APPLICABLE NONLINEAR ANALYSIS
Volume 3 (2026), No. 2, 106–113
DOI:https://doi.org/10.69829/apna-026-0302-ta03
Tulipa Opera Scholarum

THE EXTRAGRADIENT ALGORITHM FOR VARIATIONAL INEQUALITIES WITH SUMMABLE ERRORS

ALEXANDER J. ZASLAVSKI[1],*

[1] *Department of Mathematics, The Technion – Israel Institute of Technology, 32000 Haifa, Israel*

Abstract. We study, in the setting of a Hilbert space, the behavior of the sequences generated by extragradient methods for solving variational inequalities in the presence of summable computational errors. It is shown that most of iterates are good approximate solutions of variational inequalities.

Keywords. Extragradient method, Hilbert space, Iteration, Variational inequality.

1. Introduction

Gradient-type methods and variational inequalities have recently been and continue to be important topics in optimization theory and its applications. See, for example, [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13] and references mentioned therein. In the present paper we study, in the setting of a Hilbert space, the behavior of the sequences generated by the extragradient method, introduced in [12] for solving variational inequalities, in the presence of summable computational errors. It is shown that most of iterates are good approximate solutions of variational inequalities. This result is important because computational errors are always present in the practice.

Let $(X, \langle \cdot, \cdot \rangle)$ be a Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ which induces a complete norm $\| \cdot \|$. For each $x \in X$ and each $r > 0$ set

$$B(x, r) = \{y \in X : \ \|x - y\| \le r\}.$$

In the sequel we use the following well-known result [16].

Lemma 1.1. *Let D be a nonempty closed convex subset of X. Then for each $x \in X$ there is a unique point $P_D(x) \in D$ satisfying*

$$\|x - P_D(x)\| = \inf\{\|x - y\| : \ y \in D\}.$$

Moreover,

$$||P_D(x) - P_D(y)|| \le \|x - y\| \text{ for all } x, y \in X$$

and for each $x \in X$ and each $z \in D$,

$$\langle z - P_D(x), x - P_D(x)\rangle \le 0, \|z - P_D(x)\|^2 + \|x - P_D(x)\|^2 \le \|z - x\|^2.$$

Let C be a nonempty closed convex subset of X. Consider a mapping $f : X \to X$. We say that the mapping f is monotone on C if

$$\langle f(x) - f(y), x - y\rangle \ge 0 \text{ for all } x, y \in C.$$

We say that f is pseudo-monotone on C if for each $x, y \in C$ the inequality

$$\langle f(y), x - y\rangle \ge 0 \text{ implies that } \langle f(x), x - y\rangle \ge 0. \tag{1.1}$$

---

*Corresponding author.

E-mail address: ajzasl@technion.ac.il (A.-J. Zaslavski)

2020 Mathematics Subject Classification: 47H05, 47H14.

Accepted: March 06, 2026.

Clearly, if f is monotone on C, then f is pseudo-monotone on C. Denote by S the set of all $x \in C$ such that

$$\langle f(x), y - x\rangle \geq 0 \text{ for all } y \in C. \tag{1.2}$$

For each $\epsilon > 0$ denote by S_ϵ the set of all $x \in C$ such that

$$\langle f(x), y - x\rangle \geq -\epsilon\|y - x\| - \epsilon \text{ for all } y \in C. \tag{1.3}$$

Elements of S_ϵ can be considered as ϵ-approximate solutions of the variational inequality.

In this paper, in order to solve the variational inequality (to find $x \in S$), we use the algorithm known in the literature as the extragradient method [12]. In each iteration of this algorithm, in order to get the next iterate x_{k+1}, two orthogonal projections onto C are calculated, according to the following iterative step. Given the current iterate x_k calculate $y_k = P_C(x_k - \tau_k f(x_k))$ and then

$$x_{k+1} = P_C(x_k - \tau_k f(y_k)),$$

where τ_k is some positive number. It is known that this algorithm generates sequences which weakly converge to an element of S [5]. In this paper, we study the behavior of the sequences generated by the algorithm taking into account computational errors which are always present in practice. Namely, the algorithm generates sequences $\{x_k\}_{k=0}^{\infty}$ and $\{y_k\}_{k=0}^{\infty}$ such that for each integer $k \geq 0$,

$$\|y_k - P_C(x_k - \tau_k f(x_k))\| \leq \Delta_k$$

and

$$\|x_{k+1} - P_C(x_k - \tau_k f(y_k))\| \leq \Delta_k,$$

where $\{\Delta_k\}_{k=0}^{\infty}$ are summable computational errors.

We suppose that the mapping f is Lipschitz on all bounded subsets of X and that

$$\langle f(y), y - x\rangle \geq 0 \text{ for all } y \in C \text{ and all } x \in S. \tag{1.4}$$

Note that (1.4) holds if f is pseudo-monotone on C.

2. Auxiliary Results

We begin with the results obtained in [14].

Lemma 2.1. *Assume that* $\tau > 0,\ u_* \in S,\ M_0 > 0,\ M_1 > 0,\ L > 0,\ f(B(u_*, M_0)) \subset B(0, M_1)$, *and*

$$\|f(z_1) - f(z_2)\| \leq L\|z_1 - z_2\| \text{ for all } z_1, z_2 \in B(u_*, M_0 + \tau M_1).$$

Let $u \in B(u_*, M_0),\ v = P_C(u - \tau f(u)),\ T := \{w \in X :\ \langle u - \tau f(u) - v, w - v\rangle \leq 0\}$, D *be a convex and closed subset of* X *such that*

$$C \subset D \subset T$$

(note that $C \subset T$*) and let* $\tilde{u} = P_D(u - \tau f(v))$. *Then,*

$$\|\tilde{u} - u_*\|^2 \leq \|u - u_*\|^2 - (1 - \tau^2 L^2)\|u - v\|^2.$$

Lemma 2.2. *Let* $u_* \in S,\ M_0 > 0,\ M_1 > 0,\ L > 0, \delta \in (0,1), f(B(u_*, M_0)) \subset B(0, M_1)$,

$$\|f(z_1) - f(z_2)\| \leq L\|z_1 - z_2\| \text{ for all } z_1, z_2 \in B(u_*, M_0 + M_1 + 1),$$

and $\tau \in (0, 1],\ L\tau < 1$. *Assume that*

$$x \in B(u_*, M_0),\ y \in X,\ \|y - P_C(x - \tau f(x))\| \leq \delta,\ \tilde{x} \in X,\ \|\tilde{x} - P_C(x - \tau f(y))\| \leq \delta.$$

Then,

$$\|\tilde{x} - u_*\|^2 \leq 4\delta(1 + M_0) + \|x - u_*\|^2 - (1 - \tau^2 L^2)\|x - P_C(x - \tau f(x))\|^2.$$

3. Exact Iterates

We use the assumptions, definitions and the notation introduced in Section 1.1.The following theorem shows that the most exact iterates of our algorithm are approximate solutions of the variational inequality.

Theorem 3.1. *Let* $\epsilon \in (0,1)$, $M_0, M_1, L > 0$,

$$B(0, M_0) \cap S \neq \emptyset, \tag{3.1}$$

$$f(B(0, 3M_0)) \subset B(0, M_1), \tag{3.2}$$

$$\|f(z_1) - f(z_2)\| \leq L\|z_1 - z_2\| \tag{3.3}$$

for all $z_1, z_2 \in B(0, 3M_0 + M_1)$,

$$0 < \tilde{\tau} < \tau_* \leq 1,\ \tau_* L < 1 \tag{3.4}$$

and let $\epsilon_0 > 0$ *satisfy*

$$\epsilon_0(\tilde{\tau}^{-1} + M_1) < \epsilon. \tag{3.5}$$

Assume that $\{x_k\}_{k=0}^{\infty} \subset X$, $\{y_k\}_{k=0}^{\infty} \subset X$

$$\{\tau_k\}_{k=0}^{\infty} \subset [\tilde{\tau}, \tau_*], \tag{3.6}$$

$$\|x_0\| \leq M_0 \tag{3.7}$$

and that for each integer $k \geq 0$,

$$y_k = P_C(x_k - \tau_k f(x_k)), \tag{3.8}$$

$$x_{k+1} = P_C(x_k - \tau_k f(y_k)). \tag{3.9}$$

Then

$$\|x_k\| \leq 3M_0,\ k = 0, 1, \dots$$

and

$$Card(\{k \in \{0, 1, \dots\} :\ \|x_k - y_k\| > \epsilon_0\}) \leq 4M_0^2\epsilon_0^{-2}(1 - \tau_*^2 L^2)^{-1}.$$

Moreover, if $k \geq 0$ *is an integer and* $\|x_k - y_k\| \leq \epsilon_0$, *then*

$$\langle f(x_k), \xi - x_k\rangle \geq -\epsilon - \epsilon\|\xi - x_k\|,\ \xi \in C_k$$

and $x_k \in S_\epsilon$.

Proof. By (3.1), there exists

$$u_* \in S(0, B(0, M_0)). \tag{3.10}$$

Equations (3.7) and (3.10) imply that

$$\|x_0 - u_*\| \leq 2M_0. \tag{3.11}$$

Assume that $k \geq 0$ is an integer and

$$\|x_k - u_*\| \leq 2M_0. \tag{3.12}$$

(In view of (3.11), equation (3.12) holds for $k = 0$.) By (3.2), (3.8)-(3.10), (3.12) and Lemma 2.1 applied with

$$u = x_k,\ v = y_k,\ \tilde{u} = x_{k+1},\ \tau = \tau_k,$$

we have

$$\begin{aligned} \|x_{k+1} - u_*\|^2 &\leq \|x_k - u_*\|^2 - (1 - \tau_k^2 L^2)\|x_k - y_k\|^2 \\ &\leq \|x_k - u_*\|^2 - (1 - \tau_*^2 L^2)\|x_k - y_k\|^2. \end{aligned} \tag{3.13}$$

It follows from (3.4), (3.12) and (3.13) that

$$\|x_{k+1} - u_*\| \leq \|x_k - u_*\| \leq 2M_0.$$

Thus by induction (3.12) and (3.13) hold for all integers $k \geq 0$. By (3.10) and (3.12),

$$\|x_k\| \leq 3M_0,\ k = 0, 1, \dots.$$

Set

$$E = \{k \in \{0, 1, \dots, \} : \ \|x_k - y_k\| > \epsilon_0\}. \tag{3.14}$$

Let Q be a natural number. Set

$$E_Q = E \cap \{0, \dots, Q-1\}. \tag{3.15}$$

By (3.11) and (3.13)-(3.15),

$$\begin{aligned} 4M_0^2 \geq & \|x_0 - u_*\|^2 \\ \geq & \|x_0 - u_*\|^2 - \|x_Q - u_*\|^2 \\ = & \sum_{k=0}^{Q-1} (\|x_k - u_*\|^2 - \|x_{k+1} - u_*\|^2) \\ \geq & \sum \{\|x_k - u_*\|^2 - \|x_{k+1} - u_*\|^2 : \ k \in E_Q\} \\ \geq & \epsilon_0^2 \operatorname{Card}(E_Q)(1 - \tau_*^2 L^2) \end{aligned}$$

and

$$\operatorname{Card}(E_Q) \leq 4M_0^2 \epsilon_0^{-2} (1 - \tau_*^2 L^2)^{-1}.$$

Since Q is any natural number, we conclude that

$$\operatorname{Card}(E) \leq 4M_0^2 \epsilon_0^{-2} (1 - \tau_*^2 L^2)^{-1}.$$

Assume that $k \geq 0$ is an integer and

$$\|x_k - y_k\| \leq \epsilon_0. \tag{3.16}$$

By (3.8) and (3.16),

$$\|x_k - P_C(x_k - \tau_k f(x_k))\| \leq \epsilon_0. \tag{3.17}$$

It follows from (3.17) that for each $\xi \in C$,

$$\begin{aligned} 0 \geq & \langle x_k - \tau_k f(x_k) - P_C(x_k - \tau_k f(x_k)), \xi - P_C(x_k - \tau_k f(x_k)) \rangle \\ = & \langle x_k - P_C(x_k - \tau_k f(x_k)), \xi - P_C(x_k - \tau_k f(x_k)) \rangle \\ & - \tau_k \langle f(x_k), \xi - P_C(x_k - \tau_k f(x_k)) \rangle \\ \geq & - \|x_k - P_C(x_k - \tau_k f(x_k))\| (\|\xi - x_k\| + \|x_k - P_C(x_k - \tau_k f(x_k))\|) \\ & - \tau_k \langle f(x_k), \xi - x_k \rangle - \tau_k \langle f(x_k), x_k - P_C(x_k - \tau_k f(x_k)) \rangle \\ \geq & - \epsilon_0 (\|\xi - x_k\| + \epsilon_0) - \tau_k \langle f(x_k), \xi - x_k \rangle - \tau_k \|f(x_k)\| \epsilon_0. \end{aligned} \tag{3.18}$$

In view of (3.2), (3.10) and (3.12),

$$\|f(x_k)\| \leq M_1. \tag{3.19}$$

It follows from (3.5), (3.6), (3.28) and (3.19) that for each $\xi \in C$,

$$0 \geq -\epsilon_0 \|\xi - x_k\| - \epsilon_0^2 - \tau_k \langle f(x_k), \xi - x_k \rangle - \tau_k M_1 \epsilon_0$$

and

$$\begin{aligned} \langle f(x_k), \xi - x_k \rangle \geq & \tau_k^{-1} \epsilon_0 \|\xi - x_k\| - \epsilon_0^2 \tau_k^{-1} - M_1 \epsilon_0 \\ \geq & - \tilde{\tau}^{-1} \epsilon_0 \|\xi - x_k\| - \epsilon_0^2 \tilde{\tau}^{-1} - M_1 \epsilon_0 \\ \geq & \epsilon \|\xi - x_k\| - \epsilon. \end{aligned}$$

Thus, Theorem 3.1 is proved. □

4. Inexact Iterates With Summable Iterates

We use the assumptions, definitions and the notation introduced in Section 1.1. The following theorem shows that the most inexact iterates of our algorithm under the presence of summable computational errors are approximate solutions of the variational inequality.

Theorem 4.1. *Let* $\{\Delta_i\}_{i=0}^{\infty} \subset (0,1)$ *satisfy*

$$\Delta = \sum_{i=0}^{\infty} \Delta_i < \infty, \tag{4.1}$$

$\epsilon \in (0,1)$, $M_0 \geq 1, M_1, L > 0$,

$$B(0,M_0) \cap S \neq \emptyset, \tag{4.2}$$

$$f(B(0,3M_0+4\Delta+1)) \subset B(0,M_1), \tag{4.3}$$

$$\|f(z_1)-f(z_2)\| \leq L\|z_1-z_2\| \tag{4.4}$$

for all $z_1, z_2 \in B(0,3M_0+M_1+4\Delta+1)$,

$$0 < \tilde{\tau} < \tau_* \leq 1,\ \tau_* L < 1 \tag{4.5}$$

and let $\epsilon_0 > 0$ *satisfy*

$$2\epsilon_0(6\tilde{\tau}^{-1}+2M_1+2L) < \epsilon. \tag{4.6}$$

Assume that $\{x_k\}_{k=0}^{\infty} \subset X$, $\{y_k\}_{k=0}^{\infty} \subset X$

$$\{\tau_k\}_{k=0}^{\infty} \subset [\tilde{\tau}, \tau_*], \tag{4.7}$$

$$\|x_0\| \leq M_0 \tag{4.8}$$

and that for each integer $k \geq 0$,

$$\|y_k - P_C(x_k - \tau_k f(x_k))\| \leq \Delta_k, \tag{4.9}$$

$$\|x_{k+1} - P_C(x_k - \tau_k f(y_k))\| \leq \Delta_k \tag{4.10}$$

and a natural number n_0 *satisfies for each integer* $k \geq n_0$,

$$\Delta_k \leq \epsilon_0/2. \tag{4.11}$$

Then

$$\|x\| \leq 3M_0 + 4\Delta,\ k = 0,1,\dots$$

and

$$\mathit{Card}(\{k \in \{0,1,\dots\}:\ \|x_k-y_k\| > \epsilon_0\}) \leq n_0+4\epsilon_0^{-2}(1-\tau_*^2L^2)^{-1}((2M_0+4\Delta)^2+4\Delta(2M_0+4\Delta+1)).$$

Moreover, if $k \geq n_0$ *is an integer and* $\|x_k - y_k\| \leq \epsilon_0$, *then*

$$\langle f(x_k), \xi - x_k\rangle \geq -\epsilon - \epsilon\|\xi - x_k\|,\ \xi \in C$$

and $S_\epsilon \cap B(x_k, \epsilon) \neq \emptyset$.

Proof. By (4.2), there exists

$$u_* \in S \cap B(0,M_0). \tag{4.12}$$

Equations (4.8) and (4.12) imply that

$$\|x_0 - u_*\| \leq 2M_0. \tag{4.13}$$

Assume that $k \geq 0$ is an integer and

$$\|x_k - u_*\| \leq 2M_0 + 4\sum\{\Delta_i:\ i \in \{0,\dots,k\} \setminus \{k\}\}. \tag{4.14}$$

(In view of (4.13), equation (4.14) holds for $k=0$.) Set

$$\tilde{M}_k = 2M_0 + 4\sum\{\Delta_i:\ i \in \{0,\dots,k\} \setminus \{k\}\}. \tag{4.15}$$

Clearly,

$$\tilde{M}_k \leq 2M_0 + 4\Delta.$$

By (4.12), (4.15) and Lemma 2.2 applied with

$$M_0 = \tilde{M}_k,\ x = x_k,\ y = y_k,\ \tilde{x} = x_{k+1},\ \tau = \tau_k,\ \delta = \Delta_k$$

we have

$$\|x_{k+1} - u_*\|^2 \leq \|x_k - u_*\|^2 + 4\Delta_k(\tilde{M}_k + 1) - (1 - \tau_k^2 L^2)\|x_k - P_C(x_k - \tau_k f(x_k))\|^2. \tag{4.16}$$

It follows from (4.15) and (4.16) that

$$\|x_{k+1} - u_*\|^2 \leq \|x_k - u_*\|^2 + 4\Delta_k(1 + 2M_0 + 4\Delta) - (1 - \tau_k^2 L^2)\|x_k - P_C(x_k - \tau_k f(x_k))\|^2. \tag{4.17}$$

In view of (4.14)-(4.16),

$$\begin{aligned} x\|x_{k+1} - u_*\|^2 \leq & 4\Delta_k(1 + \tilde{M}_k) + \tilde{M}_k^2 \leq \tilde{M}_k^2 + 8\Delta_k \tilde{M}_k \leq (\tilde{M}_k + 4\Delta_k)^2, \\ \|x_{k+1} - u_*\| \leq & \tilde{M}_k + 4\Delta_k \leq 2M_0 + 4\sum_{i=0}^{k} \Delta_i. \end{aligned}$$

Thus, we showed by induction that (4.14) and (4.17) hold for all integers $k \geq 0$. By (4.1), (4.12) and (4.14), for each integer $k \geq 0$,

$$\|x_k - u_*\| \leq 2M_0 + 4\Delta,\ \|x_k\| \leq 3M_0 + 4\Delta. \tag{4.18}$$

Set

$$E = \{k \in \{n_0, n_0 + 1, \dots,\} :\ \|x_k - y_k\| > \epsilon_0\}. \tag{4.19}$$

In view of (4.11) and (4.19), for each integer $k \in E$,

$$\|x_k - P_C(x_k - \tau_k f(x_k))\| \geq \|x_k - y_k\| - \|y_k - P_C(x_k - \tau_k f(x_k))\| > \epsilon_0 - \Delta_k \geq \epsilon_0/2. \tag{4.20}$$

Let $Q > n_0$ be a natural number. Set

$$E_Q = E \cap \{0, \dots, Q - 1\}. \tag{4.21}$$

By (4.17) and (4.18),

$$\begin{aligned} (2M_0 + 4\Delta)^2 \geq & \|x_{n_0} - u_*\|^2 \\ \geq & \|x_{n_0} - u_*\|^2 - \|x_Q - u_*\|^2 \\ = & \sum_{k=n_0}^{Q-1} (\|x_k - u_*\|^2 - \|x_{k+1} - u_*\|^2) \\ \geq & \sum_{k=n_0}^{Q-1} ((1 - \tau_k^2 L^2)\|x_k - P_C(x_k - \tau_k f(x_k))\|^2 - 4\Delta_k(1 + 2M_0 + 4\Delta)) \end{aligned}$$

and in view of (4.7) and (4.19)-(4.21),

$$\begin{aligned} (1 - \tau_*^2 L^2)^{-1}((2M_0 + 4\Delta)^2 + 4\Delta(1 + 2M_0 + 4\Delta)) \geq & \sum_{k=n_0}^{Q-1} \|x_k - P_C(x_k - \tau_k f(x_k))\|^2 \\ \geq & \sum \{\|x_k - P_C(x_k - \tau_k f(x_k))\|^2 :\ k \in E_Q\} \\ \geq & 4^{-1}\epsilon_0^2 \mathrm{Card}(E_Q) \end{aligned}$$

and

$$\mathrm{Card}(E_Q) \leq 4\epsilon_0^{-2}(1 - \tau_*^2 L^2)^{-1}((2M_0 + 4\Delta)^2 + 14\Delta(2M_0 + 4\Delta + 1)).$$

Since Q is any natural number larger than n_0 we conclude that

$$\mathrm{Card}(E) \le 4\epsilon_0^{-2}(1-\tau_*^2 L^2)^{-1}((2M_0+4\Delta)^2 + 4\Delta(2M_0+4\Delta+1))$$

and in view of (4.19),]

$$\begin{aligned}\mathrm{Card}(\{k \in \{0,1,\dots\} :\ \|x_k - y_k\| > \epsilon_0\}) \le& n_0 + \mathrm{Card}(E)\\ \le& n_0 + 4\epsilon_0^{-2}(1-\tau_*^2 L^2)^{-1}((2M_0+4\Delta)^2 + 4\Delta(2M_0+4\Delta+1)).\end{aligned}$$

Assume that $k \ge n_0$ is an integer and

$$\|x_k - y_k\| \le \epsilon_0. \tag{4.22}$$

By (4.9), (4.11) and (4.22),

$$\begin{aligned}\|x_k - P_C(x_k - \tau_k f(x_k))\| \le& \|x_k - y_k\| + \|y_k - P_C(x_k - \tau_k f(x_k))\|\\ \le& \epsilon_0 + \Delta_k\\ \le& 2\epsilon_0.\end{aligned} \tag{4.23}$$

It follows from Lemma 1.1 that for each $\xi \in C$,

$$0 \ge \langle x_k - \tau_k f(x_k) - P_C(x_k - \tau_k f(x_k)), \xi - P_C(x_k - \tau_k f(x_k))\rangle. \tag{4.24}$$

By (4.23), (4.24) and the Cauchy-Schwartz inequality for each $\xi \in C$,

$$\begin{aligned}0 \ge& \langle x_k - P_C(x_k - \tau_k f(x_k)), \xi - P_C(x_k - \tau_k f(x_k))\rangle\\ &- \tau\langle f(x_k), \xi - P_C(x_k - \tau_k f(x_k))\rangle\\ \ge& -\|x_k - P_C(x_k - \tau_k f(x_k))\|(\|\xi - x_k\| + \|x_k - P_C(x_k - \tau_k f(x_k)\|)\\ &- \tau_k\langle f(x_k), \xi - x_k\rangle - \tau_k\langle f(x_k), x_k - P_C(x_k - \tau_k f(x_k))\rangle\\ \ge& -2\epsilon_0\|\xi - x_k\| - 4\epsilon_0^2 - \tau_k\langle f(x_k), \xi - x_k\rangle - 2\tau_k\|f(x_k)\|\epsilon_0.\end{aligned} \tag{4.25}$$

It follows from (4.3), (4.7), (4.18) and (4.25) that for each $\xi \in C$,

$$\begin{aligned}\tau_k\langle f(x_k), \xi - x_k\rangle \ge& -2\epsilon_0\|\xi - x_k\| - 4\epsilon_0^2 - 2\tau_k\epsilon_0 M_1,\\ \langle f(x_k), \xi - x_k\rangle \ge& -2\epsilon_0\tilde{\tau}^{-1}\|\xi - x_k\| - 4\epsilon_0^2\tilde{\tau}^{-1} - 4\tilde{\tau}^{-1} - 2\epsilon_0 M_1.\end{aligned} \tag{4.26}$$

By (4.6) and (4.26), for each $\xi \in C$,

$$\langle f(x_k), \xi - x_k\rangle \ge -\epsilon\|\xi - x_k\| - \epsilon.$$

Set

$$\bar{y} = P_C(x_k - \tau_k f(x_k)). \tag{4.27}$$

Clearly, $\bar{y} \in C$. In view of (4.6), (4.23) and (4.27),

$$\|x_k - \bar{y}\| \le 2\epsilon_0 < \epsilon. \tag{4.28}$$

Equations (4.3), (4.4), (4.18) and (4.28) that

$$\|f(\bar{y})\| \le M_1,\ \|f(x_k) - f(\bar{y})\| \le L\|x_k - \bar{y}\| \le 2L\epsilon_0. \tag{4.29}$$

By (4.6), (4.26), (4.28) and (4.29), for each $\xi \in C$,

$$\begin{aligned}\langle f(\bar{y}), \xi - \bar{y}\rangle \ge& \langle f(\bar{y}), \xi - x_k\rangle - \|f(\bar{y})\|\|x_k - \bar{y}\|\\ \ge& \langle f(\bar{y}), \xi - x_k\rangle - 2\epsilon_0 M_1\\ \ge& \langle f(x_k), \xi - x_k\rangle - \|f(\bar{y}) - f(x_k)\|\|\xi - x_k\| - 2\epsilon_0 M_1\\ \ge& -2\epsilon_0\tilde{\tau}^{-1}\|\xi - x_k\| - 4\epsilon_0\tilde{\tau}^{-1} - 2\epsilon_0 L\|\xi - x_k\| - 4\epsilon_0 M_1\\ \ge& -(2\tilde{\tau}^{-1}\epsilon_0 + 2L\epsilon_0)\|\xi - x_k\| - 4\epsilon_0 M_1 - 4\epsilon_0\tilde{\tau}^{-1}\\ \ge& -\epsilon\|\xi_k - \bar{y}\| - (2\tilde{\tau}^{-1}\epsilon_0 + 2L\epsilon_0)\|x_k - \bar{y}\| - 4\epsilon_0\tilde{\tau}^{-1} - 4\epsilon_0 M_1\\ \ge& -\epsilon\|\xi - \bar{y}\| - 2\epsilon_0(2\tilde{\tau}^{-1}\epsilon_0 + 2L\epsilon_0) - 4\epsilon_0\tilde{\tau}^{-1} - 4\epsilon_0 M_1\\ \ge& -\epsilon\|\xi - \bar{y}\| - \epsilon.\end{aligned}$$

Theorem 4.1 is proved. □

STATEMENTS AND DECLARATIONS

The authors declare that they have no conflict of interest, and the manuscript has no associated data.

REFERENCES

[1] Ya. I. Alber, A. N. Iusem, and M. V. Solodov. On the projected subgradient method for nonsmooth convex optimization in a Hilbert space. *Mathematical Programming*, 81:23-35, 1998.

[2] K. Barty, J.-S. Roy, and C. Strugarek. Hilbert-valued perturbed subgradient algorithms. *Mathematics of Operations Research*, 32:551-562, 2007.

[3] R. S. Burachik, L. M. Grana Drummond, A. N. Iusem, and B. F. Svaiter. Full convergence of the steepest descent method with inexact line searches. *Optimization*, 32:137-146, 1995.

[4] R. S. Burachik, J. O. Lopes, and G. J. P. Da Silva. An inexact interior point proximal method for the variational inequality. *Computational and Applied Mathematics*, 28:15-36, 2009.

[5] Y. Censor, A. Gibali, and S. Reich. The subgradient extragradient method for solving variational inequalities in Hilbert space. *Journal of Optimization Theory Application*, 148:318-335, 2011.

[6] V. F. Demyanov and L. V. Vasil'ev. *Nondifferentiable Optimization*. Nauka, Moscow, 1981.

[7] F. Facchinei and J.-S. Pang. *Finite-Dimensional Variational Inequalities and Complementarity Problems*. Volume I and Volume II, Springer-Verlag, New York, 2003.

[8] J. Gwinner and F. Raciti. On monotone variational inequalities with random data. *Journal of Mathematical Inequalities*, 3:443-453, 2009.

[9] A. Kaplan and R. Tichatschke. Bregman-like functions and proximal methods for variational problems with nonlinear constraints. *Optimization*, 56:253-265, 2007.

[10] D. Kinderlehrer and G. Stampacchia. *An Introduction to Variational Inequalities and Their Applications*. Classics in Applied Mathematics, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, 2000.

[11] I. V. Konnov. Descent method with inexact linesearch for mixed variational inequalities. *Russian Mathematics*, 53:29-35, 2009.

[12] G. M. Korpelevich. The extragradient method for finding saddle points and other problems. *Ekonomika i Matematicheskie Metody*, 12:747-756, 1976.

[13] N. Pakkaranang. Double inertial extragradient algorithms for solving variational inequality problems with convergence analysis. *Mathematical Methods in the Applied Sciences*, 47(14):11642-11669, 2024.

[14] A. J. Zaslavski. The extragradient method for solving variational inequalities in the presence of computational errors. *Journal of Optimization Theory and Applications*, 153:602-618, 2012.

[15] A. J. Zaslavski. *Approximate Solutions of Common Fixed Point Problems*. Springer Optimization and Its Applications, Springer, Cham, 2016.

[16] A. J. Zaslavski. *Numerical Optimization with Computational Errors*. Springer Optimization and Its Applications, Springer, Cham, 2016.