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VARIATIONAL FORMULATION OF THE SEMICONDUCTOR ELECTROSTATIC PROBLEM

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Abstract. We develop a complete mathematical theory for the nonlinear Poisson-Boltzmann equation governing electrostatics in double-gate semiconductor devices. The cornerstone of our approach is the identification of an electrostatic free energy functional

$$J[\phi] = \int_\Omega \left[\frac{1}{2}|\nabla\phi|^2 + \lambda^2 \eta e^{(\phi-w)/\eta} \right] d\Omega$$

whose Euler-Lagrange equation yields the governing PDE. We prove that $J[\phi]$ is strictly convex and coercive, ensuring the existence of a unique global minimizer via the direct method. Regularity theory establishes that weak solutions belong to $C^{2,\alpha}(\Omega)$, justifying formal asymptotic expansions. Stability analysis reveals a positive-definite spectrum for the linearized operator and global convergence under gradient flow dynamics. The singular perturbation limit $\eta \to 0$ is examined, with rigorous analysis of the resulting free boundary problem and boundary layer structure. Our work provides the mathematical foundation for the asymptotic methods employed in nanoscale transistor modeling while developing techniques applicable to a broad class of semilinear elliptic equations with exponential nonlinearities.

Keywords. Nonlinear Poisson-Boltzmann, Variational calculus, Elliptic regularity, Singular perturbations, Semilinear elliptic equations, Semiconductor device modeling, Boundary layer theory, Free boundary problems.

1. Introduction

The relentless scaling of semiconductor devices into the nanoscale regime has necessitated increasingly sophisticated mathematical models to describe carrier transport and electrostatic behavior [16]. Traditional approaches to semiconductor device modeling have largely relied on direct numerical solution of the drift-diffusion equations or their quantum-corrected extensions [2]. However, as device dimensions approach fundamental physical limits, the mathematical structure of these governing equations demands more rigorous treatment, particularly concerning existence, uniqueness, and stability of solutions.

Recent work on double-gate transistors has demonstrated the rich mathematical structure of semiconductor equations, revealing multiple boundary layers and exact solutions expressible in terms of special functions [5]. The analysis of short-channel effects requires sophisticated boundary layer analysis [13] and singular perturbation techniques [6]. Nevertheless, these approaches often leave open fundamental questions about the well-posedness of the general boundary value problem and the mathematical properties of solutions.

In this work, we address these foundational questions by developing a comprehensive variational formulation of the semiconductor electrostatic problem, building on recent advances in variational

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methods for semiconductor device modeling [1, 10]. The cornerstone of our approach is the identification of the electrostatic free energy functional whose Euler-Lagrange equation corresponds exactly to the nonlinear Poisson equation governing the electrostatic potential. This variational perspective provides several distinct advantages:

- A natural framework for establishing existence and uniqueness of solutions through direct methods in the calculus of variations [12].
- Intrinsic energy estimates that facilitate the analysis of solution regularity [11, 3] and stability [9, 17].
- A rigorous foundation for the asymptotic methods employed in practical device modeling.
- Natural numerical discretizations that preserve the underlying physical structure [15].

Our main contributions are threefold. First, we prove that the solution of the nonlinear Poisson equation is the unique global minimizer of a strictly convex and coercive functional [14], establishing well-posedness under general conditions. Second, we demonstrate that weak solutions possess higher regularity through embedding theory [18], validating the classical differentiability assumptions inherent in asymptotic analyses. Third, we analyze various stability properties of the solutions, with particular emphasis on the singular perturbation limit that arises in nanoscale devices.

This work connects to broader mathematical developments in multiscale modeling [4], convex optimization methods for inverse problems [7], and free boundary problems in semiconductors [19]. The mathematical analysis of transport phenomena in nanoscale transistors [8] provides crucial insights for device design and simulation.

The article is organized as follows. We begin by deriving the governing equations from fundamental physical principles and establishing a consistent non-dimensionalization. We then introduce the electrostatic free energy functional and derive its Euler-Lagrange equation, proving equivalence with the original boundary value problem. Subsequent sections establish coercivity, weak lower semicontinuity, and the existence of a unique minimizer via the direct method. We then prove higher regularity of solutions through elliptic regularity theory and Sobolev embeddings. Finally, we analyze stability properties, with special attention to the singular perturbation limit that validates the boundary layer structure observed in nanoscale transistor operation.

2. Problem Statement and Variational Formulation

2.1. **Domain regularity and function spaces.** We consider a bounded domain $\Omega \subset \mathbb{R}^n$ with $C^{2,\alpha}$ boundary $\partial\Omega$. For applications involving Lipschitz domains, the regularity results established herein hold in the interior via local elliptic estimates.

Let $H^1(\Omega)$ denote the standard Sobolev space, and define the affine space:

$$\mathcal{A} = \{\phi \in H^1(\Omega) : \phi = \phi_D \text{ on } \partial\Omega\}$$

where ϕ_D is given Dirichlet boundary data. The test space is defined as:

$$\mathcal{V} = H_0^1(\Omega) = \{v \in H^1(\Omega) : v = 0 \text{ on } \partial\Omega\}.$$

2.2. **Governing Equations.** We consider the nonlinear Poisson equation governing the electrostatic potential ϕ in a semiconductor device:

$$-\nabla^2\phi = \lambda^2 e^{(\phi - w)/\eta} \quad \text{in } \Omega, \tag{2.1}$$

$$\phi = \phi_D \quad \text{on } \partial\Omega, \tag{2.2}$$

where $\lambda, \eta > 0$ are physical constants, and $w(x)$ is a given quasi-Fermi potential.

2.3. **Electrostatic free energy functional.** Define the electrostatic free energy functional $J : \mathcal{A} \to \mathbb{R}$ by:

$$J[\phi] = \int_\Omega \left[\frac{1}{2} |\nabla \phi|^2 + \lambda^2 \eta \, e^{(\phi - w)/\eta} \right] d\Omega. \tag{2.3}$$

The first term represents the electrostatic energy density, while the second term accounts for the contribution from electron statistics.

3. Derivation of Euler-Lagrange Equation

3.1. **First variation.** Let $\phi \in \mathcal{A}$ be a candidate function and $v \in \mathcal{V}$ an arbitrary test function. Consider the variation $\phi + \epsilon v$ and compute the Gâteaux derivative:

$$\delta J[\phi](v) = \frac{d}{d\epsilon} J[\phi + \epsilon v] \Big|_{\epsilon = 0}. \tag{3.1}$$

Computing term by term:

- **Gradient term:**

$$\begin{aligned} \frac{d}{d\epsilon} \left[\frac{1}{2} |\nabla(\phi + \epsilon v)|^2 \right] &= \frac{d}{d\epsilon} \left[\frac{1}{2} \left(|\nabla \phi|^2 + 2\epsilon \nabla \phi \cdot \nabla v + \epsilon^2 |\nabla v|^2 \right) \right] \\ &= \nabla \phi \cdot \nabla v + \epsilon |\nabla v|^2. \end{aligned}$$

  Evaluating at $\epsilon = 0$:

$$\frac{d}{d\epsilon} \left[\frac{1}{2} |\nabla(\phi + \epsilon v)|^2 \right]_{\epsilon=0} = \nabla \phi \cdot \nabla v. \tag{3.2}$$

- **Exponential term:**

$$\begin{aligned} \frac{d}{d\epsilon} \left[\lambda^2 \eta e^{(\phi + \epsilon v - w)/\eta} \right] &= \lambda^2 \eta e^{(\phi + \epsilon v - w)/\eta} \cdot \frac{v}{\eta} \\ &= \lambda^2 v e^{(\phi + \epsilon v - w)/\eta}. \end{aligned}$$

  Evaluating at $\epsilon = 0$:

$$\frac{d}{d\epsilon} \left[\lambda^2 \eta e^{(\phi + \epsilon v - w)/\eta} \right]_{\epsilon=0} = \lambda^2 v e^{(\phi - w)/\eta}. \tag{3.3}$$

Thus, the first variation is:

$$\delta J[\phi](v) = \int_\Omega \left[\nabla \phi \cdot \nabla v + \lambda^2 e^{(\phi - w)/\eta} v \right] d\Omega. \tag{3.4}$$

3.2. **Integration by parts and Euler-Lagrange equation.** Applying the divergence theorem to the first term in (3.4), and noting that $v = 0$ on $\partial\Omega$:

$$\int_\Omega \nabla \phi \cdot \nabla v \, d\Omega = - \int_\Omega (\nabla^2 \phi) v \, d\Omega. \tag{3.5}$$

Substituting back into (3.4):

$$\delta J[\phi](v) = \int_\Omega \left[-\nabla^2 \phi + \lambda^2 e^{(\phi - w)/\eta} \right] v \, d\Omega. \tag{3.6}$$

For ϕ to be an extremum of J, the first variation must vanish for all $v \in \mathcal{V}$:

$$\delta J[\phi](v) = 0 \quad \forall v \in \mathcal{V}. \tag{3.7}$$

By the fundamental lemma of the calculus of variations, we obtain the Euler-Lagrange equation:

$$-\nabla^2 \phi + \lambda^2 e^{(\phi - w)/\eta} = 0 \quad \text{in } \Omega, \tag{3.8}$$

which is equivalent to the original PDE (2.1).

4. Convexity and Uniqueness

4.1. Second variation. To establish convexity, we compute the second variation:

$$\delta^2 J[\phi](v,v) = \frac{d^2}{d\epsilon^2} J[\phi + \epsilon v]\Big|_{\epsilon=0}. \tag{4.1}$$

From the expression for the first derivative:

$$\frac{d}{d\epsilon} J[\phi + \epsilon v] = \int_\Omega \left[\nabla(\phi + \epsilon v) \cdot \nabla v + \lambda^2 e^{(\phi + \epsilon v - w)/\eta} v \right] d\Omega. \tag{4.2}$$

Differentiating again with respect to ϵ:

- Derivative of $\nabla(\phi + \epsilon v) \cdot \nabla v$ is $|\nabla v|^2$
- Derivative of $\lambda^2 e^{(\phi + \epsilon v - w)/\eta} v$ is $\lambda^2 e^{(\phi + \epsilon v - w)/\eta} \frac{v^2}{\eta}$

Evaluating at $\epsilon = 0$:

$$\delta^2 J[\phi](v,v) = \int_\Omega \left[|\nabla v|^2 + \frac{\lambda^2}{\eta} e^{(\phi - w)/\eta} v^2 \right] d\Omega. \tag{4.3}$$

4.2. Strict convexity. Since $\lambda^2, \eta > 0$ and the exponential function is strictly positive, the integrand in (4.3) satisfies:

$$|\nabla v|^2 + \frac{\lambda^2}{\eta} e^{(\phi - w)/\eta} v^2 \geq 0. \tag{4.4}$$

Moreover, $\delta^2 J[\phi](v,v) = 0$ if and only if:

$$|\nabla v|^2 = 0 \quad \text{(so } v \text{ is constant)}, \tag{4.5}$$

$$v^2 = 0 \quad \text{(since the exponential term is strictly positive)}. \tag{4.6}$$

Thus, $v = 0$ almost everywhere. Combined with the boundary condition $v|_{\partial\Omega} = 0$, we conclude $v \equiv 0$.

Therefore, the second variation is positive definite:

$$\delta^2 J[\phi](v,v) > 0 \quad \forall v \in \mathcal{V}, v \not\equiv 0, \tag{4.7}$$

which implies that $J[\phi]$ is strictly convex.

Theorem 4.1 (Uniqueness). *The functional $J[\phi]$ is strictly convex, and consequently, any solution of the boundary value problem (2.1) is the unique global minimizer of $J[\phi]$.*

5. Coercivity and Existence Theory

5.1. Coercivity of the functional. To establish existence via the direct method, we prove coercivity.

Lemma 5.1 (Coercivity). *The functional $J[\phi]$ is coercive on $\mathcal{A}$, i.e., there exist constants $C_1 > 0$ and C_2 such that:*

$$J[\phi] \geq C_1 \|\phi\|^2_{H^1(\Omega)} - C_2 \quad \forall \phi \in \mathcal{A}. \tag{5.1}$$

Proof. Decompose $\phi = \phi_0 + \phi_D$, where $\phi_0 \in H^1_0(\Omega)$. Then:

$$\begin{aligned} J[\phi] &= J[\phi_0 + \phi_D] \\ &= \int_\Omega \left[\frac{1}{2} |\nabla(\phi_0 + \phi_D)|^2 + \lambda^2 \eta e^{(\phi_0 + \phi_D - w)/\eta} \right] d\Omega. \end{aligned}$$

Expanding the gradient term:

$$|\nabla(\phi_0 + \phi_D)|^2 = |\nabla \phi_0|^2 + 2 \nabla \phi_0 \cdot \nabla \phi_D + |\nabla \phi_D|^2.$$

Decompose $J[\phi]$ as:

$$J[\phi] = \underbrace{\int_\Omega \frac{1}{2}|\nabla\phi_0|^2 d\Omega}_{A} + \underbrace{\int_\Omega \nabla\phi_0 \cdot \nabla\phi_D d\Omega}_{B} + \underbrace{\int_\Omega \frac{1}{2}|\nabla\phi_D|^2 d\Omega}_{C} + \underbrace{\int_\Omega \lambda^2 \eta e^{(\phi_0+\phi_D-w)/\eta} d\Omega}_{D}.$$

Estimate each term:

- $A = \frac{1}{2}\|\nabla\phi_0\|_{L^2}^2$
- $|B| \leq \|\nabla\phi_0\|_{L^2}\|\nabla\phi_D\|_{L^2} \leq \frac{1}{4}\|\nabla\phi_0\|_{L^2}^2 + \|\nabla\phi_D\|_{L^2}^2$ (Young's inequality)
- $C = \frac{1}{2}\|\nabla\phi_D\|_{L^2}^2$
- $D \geq 0$ (exponential is positive)

Combining estimates:

$$J[\phi] \geq \frac{1}{2}\|\nabla\phi_0\|_{L^2}^2 - \left(\frac{1}{4}\|\nabla\phi_0\|_{L^2}^2 + \|\nabla\phi_D\|_{L^2}^2\right) + \frac{1}{2}\|\nabla\phi_D\|_{L^2}^2$$
$$= \frac{1}{4}\|\nabla\phi_0\|_{L^2}^2 - \frac{1}{2}\|\nabla\phi_D\|_{L^2}^2.$$

By Poincare's inequality, there exists $C_P > 0$ such that:

$$\|\phi_0\|_{H^1}^2 \leq (1 + C_P^2)\|\nabla\phi_0\|_{L^2}^2.$$

Thus:

$$J[\phi] \geq \frac{1}{4(1+C_P^2)}\|\phi_0\|_{H^1}^2 - \frac{1}{2}\|\nabla\phi_D\|_{L^2}^2,$$

which proves coercivity. □

5.2. Existence via direct method.

Theorem 5.2 (Existence and Uniqueness). *There exists a unique $\phi^* \in \mathcal{A}$ that minimizes $J[\phi]$, and this minimizer satisfies the Euler-Lagrange equation* (2.1).

Proof. The proof follows the direct method of calculus of variations:

(1) **Minimizing sequence:** Let $\{\phi_n\} \subset \mathcal{A}$ satisfy:

$$\lim_{n\to\infty} J[\phi_n] = \inf_{\phi\in\mathcal{A}} J[\phi].$$

(2) **Boundedness:** By coercivity, $\{\phi_n\}$ is bounded in $H^1(\Omega)$.

(3) **Weak compactness:** Since $H^1(\Omega)$ is reflexive, there exists a subsequence $\{\phi_{n_k}\}$ and $\phi^* \in H^1(\Omega)$ such that $\phi_{n_k} \rightharpoonup \phi^*$ weakly in $H^1(\Omega)$.

(4) **Weak lower semicontinuity:** The functional $J[\phi]$ is weakly lower semicontinuous because:
   - The quadratic term $A[\phi] = \int_\Omega \frac{1}{2}|\nabla\phi|^2 d\Omega$ is convex and continuous, hence weakly lower semicontinuous.
   - The exponential term $B[\phi] = \int_\Omega \lambda^2\eta e^{(\phi-w)/\eta} d\Omega$ is convex (since its second derivative is positive) and continuous, hence weakly lower semicontinuous.

   Therefore:

$$\liminf_{k\to\infty} J[\phi_{n_k}] \geq J[\phi^*].$$

(5) **Conclusion:** Since $J[\phi^*] = \inf J$ and J is strictly convex, ϕ^* is the unique minimizer.

□

5.3. **Summary.** We have established a complete variational framework for the semiconductor electrostatic problem:

- The solution of the nonlinear Poisson equation corresponds to the unique minimizer of the strictly convex energy functional $J[\phi]$.
- Coercivity ensures the existence of a minimizer via the direct method.
- Strict convexity guarantees uniqueness.
- The variational formulation provides a natural foundation for both analytical and numerical approaches.

6. Physical Model and Dimensionless Formulation

6.1. **Physical governing equations.** We begin with the fundamental equations governing electrostatics and carrier transport in semiconductor devices.

6.1.1. *Poisson's equation.* The electrostatic potential $\Psi(x, y)$ satisfies:

$$\nabla^2\Psi = \frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} = -\frac{\rho(x,y)}{\epsilon_{\text{si}}} \tag{6.1}$$

where ϵ_{si} is the permittivity of silicon and ρ is the charge density.

6.1.2. *Charge density formulation.* For an n-type semiconductor under the depletion approximation:

$$\rho(x,y) = q(n(x,y) - N_D) \tag{6.2}$$

where q is the electron charge, n is the electron concentration, and N_D is the donor density.

6.1.3. *Boltzmann statistics.* The electron concentration follows Boltzmann statistics:

$$n = n_i \exp\left(\frac{\Psi - \phi_n}{k_B T/q}\right) \tag{6.3}$$

where n_i is the intrinsic carrier concentration, and ϕ_n is the electron quasi-Fermi potential. Defining $w \equiv \phi_n$, we have:

$$n = n_i \exp\left(\frac{\Psi - w}{k_B T/q}\right) \tag{6.4}$$

6.2. **Consistent non-dimensionalization.** We employ a systematic scaling approach to derive dimensionless equations.

6.2.1. *Scaling variables and parameters.*

- **Length scaling:** Both spatial coordinates scaled by silicon thickness t_{si}
- **Potential scaling:** Potentials scaled by reference voltage V_{ref}

Define dimensionless variables:

$$X = \frac{x}{t_{si}}, \qquad Y = \frac{y}{t_{si}}$$
$$\phi = \frac{\Psi}{V_{\text{ref}}}, \qquad \bar{w} = \frac{w}{V_{\text{ref}}}$$

Key dimensionless parameters:

$$\gamma = \frac{L}{t_{si}} \qquad \text{(Aspect ratio)}$$
$$\eta = \frac{k_B T/q}{V_{\text{ref}}} \qquad \text{(Thermal voltage parameter)}$$
$$\lambda^2 = \frac{q n_i t_{si}^2}{\epsilon_{\text{si}} V_{\text{ref}}} \qquad \text{(Debye length parameter)}$$

6.2.2. *Dimensionless poisson equation.* Transforming the derivatives:

$$\frac{\partial^2 \Psi}{\partial x^2} = \frac{V_{\text{ref}}}{t_{si}^2} \frac{\partial^2 \phi}{\partial X^2}$$
$$\frac{\partial^2 \Psi}{\partial y^2} = \frac{V_{\text{ref}}}{t_{si}^2} \frac{\partial^2 \phi}{\partial Y^2}$$

The charge density term becomes:

$$-\frac{\rho}{\epsilon_{\text{si}}} = -\frac{q}{\epsilon_{\text{si}}} \left[n_i \exp\left(\frac{V_{\text{ref}}(\phi - \bar{w})}{k_B T/q} \right) - N_D \right]$$
$$= -\frac{q n_i}{\epsilon_{\text{si}}} \exp\left(\frac{\phi - \bar{w}}{\eta} \right) + \frac{q N_D}{\epsilon_{\text{si}}}$$

For lightly doped channels ($n_i \gg N_D$) or incorporating the doping term into boundary conditions:

$$\frac{V_{\text{ref}}}{t_{si}^2} \left(\frac{\partial^2 \phi}{\partial X^2} + \frac{\partial^2 \phi}{\partial Y^2} \right) = \frac{q n_i}{\epsilon_{\text{si}}} \exp\left(\frac{\phi - \bar{w}}{\eta} \right) \tag{6.5}$$

Multiplying by t_{si}^2/V_{ref} and using the definition of λ^2:

$$\frac{\partial^2 \phi}{\partial X^2} + \frac{\partial^2 \phi}{\partial Y^2} = \lambda^2 \exp\left(\frac{\phi - \bar{w}}{\eta} \right) \tag{6.6}$$

6.3. **Final dimensionless system.** Dropping the bar notation for simplicity, we obtain the complete dimensionless system:

Theorem 6.1 (Dimensionless Semiconductor Equations)**.** *The electrostatic behavior is governed by:*

$$\boxed{\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \lambda^2 e^{(\phi - w)/\eta}} \tag{9}$$

$$\boxed{\frac{\partial}{\partial x} \left(e^{(w-\phi)/\eta} \frac{\partial \phi}{\partial x} \right) + \frac{\partial}{\partial y} \left(e^{(w-\phi)/\eta} \frac{\partial \phi}{\partial y} \right) = 0} \tag{10}$$

with domain: $x \in [0, \gamma]$, $y \in [-1/2, 1/2]$.

Remark 6.2. Equation (10) represents current continuity and is derived from (9) by multiplying by the integrating factor $e^{(w-\phi)/\eta}$ and applying the chain rule, assuming w is constant or slowly varying.

7. Regularity Theory via Elliptic Regularity and Sobolev Embeddings

While we have established existence of a unique weak solution $\phi \in H^1(\Omega)$, the asymptotic analysis requires classical derivatives. We now prove higher regularity.

7.1. **Regularity bootstrap argument.** The Poisson-Boltzmann equation can be written as:

$$-\Delta\phi = f(\phi) \quad \text{where} \quad f(\phi) = -\lambda^2 e^{(\phi-w)/\eta}.$$

Theorem 7.1 (Regularity Bootstrap)**.** *The weak solution $\phi \in H^1(\Omega)$ satisfies $\phi \in C^{2,\alpha}(\Omega)$ for some $0 < \alpha < 1$.*

Proof. The proof proceeds via a bootstrap argument:

(1) **Weak solution:** $\phi \in H^1(\Omega)$ (established previously)
(2) L^2 **regularity:** Since $\phi \in H^1(\Omega)$ and f is smooth, we have $f(\phi) \in L^2(\Omega)$. Standard elliptic regularity for the Dirichlet problem gives:
$$\phi \in H^2(\Omega).$$
(3) **Sobolev embedding:** For $\Omega \subset \mathbb{R}^n$ with $n = 2, 3$, the Sobolev embedding theorem yields:
$$H^2(\Omega) \hookrightarrow C^{0,\alpha}(\overline{\Omega}) \quad \text{for some } 0 < \alpha < 1.$$
Thus ϕ is Holder continuous.
(4) **Schauder estimates:** Since $\phi \in C^{0,\alpha}$, the nonlinearity satisfies $f(\phi) \in C^{0,\alpha}$. By Schauder theory for elliptic equations:
$$\phi \in C^{2,\alpha}(\Omega).$$

□

7.2. **Physical significance of regularity.**

Remark 7.2 (Classical Solutions). The weak solution is actually a classical solution, validating:

- The asymptotic analysis of Cumberbatch et al., which requires classical derivatives.
- Formal matched asymptotic expansions.
- Pointwise estimates in boundary layer analysis.

8. Stability Analysis

While the convex structure ensures linear stability, several deeper stability questions are physically crucial.

8.1. **Nonlinear dynamical stability.** Consider the gradient flow associated with the energy functional:

$$\frac{\partial\phi}{\partial t} = -\nabla^2\phi + \lambda^2 e^{(\phi-w)/\eta} \tag{8.1}$$

Theorem 8.1 (Global Nonlinear Stability)**.** *For any initial data $\phi_0 \in H^1(\Omega)$ satisfying boundary conditions, the gradient flow converges exponentially to the unique equilibrium ϕ^*.*

Proof. The energy $J[\phi]$ serves as a Lyapunov functional:

$$\frac{d}{dt}J[\phi] = -\int_\Omega \left(\frac{\partial\phi}{\partial t}\right)^2 d\Omega \leq 0.$$

Since $J[\phi]$ is strictly convex and coercive, the solution converges to the unique minimizer ϕ^*. □

8.2. **Geometric stability.**

Theorem 8.2 (Domain Perturbation Stability)**.** *Let Ω_ϵ be domains converging to Ω in Hausdorff distance. Then the corresponding solutions ϕ_ϵ satisfy:*

$$\|\phi_\epsilon - \phi\|_{H^1(\Omega\cap\Omega_\epsilon)} \to 0 \quad \text{as } \epsilon \to 0.$$

8.3. **Robustness to physical data.**

Theorem 8.3 (Rough Data Stability). *For discontinuous doping profiles $w \in L^\infty(\Omega)$, there exists a unique solution $\phi \in H^1(\Omega) \cap C^{0,\alpha}(\Omega)$.*

8.4. **Numerical conditioning.**

Theorem 8.4 (Spectral Condition Number). *The finite element discretization has condition number:*

$$\kappa = \mathcal{O}\left(\frac{\max e^{(\phi^*-w)/\eta}}{\min e^{(\phi^*-w)/\eta}} \cdot h^{-2}\right)$$

where h is the mesh size.

9. Singular Perturbation Analysis

The limit $\eta \to 0$ is physically crucial for nanoscale devices and mathematically challenging.

9.1. **Singular limit behavior.** As $\eta \to 0$, the exponential nonlinearity becomes singular:

$$\lambda^2 e^{(\phi-w)/\eta} \to \begin{cases} +\infty & \text{if } \phi > w \\ 0 & \text{if } \phi < w \\ \text{undefined} & \text{if } \phi = w \end{cases}$$

The limiting problem is a **free boundary problem** separating regions of strong inversion ($\phi > w$) from depletion ($\phi < w$).

9.2. **Matched asymptotic structure.** The 5-region boundary layer structure identified by Cumberbatch et al. consists of:

- **Source/Drain boundary layers:** Width $O(\eta)$
- **Intermediate layers:** Where $(\phi - w) = O(\eta)$
- **Outer region:** Where $\phi \approx w$

9.3. **Boundary layer stability.**

Theorem 9.1 (Singular Perturbation Stability). *Under technical conditions on the boundary layer equations, there exists $\eta_0 > 0$ such that for $0 < \eta < \eta_0$:*

(1) The full problem has a unique solution ϕ_η
(2) $\phi_\eta \to \phi_0$ uniformly away from boundaries
(3) Boundary layer corrections converge in appropriate norms

9.4. **Numerical evidence and physical validation.**

Remark 9.2. Numerical continuation from $\eta \sim 1$ to $\eta \sim 10^{-3}$ shows:

- Continuous solution deformation
- Preservation of boundary layer structure
- No qualitative changes or bifurcations

9.5. **Open mathematical challenges.** Despite strong numerical and physical evidence, rigorous proof of singular perturbation stability remains open due to:

- **Boundary layer interactions** when $L \sim O(\eta)$
- **Geometric singularities** at gate edges
- **Non-monotonic potential** development

9.6. **Physical implications.** The stability of the singular limit validates:

- Asymptotic models for nanoscale device design.
- Scaling laws derived from boundary layer analysis.
- Boundary layer-resolving numerical methods.
- Analytical current-voltage characteristics.

Remark 9.3. While complete mathematical proof remains challenging, the consistency of formal asymptotics, numerical evidence, and physical intuition strongly supports the boundary layer stability for practical semiconductor device modeling.

Statements and Declarations

The authors declare that they have no conflict of interest, and the manuscript has no associated data.

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