\Omega, C^2 \otimes \mathbb{S}^-), \end{aligned} \tag{3.4}$$

where $\alpha = 0, 1$. They established Theorem 2.4 in this case to obtain compactly supported solution if g is compactly supported and satisfies the compatibility condition

$$\mathcal{D}_1 g = 0. \tag{3.5}$$

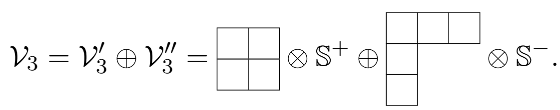
This solution implies the Hartogs' phenomenon for monogenic functions of 2 vector variables.

For $k \geq 3$, the Dirac complex in terms of the Young diagrams is



(3.6)

(cf. [13, 27, 28, 29] in the stable case, i.e. $k \leq \frac{n}{2}$), where $\rightarrow$ and $\Rightarrow$ are differential operators of first order and second order, respectively, and



For 3 vector variables with the dimension n greater than 5, Sabadini-Sommen-Struppa-Van Lancker [26] wrote down operators in the Dirac complex explicitly, with the help of the tool of megaforms and the method of computer algebra. In the stable case, the method of parabolic geometry was used to construct the Dirac complex by Krump [18] for $k = 4$ and by Salač [27]-[28] for the stable case and even n. In this case, he [27] also constructed the second order differential operators in the Dirac complex (3.6) as the sum of 4 invariant differential operators. For the unstable case, it is an open problem to construct the Dirac complex. However some results can be found in Krump [19].

To show the Hartogs-Bochner extension for monogenic functions of several vector variables, we need to solve the non-homogeneous several Dirac equations (3.1) under a compatibility condition. If we hope to use arguments in (2.29)-(2.32) to construct a compactly supported solution as in Theorem 2.4, we need at least first three operators in the Dirac complex (3.6). They were written down explicitly by Shi-Wang in [31] for any n and k.

For k ($k \geq 3$) vector variables, the first four terms of this complex can be written as

$$0 \rightarrow C^{\infty}(\Omega, \mathcal{V}_0) \xrightarrow{\mathcal{D}_0} C^{\infty}(\Omega, \mathcal{V}_1) \xrightarrow{\mathcal{D}_1} C^{\infty}(\Omega, \mathcal{V}_2) \xrightarrow[\mathcal{D}_2'']{\mathcal{D}_2'} \begin{array}{c} C^{\infty}(\Omega, \mathcal{V}_3') \\ \oplus \\ C^{\infty}(\Omega, \mathcal{V}_3'') \end{array}, \tag{3.7}$$

in the stable or unstable case, where Ω is a domain in $\mathbb{R}^{kn}$, and

$$\begin{aligned} \mathcal{V}_0 &= V_{0\ldots} \otimes \mathbb{S}^+, \\ \mathcal{V}_1 &= V_{10\ldots} \otimes \mathbb{S}^-, \\ \mathcal{V}_2 &= V_{210\ldots} \otimes \mathbb{S}^-, \\ \mathcal{V}_3' &= V_{220\ldots} \otimes \mathbb{S}^+, \\ \mathcal{V}_3'' &= V_{3110\ldots} \otimes \mathbb{S}^-. \end{aligned}$$

Obviously, $V_{0\ldots} \cong \mathbb{C}, V_{10\ldots} \cong \mathbb{C}^k$. The spaces $V_{210\ldots}, V_{220\ldots}, V_{3110\ldots}$ can be realized as subspaces of tensor products of $\mathbb{C}^k$ in terms of Young symmetrizer, which are called *Weyl modules*. They can be characterized in a very simple and elementary way as follows.

Let $\omega^A, A = 0, 1, \ldots, k-1$, be a basis of $\mathbb{C}^k$. An element h in $\otimes^l \mathbb{C}^k$ can be written as $h = h_{A_1\ldots A_l}\omega^{A_1} \otimes \cdots \otimes \omega^{A_l}$. We also identify an element h with the tuple $(h_{A_1\ldots A_l})$.

Proposition 3.1. (1) $h = (h_{ABC}) \in V_{210\ldots}$ *if and only if*

$$h_{[A\underline{B}C]} + h_{[A\underline{C}B]} = \frac{3}{2} h_{ABC}. \tag{3.8}$$

In particular, for any $h \in V_{210\ldots}$, we have

$$h_{ABC} = h_{ACB}. \tag{3.9}$$

(2) $h = (h_{DABC}) \in V_{220\ldots}$ *if and only if*

$$h_{DABC} = \frac{1}{6} \sum_{(A,D),(B,C)} \left(h_{D[A\underline{B}C]} + h_{B[C\underline{D}A]} \right). \tag{3.10}$$

(3) $h = (h_{EDABC}) \in V_{3110\ldots}$ *if and only if*

$$\sum_{(D,B,C)} h_{[E\underline{D}A\underline{B}C]} = \frac{10}{3} h_{EDABC}. \tag{3.11}$$

Here bracket $[\cdots]$ means skewsymmetrization of indices, but underlined indices in a bracket are not skewsymmetrized. $\sum_{(B,\cdots,C)}$ denotes the summation taken over all permutations $(B,\cdots,C)$.

It is convenient to use these characterization as the definition of these vector spaces.

A section in $C^\infty(\Omega, \mathcal{V}_0)$ is an $\mathbb{S}^+$-valued function on Ω, while a section in $C^\infty(\Omega, \mathcal{V}_1)$ is written as $F = (F_A)$ with F_A to be $\mathbb{S}^-$-valued functions on Ω. A section in $C^\infty(\Omega, \mathcal{V}_2)$ is written as $h = (h_{ABC})$ with h_{ABC} to be $\mathbb{S}^-$-valued functions which satisfy (3.8) on Ω. A section of $C^\infty(\Omega, \mathcal{V}_3')$ or $C^\infty(\Omega, \mathcal{V}_3'')$ is similarly defined. The operators $\mathcal{D}_0, \mathcal{D}_1, \mathcal{D}_2'$ and $\mathcal{D}_2''$ in (3.7) are given by

$$\begin{aligned}
(\mathcal{D}_0 f)_A &= \nabla_A f, \\
(\mathcal{D}_1 F)_{ABC} &= \sum_{(B,C)} \nabla_{[A} \nabla_{\underline{B}} F_{C]}, \\
(\mathcal{D}_2' h)_{DABC} &= \sum_{(A,D),(B,C)} \left(\nabla_D h_{[A\underline{B}C]} + \nabla_B h_{[C\underline{D}A]} \right), \\
(\mathcal{D}_2'' h)_{EDABC} &= \frac{1}{2} \sum_{(D,B,C)} \left(2\nabla_{[E} \nabla_{\underline{D}} h_{A]BC} + \nabla_D \nabla_{[E} h_{A]BC} + \Delta_{BC} h_{[E\underline{D}A]} \right),
\end{aligned} \tag{3.12}$$

for $f \in C^\infty(\Omega, \mathcal{V}_0), F \in C^\infty(\Omega, \mathcal{V}_1), h \in C^\infty(\Omega, \mathcal{V}_2)$, where $A, B, C, D, E = 0, 1, \ldots, k-1$. So we have the operator

$$\mathcal{D}_2 = \mathcal{D}_2' + \mathcal{D}_2'' : C^\infty(\Omega, \mathcal{V}_2) \to C^\infty(\Omega, \mathcal{V}_3), \qquad \mathcal{V}_3 = \mathcal{V}_3' \oplus \mathcal{V}_3''.$$

For 3 vector variables, the differential operators in the Dirac complex was given explicitly by [13, 26]. We can check that in this case, operators in (3.12) coincide with their formulae.

Theorem 3.2. [31, Theorem 1.1] *For $k \geq 3$, the short sequence* (3.7) *is an elliptic complex, i.e. $\mathcal{D}_{j+1} \circ \mathcal{D}_j = 0$ for $j = 0, 1$, and for any $\xi \in \mathbb{R}^{kn} \setminus \{\mathbf{0}\}$, the symbol sequence of* (3.7),

$$0 \to \mathbb{S}^+ \xrightarrow{\sigma_0(\xi)} \mathcal{V}_1 \xrightarrow{\sigma_1(\xi)} \mathcal{V}_2 \underset{\sigma_2''(\xi)}{\xrightarrow{\sigma_2'(\xi)}} \begin{matrix} \mathcal{V}_3' \\ \oplus \\ \mathcal{V}_3'' \end{matrix}, \tag{3.13}$$

is exact, where $\sigma_j(\xi)$ is the symbol of the operator $\mathcal{D}_j$ in (3.12).

3.2. **The Hartogs-Bochner extension for tangentially monogenic functions.** Without loss of generality, we can assume the defining function of the domain Ω can be written locally as

$$\rho(\mathbf{x}) = x_{01} - \phi\left(x_{02}, \dots x_{0n}, \dots, x_{(k-1)n}\right), \tag{3.14}$$

for $\mathbf{x} \in \mathbb{R}^{kn}$, with $\phi\left(0, \dots, 0\right) = 0$. Then

$$Z_A = \nabla_A - \nabla_A \rho \left(\nabla_0 \rho\right)^{-1} \nabla_0, \quad A = 1, \cdots, k-1.$$

are tangential operators, because $Z_A \rho = 0$, and

$$T = \left(\nabla_0 \rho\right)^{-1} \nabla_0 - \partial_{01},$$

is also tangential. Then

$$\begin{aligned} \nabla_0 =& \nabla_0 \rho \cdot T + \nabla_0 \rho \cdot \partial_{01}, \\ \nabla_A =& Z_A + \nabla_A \rho \cdot T + \nabla_A \rho \cdot \partial_{01}, \qquad A = 1, \dots, k-1, \end{aligned} \tag{3.15}$$

By using the construction of the boundary complex of a general differential complex (see eg. [3, 22, 23]), we get the operator induced from several Dirac operators, called the *tangential several Dirac operators*:

$$\mathscr{D}_0 : C^\infty\left(\partial\Omega, \mathcal{V}_0\right) \to C^\infty\left(\partial\Omega, \mathscr{V}_1\right),$$

where $\mathscr{V}_1$ is two copies of $\mathbb{S}^- \otimes \mathbb{C}^{k-1}$, i.e.

$$\mathscr{V}_1 \cong \left(\mathbb{S}^- \otimes \mathbb{C}^{k-1}\right) \oplus \left(\mathbb{S}^- \otimes \mathbb{C}^{k-1}\right).$$

It is given by

$$\begin{aligned} \left(\mathscr{D}_0 \hat{f}_1\right)_\mu =& = Z_\mu \hat{f}, \\ \left(\mathscr{D}_0 \hat{f}_2\right)_\mu =& - Z_\mu T \hat{f}, \qquad \mu = 1, \dots, k-1. \end{aligned} \tag{3.16}$$

$\hat{f} \in C^1\left(\partial\Omega, \mathcal{V}_0\right)$ is called *tangentially monogenic* if

$$\mathscr{D}_0 \hat{f} = 0,$$

in sense of distributions, i.e. $Z_\mu \hat{f} = 0$ and $Z_\mu T \hat{f} = 0$, $\mu = 1, \dots, k-1$. Shi-Wang [31] proved the Hartogs-Bochner extension for tangentially monogenic functions.

Theorem 3.3. [31, Theorem 1.2] *Suppose that $k \geq 2$ and Ω is a bounded domain in $\mathbb{R}^{kn}$ with smooth boundary such that $\mathbb{R}^{kn} \setminus \overline{\Omega}$ is connected. If f is a smooth tangentially monogenic function on $\partial\Omega$, then there exists $\tilde{f} \in \mathcal{O}(\Omega)$ smooth up to the boundary such that $\tilde{f} = f$ on $\partial\Omega$.*

To show the Hartogs-Bochner extension, we need an extension flat at the boundary at first.

Proposition 3.4. [31] *If $\hat{f} \in C^\infty\left(\partial\Omega, \mathcal{V}_0\right)$ is tangentially monogenic, there exists a representative $\tilde{f} \in C^\infty\left(\overline{\Omega}, \mathcal{V}_0\right)$ such that $\tilde{f}\Big|_{\partial\Omega} = \hat{f}$ and $\mathcal{D}_0 \tilde{f}$ is flat on $\partial\Omega$.*

We need again to solve the non-homogeneous several Dirac equations

$$\mathcal{D}_0 u = f, \tag{3.17}$$

under the compatibility condition

$$\mathcal{D}_1 f = 0. \tag{3.18}$$

The following theorem provide compactly supported solution.

Theorem 3.5. [31, Theorem 5.1] *Suppose that* $f \in L^2\left(\mathbb{R}^{kn}, \mathcal{V}_1\right)$ *satisfies the compatibility condition* (3.18) *in sense of distributions. Then there exists* $u \in W^{1,2}\left(\mathbb{R}^{kn}, \mathcal{V}_0\right)$ *satisfying the non-homogeneous equation* (3.17)*. Furthermore, if* $f \in C_c\left(\mathbb{R}^{kn}, \mathcal{V}_1\right)$ *satisfies the compatibility condition* (3.18) *in the sense of distributions, then there exists* $u \in C_c\left(\mathbb{R}^{kn}, \mathcal{V}_0\right) \cap W^{1,2}\left(\mathbb{R}^{kn}, \mathcal{V}_0\right)$ *satisfying* (3.17) *and vansihing on the unbounded connected component of* $\mathbb{R}^{kn} \setminus \operatorname{supp} f$.

Statements and Declarations

The authors declare that they have no conflict of interest, and the manuscript has no associated data.

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