APPLICABLE NONLINEAR ANALYSIS
Volume 3 (2026), No. 2, 76–92
DOI:https://doi.org/10.69829/apna-026-0302-ta01
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AN ALTERNATED INERTIAL PROXIMAL-TYPE SUBGRADIENT EXTRAGRADIENT METHOD FOR PSEUDO-MONOTONE VARIATIONAL INEQUALITIES

XU YU JIANG[1], GUI HUA SONG[1], MONIRUL ISLAM[3,*], AND ZAI YUN PENG[2,1]

[1] *College of Mathematics and Statistics, Chongqing Jiaotong University, Chongqing, China*
[2] *School of Mathematics, Yunnan Normal University, Kunming, China*
[3] *Department of Mathematics, Aligarh Muslim University, Aligarh-202002, India*

ABSTRACT. This paper proposes an alternated inertial proximal-type subgradient extragradient projection algorithm for solving pseudo-monotone variational inequality problems in real Hilbert spaces. We introduce an additional parameter in the second projection step to regulate the proximal-type update, which improves stability and accelerates convergence. Weak convergence of the generated sequence is established. Numerical experiments demonstrate that the proposed method outperforms several existing algorithms.

Keywords. Variational inequality, Pseudo-monotone operator, Subgradient extragradient method, Alternated Inertial algorithm.

1. INTRODUCTION

Variational inequality problems (VIPs) provide a powerful mathematical tool for describing equilibrium conditions arising in a wide range of disciplines including optimization theory, economic modeling, signal processing, and engineering applications. Let H be a real Hilbert space, $C \subset H$ be a nonempty closed convex set, and let $A : H \to H$ be a given operator. The classical variational inequality problem $\mathrm{VI}(A, C)$ consists in finding a point $x^* \in C$ satisfying

$$\langle Ax^*, x - x^* \rangle \geq 0, \qquad \forall x \in C. \tag{1.1}$$

Variational inequalities encompass numerous mathematical models such as optimality systems in constrained optimization and equilibrium problems in network analysis. When the operator A is monotone and Lipschitz continuous, a variety of projection-based algorithms have been developed and analyzed in the literature; see, for example, [1, 2, 3]. In practical applications, however, the operator is often only pseudo-monotone, which significantly increases the difficulty of both algorithmic design and convergence analysis [4, 5].

Among the numerous approaches for solving VIPs, the extragradient method introduced by Korpelevich remains one of the most influential projection-based schemes. Its main idea is to incorporate an intermediate prediction step that stabilizes the iteration and improves robustness. A generic extragradient iteration takes the form

$$\begin{cases} y_n = P_C(x_n - \tau_n A x_n), \\ x_{n+1} = P_C(x_n - \tau_n A y_n). \end{cases} \tag{1.2}$$

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[*] Corresponding author.

E-mail address: jiangxuyuw@126.com (X. Y. Jiang), songguihua0815@163.com (G. H. Song), mislam.mm@amu.ac.in (M. Islam), and pengzaiyun@126.com (Z. Peng)

2020 Mathematics Subject Classification: 47J20, 65K10, 90C33.
Accepted: March 06, 2026.

This method and its variants have been extensively investigated for both monotone and pseudo-monotone [6, 7, 8]. Several improvements have also been proposed to avoid explicit line-search procedures, among which the golden ratio algorithm provides a fully adaptive stepsize strategy [3].

Although the classical extragradient method is theoretically appealing, each iteration requires two projections onto the feasible set C, which may be computationally expensive when C has a complicated structure. To overcome this difficulty, Censor proposed the subgradient extragradient technique, in which the second projection onto C is replaced by a projection onto a suitably constructed halfspace. The resulting iteration scheme can be written as

$$\begin{cases} y_n = P_C(x_n - \tau_n A x_n), \\ H_n = \{\, w \in H : \langle x_n - \tau_n A x_n - y_n,\ w - y_n \rangle \le 0 \,\}, \\ x_{n+1} = P_{H_n}(x_n - \tau_n A y_n). \end{cases} \tag{1.3}$$

This modification preserves the correction mechanism of the extragradient method while significantly reducing computational complexity. Consequently, subgradient extragradient algorithms have attracted considerable attention and have been extended in various directions, including inertial and adaptive variants [9, 10, 11, 12, 13, 14].

To further improve numerical performance, inertial extrapolation techniques inspired by Polyak's heavy-ball method have been incorporated into extragradient-type algorithms. The basic idea is to generate an extrapolated point before performing the correction step. A typical inertial SEG scheme can be described as

$$\begin{cases} w_n = x_n + \alpha_n (x_n - x_{n-1}), \\ y_n = P_C(w_n - \tau_n A w_n), \\ x_{n+1} = P_{H_n}(w_n - \tau_n A y_n). \end{cases} \tag{1.4}$$

Such inertial strategies often lead to faster convergence in practice [15, 16, 17, 18]. However, the presence of inertia may also introduce oscillatory behavior and complicate the theoretical analysis, particularly for pseudo-monotone operators.

To balance the acceleration effect of inertia and the stability of projection-type corrections, an effective approach is to activate inertial extrapolation only at alternating iterations [19, 20]. This partially inertial mechanism preserves the stabilizing structure of the underlying method while still benefiting from extrapolation. A representative alternated inertial SEG framework can be written as

$$\begin{cases} w_n = \begin{cases} x_n, & n \text{ even}, \\ x_n + \alpha (x_n - x_{n-1}), & n \text{ odd}, \end{cases} \\ y_n = P_C(w_n - \tau_n A w_n), \\ H_n = \{\, w \in H : \langle w_n - \tau_n A w_n - y_n,\ w - y_n \rangle \le 0 \}, \\ x_{n+1} = P_{H_n}(w_n - \tau_n A y_n). \end{cases} \tag{1.5}$$

In this scheme, inertia is applied only on odd iterations, while even iterations retain the classical correction structure. Meanwhile, the projection onto the halfspace H_n replaces the second projection onto C, which leads to a lower computational burden while maintaining the key descent property of extragradient-type methods.

Motivation and proposed approach. Projection-type methods for variational inequalities can be naturally interpreted through the *proximal-point framework*. For $\gamma > 0$, the proximal-point step for $\mathrm{VI}(A, C)$ seeks $x^+ \in C$ such that

$$\langle A(x^+) + \tfrac{1}{\gamma}(x^+ - x),\, y - x^+ \rangle \ge 0, \quad \forall\, y \in C, \tag{1.6}$$

which corresponds to a regularized implicit update and provides a stabilizing effect on the iteration. However, computing this step exactly is typically expensive.

Subgradient extragradient methods can be viewed as constructing a computable approximation of (1.6). In particular, a predicted point $y_n = P_C\big(w_n - \gamma\tau_n A w_n\big)$ serves as an inexact proximal point, while the correction step is realized by projecting onto a supporting halfspace

$$H_n := \{z \in H : \langle w_n - \gamma\tau_n A w_n - y_n,\, z - y_n\rangle \le 0\},$$

thereby mimicking the proximal descent mechanism with lower computational cost.

Motivated by this proximal interpretation, we propose a new *alternated inertial subgradient extragradient algorithm with adaptive halfspace correction*. The method combines alternating inertial extrapolation, adaptive stepsize selection, and an extragradient-based correction parameter to provide a stable and efficient approximation to the proximal-point iteration. Under standard assumptions, we establish the weak convergence of the generated sequence.

The remainder of this paper is organized as follows. In Section 2, we review several preliminary results. Section 3 introduces the proposed algorithm and discusses its basic properties. The convergence analysis is presented in Section 4. Numerical experiments are reported in Section 5, and concluding remarks are given in Section 6.

2. PRELIMINARIES

Let H be a real Hilbert space endowed with the inner product $\langle\cdot,\cdot\rangle$ and the induced norm $\|\cdot\|$. Assume that $C \subset H$ is a nonempty closed convex set.

Definition 2.1. A mapping $A : H \to H$ is said to be

(a) *pseudo-monotone* on C if

$$\langle Ax,\, y - x\rangle \ge 0 \;\Rightarrow\; \langle Ay,\, y - x\rangle \ge 0, \qquad \forall x, y \in C;$$

(b) *L-Lipschitz continuous* if there exists $L > 0$ such that

$$\|Ax - Ay\| \le L\|x - y\|, \qquad \forall x, y \in H;$$

(c) *sequentially weakly continuous* if

$$x_n \rightharpoonup x \quad \Rightarrow \quad Ax_n \rightharpoonup Ax.$$

Definition 2.2. Let $D \subset H$ be nonempty, closed and convex. Given $u \in H$ and $z \in D$. Then for all $y \in D$,

(a) $z = P_D u \iff \langle u - z,\, z - y\rangle \ge 0$;
(b) $\|P_D u - P_D y\|^2 \le \langle P_D u - P_D y,\, u - y\rangle$;
(c) $\|P_D u - y\|^2 \le \|u - y\|^2 - \|u - P_D u\|^2$.

Definition 2.3. A sequence $\{x_n\} \subset H$ is said to converge weakly to $x^* \in H$ if

$$\langle x_n, z\rangle \to \langle x^*, z\rangle \qquad \forall z \in H.$$

Lemma 2.1. *Let H be a real Hilbert space. Then for all $x, y, z \in H$ and $\alpha, \beta \in \mathbb{R}$ the following identities hold:*

(a) $\|x - z\|^2 - \|y - z\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, z\rangle$;
(b) $\|\alpha x + \beta y\|^2 = \alpha(\alpha+\beta)\|x\|^2 + \beta(\alpha+\beta)\|y\|^2 - \alpha\beta\|x - y\|^2$.

Lemma 2.2 ([21])**.** *Suppose that A is pseudo-monotone on C. Let Ω denote the solution set of* $\mathrm{VI}(A, C)$. *Then Ω is closed and convex. Moreover,*

$$M(A, C) = \Omega,$$

where

$$M(A,C) := \{x \in C :\ \langle A(y),\, y-x\rangle \geq 0,\ \forall y \in C\}.$$

Lemma 2.3 ([22]). *Let $S \subset H$ be nonempty and let $\{x_n\} \subset H$ satisfy*

- *(i)* $\lim_{n\to\infty} \|x_n - z\|$ *exists for all $z \in S$;*
- *(ii) every weak cluster point of $\{x_n\}$ belongs to S.*

Then $\{x_n\}$ converges weakly to a point in S.

3. PROPOSED ALGORITHM

The proposed method incorporates an alternated inertial extrapolation strategy together with a proximal halfspace projection technique to improve the stability and convergence behavior of the algorithm. Before presenting the algorithm, we introduce the following assumptions.

Condition 1. *$C \subset H$ is nonempty, closed and convex.*

Condition 2. *$A : H \to H$ is pseudo-monotone, sequentially weakly continuous and L-Lipschitz continuous.*

Condition 3. *Solution set Ω of $\mathrm{VI}(A,C)$ is nonempty.*

Condition 4. *Choose $\gamma > 0$ and $\tau_1 > 0$ such that $\tau_1\gamma L < 1$.*

Given $x_0, x_1 \in C$, for $n \geq 1$ we perform:

Algorithm 1

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Choose $\mu \in (0,1), \gamma > 0, 0 \leq \alpha < \frac{1-\mu}{2}$, and $\{\beta_n\} \subset (0,1)$.
for $n = 1, 2, \dots$ do
    Step 1: Compute $w_n = \begin{cases} x_n, & n \text{ even}, \\ x_n + \alpha(x_n - x_{n-1}), & n \text{ odd}. \end{cases}$
    Step 2: Compute $y_n = P_C\big(w_n - \gamma\tau_n A w_n\big)$, and update the stepsize by
        $\tau_{n+1} = \begin{cases} \min\Big\{\mu \frac{\|w_n - y_n\|}{\|Aw_n - Ay_n\|},\ \tau_n\Big\}, & Aw_n \neq Ay_n, \\ \tau_n, & \text{otherwise}. \end{cases}$
    Step 3: Define $x_{n+1} := (1-\beta_n)z_n + \beta_n x_n$
        $d_n = w_n - y_n - \gamma\tau_n\big(Aw_n - Ay_n\big),$
        $H_n := \Big\{w \in H :\ \langle w_n - \gamma\tau_n A w_n - y_n,\ w - y_n\rangle \leq 0\Big\},$
        $\theta_n = \begin{cases} 0, & d_n = 0, \\ \frac{\kappa_n \|w_n - y_n\|^2}{\|d_n\|^2}, & d_n \neq 0, \end{cases}$
    and set
        $v_n := w_n - \tau_n\theta_n A y_n, \qquad z_n := P_{H_n}(v_n),$
end for
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To prove the main theorem of this section, we give the following necessary lemmas.

Lemma 3.1. *Assume Condition 2 – Condition 4 hold, let* $\{w_n\}$ *and* $\{y_n\}$ *be generated by Algorithm 1, then for all* $n \geq 1$ *with* $d_n \neq 0$*, it holds that*

$$\frac{\kappa_n}{(1+\gamma\tau_n L)^2} \;\leq\; \theta_n \;\leq\; \frac{\kappa_n}{(1-\gamma\tau_n L)^2}. \tag{3.1}$$

Consequently, if $\tau_n \leq \tau_1$ *for all* n*, then*

$$\underline{\theta} := \frac{\underline{\kappa}}{(1+\gamma\tau_1 L)^2} \;\leq\; \theta_n \;\leq\; \frac{\overline{\kappa}}{(1-\gamma\tau_1 L)^2} =: \overline{\theta}, \qquad \forall n \text{ with } d_n \neq 0. \tag{3.2}$$

Proof. Fix $n \geq 1$ and assume $d_n \neq 0$. Recall that

$$d_n = w_n - y_n - \gamma\tau_n\big(Aw_n - Ay_n\big).$$

By the triangle inequality and L-Lipschitz continuity of A,

$$\|d_n\| \geq \|w_n - y_n\| - \gamma\tau_n\|Aw_n - Ay_n\| \geq (1-\gamma\tau_n L)\|w_n - y_n\|,$$

and similarly,

$$\|d_n\| \leq \|w_n - y_n\| + \gamma\tau_n\|Aw_n - Ay_n\| \leq (1+\gamma\tau_n L)\|w_n - y_n\|.$$

Hence,

$$\frac{1}{(1+\gamma\tau_n L)^2} \leq \frac{\|w_n - y_n\|^2}{\|d_n\|^2} \leq \frac{1}{(1-\gamma\tau_n L)^2}.$$

Multiplying by κ_n yields (3.1). If $\tau_n \leq \tau_1$ for all n and $\underline{\kappa} \leq \kappa_n \leq \overline{\kappa}$, then (3.2) follows immediately. ☐

Lemma 3.2. *Let Condition 1 – Condition 4 hold and that* $\{\tau_n\}$ *is generated by the stepsize update rule in Algorithm 1. Then* $\{\tau_n\}$ *is nonincreasing and there exists a constant*

$$\underline{\tau} := \min\{\tau_1, \mu/L\} > 0$$

such that

$$0 < \underline{\tau} \leq \tau_n \leq \tau_1, \qquad \forall n \geq 1. \tag{3.3}$$

Proof. We first show that the sequence $\{\tau_n\}$ is nonincreasing. According to the update rule in Algorithm 1, two cases may occur.

Case 1. If $Aw_n = Ay_n$, then the rule directly yields

$$\tau_{n+1} = \tau_n.$$

Hence the sequence does not increase at this step.

Case 2. If $Aw_n \neq Ay_n$, the stepsize is updated by

$$\tau_{n+1} = \min\left\{\mu\frac{\|w_n - y_n\|}{\|Aw_n - Ay_n\|},\, \tau_n\right\}.$$

By Condition 2, the operator A is L-Lipschitz continuous, that is,

$$\|Aw_n - Ay_n\| \leq L\|w_n - y_n\|.$$

Consequently,

$$\mu\frac{\|w_n - y_n\|}{\|Aw_n - Ay_n\|} \geq \frac{\mu}{L}.$$

Substituting this estimate into the update rule gives

$$\tau_{n+1} \geq \min\left\{\frac{\mu}{L},\, \tau_n\right\}, \qquad \tau_{n+1} \leq \tau_n.$$

The second inequality immediately shows that $\{\tau_n\}$ is nonincreasing.

Next we derive a uniform lower bound for τ_n. Since $\tau_1 > 0$ and $\mu/L > 0$, define

$$\underline{\tau} := \min\{\tau_1, \mu/L\} > 0.$$

From the previous inequality we have

$$\tau_{n+1} \geq \min\left\{\frac{\mu}{L},\, \tau_n\right\} \geq \underline{\tau}.$$

Thus, by induction it follows that

$$\underline{\tau} \leq \tau_n \leq \tau_1, \qquad \forall n \geq 1.$$

This completes the proof of (3.3). □

4. CONVERGENCE ANALYSIS

Throughout this section, let Ω be the solution set of $\mathrm{VI}(A, C)$ and fix $x^* \in \Omega$.

Lemma 4.1. *Assume that Condition 1–Condition 4 hold. Let $\{x_n\}$ be generated by Algorithm 1. Then the following assertions are valid:*

- *(i) The even subsequence $\{x_{2n}\}$ is bounded and, for every $x^* \in \Omega$, the limit $\lim_{n\to\infty} \|x_{2n} - x^*\|$ exists;*
- *(ii) $\lim_{n\to\infty} \|x_{2n} - y_{2n}\| = 0$.*

Proof. By construction, H_n is a supporting halfspace of C at y_n, and hence $C \subset H_n$. Fix an arbitrary $x^* \in \Omega$ and define

$$v_n := w_n - \tau_n\theta_n A y_n, \qquad z_n := P_{H_n}(v_n).$$

Since $x^* \in C \subset H_n$ and $z_n = P_{H_n}(v_n)$, applying the projection inequality (Definition 2.2) gives

$$\begin{aligned}
\|z_n - x^*\|^2 &\leq \|v_n - x^*\|^2 - \|v_n - z_n\|^2 \\
&\leq \|w_n - \tau_n\theta_n A y_n - x^*\|^2 - \|w_n - \tau_n\theta_n A y_n - z_n\|^2 \\
&\leq \|w_n - x^*\|^2 - \|w_n - z_n\|^2 \\
&\quad + \tau_n^2\theta_n^2\|Ay_n\|^2 - \tau_n^2\theta_n^2\|Ay_n\|^2 \\
&\quad + 2\tau_n\theta_n\langle Ay_n,\, w_n - z_n\rangle - 2\tau_n\theta_n\langle Ay_n,\, w_n - x^*\rangle \\
&\leq \|w_n - x^*\|^2 - \|w_n - z_n\|^2 + 2\tau_n\theta_n\langle Ay_n,\, x^* - z_n\rangle.
\end{aligned} \tag{4.1}$$

Next, since $x^* \in \Omega$ and $y_n \in C$, the definition of $\mathrm{VI}(A, C)$ yields $\langle Ax^*,\, y_n - x^*\rangle \geq 0$. By the pseudo-monotonicity of A on C, we further have $\langle Ay_n,\, y_n - x^*\rangle \geq 0$. Therefore,

$$\begin{aligned}
\langle Ay_n,\, x^* - z_n\rangle &= \langle Ay_n,\, y_n - z_n\rangle - \langle Ay_n,\, y_n - x^*\rangle \\
&\leq \langle Ay_n,\, y_n - z_n\rangle.
\end{aligned} \tag{4.2}$$

Applying (4.2) to (4.1) leads to

$$\|z_n - x^*\|^2 \leq \|w_n - x^*\|^2 - \|w_n - z_n\|^2 + 2\tau_n\theta_n\langle Ay_n,\, y_n - z_n\rangle. \tag{4.3}$$

The last term

$$\langle Ay_n,\, y_n - z_n\rangle = \langle Ay_n - Aw_n,\, y_n - z_n\rangle + \langle Aw_n,\, y_n - z_n\rangle. \tag{4.4}$$

Since $y_n = P_C(w_n - \gamma\tau_n A w_n)$ and $z_n \in H_n$, by the definition of H_n we have

$$\langle w_n - \gamma\tau_n A w_n - y_n,\; z_n - y_n\rangle \leq 0. \tag{4.5}$$

It follows that

$$\langle Aw_n,\, y_n - z_n\rangle \leq \frac{1}{\gamma\tau_n}\langle w_n - y_n,\, y_n - z_n\rangle. \tag{4.6}$$

Using Young's inequality, we obtain

$$\begin{aligned}\langle Ay_n, y_n - z_n\rangle &= \langle Ay_n - Aw_n, y_n - z_n\rangle + \langle Aw_n, y_n - z_n\rangle\\ &\le \|Ay_n - Aw_n\|\,\|y_n - z_n\| + \langle Aw_n, y_n - z_n\rangle\\ &\le L\|y_n - w_n\|\,\|y_n - z_n\| + \langle Aw_n, y_n - z_n\rangle\\ &\le \frac{L^2}{2\varepsilon}\|w_n - y_n\|^2 + \frac{\varepsilon}{2}\|y_n - z_n\|^2 + \langle Aw_n, y_n - z_n\rangle, \quad \forall\varepsilon > 0.\end{aligned} \tag{4.7}$$

Moreover,

$$\|w_n - z_n\|^2 = \|w_n - y_n\|^2 + \|y_n - z_n\|^2 + 2\langle w_n - y_n, y_n - z_n\rangle. \tag{4.8}$$

Substituting (4.6)–(4.8) into (4.3), we obtain

$$\begin{aligned}\|z_n - x^*\|^2 \le \|w_n - x^*\|^2 &- \|w_n - y_n\|^2 - \|y_n - z_n\|^2\\ &- 2\langle w_n - y_n, y_n - z_n\rangle\\ &+ 2\tau_n\theta_n\left(\frac{L^2}{2\varepsilon}\|w_n - y_n\|^2\right)\\ &+ 2\tau_n\theta_n\left(\frac{\varepsilon}{2}\|y_n - z_n\|^2\right)\\ &+ 2\tau_n\theta_n\left(\frac{1}{\gamma\tau_n}\langle w_n - y_n, y_n - z_n\rangle\right).\end{aligned} \tag{4.9}$$

Using Young's inequality, we derive

$$\begin{aligned}\|z_n - x^*\|^2 &\le \|w_n - x^*\|^2 - 2(1 - \tfrac{\theta_n}{\gamma})\langle w_n - y_n, y_n - z_n\rangle\\ &\quad - (1 - \tau_n\theta_n\varepsilon)\|y_n - z_n\|^2\\ &\quad - (1 - \tau_n\theta_n\tfrac{L^2}{\varepsilon})\|w_n - y_n\|^2\\ &\le \|w_n - x^*\|^2 + (1 - \tfrac{\theta_n}{\gamma})\left(\delta\|w_n - y_n\|^2 + \frac{1}{\delta}\|y_n - z_n\|^2\right)\\ &\quad - (1 - \tau_n\theta_n\tfrac{L^2}{\varepsilon})\|w_n - y_n\|^2\\ &\quad - (1 - \tau_n\theta_n\varepsilon)\|y_n - z_n\|^2\\ &\le \|w_n - x^*\|^2 - [1 - \tau_n\theta_n\tfrac{L^2}{\varepsilon} - (1 - \tfrac{\theta_n}{\gamma})\delta]\|w_n - y_n\|^2\\ &\quad - \left[1 - \tau_n\theta_n\varepsilon - (1 - \tfrac{\theta_n}{\gamma})\tfrac{1}{\delta}\right]\|y_n - z_n\|^2\\ &\le \|w_n - x^*\|^2 - a_n\|w_n - y_n\|^2 - b_n\|y_n - z_n\|^2,\end{aligned} \tag{4.10}$$

where

$$a_n := 1 - \tau_n\theta_n\frac{L^2}{\varepsilon} - \left(1 - \frac{\theta_n}{\gamma}\right)\delta, \qquad b_n := 1 - \tau_n\theta_n\varepsilon - \left(1 - \frac{\theta_n}{\gamma}\right)\frac{1}{\delta}.$$

On the other hand, Algorithm 1 gives

$$x_{n+1} = (1 - \beta_n)z_n + \beta_n x_n.$$

Invoking the Hilbert space identity (Lemma 2.1), we obtain, for any $\beta_n \in (0, 1)$,

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= (1 - \beta_n)\|z_n - x^*\|^2 + \beta_n\|x_n - x^*\|^2 - \beta_n(1 - \beta_n)\|z_n - x_n\|^2\\ &\le (1 - \beta_n)\|z_n - x^*\|^2 + \beta_n\|x_n - x^*\|^2.\end{aligned} \tag{4.11}$$

Combining (4.11) with (4.10), we obtain

$$\begin{aligned}\|x_{n+1} - x^*\|^2 \le (1 - \beta_n)\|w_n - x^*\|^2 &+ \beta_n\|x_n - x^*\|^2\\ &- (1 - \beta_n)a_n\|w_n - y_n\|^2 - (1 - \beta_n)b_n\|y_n - z_n\|^2.\end{aligned} \tag{4.12}$$

Setting $n \mapsto 2n$ in (4.12) and using $w_{2n} = x_{2n}$, we have

$$\|x_{2n+1} - x^*\|^2 \leq \|x_{2n} - x^*\|^2 - (1-\beta_{2n})a_{2n}\|x_{2n} - y_{2n}\|^2 - (1-\beta_{2n})b_{2n}\|y_{2n} - z_{2n}\|^2. \tag{4.13}$$

Similarly, taking $n \mapsto 2n+1$ in (4.12) yields

$$\begin{aligned}\|x_{2n+2} - x^*\|^2 \leq{}& (1-\beta_{2n+1})\|w_{2n+1} - x^*\|^2 + \beta_{2n+1}\|x_{2n+1} - x^*\|^2 \\ &- (1-\beta_{2n+1})a_{2n+1}\|w_{2n+1} - y_{2n+1}\|^2 - (1-\beta_{2n+1})b_{2n+1}\|y_{2n+1} - z_{2n+1}\|^2.\end{aligned} \tag{4.14}$$

By Lemma 2.1, we have

$$\begin{aligned}\|w_{2n+1} - x^*\|^2 &= \big\|x_{2n+1} + \alpha(x_{2n+1} - x_{2n}) - x^*\big\|^2 \\ &= (1+\alpha)\|x_{2n+1} - x^*\|^2 - \alpha\|x_{2n} - x^*\|^2 + \alpha(1+\alpha)\|x_{2n+1} - x_{2n}\|^2.\end{aligned} \tag{4.15}$$

Substituting (4.15) into (4.14) and then using (4.13), we obtain

$$\begin{aligned}\|x_{2n+2} - x^*\|^2 \leq{}& (1-\beta_{2n+1})\Big[(1+\alpha)\|x_{2n+1} - x^*\|^2 && (4.16)\\ &- (1-\beta_{2n+1})b_{2n+1}\|y_{2n+1} - z_{2n+1}\|^2 \\ &- \alpha\|x_{2n} - x^*\|^2 + \alpha(1+\alpha)\|x_{2n+1} - x_{2n}\|^2\Big] \\ &+ \beta_{2n+1}\|x_{2n+1} - x^*\|^2 - (1-\beta_{2n+1})a_{2n+1}\|w_{2n+1} - y_{2n+1}\|^2 \\ \leq{}& \Big(1 + \alpha - \alpha\beta_{2n+1}\Big)\|x_{2n+1} - x^*\|^2 - \alpha(1-\beta_{2n+1})\|x_{2n} - x^*\|^2 \\ &+ \alpha(1+\alpha)(1-\beta_{2n+1})\|x_{2n+1} - x_{2n}\|^2 \\ &- (1-\beta_{2n+1})a_{2n+1}\|w_{2n+1} - y_{2n+1}\|^2 \\ &- (1-\beta_{2n+1})b_{2n+1}\|y_{2n+1} - z_{2n+1}\|^2. && (4.17)\end{aligned}$$

It follows that

$$\begin{aligned}\|x_{2n+2} - x^*\|^2 \leq{}& \|x_{2n} - x^*\|^2 - (1+\alpha)\big((1-\beta_{2n})a_{2n} - 2\alpha\big)\|x_{2n} - y_{2n}\|^2 \\ &- (1+\alpha)\big((1-\beta_{2n})b_{2n} - 2\alpha\big)\|y_{2n} - x_{2n+1}\|^2 \\ &- (1-\beta_{2n+1})a_{2n+1}\|w_{2n+1} - y_{2n+1}\|^2 \\ &- (1-\beta_{2n+1})b_{2n+1}\|y_{2n+1} - z_{2n+1}\|^2.\end{aligned} \tag{4.18}$$

Assume that $\bar{\beta} := \sup_{n\geq 1}\beta_n < 1$ and choose ε, δ (as in Remark 4.1) such that

$$(1-\bar{\beta})a_n \geq 2\alpha \geq 0, \qquad (1-\bar{\beta})b_n \geq 2\alpha \geq 0, \qquad \forall n \text{ with } d_n \neq 0.$$

Then (4.18) implies

$$\|x_{2n+2} - x^*\|^2 \leq \|x_{2n} - x^*\|^2. \tag{4.19}$$

In particular, $\{\|x_{2n} - x^*\|\}$ is nonincreasing and bounded below, hence it converges. This proves (i).

Rearranging (4.18) yields

$$\begin{aligned}&(1+\alpha)\big((1-\beta_{2n})a_{2n} - 2\alpha\big)\|x_{2n} - y_{2n}\|^2 + (1+\alpha)\big((1-\beta_{2n})b_{2n} - 2\alpha\big)\|y_{2n} - x_{2n+1}\|^2 \\ &\quad + (1-\beta_{2n+1})a_{2n+1}\|w_{2n+1} - y_{2n+1}\|^2 + (1-\beta_{2n+1})b_{2n+1}\|y_{2n+1} - z_{2n+1}\|^2 \\ &\leq \|x_{2n} - x^*\|^2 - \|x_{2n+2} - x^*\|^2.\end{aligned} \tag{4.20}$$

Summing (4.20) from $n = N$ to $N + m$, we obtain

$$\begin{aligned}
&\sum_{n=N}^{N+m} (1+\alpha)\big((1-\beta_{2n})a_{2n} - 2\alpha\big)\|x_{2n} - y_{2n}\|^2 \\
&\quad + \sum_{n=N}^{N+m} (1+\alpha)\big((1-\beta_{2n})b_{2n} - 2\alpha\big)\|y_{2n} - x_{2n+1}\|^2 \\
&\quad + \sum_{n=N}^{N+m} (1-\beta_{2n+1})a_{2n+1}\|w_{2n+1} - y_{2n+1}\|^2 \\
&\quad + \sum_{n=N}^{N+m} (1-\beta_{2n+1})b_{2n+1}\|y_{2n+1} - z_{2n+1}\|^2 \\
&\le \sum_{n=N}^{N+m} \Big(\|x_{2n} - x^*\|^2 - \|x_{2n+2} - x^*\|^2\Big) \\
&\le \big(\|x_{2N} - x^*\|^2 - \|x_{2N+2} - x^*\|^2\big) \\
&\quad + \cdots \\
&\quad + \big(\|x_{2N+2m} - x^*\|^2 - \|x_{2N+2m+2} - x^*\|^2\big) \\
&= \|x_{2N} - x^*\|^2 - \|x_{2N+2m+2} - x^*\|^2 \\
&\le \|x_{2N} - x^*\|^2.
\end{aligned} \tag{4.21}$$

Letting $m \to \infty$ in (4.21) and using the nonnegativity of the coefficients for $n \ge N$, we conclude that the above four series are convergent. In particular,

$$\sum_{n=N}^{\infty} \|x_{2n} - y_{2n}\|^2 < \infty,$$

and hence $\|x_{2n} - y_{2n}\| \to 0$ as $n \to \infty$, which proves (ii). □

Remark 4.1. Recall from Lemma 3.1 that there exist constants $0 < \underline{\theta} \le \theta_n \le \overline{\theta}$ for all $n \ge 1$. In the proof of Lemma 4.1, the parameters ε, δ, μ, α and $\{\tau_n\}$ are required to satisfy

$$(1-\bar\beta)a_n \ge 2\alpha, \qquad (1-\bar\beta)b_n \ge 2\alpha,$$

where $\bar\beta := \sup_{n\ge1} \beta_n < 1$.

To see that this parameter region is nonempty, we provide an explicit admissible choice. Let

$$\mu = 0.7, \quad \alpha = \frac{1-\mu}{4} = 0.075, \quad \bar\beta = 0.5, \quad \varepsilon = L, \quad \delta = 1,$$

and assume $\theta := \inf_{n\ge1} \theta_n \ge 1$. Choose

$$\tau_n = \frac{1}{20\gamma L} \qquad (\forall n \ge 1).$$

Then

$$a_n = b_n = \theta_n\Big(\frac{1}{\gamma} - \tau_n L\Big) \ge \theta\Big(\frac{1}{\gamma} - \frac{1}{20\gamma}\Big) = \theta\frac{19}{20\gamma} \ge \frac{19}{20\gamma}.$$

Hence,

$$(1-\bar\beta)a_n \ge 0.5 \times \frac{19}{20\gamma} = \frac{19}{40\gamma}, \qquad 2\alpha = 2 \times 0.075 = 0.15.$$

Therefore, if

$$\frac{19}{40\gamma} \geq 0.15 \quad \Longleftrightarrow \quad \gamma \leq \frac{19}{6},$$

then $(1-\bar{\beta})a_n \geq 2\alpha$ and $(1-\bar{\beta})b_n \geq 2\alpha$ hold.

Lemma 4.2. *Assume that Condition 2–Condition 4 hold. Let $\{x_n\}$ be generated by Algorithm 1. Then every weak cluster point of the even subsequence $\{x_{2n}\}$ belongs to Ω.*

Proof. By Lemma 4.1, $\{x_{2n}\}$ is bounded. Hence there exist a subsequence $\{x_{2n_k}\}$ and $\bar{x} \in H$ such that

$$x_{2n_k} \rightharpoonup \bar{x} \qquad (k \to \infty).$$

For even indices, $w_{2n_k} = x_{2n_k}$. Moreover, Lemma 4.1(ii) gives $\|x_{2n} - y_{2n}\| \to 0$, and thus $y_{2n_k} \rightharpoonup \bar{x}$. Fix any $x \in C$. Since $y_n = P_C(w_n - \gamma\tau_n A w_n)$, taking $n = 2n_k$ and using $w_{2n_k} = x_{2n_k}$ to the projection inequality, we obtain

$$\langle Ax_{2n_k}, y_{2n_k} - x\rangle \leq \frac{1}{\gamma\tau_{2n_k}}\langle x_{2n_k} - y_{2n_k}, x - y_{2n_k}\rangle.$$

Since $\{x_{2n_k}\}$ and $\{y_{2n_k}\}$ are bounded and $\|x_{2n_k} - y_{2n_k}\| \to 0$, it follows that

$$\limsup_{k\to\infty}\langle Ax_{2n_k}, y_{2n_k} - x\rangle \leq 0, \qquad \forall x \in C.$$

Writing

$$\langle Ax_{2n_k}, y_{2n_k} - x\rangle = \langle Ax_{2n_k}, x_{2n_k} - x\rangle + \langle Ax_{2n_k}, y_{2n_k} - x_{2n_k}\rangle,$$

and using $\|y_{2n_k} - x_{2n_k}\| \to 0$, we deduce

$$\liminf_{k\to\infty}\langle Ax_{2n_k}, x - x_{2n_k}\rangle \geq 0, \qquad \forall x \in C. \tag{4.22}$$

Let $\{\varepsilon_k\} \subset (0,\infty)$ be decreasing with $\varepsilon_k \to 0$. For each k, choose N_k to be the smallest integer such that

$$\langle Ax_{2n_j}, x - x_{2n_j}\rangle + \varepsilon_k \geq 0, \qquad \forall j \geq N_k, \tag{4.23}$$

which is possible by (4.22). Clearly, $\{N_k\}$ is increasing.

If $Ax_{2n_{N_{k_0}}} = 0$ for some k_0, then $x_{2n_{N_{k_0}}} \in \Omega$ and we are done. Otherwise assume $Ax_{2n_{N_k}} \neq 0$ for all k and define

$$v_{2n_{N_k}} := \frac{Ax_{2n_{N_k}}}{\|Ax_{2n_{N_k}}\|^2}, \qquad \langle Ax_{2n_{N_k}}, v_{2n_{N_k}}\rangle = 1.$$

Then (4.23) implies

$$\langle Ax_{2n_{N_k}}, x + \varepsilon_k v_{2n_{N_k}} - x_{2n_{N_k}}\rangle \geq 0,$$

and by the pseudo-monotonicity of A we infer

$$\langle A(x + \varepsilon_k v_{2n_{N_k}}), x + \varepsilon_k v_{2n_{N_k}} - x_{2n_{N_k}}\rangle \geq 0. \tag{4.24}$$

Since $x_{2n_k} \rightharpoonup \bar{x}$, we also have $x_{2n_{N_k}} \rightharpoonup \bar{x}$. If $A\bar{x} = 0$, then $\bar{x} \in \Omega$ and we are done. Otherwise $\|A\bar{x}\| > 0$ and, by Condition 2,

$$0 < \|A\bar{x}\| \leq \liminf_{k\to\infty}\|Ax_{2n_{N_k}}\|.$$

Consequently,

$$\|\varepsilon_k v_{2n_{N_k}}\| = \frac{\varepsilon_k}{\|Ax_{2n_{N_k}}\|} \to 0.$$

Letting $k \to \infty$ in (4.24) and using $x_{2n_{N_k}} \rightharpoonup \bar{x}$ and $\varepsilon_k v_{2n_{N_k}} \to 0$, we obtain

$$\langle Ax, x - \bar{x}\rangle \geq 0, \qquad \forall x \in C.$$

By Lemma 2.2, it follows that $\bar{x} \in \Omega$. □

Theorem 4.1. *Assume that Condition 1–Condition 4 hold. Let* $\{x_n\}$ *be the sequence generated by Algorithm 1. Then* $\{x_n\}$ *converges weakly to some point in* Ω.

Proof. We now use Opial's lemma[22] to prove the weak convergence of the generated sequence. (i) By Lemma 4.1, for every $x^* \in \Omega$, the limit $\lim_{n\to\infty} \|x_{2n} - x^*\|$ exists. Hence, condition (i) of Lemma 2.3 is satisfied.

(ii) By Lemma 4.2, every weak cluster point of $\{x_{2n}\}$ lies in Ω. Thus, condition (ii) of Lemma 2.3 holds.

Therefore, Opial's lemma yields the existence of $\bar{x} \in \Omega$ such that

$$x_{2n} \rightharpoonup \bar{x} \qquad (n \to \infty).$$

It remains to show that the whole sequence $\{x_n\}$ converges to the same limit. From Lemma 4.1 and the relation $x_{n+1} = (1-\beta_n)z_n + \beta_n x_n$, we have

$$\|x_{2n+1} - x_{2n}\| \to 0 \qquad (n \to \infty).$$

Let $u \in H$ be arbitrary. Then

$$\langle x_{2n+1} - \bar{x},\, u\rangle = \langle x_{2n} - \bar{x},\, u\rangle + \langle x_{2n+1} - x_{2n},\, u\rangle.$$

Since $x_{2n} \rightharpoonup \bar{x}$, the first term converges to 0, while the second term satisfies $|\langle x_{2n+1} - x_{2n}, u\rangle| \le \|x_{2n+1} - x_{2n}\|\,\|u\| \to 0$. Hence $x_{2n+1} \rightharpoonup \bar{x}$.

Consequently, both subsequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ converge weakly to $\bar{x}$, and thus the whole sequence $\{x_n\}$ converges weakly to $\bar{x} \in \Omega$. □

5. NUMERICAL EXPERIMENTS

In this section, we present several numerical experiments to evaluate the performance of the proposed alternated inertial subgradient extragradient projection algorithm. All experiments were implemented in MATLAB R2024a on a PC with an AMD Ryzen 9 7940H CPU @ 4.00 GHz and 16 GB RAM.

Example 5.1. Let the feasible set be the polyhedral convex set

$$C := \{x \in \mathbb{R}^m \mid Qx \le b\},$$

where $Q \in \mathbb{R}^{\ell\times m}$ is a randomly generated matrix and $b \in \mathbb{R}^{\ell}_{+}$ is a nonnegative vector.

The operator $F : \mathbb{R}^m \to \mathbb{R}^m$ is defined by $F(x) = Mx$, where

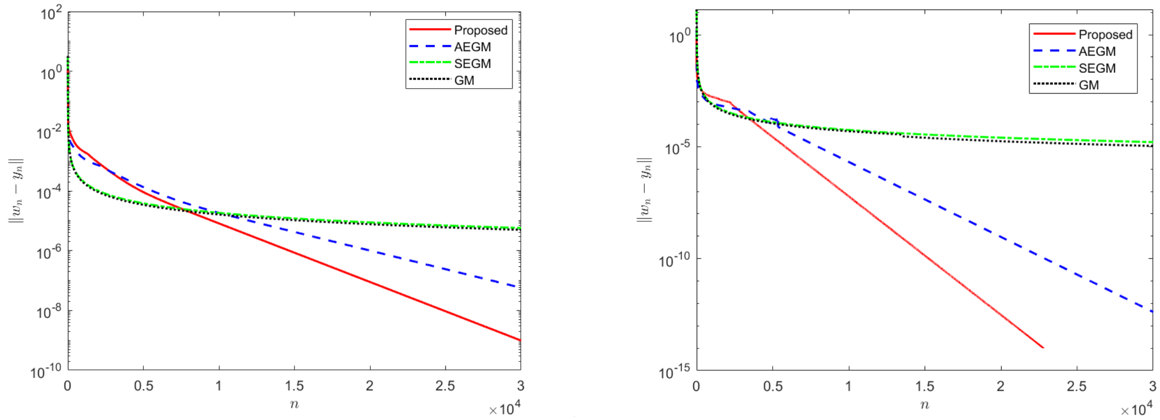
$$M = B_M B_M^\top + S + D.$$

Here $B_M \in \mathbb{R}^{m\times m}$ is a randomly generated matrix whose entries follow a uniform distribution on $(0,1)$, $S = T - T^\top$ is a skew-symmetric matrix with $T \in \mathbb{R}^{m\times m}$ being randomly generated, and D is a diagonal positive definite matrix whose diagonal elements are randomly chosen from $(0,1)$. It is easy to verify that the operator F is linear and Lipschitz continuous. Moreover, due to the presence of the skew-symmetric component S, the operator F is generally non-potential. However, the matrix $B_M B_M^\top + D$ is symmetric positive definite, which ensures that the operator F is pseudomonotone. Consequently. The parameters of all algorithms are set as follows.

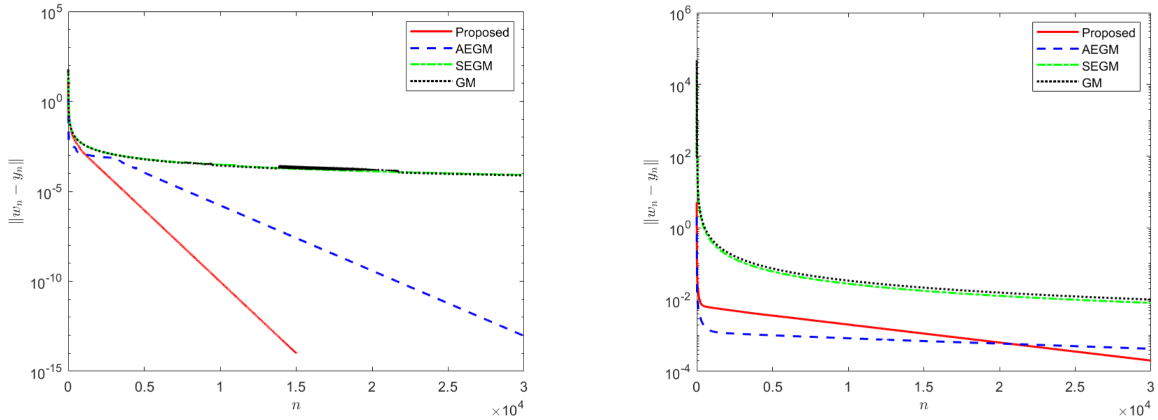
- **Proposed.** Set $\mu = 0.5$, $\alpha = 0.99(1-\mu)/2$, $\beta = 0.1$, $\gamma = 0.9$, $\tau_1 = 5 \times 10^{-3}$, $\varepsilon = 10^{-6}$.
- **AEGM[19].** Set $\mu = 0.5$, $\alpha = 0.99(1-\mu)/2$, $\tau_1 = 5 \times 10^{-3}$, $\varepsilon = 10^{-6}$.
- **SEGM[14].** Set $\delta = 0.1$, $\gamma_n = \frac{1}{n+1}$, $\tau = 0.8$, $\varepsilon = 10^{-6}$.
- **GM[6].** Set $\delta = 0.1$, $\gamma_n = \frac{1}{n+1}$, $\tau = 0.8$, $\varepsilon = 10^{-6}$.

Table 5.1 Comparison of all algorithms for different problem sizes in Example 5.1

Size	m	Proposed		AEGM		SEGM		GM	
		Iter	CPU time	Iter	CPU time	Iter	CPU time	Iter	CPU time
$l = 10$	5	8553	21.4317	6344	13.1390	Max	109.2130	Max	106.8747
	10	20098	45.6499	14486	32.4652	Max	104.5998	Max	112.1499
	15	11853	29.0662	9052	21.0344	Max	112.4617	Max	108.1029
	20	Max	97.8806	Max	86.5437	Max	84.0050	Max	83.7767
	30	Max	104.8138	Max	91.0431	Max	1395.9429	Max	1414.7789



(a) m=5 **(b)** m=10



(c) m=15 **(d)** m=20

Figure 5.1 Comparison of all algorithms for different problem sizes in Example 5.1

Example 5.2. The operator $F : \mathbb{R}^m \to \mathbb{R}^m$ is defined by

$$F(x) = Mx + q,$$

where $q = 0$ and

$$M = B_M B_M^{\top} + S + D.$$

Here $B_M \in \mathbb{R}^{m \times m}$ is a randomly generated matrix whose entries are drawn from a uniform distribution on $(0,1)$, $S = T - T^\top$ is a skew-symmetric matrix with $T \in \mathbb{R}^{m\times m}$ being randomly generated, and D is a diagonal positive definite matrix whose diagonal elements are chosen from $(0,1)$.

The feasible set is the polyhedral convex set

$$C := \{x \in \mathbb{R}^m \mid \tilde{B}x \leq \tilde{b}\},$$

where $\tilde{B} \in \mathbb{R}^{\ell\times m}$ is a randomly generated matrix and $\tilde{b} \in \mathbb{R}^\ell_+$ is a nonnegative vector.

It is straightforward to verify that the operator F is linear and Lipschitz continuous. Moreover, due to the skew-symmetric component S, the operator F is generally non-potential. However, the symmetric part $B_M B_M^\top + D$ is positive definite, which guarantees that F is pseudomonotone. Therefore, the associated variational inequality problem admits a unique solution. The parameters of all algorithms are set as follows.

- **Proposed:** $\mu = 0.5$, $\gamma = 0.5$, $\alpha = 0.1, \tau_1 = 0.7, \beta = 0.1$, and $\varepsilon = 10^{-3}$.
- **AEGM** [19]**:** $\mu = 0.5$, $\gamma = 0.5$, $\alpha = 0.1, \tau_1 = 0.7$, and $\varepsilon = 10^{-3}$.
- **SEGM** [14]**:** $\delta = 0.01$, $\mu = 0.5$, and $\varepsilon = 10^{-3}$.
- **AM** [3]**:** $\phi = 0.9\frac{\sqrt{5}+1}{2}$, $\tau_0 = 1$, and $\varepsilon = 10^{-3}$.

Table 5.2 Comparison of all algorithms for different problem sizes in Example 5.2

Size	m	Proposed		AEGM		SEGM		AM	
		Iter	CPU time	Iter	CPU time	Iter	CPU time	Iter	CPU time
$l = 20$	10	128	0.0383	329	0.0214	873	0.0461	830	0.0429
	20	249	0.0210	1093	0.0393	1930	0.1001	4096	0.1669
	30	159	0.0127	2096	0.0856	6777	0.3387	10103	0.4402
	40	324	0.0203	3219	0.1211	9909	0.5266	19148	0.8696
	50	296	0.0225	4900	0.2263	16477	1.1037	29558	1.4071
	60	429	0.0294	5392	0.2301	23536	1.8080	43268	3.1333
$l = 50$	10	67	0.0192	336	0.0263	899	0.0920	1001	0.1130
	20	81	0.0247	1131	0.1018	2192	0.2591	3624	0.3902
	30	104	0.0239	1686	0.1999	3875	0.5884	8717	1.1426
	40	153	0.0405	3609	0.4459	8813	1.5967	18997	3.0092
	50	189	0.0570	5170	0.8016	14502	3.0359	30867	5.1135
	60	186	0.0635	6632	1.0912	23595	6.0426	43466	7.7622

Example 5.3. The third numerical experiment considers a variational inequality problem in the Hilbert space $H = L^2([0,1])$, which is equipped with the inner product

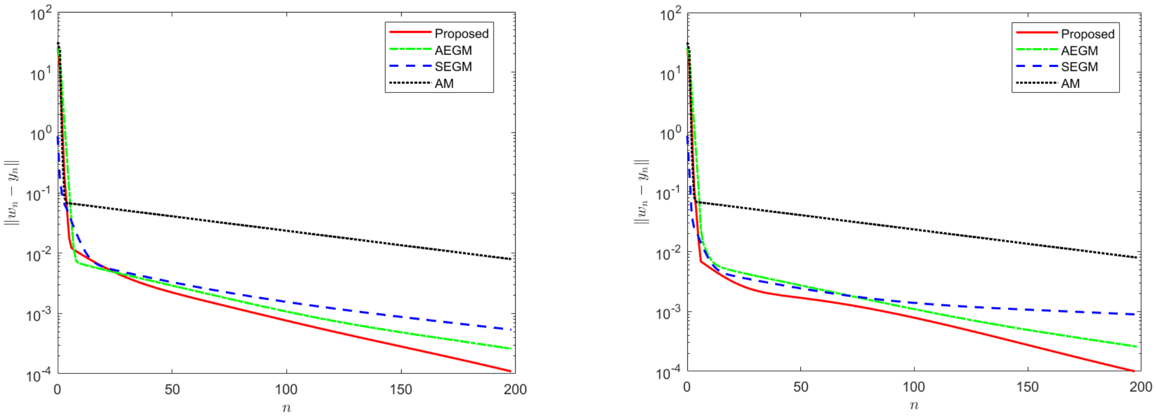
$$\langle x, y\rangle := \int_0^1 x(t)y(t)\,dt$$

and the induced norm

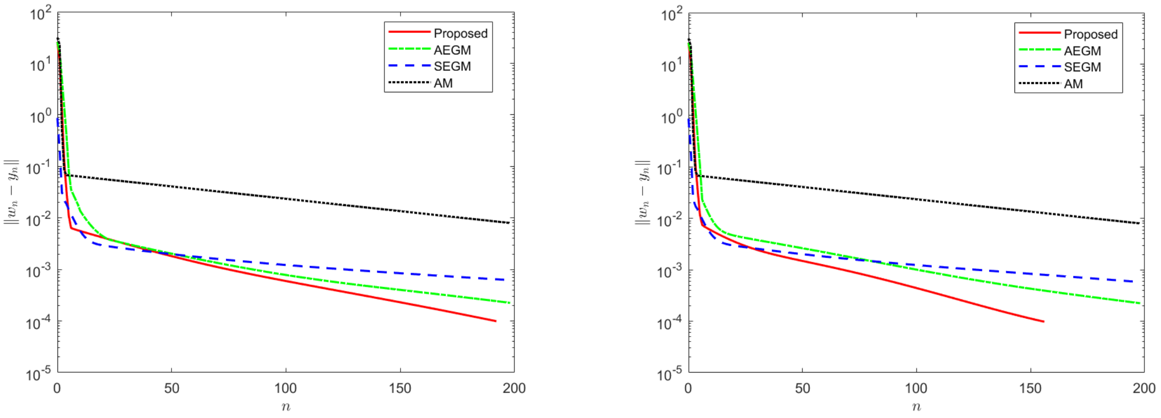
$$\|x\| := \left(\int_0^1 |x(t)|^2 dt\right)^{1/2}.$$

The feasible set is defined as the closed convex ball

$$C := \{x \in H \mid \|x\| \leq 2\}.$$



(a) l=10, m=10 **(b)** l=30, m=10



(c) l=40, m=10 **(d)** l=60, m=60

Figure 5.2 Comparison of all algorithms for different problem sizes in Example 5.2

Let $g : C \to \mathbb{R}$ be defined by

$$g(s) = \frac{1}{1 + \|s\|^2}, \quad s \in C.$$

Consider the Volterra integral operator $F : H \to H$ defined by

$$(Fs)(t) = \int_0^t s(x)\,dx, \quad s \in H,\ t \in [0,1].$$

It is well known that F is a bounded linear operator and is monotone on H.

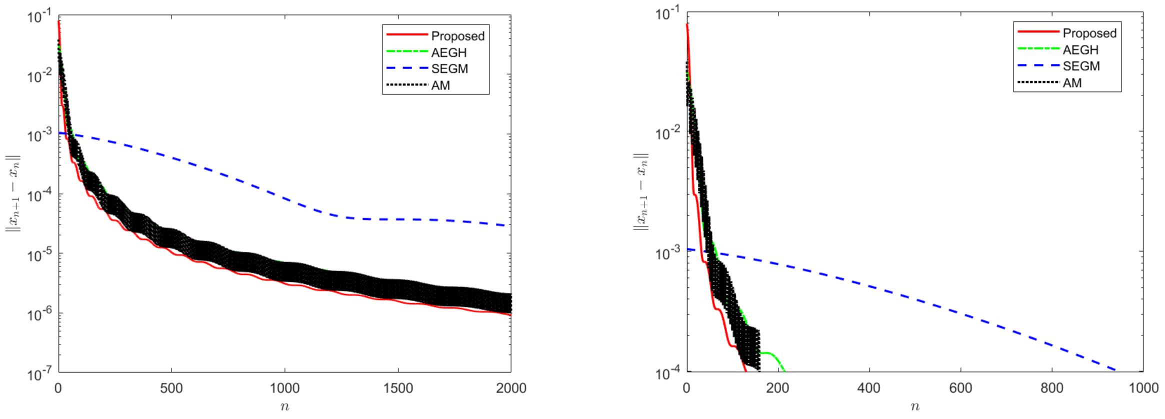
Define the operator $A : C \to H$ by

$$(As)(t) = g(s)(Fs)(t), \quad s \in C,\ t \in [0,1].$$

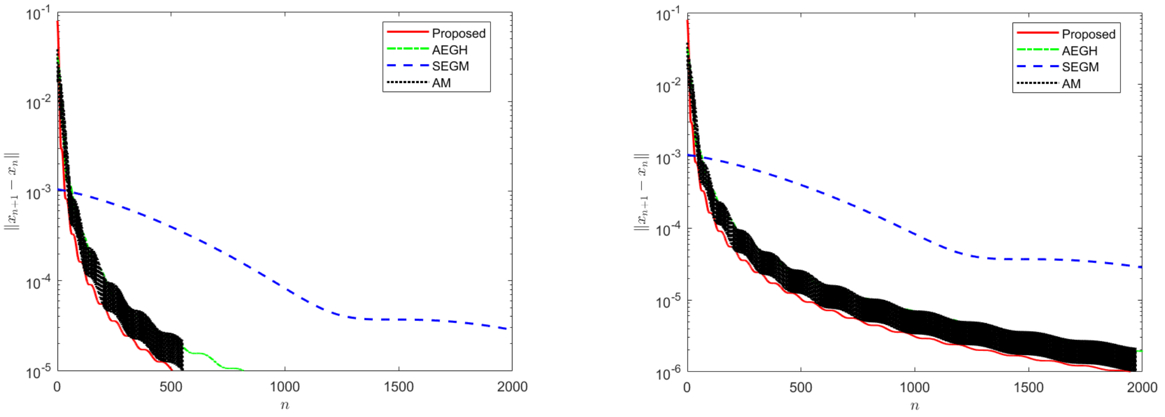
The function g is Lipschitz continuous on C with Lipschitz constant $L_g = \frac{16}{25}$ and satisfies $g(s) \in [\frac{1}{5}, 1]$ for all $s \in C$. As a consequence, the operator A is Lipschitz continuous with constant $L = \frac{82}{\pi}$. Although the operator A is generally not monotone, it can be shown that A is pseudomonotone on C. Therefore, the corresponding variational inequality problem is well posed. The parameters of all algorithms are set as follows.

- **Proposed.** Set $\mu = \frac{1}{\sqrt{2}}$, $\gamma = 0.5$, $\alpha = 0.1\frac{\sqrt{2-\mu^2}-1}{\sqrt{2-\mu^2}+1}$, $\tau_1 = 0.3$, $\beta = 0.1$.

- **AEGM** [19]**.** Set $\mu = \frac{1}{\sqrt{2}}$, $\alpha = 0.1\frac{\sqrt{2-\mu^2}-1}{\sqrt{2-\mu^2}+1}$, $\tau_1 = 0.3$.
- **SEGM** [14]**.** Set $\delta = 0.01$, $\mu_1 = 0.4$.
- **AM** [6]**.** Set $\tau_0 = 1$, $\eta = 0.9\frac{\sqrt{5}+1}{2}$.



(a) $\varepsilon_n = 10^{-3}$ **(b)** $\varepsilon_n = 10^{-4}$



(c) $\varepsilon_n = 10^{-5}$ **(d)** $\varepsilon_n = 10^{-6}$

Figure 5.3 Convergence behavior under different stopping criteria in Example 5.3

Table 5.3 Algorithm performance under different stopping criteria in Example 5.3

ε_n	Proposed		AEGH		SEGM		AM	
	Iter	CPU time	Iter	CPU time	Iter	CPU time	Iter	CPU time
10^{-3}	33	0.0033	65	0.0090	42	0.0089	45	0.0080
10^{-4}	131	0.0100	215	0.0255	947	0.1515	161	0.0177
10^{-5}	505	0.0340	817	0.0957	3429	0.5345	553	0.0629
10^{-6}	1947	0.1284	2981	0.3402	12274	1.9941	1973	0.2200

6. CONCLUSION

In this paper, an alternated inertial proximal-type subgradient extragradient projection algorithm with adaptive stepsize is proposed for solving pseudo-monotone variational inequality problems in real Hilbert spaces. Under suitable assumptions, the weak convergence of the sequence generated by the proposed algorithm is established. From the figures and tables, it can be observed that the proposed algorithm is more competitive than other existing methods of the same type.

STATEMENTS AND DECLARATIONS

The authors declare that they have no conflict of interest, and the manuscript has no associated data.

FUNDING

The first author supported by the Chongqing Jiaotong University Postgraduate Research Innovation Project (Grant No. 2025ST002).

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