APPLICABLE NONLINEAR ANALYSIS
Volume 3 (2026), No. 2, 93–105
DOI:https://doi.org/10.69829/apna-026-0302-ta02
Tulipa Opera Scholarum

IMPLICIT FUNCTIONAL-INTEGRAL EQUATIONS ASSOCIATED WITH UNBOUNDED DISCONTINUOUS FUNCTIONS

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Abstract. Let $I = [0, 1]$. We deal with the existence of solutions $u \in L^p(I)$ of the implicit functional-integral equation

$$\psi(u(t)) = f\Big(t, \int_I k(t,s)\, u(\varphi(s))\, ds\Big) \quad \text{for a.e.} \quad t \in I,$$

where $Y \subseteq \mathbf{R}$ is a closed interval, $p \in]1, +\infty[$, and $\psi : Y \to \mathbf{R}$, $f : I \times \mathbf{R} \to \mathbf{R}$, $k : I \times I \to [0, +\infty[$ and $\varphi : I \to I$ are given functions. We prove an existence result where the function f can be discontinuous, with respect to the second variable, even at all points $x \in \mathbf{R}$. Our result improve in several aspects a very recent result in the field. In particular, we impose a linear growth condition for the function $\psi^{-1}(f(t,\cdot))$, meaningfully weaker than boundedness condition which was previously imposed. As regards the function ψ, we only require that it is continuous and non-constant on intervals.

Keywords. Functional-integral equations, discontinuity, discontinuous selections, lower semicontinuity, operator inclusions.

1. Introduction

Let $Y \subseteq \mathbf{R}$ be a closed interval, $p \in]1, +\infty[$, $I := [0, 1]$, and $\mathbf{R}_0^+ := [0, +\infty[$. In this paper we deal with the existence of solutions $u \in L^p(I)$ of the implicit functional-integral equation

$$\psi(u(t)) = f\Big(t, \int_I k(t,s)\, u(\varphi(s))\, ds\Big) \quad \text{for a.e.} \quad t \in I, \tag{1.1}$$

where $\psi : Y \to \mathbf{R}$, $f : I \times \mathbf{R} \to \mathbf{R}$, $k : I \times I \to \mathbf{R}_0^+$ and $\varphi : I \to I$ are given functions. Equation (1.1) has been deeply studied even in recent times, both in the explicit and in the implicit form, and even in some more general forms, including the Hammerstein and the Urysohn functional-integral equations (see, for instance, [1, 2, 10, 13, 14, 15, 16, 17, 20]). The reason of this fact resides in the wide range of applications of equation (1.1), including a variety of problems arising from physics, engineering, economics, and partial differential equations. In this connection, we refer to the above quoted paper and to the references therein for motivations for studying equation (1.1).

As regards the function f, a systematical assumption is to require that it satisfies the Carathéodory conditions in $I \times \mathbf{R}$. That is, it is assumed that $f(\cdot, x)$ is Lebesgue measurable for each $x \in \mathbf{R}$, and $f(t,\cdot)$ is continuous for almost every $t \in I$. Together with Carathéodory conditions, various other assumptions, including monotonicity or growth conditions, are commonly assumed on f.

A first attempt to weaken the Carathéodory assumptions on f, allowing f to be discontinuous with respect to the second variable, was made in the paper [5], where an autonomous version of equation

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2020 Mathematics Subject Classification: 45P05, 47G10.
Accepted: March 06, 2026.

(1.1), where φ was the identity mapping and f did not depend on t, was considered. Actually, a function $f : \mathbf{R} \to \mathbf{R}$ satisfing the assumptions of [5] could be discontinuous even at all points $x \in \mathbf{R}$. Since then, some other results have been obtained in this direction in recent times, which have progressively refined the assumptions on f, and have also allowed to consider more general equations. Among these, the most refined result seems to be Theorem 3.1 of [9] (dealing with a slightly more general version of equation (1.1)), which, in turn, extends in some aspects another very recent result, Theorem 3.1 of [8]. For a detailed bibliography of various intermediate results in the field, we refer to [9] and to the references therein.

We observe that, in the paper [9], the following assumptions are made on the function f:

- (i_1) there exists a null-measure set $E \subseteq \mathbf{R}$, such that for all $x \in \mathbf{R} \setminus E$ the function $f(\cdot, x)$ is measurable, and for almost every $t \in I$ the function $f(t,\cdot)|_{\mathbf{R}\setminus E}$ is continuous;
- (i_2) there exists a function $\beta \in L^p(I)$ such that for almost every $t \in I$, one has

$$\psi^{-1}(f(t, \mathbf{R} \setminus E)) \subseteq [-\beta(t), \beta(t)]. \tag{1.2}$$

As showed in [9], a function f satisfying the assumptions of Theorem 3.1 of [9] can be discontinuous, with respect to the second variable, even at all points $x \in \mathbf{R}$. On the other side, the assumption (i_2) seems to be a little bit restrictive. Indeed, if we consider the explicit case where ψ is the identity mapping, assumption (i_2) takes the following form:

$$|f(t,x)| \leq \beta(t) \quad \text{for a.e. } t \in I \text{ and for all } x \in \mathbf{R} \setminus E. \tag{1.3}$$

Hence, in particular, for a.e. $t \in I$ the function $f(t,\cdot)$ must be bounded in $\mathbf{R} \setminus E$. In the autonomous case where f does not depend on t explicitly, this last condition forces the whole f to be bounded on $\mathbf{R} \setminus E$. Even if condition (1.3) is assumed in part of the literature, even in stronger forms (see, for instance, [1]), it is clearly stronger then the growth condition

$$|f(t,x)| \leq \beta(t) + c|x| \tag{1.4}$$

which is required in other papers (see, for instance, [14, 16]). We point out that that the requirement (1.3) (or, in the implicit case, the requirement (1.2)) has been made necessary by the approach used in [9], based on set-valued analysis and on a deep result on operator inclusions (stated as Theorem 2.3 below).

At this point, it is natural to ask if, in the results of [9], condition (i_2) can be weakened to a linear growth condition, in order to replace, in the explicit case, condition (1.3) by the weaker condition (1.4). The aim of this paper is exactly to provide such an improvement, by means of a very different technical construction. As a matter of fact, we are able to replace assumption (i_2) with the following one:

- (i_3) there exist a function $\beta \in L^p(I)$ and $C > 0$, such that one has

$$\sup \psi^{-1}(f(t,x)) \leq \beta(t) + Cx$$

  for almost every $t \in I$ and for all $x \in \mathbf{R}_0^+ \setminus E$.

The main result of this paper (stated as Theorem 3.1 below) will be proved in Section 3, together with some consequences. As in [9], a function f satisfying the assumptions of our results can be discontinuous, with respect to the second variable, even at all points $x \in \mathbf{R}$. We also point out that our results improve the results of [9] in several other aspects, as we shall discuss in detail at the end of Section 3. In Section 2, conversely, we shall fix some notations and preliminary results, and recall some known results which will be fundamental in the sequel.

2. Preliminaries

In what follows, we put $I = [0,1]$ and $\mathbf{R}_0^+ := [0,+\infty[$. Moreover, we shall denote by m_1 the usual Lebesgue measure in $\mathbf{R}$. If $A \subseteq \mathbf{R}$, we shall denote by $\overline{\mathrm{conv}}(A)$ the closed convex hull of A.

Let $p \in [1,+\infty]$ and let $[a,b] \subseteq \mathbf{R}$ be any compact interval. The space $L^p([a,b])$ is considered with the usual norm

$$\|u\|_{L^p([a,b])} := \begin{cases} \left(\int_{[a,b]} |u(t)|^p \, dt \right)^{\frac{1}{p}} & \text{if } p < +\infty, \\ \operatorname{ess\,sup}_{t\in[a,b]} |u(t)| & \text{if } p = +\infty. \end{cases}$$

If $p \in]1,+\infty[$, we shall denote by $p' = p/(p-1)$ the conjugate exponent of p. We also denote by $\mathcal{L}([a,b])$ the family of all Lebesgue-measurable subsets of the interval $[a,b]$. We also denote by $AC([a,b])$ the space of all absolutely continuous real functions in $[a,b]$. Finally, "measurable set" (resp., "measurable function") will mean "Lebesgue measurable set" (resp., "Lebesgue measurable function").

Our terminology concerning multifunctions is standard. For what concerns the basic definitions and facts about multifunctions, we refer to [11, 22]. We also refer to the basic paper [19] for the basic properties of measurable multifunction. For the sake of a clearer exposition, here we recall that if X and Y are topological spaces and $F : X \to 2^Y$ is a multifunction, then F is said to be *lower semicontinuous (resp., upper semicontinuous) at* $x_0 \in X$ if for any open set $V \subseteq Y$, with $F(x_0) \cap V \neq \emptyset$ (resp., $F(x_0) \subseteq V$), there exists an open set $U \subseteq X$, with $x_0 \in U$, such that $F(x) \cap V \neq \emptyset$ (resp., $F(x) \subseteq V$) for all $x \in U$. The multifunction F is said to be lower semicontinuous (resp., upper semicontinuous) in X if it is lower semicontinuous (resp., upper semicontinuous) at every point $x \in X$. We recall (see [22], Theorems 7.1.4 and 7.1.7) that F is lower semicontinuous (resp., upper semicontinuous) in X if and only if the set

$$F^-(B) := \{x \in X : \ F(x) \cap B \neq \emptyset\}$$

is open (resp., closed) for every open (resp., closed) set $B \subseteq Y$. The graph of F is the set $\{(x,y) \in X \times Y : y \in F(x)\}$. We also recall that if $(T, \mathcal{T})$ is a measurable space and Y is a topological space, a multifunction $F : T \to 2^Y$ is said to be *measurable* (resp., *weakly measurable*) in T if for every closed (resp., open) set $B \subseteq Y$ one has $F^-(B) \in \mathcal{A}$.

For the reader's convenience, we now state explicitly some known results which will be fundamental in the sequel. Firstly, we state the following proposition, recently proved in [6].

Proposition 2.1. (Proposition 2.2 of [6]) *Let $A \subseteq \mathbf{R}^n$ be a nonempty measurable set, and let $\Omega_1 \subseteq A$, with $m_n(\Omega_1) = 0$. Let $f : A \times \mathbf{R}_0^+ \times \mathbf{R}_0^+ \to \mathbf{R}_0^+$ be a given function, and let $H \subseteq \mathbf{R}_0^+$, with $m_1(H) = 0$. Let $D \subseteq \mathbf{R}_0^+ \setminus H$ be a countable set, dense in $\mathbf{R}_0^+$. Finally, let $\{r_k\}$ be an increasing unbounded sequence of positive real numbers. Assume that:*

(i) *for each $k \in \mathbf{N}$, there exists a function $\mu_k : A \to \mathbf{R}$ such that*

$$\sup\{f(x,z,\lambda) : \ z \in [0,r_k] \setminus H, \lambda \in [0,r_k]\} \le \mu_k(x) \quad \text{for all} \quad x \in A \setminus \Omega_1.$$

(ii) *for each $z \in D$, and each $\lambda \in \mathbf{R}_0^+$, the function $f(\,\cdot\,,z,\lambda)$ is measurable;*

(iii) *for each $x \in A \setminus \Omega_1$, the function $\lambda \to f(x,z,\lambda)$ is continuous in $\mathbf{R}_0^+$, locally uniformly in $z \in D$.*

Let $F : A \times \mathbf{R}_0^+ \times \mathbf{R}_0^+ \to 2^{\mathbf{R}_0^+}$ be the multifunction defined by putting, for each $(x,z,\lambda) \in A \times \mathbf{R}_0^+ \times \mathbf{R}_0^+$,

$$F(x,z,\lambda) = \begin{cases} \bigcap_{m\in\mathbf{N}} \overline{\mathrm{conv}}(f(x,[z-\frac{1}{m}, z+\frac{1}{m}] \cap D, \lambda)) & \text{if } x \in A \setminus \Omega_1 \\ V & \text{if } x \in \Omega_1, \end{cases}$$

where $V \subseteq \mathbf{R}_0^+$ is any nonempty closed interval. Then, one has:

(1) *F has nonempty closed convex values;*

(2) *one has* $F(x,z,\lambda) \subseteq [0, \mu_k(x)]$ *for every* $(x,z,\lambda) \in (A \setminus \Omega_1) \times \mathbf{R}_0^+ \times \mathbf{R}_0^+$, *and for each* $k \in \mathbf{N}$ *such that* $r_k > \max\{z, \lambda\}$*;*
(3) *for each* $(z,\lambda) \in \mathbf{R}_0^+ \times \mathbf{R}_0^+$, *the multifunction* $F(\,\cdot\,, z, \lambda)$ *is measurable;*
(4) *for each* $x \in A \setminus \Omega_1$, *the multifunction* $F(x, \cdot\,, \cdot\,)$ *is upper semicontinuous in* $\mathbf{R}_0^+ \times \mathbf{R}_0^+$*;*
(5) *If* $x \in A \setminus \Omega_1$, $\lambda \in \mathbf{R}_0^+$ *and* $f(x,\cdot,\lambda)|_{\mathbf{R}_0^+ \setminus H}$ *is continuous at* $z \in \mathbf{R}_0^+ \setminus H$, *then one has*

$$F(x,z,\lambda) = \{f(x,z,\lambda)\}.$$

The following proposition recollects some simple known facts. We state it explicitly, together with a short proof, for further reference.

Proposition 2.2. *Let* $p \in [1, +\infty[$. *Let* $\varphi : I \to I$ *be an absolutely continuous, and assume that there exists* $B > 0$ *such that* $\varphi'(s) \geq B$ *for a.e.* $s \in I$. *Let* $J := \varphi(I)$. *Then, one has:*

(1) *for every* $u \in L^p(J)$, *one has* $u(\varphi(\cdot)) \in L^p(I)$*;*
(2) *one has*

$$\|u(\varphi)\|_{L^p(I)} \leq B^{-1/p} \|u\|_{L^p(J)} \quad \textit{for all} \quad u \in L^p(J).$$

Proof. Fix $u \in L^p(J)$. By Corollary 5.4.4 of [3], the function $|u(\varphi(\cdot))|^p \varphi'(\cdot)$ is summable in I, and

$$\int_J |u(t)|^p \, dt = \int_I |u(\varphi(s))|^p \varphi'(s) \, ds.$$

Since by assumption we have

$$|u(\varphi(s))|^p \leq \frac{1}{B} \, \varphi'(s) \, |u(\varphi(s))|^p \quad \text{for a.e.} \quad s \in I,$$

it follows that the measurable function $|u(\varphi(\cdot))|^p$ is summable in I, hence conclusion (1) follows. Conclusion (2) follows at once. ☐

Finally, for the sake of easy reference, we state the following deep result on operator inclusions, which will be one of the main tools in the sequel.

Theorem 2.3. (Theorem 1 of [25]). *Let* $(T, \mathcal{F}, \mu)$ *be a finite non-atomic complete measure space;* V *a nonempty set;* $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ *two separable real Banach spaces, with* Y *finite-dimensional;* $p, q, s \in [1, +\infty]$, *with* $q < +\infty$ *and* $q \leq p \leq s$*;* $\Psi : V \to L^s(T, Y)$ *a surjective and one-to-one operator;* $\Phi : V \to L^1(T, X)$ *an operator such that, for every* $v \in L^s(T, Y)$ *and every sequence* $\{v_n\}$ *in* $L^s(T, Y)$ *weakly converging to* v *in* $L^q(T, Y)$, *the sequence* $\{\Phi(\Psi^{-1}(v_n))\}$ *converges strongly to* $\Phi(\Psi^{-1}(v))$ *in* $L^1(T, X)$*;* $\varphi : [0, +\infty[\to [0, +\infty]$ *a non-decreasing function such that*

$$\operatorname{ess\,sup}_{t \in T} \|\Phi(u)(t)\|_X \leq \varphi(\|\Psi(u)\|_{L^p(T,Y)})$$

for all $u \in V$.

Further, let $F : T \times X \to 2^Y$ *be a multifunction, with nonempty closed convex values, satisfying the following conditions:*

(i) *for* μ*-almost every* $t \in T$, *the multifunction* $F(t, \cdot)$ *has closed graph;*
(ii) *the set*

$$\{x \in X : \textit{ the multifunction } F(\cdot, x) \textit{ is weakly measurable}\}$$

is dense in X*;*

(iii) *there exists a number* $r > 0$ *such that the function*

$$t \to \sup_{\|x\|_X \leq \varphi(r)} d(0_Y, F(t,x))$$

belongs to $L^s(T)$ *and its norm in* $L^p(T)$ *is less or equal to* r.

Under such hypotheses, there exists $\tilde{u} \in V$ *such that*

$$\Psi(\tilde{u})(t) \in F(t, \Phi(\tilde{u})(t)) \quad \mu\text{-a.e.},$$

$$\|\Psi(\tilde{u})(t)\|_Y \leq \sup_{\|x\|_X \leq \varphi(r)} d(0_Y, F(t,x)) \quad \mu\text{-a.e. in } T.$$

3. Existence Results

The following is our main result.

Theorem 3.1. *Let* $Y \subseteq]0, +\infty[$ *be a closed interval, and let* $\psi : Y \to \mathbf{R}$ *be a continuous function. Let* $f : I \times \mathbf{R}_0^+ \to \mathbf{R}$, $k : I \times I \to \mathbf{R}_0^+$ *and* $\varphi : I \to I$ *be three given functions. Let* $p \in]1, +\infty[$, *let* $\gamma_0, \gamma_1 \in L^{p'}(I)$ *be two functions, and let* $E \subseteq \mathbf{R}_0^+$ *be a measurable set, such that* $m_1(E) = 0$. *Finally, let* $\{r_n\}$ *be an increasing unbounded sequence of positive real numbers. Assume that:*

(i) φ *is absolutely continuous, and there exists* $B > 0$ *such that* $\varphi'(s) \geq B$ *for a.e.* $s \in I$;

(ii) ψ *is non-constant on intervals;*

(iii) *for every* $x \in \mathbf{R}_0^+ \setminus E$, *the function* $f(\cdot, x)$ *is measurable;*

(iv) *for a.e.* $t \in I$, *the function* $f(t, \cdot)|_{\mathbf{R}_0^+ \setminus E}$ *is continuous and* $f(t, \mathbf{R}_0^+ \setminus E) \subseteq \psi(Y)$

(v) *for each* $n \in \mathbf{N}$, *there exists a function* $\mu_n \in L^p(I)$ *such that for a.e.* $t \in I$ *one has*

$$\sup \psi^{-1}(f(t, [0, r_n] \setminus E)) \leq \mu_n(t), \tag{3.1}$$

and one also has

$$\|\gamma_0\|_{L^{p'}(I)} < \sup_{n \in \mathbf{N}} \frac{r_n B^{1/p}}{\|\mu_n\|_{L^p(I)}};$$

(vi) *for every* $t \in I$, $k(t, \cdot)$ *is measurable;*

(vii) *for a.e.* $s \in I$, *the function* $k(\cdot, s)$ *is continuous in* I, *differentiable in* $]0,1[$, *and*

$$k(t,s) \leq \gamma_0(s), \quad 0 < \frac{\partial k}{\partial t}(t,s) \leq \gamma_1(s) \quad \text{for all} \quad t \in]0,1[\,.$$

Then,there exists $u \in L^p(I)$ *such that*

$$u(t) \in Y \quad \text{and} \quad \int_I k(t,s)\, u(\varphi(s))\, ds \in \mathbf{R}_0^+ \setminus E \quad \text{for a.e.} \quad t \in I,$$

and

$$\psi(u(t)) = f\Big(t, \int_I k(t,s)\, u(\varphi(s))\, ds\Big) \quad \text{for a.e.} \quad t \in I.$$

Proof. Without loss of generality, we can assume that (iv) holds for every $t \in I$. Of course, assumption (vii) implies that $\gamma_0(s) > 0$ for almost every $s \in I$. By assumption (i), the function φ is strictly increasing in I. Put $J := \varphi(I)$. By assumption (i) and Theorem 2 of [27], we have that the function $\varphi^{-1} : J \to I$ is absolutely continuous.

Let $V \in \mathcal{B}(\mathbf{R}_0^+)$ be such that $E \subseteq V$ and $m_1(V) = 0$. By assumption (ii) and by Theorem 2.4 of [26], there exists a set $Z \subseteq Y$ such that $\psi(Z) = \psi(Y)$ and the restriction $\psi|_Z : Z \to \psi(Y)$ is open (it maps open subsets of Z onto open subsets of $\psi(Z) = \psi(Y)$). Thus, it is routine matter to check that the multifunction $P : \psi(Y) \to 2^Z$ defined by setting, for each $w \in \psi(Y)$,

$$P(w) := \psi^{-1}(w) \cap Z,$$

is lower semicontinuous in $\psi(Y)$, with nonempty values.

Let $M : I \times (\mathbf{R}_0^+ \setminus V) \to 2^Z$ be the multifunction defined by setting, for each $(t,x) \in I \times (\mathbf{R}_0^+ \setminus V)$,

$$M(t,x) := P(f(t,x)) = \psi^{-1}(f(t,x)) \cap Z.$$

The multifunction M is well-defined by assumption (iv). Moreover, by assumption (iv), taking into account the lower semicontinuity of P and Theorem 7.3.11 of [22], we get that for each fixed $t \in I$, the multifunction $M(t, \cdot)$ is lower semicontinuous in $\mathbf{R}_0^+ \setminus V$.

Let $\overline{M} : I \times (\mathbf{R}_0^+ \setminus V) \to 2^Y$ be the multifunction defined by

$$\overline{M}(t,x) := \overline{M(t,x)} = \overline{(\psi^{-1}(f(t,x)) \cap Z)} \, .$$

for each $(t,x) \in I \times (\mathbf{R}_0^+ \setminus V)$. By Proposition 7.3.3 of [22], for each $t \in I$ the multifunction $\overline{M}(t, \cdot\,)$ is lower semicontinuous in $\mathbf{R}_0^+ \setminus V$, with nonempty closed (both in Y and in $\mathbf{R}$) values.

Thanks to our assumptions (iii) and (iv), and to the Lemma at p.198 of [23], the function $f|_{I \times ([\mathbf{R}_0^+ \setminus V)}$ is $\mathcal{L}(I) \otimes \mathcal{B}(\mathbf{R}_0^+ \setminus V)$-measurable. Therefore, by the lower semicontinuity of P, the multifunction M is $\mathcal{L}(I) \otimes \mathcal{B}(\mathbf{R}_0^+ \setminus V)$-weakly measurable. By Proposition 2.6 of [19], the multifunction $\overline{M}$ is also $\mathcal{L}(I) \otimes \mathcal{B}(\mathbf{R}_0^+ \setminus V)$-weakly measurable.

Now, we apply Theorem 2.1 of [7]. Hence, there exist sets $U_0 \in \mathcal{L}(I)$ and $V_1 \in \mathcal{B}(\mathbf{R}_0^+)$, with $m_1(U_0) = m_1(V_1) = 0$, and a function $\eta : I \times (\mathbf{R}_0^+ \setminus V) \to \mathbf{R}$ such that:

(a_1) for every $(t,x) \in I \times (\mathbf{R}_0^+ \setminus V)$, one has $\eta(t,x) \in \overline{M}(t,x) \subseteq Y$;
(a_2) for every $x \in \mathbf{R}_0^+ \setminus (V \cup V_1)$, the function $\eta(\cdot, x)$ is measurable in I;
(a_3) for every $t \in I \setminus U_0$, the function $\eta(t, \cdot)$ is continuous at every point $x \in \mathbf{R}_0^+ \setminus (V \cup V_1)$.

For each $(t,x) \in I \times (\mathbf{R}_0^+ \setminus V)$, taking into account the property (a_1), the continuity of ψ and the closedness of Y in $\mathbf{R}$, we get

$$\begin{aligned} \eta(t,x) \in \overline{M}(t,x) &= \overline{\psi^{-1}(f(t,x)) \cap Z} \subseteq \\ &\subseteq \overline{\psi^{-1}(f(t,x))} = \\ &= \psi^{-1}(f(t,x)) \end{aligned} \tag{3.2}$$

(where closures are taken in $\mathbf{R}$). Let $\sigma := \min Y \ > 0$. By (3.2), we have that

$$\eta(t,x) \geq \sigma \quad \text{for all} \quad (t,x) \in I \times (\mathbf{R}_0^+ \setminus V). \tag{3.3}$$

By assumption (v) and condition (3.1), there exists a set $\Omega_1 \subseteq I$, with $m_1(\Omega_1) = 0$, such that

$$\sup \psi^{-1}(f(t,[0,r_n] \setminus E)) \leq \mu_n(t) \quad \text{for all } t \in I \setminus \Omega_1 \text{ and all } n \in \mathbf{N}.$$

Hence, for each fixed $n \in \mathbf{N}$, taking into account (3.2), we get that

$$\eta(t,x) \in [\sigma, \mu_n(x)] \quad \text{for all } t \in I \setminus \Omega_1 \text{ and all } x \in [0,r_n] \setminus V. \tag{3.4}$$

Now, let us consider the function $\eta^* : I \times \mathbf{R}_0^+ \to \mathbf{R}_0^+$ defined by

$$\eta^*(t,x) = \begin{cases} \eta(t,x) & \text{if } t \in I \text{ and } x \in \mathbf{R}_0^+ \setminus V, \\ \sigma & \text{if } t \in I \text{ and } x \in V\, . \end{cases}$$

By (3.4), for each $n \in \mathbf{N}$ we get

$$\eta^*(t,x) \in [\sigma, \mu_n(x)] \quad \text{for all } t \in I \setminus \Omega_1 \text{ and all } x \in [0,r_n]. \tag{3.5}$$

Let D be a countable subset of $\mathbf{R}_0^+ \setminus (V \cup V_1)$, dense in $\mathbf{R}_0^+$. Such a set D exists since $m_1(V \cup V_1) = 0$. By the property (a_2), the function $\eta^*(\cdot, x)$ is measurable in I for each $x \in D$.

Let $G : I \times \mathbf{R}_0^+ \to 2^{\mathbf{R}_0^+}$ be defined by putting, for each $(t,x) \in I \times \mathbf{R}_0^+$,

$$G(t,x) = \begin{cases} \bigcap_{m \in \mathbf{N}} \overline{\text{conv}}\, (\eta^*(t, [x - \frac{1}{m}, x + \frac{1}{m}] \cap D)) & \text{if } t \in I \setminus \Omega_1, \\ \\ \{\sigma\} & \text{if } t \in \Omega_1. \end{cases}$$

By Proposition 2.1, we have that:

(b_1) G has nonempty closed convex values;
(b_2) one has $G(t,x) \subseteq [0,\mu_n(t)]$ for every $(t,x) \in (I \setminus \Omega_1) \times \mathbf{R}_0^+$, and for each $n \in \mathbf{N}$ such that $r_n > x$;
(b_3) for each $x \in \mathbf{R}_0^+$, the multifunction $G(\cdot,x)$ is $\mathcal{L}(I)$-measurable;
(b_4) for every $t \in I \setminus \Omega_1$, the multifunction $G(t,\cdot)$ is upper semicontinuous in $\mathbf{R}_0^+$;
(b_5) if $t \in I \setminus \Omega_1$, and $\eta^*(t,\cdot)|_{\mathbf{R}_0^+ \setminus V}$ is continuous at $x \in \mathbf{R}_0^+ \setminus V$, then one has $G(t,x) = \{\eta^*(t,x)\}$.

By properties (a_3) and (b_5), taking into account the definition of η^*, we have that

$$G(t,x) = \{\eta(t,x)\} \quad \text{for all } t \in I \setminus (U_0 \cup \Omega_1) \text{ and all } x \in \mathbf{R}_0^+ \setminus (V \cup V_1). \tag{3.6}$$

Moreover, by (3.3) and the definition of G, we get

$$G(t,x) \subseteq [\sigma,+\infty[\quad \text{for all} \quad (t,x) \in I \times \mathbf{R}_0^+.$$

By assumption (v), there exists $n^* \in \mathbf{N}$ such that

$$\|\gamma_0\|_{L^{p'}(I)} < \frac{r_{n^*} B^{1/p}}{\|\mu_{n^*}\|_{L^p(I)}}.$$

Choose $\lambda > 0$ in such a way that

$$B^{-1/p}\|\mu_{n^*}\|_{L^p(I)} \cdot \|\gamma_0\|_{L^{p'}(I)} < \lambda < r_{n^*}.$$

By property (b_2) and by the definition of G we get

$$G(t,x) \subseteq [\sigma,\mu_{n^*}(t)] \quad \text{for all} \quad (t,x) \in I \times [0,\lambda]. \tag{3.7}$$

Let $F : I \times \mathbf{R} \to \mathbf{R}$ be the multifunction defined by

$$F(t,x) = \begin{cases} G(t,x) & \text{if } t \in I \text{ and } x \in]0,\lambda[, \\ [\sigma,\mu_{n^*}(t)] & \text{if } t \in I \text{ and } x \in]-\infty,0] \cup [\lambda,+\infty[. \end{cases} \tag{3.8}$$

By (b_1), (b_4) and (3.7), it is easy to check that for each fixed $t \in I \setminus \Omega_1$ the multifunction $F(t,\cdot)$ is upper semicontinuous in $\mathbf{R}$ with nonempty closed convex values, hence it has closed graph (see Theorem 71.1.15 of [22]).

By Theorem 9.1 and Proposition 2.3 of [19], the multifunction

$$t \in I \to [\sigma,\mu_{n^*}(t)]$$

is weakly measurable. This easily implies that for every $x \in \mathbf{R}$, the multifunction $F(\cdot,x)$ is weakly measurable.

Now we want to apply Theorem 2.3, by choosing $T = I$ (with the usual Lebesgue measure), $X = Y = \mathbf{R}$, $s = q = p$, $V = L^p(I)$, $\Psi(u) = u$, $r = \|\mu_{n^*}\|_{L^p(I)}$, $\varphi \equiv +\infty$, and

$$\Phi(u)(t) = \int_I k(t,s)\, u(\varphi(s))\, ds$$

for each $u \in L^p(I)$ and $t \in I$. To this aim, in addition to the above construction, we observe the following facts.

(c_1) For every fixed $u \in L^p(I)$, one has that $\Phi(u) \in AC(I)$, and

$$\Phi(u)'(t) = \int_I \frac{\partial k}{\partial t}(t,s)\, u(\varphi(s))\, ds \quad \text{for a.e.} \quad t \in I. \tag{3.9}$$

To see this, fix $u \in L^p(I)$. By Proposition 2.2 we get $u(\varphi(\cdot)) \in L^p(I)$, hence Φ is well-defined by assumptions (vi) and (vii). Now we want to apply Proposition 2.6 of [28]. To this aim, let $S_0 \subseteq I$ be such that $m_1(S_0) = 0$ and assumption (vii) is satisfied for all $s \in I \setminus S_0$. Let $\omega : I \times I \to \mathbf{R}$ be defined by

$$\omega(t,s) = k(t,s)\, u(\varphi(s)).$$

We have already observed that $\omega(t,\cdot) \in L^1(I)$ for every $t \in I$. Moreover, for every $s \in I \setminus S_0$, one has $\omega(\cdot, s) \in AC(I)$. This follows by assumption (vii), since the function $k(\cdot, s)$ is Lipschitzian in I. Now, observe that the partial derivative $\frac{\partial \omega}{\partial t}(t,s)$ exists almost everywhere in $I \times I$, and one has

$$\frac{\partial \omega}{\partial t}(t,s) \in L^1(I \times I).$$

Indeed, the partial derivative $\frac{\partial k}{\partial t}(t,s)$ (which exists on the set $]0,1[\times(I \setminus S_0)$, hence almost everywhere in $I \times I$) is measurable in $I \times I$ by a classical result (see [24], p. 236]). By assumption (vii), our claim follows. Consequently, all the assumptions of Proposition 2.6 of [28] are satisfied. Hence the function $\Phi(u)$ is absolutely continuous in I, and (3.9) holds.

(c_2) If $v \in L^p(I)$ and $\{v_m\}$ is a sequence in $L^p(I)$, weakly convergent to v in $L^p(I)$, then the sequence $\{\Phi(v_m)\}$ converges to $\Phi(v)$ strongly in $L^1(I)$. To see this, consider the linear operators

$$\Gamma(u)(s) := u(\varphi(s)), \qquad s \in I$$

and

$$K(v)(t) := \int_I k(t,s)\, v(s)\, ds, \qquad t \in I.$$

By Proposition 2.2 it follows that Γ is a bounded operator from $L^p(I)$ into itself. Moreover, since k is p'-th power summable in $I \times I$, then by Theorem 2 at p. 326 of [21] the operator K is compact between $L^p(I)$ and $L^1(I)$ (the measurability of k on $I \times I$ follows by the classical Scorza-Dragoni's theorem [23, 12]). Hence, the linear operator $\Phi = K \circ \Gamma$ is compact between $L^p(I)$ and $L^1(I)$. At this point, the conclusion follows easily by well-known facts (see [4], pp. 171-172).

(c_3) Let $\delta : I \to [0, +\infty]$ be defined by

$$\delta(t) = \sup_{x \in \mathbf{R}} \inf_{y \in F(t,x)} |y|.$$

By (3.7) and (3.8), we have $\delta(t) \leq \mu_{n^*}(t)$ for every $t \in I$. Hence, we get that $\delta \in L^p(I)$ and $\|\delta\|_{L^p(I)} \leq \|\mu_{n^*}\|_{L^p(I)}$ (as regard the measurability of δ, we refer to [25], p. 262).

Therefore, by properties (c_1)–(c_3) and by the above construction, we have that all the assumptions of Theorem 2.3 are satisfied. Consequently, there exist a function $u^* \in L^p(I)$ and a set $\Omega_2 \subseteq I$, with $m_1(\Omega_2) = 0$, such that

$$u^*(t) \in F(\,t\,, \Phi(u^*)(t)) = F\Big(t, \int_I k(t,s)\, u^*(\varphi(s))\, ds\Big) \quad \text{for all} \quad t \in I \setminus \Omega_2. \tag{3.10}$$

By (3.7), (3.8)and (3.10) we get

$$u^*(t) \in [\sigma, \mu_{n^*}(t)] \quad \text{for all} \quad t \in I \setminus \Omega_2. \tag{3.11}$$

By (3.11) we get

$$\sigma \leq u^*(\varphi(s)) \leq \mu_{n^*}(\varphi(s)) \quad \text{for all} \quad s \in I \setminus \varphi^{-1}(\Omega_2 \cap J).$$

Since the function $\varphi^{-1} : J \to I$ is absolutely continuous, by Theorem 18.25 of [18] we get $m_1(\varphi^{-1}(\Omega_2 \cap J)) = 0$, hence

$$\sigma \leq u^*(\varphi(s)) \leq \mu_{n^*}(\varphi(s)) \quad \text{for a.e.} \quad s \in I. \tag{3.12}$$

Now, observe that by Proposition 2.2 we get $\mu_{n^*}(\varphi) \in L^p(I)$. Consequently, by assumption (vii), (3.12) and Proposition 2.2 we get

$$\begin{aligned} 0 &\le \Phi(u^*)(t) \\ &\le \int_I \gamma_0(s)\, u^*(\varphi(s))\, ds \\ &\le \|\gamma_0\|_{L^{p'}(I)}\, \|\mu_{n^*}(\varphi)\|_{L^p(I)} \\ &\le B^{-1/p} \|\gamma_0\|_{L^{p'}(I)}\, \|\mu_{n^*}\|_{L^p(I)} \\ &\le \lambda \end{aligned} \tag{3.13}$$

for all $t \in I$. On the other hand, by (3.9), (3.12) and assumption (vii), we have

$$\Phi(u^*)'(t) = \int_I \frac{\partial k}{\partial t}(t,s)\, u^*(\varphi(s))\, ds > 0 \quad \text{for a.e.} \quad t \in I,$$

hence the absolutely continuous function $\Phi(u^*)$ is strictly increasing in I. By the previous inequality, and by Theorem 2 of [27], the function

$$\Phi(u^*)^{-1} : \Phi(u^*)(I) \to I$$

is absolutely continuous. Consequently, if we put

$$\Omega_3 := \Phi(u^*)^{-1}\Big((V \cup V_1 \cup \{0, \lambda\}) \cap \Phi(u^*)(I)\Big)$$

by Theorem 18.25 of [18] we get $m_1(\Omega_3) = 0$. Taking into account (3.13), we get that

$$\Phi(u^*)(t) \in]0, \lambda[\setminus (V \cup V_1) \quad \text{for all} \quad t \in I \setminus \Omega_3. \tag{3.14}$$

Now, fix $t \in I \setminus (U_0 \cup \Omega_1 \cup \Omega_2 \cup \Omega_3)$. By (3.6), (3.10), (3.14) and by the definition of F we get

$$u^*(t) \in F(t, \Phi(u^*)(t)) = G(t, \Phi(u^*)(t)) = \{\eta(t, \Phi(u^*)(t))\}.$$

In particular, since by property (a_1) the function η takes its values in Y, we get $u^*(t) \in Y$. Moreover, by (3.2) we get

$$u^*(t) \in \psi^{-1}(f(t, \Phi(u^*)(t))),$$

hence

$$\psi(u^*(t)) = f(t, \Phi(u^*)(t)) = f\Big(t, \int_I k(t,s)\, u^*(\varphi(s))\, ds\Big).$$

Since the set $C_0 \cup \bigcup_{j=1}^3 \Omega_j$ has null Lebesgue measure, we have actually proved that

$$u^*(t) \in Y \quad \text{and} \quad \psi(u^*(t)) = f\Big(t, \int_I k(t,s)\, u^*(\varphi(s))\, ds\Big) \quad \text{for a.e.} \quad t \in I.$$

The fact that

$$\int_I k(t,s)\, u^*(\varphi(s))\, ds \in \mathbf{R}_0^+ \setminus E \quad \text{for a.e.} \quad t \in I$$

follows at once by (3.14), since $E \subseteq V$. The proof is now complete. □

Before discussing in details the various features of Theorem 3.1, we derive some interesting consequences of this latter results. Firstly, we proof the following result.

Theorem 3.2. *Let $Y \subseteq]0, +\infty[$ be a closed interval, and let $\psi : Y \to \mathbf{R}$ be a continuous function. Let $f : I \times \mathbf{R}_0^+ \to \mathbf{R}$, $k : I \times I \to \mathbf{R}_0^+$ and $\varphi : I \to I$ be three given functions. Let $p \in]1, +\infty[$, let $\gamma_0, \gamma_1 \in L^{p'}(I)$ be two functions, and let $E \subseteq \mathbf{R}_0^+$ be a measurable set such that $m_1(E) = 0$.*

Assume that:

- (i) *φ is absolutely continuous, and there exists $B > 0$ such that $\varphi'(s) \ge B$ for a.e. in $s \in I$;*
- (ii) *ψ is non-constant on intervals;*

(iii) *for every* $x \in \mathbf{R}_0^+ \setminus E$, *the function* $f(\cdot, x)$ *is measurable;*

(iv) *for a.e.* $t \in I$, *the function* $f(t, \cdot)|_{\mathbf{R}_0^+ \setminus E}$ *is continuous and* $f(t, \mathbf{R}_0^+ \setminus E) \subseteq \psi(Y)$;

(v) *there exist a function* $\omega \in L^p(I)$ *and* $C > 0$, *with*

$$\|\gamma_0\|_{L^{p'}(I)} < \frac{B^{1/p}}{C},$$

such that one has

$$\sup \psi^{-1}(f(t,x)) \leq \omega(t) + Cx;$$

for a.e. $t \in I$ *and for all* $x \in \mathbf{R}_0^+ \setminus E$.

(vi) *for every* $t \in I$, $k(t, \cdot)$ *is measurable;*

(vii) *for a.e.* $s \in I$, *the function* $k(\cdot, s)$ *is continuous in* I, *differentiable in* $]0, 1[$, *and*

$$k(t,s) \leq \gamma_0(s), \quad 0 < \frac{\partial k}{\partial t}(t,s) \leq \gamma_1(s) \quad \textit{for all} \quad t \in]0,1[\,.$$

Then,there exists $u \in L^p(I)$ *such that*

$$u(t) \in Y \quad \textit{and} \quad \int_I k(t,s)\, u(\varphi(s))\, ds \in \mathbf{R}_0^+ \setminus E \quad \textit{for a.e.} \quad t \in I,$$

and

$$\psi(u(t)) = f\Big(t, \int_I k(t,s)\, u(\varphi(s))\, ds\Big) \quad \textit{for a.e.} \quad t \in I. \tag{3.15}$$

Proof. For each $n \in \mathbf{N}$, choose $r_n := n$ and $\mu_n(t) := \omega(t) + Cn$.Of course, $\mu_n \in L^p(I)$ for all $n \in \mathbf{N}$. By assumption (v), for almost every $t \in I$ and for each $n \in \mathbf{N}$ we get

$$\sup \psi^{-1}(f(t, [0,n] \setminus E)) \leq \omega(t) + Cn = \mu_n(t).$$

Since by assumption (iv) we have

$$\lim_{n \to +\infty} \frac{nB^{1/p}}{\|\omega\|_{L^p(I)} + Cn} = \frac{B^{1/p}}{C} > \|\gamma_0\|_{L^{p'}(I)},$$

there exists $n^* \in \mathbf{N}$ such that

$$\frac{n^* B^{1/p}}{\|\omega\|_{L^p(I)} + Cn^*} > \|\gamma_0\|_{L^{p'}(I)}.$$

Thus, we get

$$\frac{n^* B^{1/p}}{\|\mu_{n^*}\|_{L^p(I)}} \geq \frac{n^* B^{1/p}}{\|\omega\|_{L^p(I)} + Cn^*} > \|\gamma_0\|_{L^{p'}(I)}.$$

Consequently, all the assumptions of Theorem 3.1 are satisfied, and our conclusion follows. □

For the special explicit case where ψ is the identity mapping, Theorem 3.2 gives immediately the following result.

Theorem 3.3. *Let* $f : I \times \mathbf{R}_0^+ \to \mathbf{R}$, $k : I \times I \to \mathbf{R}_0^+$ *and* $\varphi : I \to I$ *be three given functions. Let* $p \in]1, +\infty[$, *let* $\gamma_0, \gamma_1 \in L^{p'}(I)$ *be two functions, and let* $E \subseteq \mathbf{R}_0^+$ *be a measurable set, such that* $m_1(E) = 0$. *Assume that:*

(i) φ *is absolutely continuous, and there exists* $B > 0$ *such that* $\varphi'(s) \geq B$ *for a.e.* $s \in I$;

(ii) *for every* $x \in \mathbf{R}_0^+ \setminus E$, *the function* $f(\cdot, x)$ *is measurable;*

(iii) *for a.e.* $t \in I$, *the function* $f(t, \cdot)|_{\mathbf{R}_0^+ \setminus E}$ *is continuous;*

(iv) *there exist a function* $\omega \in L^p(I)$ *and two constants* $\sigma > 0$, $C > 0$, *with*

$$\|\gamma_0\|_{L^{p'}(I)} < \frac{B^{1/p}}{C},$$

such that for a.e. $t \in I$ *and for all* $x \in \mathbf{R}_0^+ \setminus E$ *one has*

$$\sigma \leq f(t,x) \leq \omega(t) + Cx;$$

(v) *for every* $t \in I$, $k(t,\cdot)$ *is measurable;*

(vi) *for a.e.* $s \in I$, *the function* $k(\cdot,s)$ *is continuous in* I, *differentiable in* $]0,1[$, *and*

$$k(t,s) \leq \gamma_0(s), \quad 0 < \frac{\partial k}{\partial t}(t,s) \leq \gamma_1(s) \quad \textit{for all} \quad t \in]0,1[\,.$$

Then,there exists $u \in L^p(I)$ *such that*

$$u(t) \geq \sigma \quad \textit{and} \quad \int_I k(t,s)\, u(\varphi(s))\, ds \in \mathbf{R}_0^+ \setminus E \quad \textit{for a.e.} \quad t \in I,$$

and

$$u(t) = f\Big(t, \int_I k(t,s)\, u(\varphi(s))\, ds\Big) \quad \textit{for a.e.} \quad t \in I.$$

Before concluding, we discuss in detail some features of our results. To this aim, we have the following remark.

Remark 3.4. (i) As remarked in the Introduction, the assumptions on the function f in Theorems 3.1, 3.2 and 3.3 do not imply the continuity of f with respect to the second variable. Indeed, a function f satisfying our assumptions can be discontinuous, with respect to the second variable, even at all points $x \in \mathbf{R}_0^+$. To see this, fix $p \in]1,+\infty[$ and a non-negative function $g \in L^p(I)$, and consider the function $f : I \times \mathbf{R}_0^+ \to \mathbf{R}$ defined by putting

$$f(t,x) = \begin{cases} g(t) + 3 + x|\sin x| & \text{if } x \in \mathbf{R}_0^+ \setminus \mathbf{Q}, \\ 1 & \text{if } x \in \mathbf{R}_0^+ \cap \mathbf{Q}, \end{cases}$$

where $\mathbf{Q}$ denotes the set of all rational real numbers. It is immediate to check that f satisfies assumptions (ii)–(iv) of Theorem 3.1, with $E = \mathbf{Q} \cap \mathbf{R}_0^+$, $\sigma = 3$, $C = 1$ and $\omega(t) = g(t) + 3$. At the same time, it is immediate to check that for each $t \in I$ the function $f(t,\cdot)$ (considered on the whole $\mathbf{R}_0^+$) is discontinuous *at every point* $x \in \mathbf{R}_0^+$.

(ii) Besides the possibility of the function $f(t,\cdot)$ to be unbounded on $\mathbf{R}_0^+ \setminus E$, we observe that Theorems 3.1, 3.2 and 3.3 improve some other aspects of the results of [8, 9]. Firstly, we observe that in our results we assume that, for every $t \in I$, the function $k(t,\cdot)$ is Lebesgue measurable. Conversely, in [8, 9], it is required that $k(t,\cdot)$ is a Borel function for every $t \in I$. Moreover, in Theorems 3.1–3.3 above, the functions $\gamma_0(s)$ and $\gamma_1(s)$ (which represent upper bounds for the functions $k(t,s)$ and $\frac{\partial k}{\partial t}(t,s)$, respectively) are supposed to belong to the space $L^{p'}(I)$. This is a more general condition than the one assumed in [9], where, in the case $p < +\infty$, it is required that $\gamma_0(\varphi^{-1})$ and $\gamma_1(\varphi^{-1})$ belong to $L^{p'}(J)$, with $J := \varphi(I)$. Indeed, if we assume that assumption (i) of Theorems 3.1–3.3 holds, taking into account Corollary 5.4.4 of [3], we have that the assumption $\gamma_0(\varphi^{-1}), \gamma_1(\varphi^{-1}) \in L^{p'}(J)$ implies that $\gamma_0, \gamma_1 \in L^{p'}(I)$, but the converse implication is not necessarily true. Such circumstances are discussed in detail in Remark 3.2 of [8], where a counter-example is given.

(iii) We point out that, in our results, the behaviour of the function f over the set $I \times E$ plays no role at all. Actually, the function f could be even defined only over the set $I \times (\mathbf{R}_0^+ \setminus E)$. Moreover, we point out the following peculiarity of our results. Assume that the function $f : I \times \mathbf{R}_0^+ \to \mathbf{R}$ satisfies the assumptions of Theorem 3.2 with respect to a null-measure set $E \subseteq \mathbf{R}_0^+$, and that $V \subseteq \mathbf{R}_0^+$

is any other null-measure set. Then, the function f satisfies the assumptions of the same Theorem 3.2 even with respect to the null-measure set $E \cup V$. Consequently, there exists a solution $u_V \in L^p(I)$ of equation (3.15) such that, in particular, one has

$$\int_I k(t,s)\, u_V(\varphi(s))\, ds \in \mathbf{R}_0^+ \setminus (E \cup V) \quad \text{for a.e.} \quad t \in I.$$

That is, if f satisfies the assumptions of Theorem 2.2, then for every further set $V \subseteq \mathbf{R}_0^+$, with $m_1(V) = 0$, there exists a solution $u_V \in L^p(I)$ of the equation (3.15) such that

$$\int_I k(t,s)\, u_V(\varphi(s))\, ds \in \mathbf{R}_0^+ \setminus V \quad \text{for a.e.} \quad t \in I.$$

The same argument applies to Theorems 3.1 and 3.3.

(iv) Finally, as regards possible improvements of our results, the example in Remark 3.3 of [9] shows that Theorems 3.1, 3.2, and 3.3 are no longer true if we replace the assumption

$$0 < \frac{\partial k}{\partial t}(t,s) \leq \gamma_1(s) \quad \text{for a.e. } s \in I \text{ and for all } t \in]0,1[$$

with the weaker assumption

$$0 \leq \frac{\partial k}{\partial t}(t,s) \leq \gamma_1(s) \quad \text{for a.e. } s \in I \text{ and for all } t \in]0,1[.$$

4. CONCLUSION

As regards further possible developments of the present research, it would be nice, on one hand (if possible), to extend our results to the vector case, where $f : I \times \mathbf{R}^n \to \mathbf{R}$. On the other hand, it would be interesting to extend our results to the case where the function ψ can depend on t also explicitly, both in the scalar and in the vector case. We leave these as open problems.

STATEMENTS AND DECLARATIONS

The authors declare that they have no conflict of interest, and the manuscript has no associated data.

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