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THE BOUNDS OF THE SOLUTION SET FOR THE EXTENDED VERTICAL TENSOR COMPLEMENTARITY PROBLEM

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ABSTRACT. In this paper, we investigate the bounds of the solution set for the extended vertical tensor complementarity problem (EVTCP). We first introduce several generalized structural constants for a system of l tensors, utilizing absolute value products and nested minimum-maximum operators to preserve strict algebraic homogeneity and sign consistency. Based on these newly defined constants, we establish several explicit upper bounds for the solution set of the EVTCP under the conditions that the underlying tensor tuple is an EVP-tensor tuple or an EVR_0-tensor tuple. Furthermore, we rigorously compare the sharpness of these proposed upper bounds. The theoretical results elegantly extend and refine the existing bounds from the two-tensor vertical complementarity problem to the generalized multi-tensor case. The results obtained in this paper are extensions of those proposed by Wang-Fu-Wu (J. Optim. Theory Appl., 2025, 204:2) from the vertical tensor complementarity problem (VTCP) to EVTCP.

Keywords. Extended vertical tensor complementarity problem, EVP-tensor tuple, EVR_0-tensor tuple, Bounds of the solution set.

1. INTRODUCTION

The complementarity problem(CP) is a long-standing and well-established topic within mathematical programming, with a rich history and significant impact across various fields such as economics, engineering, operations research, and applied mathematics. The classical linear complementarity problem (LCP) has been extensively studied over the past several decades, and its theoretical and algorithmic developments have laid the foundation for addressing more complex and generalized forms of complementarity problems. We refer the reader to monographs [3, 8] and surveys [16, 9] for comprehensive coverage of the well-developed basic theory, numerical algorithms, and applications of complementarity problems.

As a natural extension of the linear complementarity problem, Gowda-Sznajder [10] introduced the extended vertical linear complementarity problem (EVLCP). Given a matrix set $\hat{A} = \{A_1, A_2, \ldots, A_l\} \subset \mathbb{R}^{n\times n}$ and a vector set $\hat{\boldsymbol{q}} = \{\boldsymbol{q}_1, \boldsymbol{q}_2, \ldots, \boldsymbol{q}_l\} \subset \mathbb{R}^n$, the EVLCP$(\hat{A}, \hat{\boldsymbol{q}})$ seeks a vector $\boldsymbol{x} \in \mathbb{R}^n$ satisfying

$$(A_1\boldsymbol{x} + \boldsymbol{q}_1) \wedge (A_2\boldsymbol{x} + \boldsymbol{q}_2) \wedge \cdots \wedge (A_l\boldsymbol{x} + \boldsymbol{q}_l) = \mathbf{0},$$

where "$\wedge$" denotes the pointwise minimum operator. The EVLCP has found wide applications in optimization theory, generalized bimatrix games, and stochastic games [11, 37]. Consequently, the existence of solutions and numerical algorithms for this problem have been extensively studied, see [10, 17, 23, 31, 34, 38].

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An mth-order n-dimensional real tensor $\mathcal{A} = (a_{i_1 i_2 \ldots i_m})$ is a multidimensional array with entries $a_{i_1 i_2 \ldots i_m} \in \mathbb{R}$, where $i_j \in [n]$ for $j \in [m]$. The set of all such tensors is denoted by $\mathbb{R}^{[m,n]}$. Song-Qi [28] first proposed the tensor complementarity problem (TCP): for $\mathcal{A} \in \mathbb{R}^{[m,n]}$ and $\boldsymbol{q} \in \mathbb{R}^n$, denoted by TCP($\mathcal{A}, \boldsymbol{q}$), it aims to find $\boldsymbol{x} \in \mathbb{R}^n$ such

$$\boldsymbol{x} \geq \mathbf{0}, \quad \mathcal{A}\boldsymbol{x}^{m-1} + \boldsymbol{q} \geq \mathbf{0}, \quad \boldsymbol{x}^\top(\mathcal{A}\boldsymbol{x}^{m-1} + \boldsymbol{q}) = 0.$$

Equivalently, this can be expressed via the minimum map as $\boldsymbol{x} \wedge (\mathcal{A}\boldsymbol{x}^{m-1} + \boldsymbol{q}) = \mathbf{0}$.

When $m = 2$, the TCP($\mathcal{A}, \boldsymbol{q}$) reduces to the classical LCP(A, q). As a special case of the generalized complementarity problem CP(F), the TCP has attracted substantial research interest over the past decade due to its rich algebraic structure and broad applicability. Numerous studies have investigated its theoretical properties, including the nonemptiness and compactness of the solution set [1, 4, 18, 33], solution bounds [22, 25, 29, 35, 36], problem feasibility [12, 30], and numerical methods [5, 6, 13]. Moreover, practical applications of TCP have emerged in fields such as nonlinear compressed sensing, hypergraph clustering, communications, DNA microarrays, and n-person noncooperative games [4, 13]. For comprehensive surveys on TCP theory and applications, see [14, 15].

In particular, regarding solution bounds, Song-Qi [27] introduced two structural constants $\alpha(T_\mathcal{A})$ and $\alpha(F_\mathcal{A})$ to establish upper and lower bounds for TCP with a $\mathcal{P}$-tensor. Later, Zheng-Zhang-Huang [39] employed these constants to derive global error bounds. Subsequent works have further refined bound estimates: Song-Yu [30] and Song-Qi [29] obtained upper and lower bounds for strictly semi-positive tensors; Song-Mei [18] derived lower bounds for $\mathcal{B}$-tensors; Mei-Yang [22] studied boundedness for nonnegative $\mathcal{Q}$-tensors; Xu-Huang [36] introduced a quantity $\beta(\mathcal{A})$ to bound R_0-tensors; and Li-Li [18] estimated upper bounds for nonsingular $\mathcal{H}$-tensors.

Bridging the concepts of extended complementarity and tensor spaces, the vertical tensor complementarity problem (VTCP) was later introduced. For a tensor pair $\widetilde{\mathcal{A}} = \{\mathcal{A}_1, \mathcal{A}_2\} \subset \mathbb{R}^{[m,n]}$ and a vector pair $\widetilde{\boldsymbol{q}} = \{\boldsymbol{q}_1, \boldsymbol{q}_2\} \subset \mathbb{R}^n$, the VTCP($\widetilde{\mathcal{A}}, \widetilde{\boldsymbol{q}}$) seeks $\boldsymbol{x} \in \mathbb{R}^n$ such that

$$\mathcal{A}_1\boldsymbol{x}^{m-1} + \boldsymbol{q}_1 \geq \mathbf{0}, \quad \mathcal{A}_2\boldsymbol{x}^{m-1} + \boldsymbol{q}_2 \geq \mathbf{0}, \quad (\mathcal{A}_1\boldsymbol{x}^{m-1} + \boldsymbol{q}_1)^\top(\mathcal{A}_2\boldsymbol{x}^{m-1} + \boldsymbol{q}_2) = 0,$$

which is equivalent to $(\mathcal{A}_1\boldsymbol{x}^{m-1} + \boldsymbol{q}_1) \wedge (\mathcal{A}_2\boldsymbol{x}^{m-1} + \boldsymbol{q}_2) = \mathbf{0}$.

Recently, Che-Qi-Wei [2] introduced an unsolved problem, referred to in this paper as the extended vertical tensor complementarity problem, which seeks a vector $\boldsymbol{x} \in \mathbb{R}^n$ such that

$$(\boldsymbol{q}_1 + \mathcal{A}_1\boldsymbol{x}^{m-1}) \wedge (\boldsymbol{q}_2 + \mathcal{A}_2\boldsymbol{x}^{m-1}) \wedge \cdots \wedge (\boldsymbol{q}_l + \mathcal{A}_l\boldsymbol{x}^{m-1}) = \mathbf{0},$$

where $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_l$ are mth-order n-dimensional tensors and $\boldsymbol{q}_1, \boldsymbol{q}_2, \ldots, \boldsymbol{q}_l \in \mathbb{R}^n$. We denote it by EVTCP($\hat{\mathcal{A}}, \hat{\boldsymbol{q}}$) with $\hat{\mathcal{A}} = \{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_l\}$ and $\hat{\boldsymbol{q}} = \{\boldsymbol{q}_1, \boldsymbol{q}_2, \ldots, \boldsymbol{q}_l\}$. When $\hat{\mathcal{A}}$ is a set of matrices, EVTCP reduces to the extended generalized order linear complementarity problem (EGOLCP) [10]. In [24], the equivalence of EVTCP (with $\hat{\mathcal{A}} = \{\mathcal{A}_1, \mathcal{A}_2\}$) to the generalized tensor absolute value equation (GTAVE) is established, and the significance of GTAVE in various fields such as linear programming and quadratic programming is highlighted. Li-Jiang-Yang-Tang [20] introduced vertical non-degenerate (VND) and strong vertical non-degenerate (SVND) tensor tuples, and extended the finiteness property of solutions from the standard TCP to the EVTCP framework. Li-Wu-Li [19] introduces several structured tensor tuples, including type EVR_0, type EVE, type EVP and type strong EVP, and establishes sufficient conditions for the boundedness, existence and uniqueness of solutions to the EVTCP.

Furthermore, EVTCP($\hat{\mathcal{A}}, \hat{\boldsymbol{q}}$) is important for two reasons. Firstly, the special case of EVTCP($\hat{\mathcal{A}}, \hat{\boldsymbol{q}}$) with $l = 2$ is equivalent to the tensor absolute value equation, as demonstrated in [7]. Secondly, EVTCP($\hat{\mathcal{A}}, \hat{\boldsymbol{q}}$) is a generalization of the tensor complementarity problem TCP($\mathcal{A}, \boldsymbol{q}$). To the best of our knowledge, there is a lack of results on the boundedness of the solution set for EVTCP($\hat{\mathcal{A}}, \hat{\boldsymbol{q}}$). This gap motivates our research, which aims to investigate the boundedness of the solution set for EVTCP($\hat{\mathcal{A}}, \hat{\boldsymbol{q}}$).

The remainder of this paper is organized as follows. Section 2 introduces preliminary concepts, including basic notations and known results. In Section 3, we define two generalized quantities of a tensor tuple, explore their interrelationships, and establish an equivalent condition for EVR_0-tensor tuples. Furthermore, upper bounds for these quantities are also provided. Section 4 investigates the lower bounds of the solution set for the EVTCP under a general non-zero tensor tuple, provided that the solution set is nonempty. Section 5 estimates the upper bounds of the solution set for the EVTCP equipped with an EVP-tensor tuple and an EVR_0-tensor tuple. Finally, Section 6 summarizes our concluding remarks.

2. Preliminaries

In this section, we review some definitions and notations used in the later discussion. Assume that m, l, n are positive integers with $m, m_1, m_2, n \geq 2$ and denote $[n] := \{1, 2, \dots, n\}$. For a given vector $\boldsymbol{x} = (x_1, x_2, \dots, x_n)^\top$, define

$$(\boldsymbol{x})_+ = (\max\{x_1, 0\}, \max\{x_2, 0\}, \dots, \max\{x_n, 0\})^\top,$$
$$\boldsymbol{x}^{[m]} = (x_1^m, x_2^m, \dots, x_n^m)^\top.$$

It is said to be $\boldsymbol{x} \geq \mathbf{0}$, if all components of $\boldsymbol{x}$ are nonnegative. For two n-dimensional vectors $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$, their Hadamard product is defined by

$$\boldsymbol{q}_1 \circ \boldsymbol{q}_2 = \big((\boldsymbol{q}_1)_1(\boldsymbol{q}_2)_1, (\boldsymbol{q}_1)_2(\boldsymbol{q}_2)_2, \dots, (\boldsymbol{q}_1)_n(\boldsymbol{q}_2)_n\big)^\top.$$

An m-th order n-dimensional tensor $\mathcal{A}$ is a multidimensional array of real elements $a_{i_1 i_2 \dots i_m}$, where $i_j \in [n]$ for all $j \in [m]$. The tensor $\mathcal{A}$ is said to be nonnegative, if all its entries are nonnegative; the tensor $\mathcal{A}$ is said to be symmetric, if all its entries do not change under any permutation of its indices.

We use the following norms. For an n-dimensional vector $\boldsymbol{x}$ and $p \geq 1$, define

$$\|\boldsymbol{x}\|_\infty = \max\{|x_i| : i \in [n]\} \quad \text{and} \quad \|\boldsymbol{x}\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}}.$$

For a given tensor $\mathcal{A} \in \mathbb{R}^{[m,n]}$, define

$$\|\mathcal{A}\|_\infty = \max_{i \in [n]} \sum_{j_2, \dots, j_m = 1}^n |a_{i j_2 \dots i_m}|.$$

For an n-dimensional vector $\boldsymbol{x}$ and $p \geq 1$, it is obvious that

$$\|\boldsymbol{x}\|_\infty \leq \|\boldsymbol{x}\|_p \leq n^{\frac{1}{p}} \|\boldsymbol{x}\|_\infty,$$
$$n^{-\frac{1}{p}} \|\boldsymbol{x}\|_p \leq \|\boldsymbol{x}\|_\infty \leq \|\boldsymbol{x}\|_p.$$

Thus, we conclude that two norms $\|\cdot\|_\infty$ and $\|\cdot\|_p$ of a vector are equivalent.

For a tensor $\mathcal{A}$ and a vector $\boldsymbol{x}$, Song-Qi [26] proposed two positively homogeneous operators $T_\mathcal{A}$ and $F_\mathcal{A}$. The operators are defined as follows

$$T_\mathcal{A}\boldsymbol{x} = \|\boldsymbol{x}\|_2^{2-m} \mathcal{A}\boldsymbol{x}^{m-1}.$$

If m is an even number, then

$$F_\mathcal{A}\boldsymbol{x} = (\mathcal{A}\boldsymbol{x}^{m-1})^{[\frac{1}{m-1}]}.$$

Some of their norms are defined as follows:

$$\|T_\mathcal{A}\|_\infty = \max_{\|\boldsymbol{x}\|_\infty = 1} \|T_\mathcal{A}\boldsymbol{x}\|_\infty,$$
$$\|F_\mathcal{A}\|_\infty = \max_{\|\boldsymbol{x}\|_\infty = 1} \|F_\mathcal{A}\boldsymbol{x}\|_\infty,$$

$$\|T_{\mathcal{A}}\|_p = \max_{\|\boldsymbol{x}\|_p=1} \|T_{\mathcal{A}}\boldsymbol{x}\|_p,$$

$$\|F_{\mathcal{A}}\|_p = \max_{\|\boldsymbol{x}\|_p=1} \|F_{\mathcal{A}}\boldsymbol{x}\|_p.$$

Definition 2.1. [19, Definition 1] Given a tensor tuple $\hat{\mathcal{A}} = \{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_l\}$ with $\mathcal{A}_t \in \mathbb{R}^{[m,n]}$ for all $t \in [l]$. $\hat{\mathcal{A}}$ is said to be an EVP-tensor tuple, iff

$$\min\{\mathcal{A}_1\boldsymbol{x}^{m-1}, \cdots, \mathcal{A}_l\boldsymbol{x}^{m-1}\} \leq \mathbf{0} \leq \max\{\mathcal{A}_1\boldsymbol{x}^{m-1}, \cdots, \mathcal{A}_l\boldsymbol{x}^{m-1}\} \Rightarrow \boldsymbol{x} = \mathbf{0}.$$

Definition 2.2. [19, Definition 1] Given a tensor tuple $\hat{\mathcal{A}} = \{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_l\}$ with $\mathcal{A}_t \in \mathbb{R}^{[m,n]}$ for all $t \in [l]$. $\hat{\mathcal{A}}$ is said to be an EVR_0-tensor tuple, iff

$$\min\{\mathcal{A}_1\boldsymbol{x}^{m-1}, \mathcal{A}_2\boldsymbol{x}^{m-1}, \cdots, \mathcal{A}_l\boldsymbol{x}^{m-1}\} = \mathbf{0} \Rightarrow \boldsymbol{x} = \mathbf{0}.$$

Several basic results in this study are given as follows. For a tensor tuple $\hat{\mathcal{A}} = \{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_l\}$ with $\mathcal{A}_t \in \mathbb{R}^{[m,n]}$ for all $t \in [l]$, we introduce two quantities as follows.

$$\alpha(T_{\hat{\mathcal{A}}}) := \min_{\|\boldsymbol{x}\|_\infty=1} \max_{i\in[n]} \prod_{t=1}^{l} |(T_{\mathcal{A}_t}\boldsymbol{x})_i|\,,$$

and if m is an even number

$$\alpha(F_{\hat{\mathcal{A}}}) := \min_{\|\boldsymbol{x}\|_\infty=1} \max_{i\in[n]} \prod_{t=1}^{l} |(F_{\mathcal{A}_t}\boldsymbol{x})_i|\,.$$

Lemma 2.3. [26, Theorems 4.3 and 4.4] *If $\mathcal{A} \in \mathbb{R}^{[m,n]}$, $\boldsymbol{x} \in \mathbb{R}^n$, then*

(i) $\|T_{\mathcal{A}}(\boldsymbol{x})\|_\infty \leq \|T_{\mathcal{A}}\|_\infty \|\boldsymbol{x}\|_\infty$ *and* $\|T_{\mathcal{A}}(\boldsymbol{x})\|_p \leq \|T_{\mathcal{A}}\|_p \|\boldsymbol{x}\|_p$;

(ii) $\|F_{\mathcal{A}}(\boldsymbol{x})\|_\infty \leq \|F_{\mathcal{A}}\|_\infty \|\boldsymbol{x}\|_\infty$ *and* $\|F_{\mathcal{A}}(\boldsymbol{x})\|_p \leq \|F_{\mathcal{A}}\|_p \|\boldsymbol{x}\|_p$;

(iii) $\|T_{\mathcal{A}}\|_\infty \leq \|\mathcal{A}\|_\infty$;

(iv) $\|F_{\mathcal{A}}\|_\infty \leq \|\mathcal{A}\|_\infty^{\frac{1}{m-1}}$, *for m being an even number;*

(v) $\|F_{\mathcal{A}}\|_\infty = \|\mathcal{A}\|_\infty^{\frac{1}{m-1}}$, *for m being even and $\mathcal{A}$ being a nonnegative tensor;*

(vi) $n^{\frac{2-m}{2}}\|\mathcal{A}\|_\infty \leq \|T_{\mathcal{A}}\|_\infty \leq \|\mathcal{A}\|_\infty$, *for $\mathcal{A}$ being a nonnegative tensor.*

Lemma 2.4. [21, Lemma 3.1] *If $\mathcal{A} \in \mathbb{R}^{[m,n]}$, $\boldsymbol{x} \in \mathbb{R}^n$, then*

$$\|\mathcal{A}\boldsymbol{x}^{m-1}\|_\infty \leq \|\mathcal{A}\|_\infty \|\boldsymbol{x}\|_\infty^{m-1} \text{ and } \|\mathcal{A}\boldsymbol{x}^{m-1}\|_2 \leq \|\mathcal{A}\|_2 \|\boldsymbol{x}\|_2^{m-1}.$$

Lemma 2.5. *For a tensor tuple $\hat{\mathcal{A}} = \{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_l\}$ with $\mathcal{A}_t \in \mathbb{R}^{[m,n]}$ for all $t \in [l]$,*

(i) *if $\hat{\mathcal{A}}$ is an EVP-tensor tuple, then $\alpha(T_{\hat{\mathcal{A}}}) > 0$;*

(ii) *if $\hat{\mathcal{A}}$ is an EVP-tensor tuple and m is even, then $\alpha(F_{\hat{\mathcal{A}}}) > 0$.*

Proof. (i) Suppose on the contrary that $\alpha(T_{\hat{\mathcal{A}}}) = 0$. Since $\{\boldsymbol{x} \in \mathbb{R}^n \mid \|\boldsymbol{x}\|_\infty = 1\}$ is a compact set and the objective function is continuous, there exists $\boldsymbol{x}^* \in \mathbb{R}^n$ with $\|\boldsymbol{x}^*\|_\infty = 1$ (i.e., $\boldsymbol{x}^* \neq \mathbf{0}$) such that

$$\max_{i\in[n]} \prod_{t=1}^{l} |(T_{\mathcal{A}_t}\boldsymbol{x}^*)_i| = 0.$$

This implies that $\prod_{t=1}^{l} |(T_{\mathcal{A}_t}\boldsymbol{x}^*)_i| = 0$ for all $i \in [n]$.

On the other hand, since $\hat{\mathcal{A}}$ is an EVP-tensor tuple and $\boldsymbol{x}^* \neq \mathbf{0}$, there must exist an index $k \in [n]$ such that the components $(\mathcal{A}_t(\boldsymbol{x}^*)^{m-1})_k$ are strictly of the same sign for all $t \in [l]$. By the definition of the operator $T_{\mathcal{A}_t}$, we have $|(T_{\mathcal{A}_t}\boldsymbol{x}^*)_k| > 0$ for all $t \in [l]$, which yields

$$\prod_{t=1}^{l} |(T_{\mathcal{A}_t}\boldsymbol{x}^*)_k| > 0.$$

This contradicts the fact that $\prod_{t=1}^{l} |(T_{\mathcal{A}_t}\boldsymbol{x}^*)_i| = 0$ for all $i \in [n]$. Thus, we conclude that $\alpha(T_{\hat{\mathcal{A}}}) > 0$.

(ii) Suppose on the contrary that $\alpha(F_{\hat{\mathcal{A}}}) = 0$. There exists $\bar{\boldsymbol{x}} \in \mathbb{R}^n$ with $\|\bar{\boldsymbol{x}}\|_\infty = 1$ such that

$$\prod_{t=1}^{l} |(F_{\mathcal{A}_t}\bar{\boldsymbol{x}})_i| = 0, \quad \text{for all } i \in [n].$$

However, since $\hat{\mathcal{A}}$ is an EVP-tensor tuple and $\bar{\boldsymbol{x}} \neq \mathbf{0}$, there exists an integer $k \in [n]$ such that $|(\mathcal{A}_t\bar{\boldsymbol{x}}^{m-1})_k| > 0$ for all $t \in [l]$. By the definition of the operator $F_{\mathcal{A}_t}$, we obtain

$$|(F_{\mathcal{A}_t}\bar{\boldsymbol{x}})_k| = |(\mathcal{A}_t\bar{\boldsymbol{x}}^{m-1})_k|^{\frac{1}{m-1}} > 0, \quad \text{for all } t \in [l].$$

This yields $\prod_{t=1}^{l} |(F_{\mathcal{A}_t}\bar{\boldsymbol{x}})_k| > 0$, which sharply contradicts the deduction that the product is zero for all dimensions. Hence, $\alpha(F_{\hat{\mathcal{A}}}) > 0$. This completes the proof. □

Lemma 2.6. [19, Theorem 1] *Given a tensor tuple* $\hat{\mathcal{A}} = \{\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_l\}$ *with* $\mathcal{A}_t \in \mathbb{R}^{[m,n]}$ *for all* $t \in [l]$. $\hat{\mathcal{A}}$ *is a* EVR_0*-tensor tuple, iff the solution set of EVTCP(*$\hat{\mathcal{A}}, \hat{\boldsymbol{q}}$*) is a bounded closed set for any* $\hat{\boldsymbol{q}} = (q_1, q_2, \dots, q_l)$ *with* $\boldsymbol{q}_t \in \mathbb{R}^n$ *for all* $t \in [l]$.

Lemma 2.7. [19, Theorem 5] *Given a tensor tuple* $\hat{\mathcal{A}} = \{\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_l\}$ *with* $\mathcal{A}_t \in \mathbb{R}^{[m,n]}$. *If* $\hat{\mathcal{A}}$ *is an* EVP*-tensor tuple, then the EVTCP(*$\hat{\mathcal{A}}, \hat{\boldsymbol{q}}$*) has a solution for any* $\hat{\boldsymbol{q}} = (q_1, q_2, \dots, q_l)$ *with* $\boldsymbol{q}_t \in \mathbb{R}^n$ *for all* $t \in [l]$.

Lemma 2.8. *Let* $\hat{\mathcal{A}} = \{\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_l\}$ *be an* EVP*-tensor tuple. For any nonzero vector* $\boldsymbol{x} \in \mathbb{R}^n$*, there must exist an index* $k \in [n]$ *such that all the components* $(\mathcal{A}_t\boldsymbol{x}^{m-1})_k$ *for* $t \in [l]$ *strictly share the same sign.*

Proof. Suppose for contradiction that for a nonzero vector $\boldsymbol{x} \in \mathbb{R}^n$, the terms $(\mathcal{A}_t\boldsymbol{x}^{m-1})_i$ for $t \in [l]$ do not share the same strict sign for any index $i \in [n]$. This directly implies that for all $i \in [n]$, we have

$$\min_{t\in[l]}(\mathcal{A}_t\boldsymbol{x}^{m-1})_i \le 0 \le \max_{t\in[l]}(\mathcal{A}_t\boldsymbol{x}^{m-1})_i.$$

By the definition of an EVP-tensor tuple, this inequality system yields $\boldsymbol{x} = 0$, which contradicts the assumption that $\boldsymbol{x} \neq 0$. Thus, the original claim holds. □

3. Two New Quantities of a Tensor Tuple

In this section, we introduce two new quantities of a tensor tuple, discuss some properties of them, and give the upper bounds of all the quantities. Two quantities are defined as follows.

$$\beta(\hat{\mathcal{A}}) := \min_{\|\boldsymbol{x}\|_\infty=1} \max_{i\in[n]} \prod_{t=1}^{l} \left|(\mathcal{A}_t\boldsymbol{x}^{m-1})_i\right|,$$

$$\gamma(\hat{\mathcal{A}}) := \min_{\|\boldsymbol{x}\|_\infty=1} \max\left\{\max_{i\in[n]} \min_{t\in[l]}(\mathcal{A}_t\boldsymbol{x}^{m-1})_i,\ \max_{t\in[l]} \left\|(-\mathcal{A}_t\boldsymbol{x}^{m-1})_+\right\|_\infty\right\}.$$

Remark 3.1. Since $\hat{\mathcal{A}}$ is an EVP-tensor tuple, for any non-zero vector $\boldsymbol{x} \in \mathbb{R}^n$, there exists an index $k \in [n]$ such that all $(\mathcal{A}_t \boldsymbol{x}^{m-1})_k$ share the same strict non-zero sign. Thus, none of these terms are zero at index k, which guarantees that the absolute value product is strictly positive. Consequently, the minimum over the compact unit sphere $\|\boldsymbol{x}\|_\infty = 1$ ensures that $\alpha(T_{\hat{\mathcal{A}}}) > 0$, $\beta(\hat{\mathcal{A}}) > 0$, and $\alpha(F_{\hat{\mathcal{A}}}) > 0$.

Lemma 3.2. *If $\hat{\mathcal{A}} = \{\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_l\}$ is an EVP-tensor tuple with $\mathcal{A}_t \in \mathbb{R}^{[m,n]}$ for all $t \in [l]$, then one has*

$$n^{\frac{l(2-m)}{2}} \beta(\hat{\mathcal{A}}) \leq \alpha(T_{\hat{\mathcal{A}}}) \leq \beta(\hat{\mathcal{A}})$$

Proof. Since $\hat{\mathcal{A}} = \{\mathcal{A}_1, \dots, \mathcal{A}_l\}$ is an EVP-tensor tuple, for each nonzero vector $\boldsymbol{x} \in \mathbb{R}^n$, there exists an integer $i \in [n]$ such that all $(\mathcal{A}_t \boldsymbol{x}^{m-1})_i$ for $t \in [l]$ have the same strict sign. Then, for each nonzero vector $\boldsymbol{x} \in \mathbb{R}^n$, we have

$$\max_{i \in [n]} \prod_{t=1}^{l} |(\mathcal{A}_t \boldsymbol{x}^{m-1})_i| > 0.$$

For $m \geq 2$, we have $l(2-m) \leq 0$. Since

$$0 \leq \|\boldsymbol{x}\|_\infty \leq \|\boldsymbol{x}\|_2 \leq \sqrt{n}\|\boldsymbol{x}\|_\infty,$$

we get

$$\|\boldsymbol{x}\|_\infty^{l(2-m)} \geq \|\boldsymbol{x}\|_2^{l(2-m)} \geq (\sqrt{n}\|\boldsymbol{x}\|_\infty)^{l(2-m)}.$$

Therefore,

$$\begin{aligned}
\alpha(T_{\hat{\mathcal{A}}}) &= \min_{\|\boldsymbol{x}\|_\infty = 1} \max_{i \in [n]} \prod_{t=1}^{l} |(T_{\mathcal{A}_t} \boldsymbol{x})_i| \\
&= \min_{\|\boldsymbol{x}\|_\infty = 1} \max_{i \in [n]} \prod_{t=1}^{l} \big| \|\boldsymbol{x}\|_2^{2-m} (\mathcal{A}_t \boldsymbol{x}^{m-1})_i \big| \\
&= \min_{\|\boldsymbol{x}\|_\infty = 1} \|\boldsymbol{x}\|_2^{l(2-m)} \max_{i \in [n]} \prod_{t=1}^{l} |(\mathcal{A}_t \boldsymbol{x}^{m-1})_i| \\
&\leq \min_{\|\boldsymbol{x}\|_\infty = 1} \|\boldsymbol{x}\|_\infty^{l(2-m)} \max_{i \in [n]} \prod_{t=1}^{l} |(\mathcal{A}_t \boldsymbol{x}^{m-1})_i| \\
&= \min_{\|\boldsymbol{x}\|_\infty = 1} \max_{i \in [n]} \prod_{t=1}^{l} |(\mathcal{A}_t \boldsymbol{x}^{m-1})_i| \\
&= \beta(\hat{\mathcal{A}}),
\end{aligned}$$

and

$$\begin{aligned}
\alpha(T_{\hat{\mathcal{A}}}) &= \min_{\|\boldsymbol{x}\|_\infty = 1} \|\boldsymbol{x}\|_2^{l(2-m)} \max_{i \in [n]} \prod_{t=1}^{l} |(\mathcal{A}_t \boldsymbol{x}^{m-1})_i| \\
&\geq \min_{\|\boldsymbol{x}\|_\infty = 1} (\sqrt{n}\|\boldsymbol{x}\|_\infty)^{l(2-m)} \max_{i \in [n]} \prod_{t=1}^{l} |(\mathcal{A}_t \boldsymbol{x}^{m-1})_i| \\
&= n^{\frac{l(2-m)}{2}} \min_{\|\boldsymbol{x}\|_\infty = 1} \max_{i \in [n]} \prod_{t=1}^{l} |(\mathcal{A}_t \boldsymbol{x}^{m-1})_i| \\
&= n^{\frac{l(2-m)}{2}} \beta(\hat{\mathcal{A}}).
\end{aligned}$$

This yields the desired conclusion. □

Lemma 3.3. *For a tensor tuple $\hat{\mathcal{A}} = \{\mathcal{A}_1, \ldots, \mathcal{A}_l\}$ with $\mathcal{A}_t \in \mathbb{R}^{[m,n]}$ for all $t \in [l]$, one has*

(i) $\gamma(\hat{\mathcal{A}}) \geq 0$;

(ii) $\gamma(\hat{\mathcal{A}}) > 0$ *if and only if $\hat{\mathcal{A}}$ is an EVR_0-tensor tuple;*

(iii) $\gamma(\hat{\mathcal{A}}) \geq \beta(\hat{\mathcal{A}})$.

Proof. (i) It is obvious that

$$\begin{aligned}\gamma(\hat{\mathcal{A}}) &= \min_{\|\boldsymbol{x}\|_\infty=1} \max\left\{\max_{i\in[n]}\min_{t\in[l]}(\mathcal{A}_t\boldsymbol{x}^{m-1})_i,\ \max_{t\in[l]}\left\|[-\mathcal{A}_t\boldsymbol{x}^{m-1}]_+\right\|_\infty\right\}\\ &\geq \min_{\|\boldsymbol{x}\|_\infty=1}\max_{t\in[l]}\left\|[-\mathcal{A}_t\boldsymbol{x}^{m-1}]_+\right\|_\infty \geq 0.\end{aligned}$$

(ii) If $\gamma(\hat{\mathcal{A}}) > 0$, then for each vector $\boldsymbol{y} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$,

$$\max\left\{\max_{i\in[n]}\min_{t\in[l]}(\mathcal{A}_t\boldsymbol{y}^{m-1})_i,\ \max_{t\in[l]}\|[-\mathcal{A}_t\boldsymbol{y}^{m-1}]_+\|_\infty\right\} > 0.$$

That is, there exists an integer $k \in [n]$ such that $\min_{t\in[l]}(\mathcal{A}_t\boldsymbol{y}^{m-1})_k > 0$, or there exist $t \in [l]$ and $k \in [n]$ such that $([-\mathcal{A}_t\boldsymbol{y}^{m-1}]_+)_k > 0$. The latter implies $(\mathcal{A}_t\boldsymbol{y}^{m-1})_k < 0$, which yields $\min_{t\in[l]}(\mathcal{A}_t\boldsymbol{y}^{m-1})_k < 0$. Consequently, $\min_{t\in[l]}(\mathcal{A}_t\boldsymbol{y}^{m-1})_k \neq 0$. This means $\bigwedge_{t=1}^{l}(\mathcal{A}_t\boldsymbol{y}^{m-1}) \neq \mathbf{0}$. By the definition of EVR_0-tensor tuple, $\hat{\mathcal{A}}$ is an EVR_0-tensor tuple.

Conversely, if $\hat{\mathcal{A}}$ is an EVR_0-tensor tuple, then for any $\boldsymbol{x} \in \mathbb{R}^n$ with $\|\boldsymbol{x}\|_\infty = 1$, there exists an integer $k \in [n]$ such that $\min_{t\in[l]}(\mathcal{A}_t\boldsymbol{x}^{m-1})_k \neq 0$. If $\min_{t\in[l]}(\mathcal{A}_t\boldsymbol{x}^{m-1})_k > 0$, we get $\max_{i\in[n]}\min_{t\in[l]}(\mathcal{A}_t\boldsymbol{x}^{m-1})_i > 0$. If $\min_{t\in[l]}(\mathcal{A}_t\boldsymbol{x}^{m-1})_k < 0$, there exists $t \in [l]$ such that $(\mathcal{A}_t\boldsymbol{x}^{m-1})_k < 0$, meaning $([-\mathcal{A}_t\boldsymbol{x}^{m-1}]_+)_k > 0$. We get $\max_{t\in[l]}\|[-\mathcal{A}_t\boldsymbol{x}^{m-1}]_+\|_\infty > 0$. Since the unit sphere $\|\boldsymbol{x}\|_\infty = 1$ is compact, $\gamma(\hat{\mathcal{A}}) > 0$.

(iii) For any vector $\boldsymbol{x} \in \mathbb{R}^n$ with $\|\boldsymbol{x}\|_\infty = 1$, there exists an integer $k \in [n]$ such that

$$\min_{t\in[l]}\left|(\mathcal{A}_t\boldsymbol{x}^{m-1})_k\right| = \max_{i\in[n]}\min_{t\in[l]}\left|(\mathcal{A}_t\boldsymbol{x}^{m-1})_i\right|.$$

If $(\mathcal{A}_t\boldsymbol{x}^{m-1})_k \geq 0$ for all $t \in [l]$, then we get

$$\max_{i\in[n]}\min_{t\in[l]}(\mathcal{A}_t\boldsymbol{x}^{m-1})_i \geq \min_{t\in[l]}(\mathcal{A}_t\boldsymbol{x}^{m-1})_k = \max_{i\in[n]}\min_{t\in[l]}\left|(\mathcal{A}_t\boldsymbol{x}^{m-1})_i\right|.$$

If there exists an integer $t' \in [l]$ such that $(\mathcal{A}_{t'}\boldsymbol{x}^{m-1})_k < 0$, we get

$$([-\mathcal{A}_{t'}\boldsymbol{x}^{m-1}]_+)_k = \left|(\mathcal{A}_{t'}\boldsymbol{x}^{m-1})_k\right| \geq \min_{t\in[l]}\left|(\mathcal{A}_t\boldsymbol{x}^{m-1})_k\right| = \max_{i\in[n]}\min_{t\in[l]}\left|(\mathcal{A}_t\boldsymbol{x}^{m-1})_i\right|.$$

This implies that

$$\max_{t\in[l]}\|[-\mathcal{A}_t\boldsymbol{x}^{m-1}]_+\|_\infty \geq \max_{i\in[n]}\min_{t\in[l]}\left|(\mathcal{A}_t\boldsymbol{x}^{m-1})_i\right|.$$

Therefore,

$$\max\left\{\max_{i\in[n]}\min_{t\in[l]}(\mathcal{A}_t\boldsymbol{x}^{m-1})_i,\ \max_{t\in[l]}\|[-\mathcal{A}_t\boldsymbol{x}^{m-1}]_+\|_\infty\right\} \geq \max_{i\in[n]}\min_{t\in[l]}\left|(\mathcal{A}_t\boldsymbol{x}^{m-1})_i\right|.$$

Taking the minimum over $\|\boldsymbol{x}\|_\infty = 1$, we have

$$\gamma(\hat{\mathcal{A}}) \geq \beta(\hat{\mathcal{A}}).$$

This yields the desired conclusion. □

Theorem 3.4. *For an EVP-tensor tuple $\hat{\mathcal{A}} = \{\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_l\}$ with $\mathcal{A}_t \in \mathbb{R}^{[m,n]}$ for all $t \in [l]$, the following inequalities hold:*

$$0 < \alpha(T_{\hat{\mathcal{A}}}) \leq \min_{i\in[n]} \prod_{t=1}^{l} \big|(\mathcal{A}_t)_{ii\dots i}\big|,$$

$$0 < \alpha(F_{\hat{\mathcal{A}}}) \leq \min_{i\in[n]} \prod_{t=1}^{l} \big|(\mathcal{A}_t)_{ii\dots i}\big|^{\frac{1}{m-1}}, \quad \textit{if } m \textit{ is even},$$

$$0 < \beta(\hat{\mathcal{A}}) \leq \min_{i\in[n]} \prod_{t=1}^{l} \big|(\mathcal{A}_t)_{ii\dots i}\big|.$$

Proof. Since $\hat{\mathcal{A}}$ is an EVP-tensor tuple, we have $\alpha(T_{\hat{\mathcal{A}}}) > 0$, $\alpha(F_{\hat{\mathcal{A}}}) > 0$, and $\beta(\hat{\mathcal{A}}) > 0$.

For any $i \in [n]$, let $\boldsymbol{e}_i \in \mathbb{R}^n$ be the i-th unit vector, where the i-th component is 1 and all others are 0. It is obvious that $\|\boldsymbol{e}_i\|_\infty = 1$ and $\|\boldsymbol{e}_i\|_2 = 1$. Then we obtain

$$\begin{aligned}(\mathcal{A}_t \boldsymbol{e}_i^{m-1})_i &= (\mathcal{A}_t)_{ii\dots i},\\ (T_{\mathcal{A}_t}\boldsymbol{e}_i)_i &= (\mathcal{A}_t)_{ii\dots i},\\ (F_{\mathcal{A}_t}\boldsymbol{e}_i)_i &= (\mathcal{A}_t)_{ii\dots i}^{\frac{1}{m-1}}, \quad \text{if } m \text{ is even}.\end{aligned}$$

Consequently,

$$\alpha(T_{\hat{\mathcal{A}}}) = \min_{\|\boldsymbol{x}\|_\infty=1} \max_{j\in[n]} \prod_{t=1}^{l} |(T_{\mathcal{A}_t}\boldsymbol{x})_j| \leq \max_{j\in[n]} \prod_{t=1}^{l} |(T_{\mathcal{A}_t}\boldsymbol{e}_i)_j| = \prod_{t=1}^{l} \big|(\mathcal{A}_t)_{ii\dots i}\big|,$$

$$\alpha(F_{\hat{\mathcal{A}}}) = \min_{\|\boldsymbol{x}\|_\infty=1} \max_{j\in[n]} \prod_{t=1}^{l} |(F_{\mathcal{A}_t}\boldsymbol{x})_j| \leq \max_{j\in[n]} \prod_{t=1}^{l} |(F_{\mathcal{A}_t}\boldsymbol{e}_i)_j| = \prod_{t=1}^{l} \big|(\mathcal{A}_t)_{ii\dots i}\big|^{\frac{1}{m-1}}, \quad \text{if } m \text{ is even},$$

$$\beta(\hat{\mathcal{A}}) = \min_{\|\boldsymbol{x}\|_\infty=1} \max_{j\in[n]} \prod_{t=1}^{l} |(\mathcal{A}_t\boldsymbol{x}^{m-1})_j| \leq \max_{j\in[n]} \prod_{t=1}^{l} |(\mathcal{A}_t\boldsymbol{e}_i^{m-1})_j| = \prod_{t=1}^{l} \big|(\mathcal{A}_t)_{ii\dots i}\big|.$$

Since the above inequalities hold for every $i \in [n]$, taking the minimum over i yields

$$\begin{aligned}\alpha(T_{\hat{\mathcal{A}}}) &\leq \min_{i\in[n]} \prod_{t=1}^{l} \big|(\mathcal{A}_t)_{ii\dots i}\big|,\\ \alpha(F_{\hat{\mathcal{A}}}) &\leq \min_{i\in[n]} \prod_{t=1}^{l} \big|(\mathcal{A}_t)_{ii\dots i}\big|^{\frac{1}{m-1}}, \qquad \text{if } m \text{ is even},\\ \beta(\hat{\mathcal{A}}) &\leq \min_{i\in[n]} \prod_{t=1}^{l} \big|(\mathcal{A}_t)_{ii\dots i}\big|.\end{aligned}$$

Combining these with the already established positivity of the constants completes the proof. □

Theorem 3.5. *For an EVR_0-tensor tuple $\hat{\mathcal{A}} = \{\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_l\}$ with $\mathcal{A}_t \in \mathbb{R}^{[m,n]}$ for all $t \in [l]$, one has*

$$0 < \gamma(\hat{\mathcal{A}}) \leq \min_{i\in[n]} \max \left\{ \max_{t\in[l]} \min_{j\in[n]} (\mathcal{A}_t)_{ji\dots i},\ \max_{t\in[l]} \max_{j\in[n]} \big([-\mathcal{A}_t]_{ji\dots i}\big)_+ \right\}.$$

Proof. Since $\hat{\mathcal{A}}$ is an EVR_0-tensor tuple, we have $\gamma(\hat{\mathcal{A}}) > 0$.

For any $i \in [n]$, let $\boldsymbol{e}_i \in \mathbb{R}^n$ be the i-th unit vector, where the i-th component is 1 and all others are 0. It is obvious that $\|\boldsymbol{e}_i\|_\infty = 1$. Then we obtain

$$(\mathcal{A}_t \boldsymbol{e}_i^{m-1})_j = (\mathcal{A}_t)_{ji\dots i}, \quad \text{and} \quad \big([-\mathcal{A}_t \boldsymbol{e}_i^{m-1}]_+\big)_j = \big([-\mathcal{A}_t]_{ji\dots i}\big)_+.$$

Therefore,

$$\begin{aligned}\gamma(\hat{\mathcal{A}}) &= \min_{\|\boldsymbol{x}\|_\infty=1} \max\left\{\max_{t\in[l]}\min_{j\in[n]}(\mathcal{A}_t\boldsymbol{x}^{m-1})_j,\ \max_{t\in[l]}\big\|[-\mathcal{A}_t\boldsymbol{x}^{m-1}]_+\big\|_\infty\right\} \\ &\le \max\left\{\max_{t\in[l]}\min_{j\in[n]}(\mathcal{A}_t\boldsymbol{e}_i^{m-1})_j,\ \max_{t\in[l]}\max_{j\in[n]}\big([-\mathcal{A}_t\boldsymbol{e}_i^{m-1}]_+\big)_j\right\} \\ &= \max\left\{\max_{t\in[l]}\min_{j\in[n]}(\mathcal{A}_t)_{ji\dots i},\ \max_{t\in[l]}\max_{j\in[n]}\big([-\mathcal{A}_t]_{ji\dots i}\big)_+\right\}.\end{aligned}$$

Since this inequality holds for every $i \in [n]$, taking the minimum over i yields

$$0 < \gamma(\hat{\mathcal{A}}) \le \min_{i\in[n]}\max\left\{\max_{t\in[l]}\min_{j\in[n]}(\mathcal{A}_t)_{ji\dots i},\ \max_{t\in[l]}\max_{j\in[n]}\big([-\mathcal{A}_t]_{ji\dots i}\big)_+\right\}.$$

This completes the proof. □

4. Lower Bounds of The Solution Set for The EVTCP

This section deals with the lower bounds of the solution set for the EVTCP($\hat{\mathcal{A}}, \hat{\boldsymbol{q}}$). Before considering the lower bounds of the solution set for the EVTCP($\hat{\mathcal{A}}, \hat{\boldsymbol{q}}$), we give the following result.

Theorem 4.1. *Let $\hat{\boldsymbol{q}} = \{\boldsymbol{q}_1, \boldsymbol{q}_2, \dots, \boldsymbol{q}_l\} \subset \mathbb{R}^n$ and $\hat{\mathcal{A}} = \{\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_l\} \subset \mathbb{R}^{[m,n]}$ with $\mathcal{A}_t \neq \mathcal{O}$ for all $t \in [l]$. If $\boldsymbol{x}$ is a solution of the EVTCP($\hat{\mathcal{A}}, \hat{\boldsymbol{q}}$), then one has*

(i) $\|\boldsymbol{x}\|_\infty \ge \max\limits_{t\in[l]}\left(\dfrac{\|(-\boldsymbol{q}_t)_+\|_\infty}{n^{\frac{m-2}{2}}\|T_{\mathcal{A}_t}\|_\infty}\right)^{\frac{1}{(m-1)}}$;

(ii) $\|\boldsymbol{x}\|_2 \ge \max\limits_{t\in[l]}\left(\dfrac{\|(-\boldsymbol{q}_t)_+\|_2}{\|T_{\mathcal{A}_t}\|_2}\right)^{\frac{1}{(m-1)}}$.

Proof. (i) Since $\boldsymbol{x}$ is a solution of the EVTCP($\hat{\mathcal{A}}, \hat{\boldsymbol{q}}$), we get

$$(\boldsymbol{q}_1 + \mathcal{A}_1\boldsymbol{x}^{m-1}) \wedge (\boldsymbol{q}_2 + \mathcal{A}_2\boldsymbol{x}^{m-1}) \wedge \dots \wedge (\boldsymbol{q}_l + \mathcal{A}_l\boldsymbol{x}^{m-1}) = \boldsymbol{0},$$

which implies that for any $t \in [l]$,

$$(\mathcal{A}_t\boldsymbol{x}^{m-1})_i \ge (-\boldsymbol{q}_t)_i, \quad \text{for all } i \in [n].$$

Thus, we have

$$\begin{aligned}\|(-\boldsymbol{q}_t)_+\|_\infty &\le \|\mathcal{A}_t\boldsymbol{x}^{m-1}\|_\infty \\ &= \|\boldsymbol{x}\|_2^{m-2}\big\|\|\boldsymbol{x}\|_2^{2-m}\mathcal{A}_t\boldsymbol{x}^{m-1}\big\|_\infty \\ &= \|\boldsymbol{x}\|_2^{m-2}\|T_{\mathcal{A}_t}(\boldsymbol{x})\|_\infty \\ &\le \|\boldsymbol{x}\|_2^{m-2}\|\boldsymbol{x}\|_\infty\|T_{\mathcal{A}_t}\|_\infty \\ &\le \Big(n^{\frac{1}{2}}\|\boldsymbol{x}\|_\infty\Big)^{m-2}\|\boldsymbol{x}\|_\infty\|T_{\mathcal{A}_t}\|_\infty \\ &= n^{\frac{m-2}{2}}\|\boldsymbol{x}\|_\infty^{m-1}\|T_{\mathcal{A}_t}\|_\infty.\end{aligned}$$

This implies that

$$\|\boldsymbol{x}\|_\infty \geq \left(\frac{\|(-\boldsymbol{q}_t)_+\|_\infty}{n^{\frac{m-2}{2}}\|T_{\mathcal{A}_t}\|_\infty}\right)^{\frac{1}{m-1}}, \quad \text{for all } t \in [l].$$

Thus, item (i) of the theorem holds.

(ii) Since $\boldsymbol{x}$ is a solution of the EVTCP($\hat{\mathcal{A}}, \hat{\boldsymbol{q}}$), for all $t \in [l]$ we get

$$\begin{aligned}\|(-\boldsymbol{q}_t)_+\|_2 &\leq \|\mathcal{A}_t\boldsymbol{x}^{m-1}\|_2 \\ &= \|\boldsymbol{x}\|_2^{m-2}\|T_{\mathcal{A}_t}(\boldsymbol{x})\|_2 \\ &\leq \|\boldsymbol{x}\|_2^{m-2}\|\boldsymbol{x}\|_2\|T_{\mathcal{A}_t}\|_2 \\ &= \|\boldsymbol{x}\|_2^{m-1}\|T_{\mathcal{A}_t}\|_2.\end{aligned}$$

Then, we get

$$\|\boldsymbol{x}\|_2 \geq \left(\frac{\|(-\boldsymbol{q}_t)_+\|_2}{\|T_{\mathcal{A}_t}\|_2}\right)^{\frac{1}{m-1}}, \quad \text{for all } t \in [l].$$

Thus, item (ii) of the theorem holds. This completes the proof. □

Remark 4.2. The lower bound estimation in our work is inspired by the elegant algebraic techniques in Theorem 4.1 of Wang-Fu-Wu [32]. While their original result was established specifically for VR_0-tensor tuples, we observe that the derivation fundamentally relies on tensor norm properties rather than the VR_0 topological structure. Building upon their foundational work, we successfully remove this restrictive assumption, thereby extending the applicability of the lower bound to any general tensor tuple with non-zero operators.

Theorem 4.3. *Let $\hat{\boldsymbol{q}} = \{\boldsymbol{q}_1, \boldsymbol{q}_2, \ldots, \boldsymbol{q}_l\} \subset \mathbb{R}^n$, $\hat{\mathcal{A}} = \{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_l\} \subset \mathbb{R}^{[m,n]}$ with $\mathcal{A}_t \neq \mathcal{O}$ for all $t \in [l]$, and let p and m be even integers. If $\boldsymbol{x}$ is a solution of the* $\mathrm{EVTCP}(\hat{\mathcal{A}}, \hat{\boldsymbol{q}})$*, then one has*

(i) $\|\boldsymbol{x}\|_\infty \geq \max_{t\in[l]} \frac{\|(-\boldsymbol{q}_t)_+\|_\infty^{\frac{1}{m-1}}}{\|F_{\mathcal{A}_t}\|_\infty}$;

(ii) $\|\boldsymbol{x}\|_p \geq \max_{t\in[l]} \frac{\|(-\boldsymbol{q}_t)_+\|_p^{\frac{1}{m-1}}}{\|F_{\mathcal{A}_t}\|_p}$.

Proof. Since $\boldsymbol{x}$ is a solution of the $\mathrm{EVTCP}(\hat{\mathcal{A}}, \hat{\boldsymbol{q}})$, we get

$$(\boldsymbol{q}_1 + \mathcal{A}_1\boldsymbol{x}^{m-1}) \wedge (\boldsymbol{q}_2 + \mathcal{A}_2\boldsymbol{x}^{m-1}) \wedge \cdots \wedge (\boldsymbol{q}_l + \mathcal{A}_l\boldsymbol{x}^{m-1}) = \mathbf{0}.$$

This implies that for any $t \in [l]$, we have:

$$(\mathcal{A}_t\boldsymbol{x}^{m-1})_i \geq (-\boldsymbol{q}_t)_i, \quad \text{for all } i \in [n].$$

Thus, for any $t \in [l]$ and for all $i \in [n]$,

$$\begin{aligned}\left|(\mathcal{A}_t\boldsymbol{x}^{m-1})_i^{\frac{1}{m-1}}\right|^{m-1} &= \left|(\mathcal{A}_t\boldsymbol{x}^{m-1})_i\right| \\ &\geq \left((\mathcal{A}_t\boldsymbol{x}^{m-1})_+\right)_i \\ &\geq ((-\boldsymbol{q}_t)_+)_i \\ &\geq 0.\end{aligned}$$

That is,

$$\begin{aligned}\|(-\boldsymbol{q}_t)_+\|_\infty &\le \left\|(\mathcal{A}_t\boldsymbol{x}^{m-1})^{[\frac{1}{m-1}]}\right\|_\infty^{m-1}\\ &= \|F_{\mathcal{A}_t}(\boldsymbol{x})\|_\infty^{m-1}\\ &\le \|\boldsymbol{x}\|_\infty^{m-1}\|F_{\mathcal{A}_t}\|_\infty^{m-1},\end{aligned}$$

and

$$\begin{aligned}\|(-\boldsymbol{q}_t)_+\|_p &\le \left\|(\mathcal{A}_t\boldsymbol{x}^{m-1})^{[\frac{1}{m-1}]}\right\|_p^{m-1}\\ &= \|F_{\mathcal{A}_t}(\boldsymbol{x})\|_p^{m-1}\\ &\le \|\boldsymbol{x}\|_p^{m-1}\|F_{\mathcal{A}_t}\|_p^{m-1}.\end{aligned}$$

Thus, for any $t \in [l]$ we get

$$\|\boldsymbol{x}\|_\infty \ge \frac{\|(-\boldsymbol{q}_t)_+\|_\infty^{\frac{1}{m-1}}}{\|F_{\mathcal{A}_t}\|_\infty} \quad \text{and} \quad \|\boldsymbol{x}\|_p \ge \frac{\|(-\boldsymbol{q}_t)_+\|_p^{\frac{1}{m-1}}}{\|F_{\mathcal{A}_t}\|_p}.$$

Therefore, we can easily obtain that

$$\|\boldsymbol{x}\|_\infty \ge \max_{t\in[l]} \frac{\|(-\boldsymbol{q}_t)_+\|_\infty^{\frac{1}{m-1}}}{\|F_{\mathcal{A}_t}\|_\infty} \quad \text{and} \quad \|\boldsymbol{x}\|_p \ge \max_{t\in[l]} \frac{\|(-\boldsymbol{q}_t)_+\|_p^{\frac{1}{m-1}}}{\|F_{\mathcal{A}_t}\|_p}.$$

This yields the desired conclusion. □

Remark 4.4. Theorem 4.3 generalizes the theoretical bounds established in Theorem 4.2 of Wang-Fu-Wu [32] to the multi-tensor EVTCP framework ($l \ge 2$). Notably, when the system is restricted to the classical two-tensor case ($l = 2$), our generalized formulation naturally reduces to the exact result presented in [32].

Theorem 4.5. *Let $\hat{\boldsymbol{q}} = \{\boldsymbol{q}_1, \boldsymbol{q}_2, \dots, \boldsymbol{q}_l\} \subset \mathbb{R}^n$ and $\hat{\mathcal{A}} = \{\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_l\} \subset \mathbb{R}^{[m,n]}$ with $\mathcal{A}_t \ne \mathcal{O}$ for all $t \in [l]$. If $\boldsymbol{x}$ is a solution of the* $\mathrm{EVTCP}(\hat{\mathcal{A}}, \hat{\boldsymbol{q}})$*, then one has*

$$\|\boldsymbol{x}\|_\infty \ge \max_{t\in[l]} \left(\frac{\|(-\boldsymbol{q}_t)_+\|_\infty}{\|\mathcal{A}_t\|_\infty}\right)^{\frac{1}{m-1}}.$$

Proof. Since $\boldsymbol{x}$ is a solution of the $\mathrm{EVTCP}(\hat{\mathcal{A}}, \hat{\boldsymbol{q}})$, we get

$$(\boldsymbol{q}_1 + \mathcal{A}_1\boldsymbol{x}^{m-1}) \wedge (\boldsymbol{q}_2 + \mathcal{A}_2\boldsymbol{x}^{m-1}) \wedge \cdots \wedge (\boldsymbol{q}_l + \mathcal{A}_l\boldsymbol{x}^{m-1}) = \boldsymbol{0}.$$

So, for any $t \in [l]$ we have

$$(\mathcal{A}_t\boldsymbol{x}^{m-1})_i \ge (-\boldsymbol{q}_t)_i, \quad \text{for all } i \in [n].$$

Then,

$$|(\mathcal{A}_t\boldsymbol{x}^{m-1})_i| \ge ((\mathcal{A}_t\boldsymbol{x}^{m-1})_+)_i \ge ((-\boldsymbol{q}_t)_+)_i, \quad \text{for all } i \in [n], t \in [l].$$

Thus, we get

$$\|\mathcal{A}_t\boldsymbol{x}^{m-1}\|_\infty \ge \|(-\boldsymbol{q}_t)_+\|_\infty, \quad \text{for all } t \in [l].$$

By Lemma 2.4, we get

$$\|\boldsymbol{x}\|_\infty^{m-1} \ge \frac{\|(-\boldsymbol{q}_t)_+\|_\infty}{\|\mathcal{A}_t\|_\infty}, \quad \text{for all } t \in [l].$$

This implies that

$$\|\boldsymbol{x}\|_\infty \ge \max_{t\in[l]} \left(\frac{\|(-\boldsymbol{q}_t)_+\|_\infty}{\|\mathcal{A}_t\|_\infty}\right)^{\frac{1}{m-1}}.$$

This completes the proof. □

Remark 4.6. By using items (iv) and (vi) of Lemma 2.3, we compare the lower bounds obtained above. The lower bound in part (i) of Theorem 4.3 is sharper than that in Theorem 4.5. If $\mathcal{A}_t$ ($t \in [l]$) are all nonnegative tensors, then the bounds in Theorems 4.3 and 4.5 coincide, and in this case the bounds of Theorem 4.5 is sharper than the one in item (i) of Theorem 4.1.

5. Upper Bounds of The Solution Set for The EVTCP($\hat{\mathcal{A}}, \hat{\boldsymbol{q}}$)

In this section, we propose several upper bounds of the solution set for the $\mathrm{EVTCP}(\hat{\mathcal{A}}, \hat{\boldsymbol{q}})$.

Theorem 5.1. *For an EVP-tensor tuple $\hat{\mathcal{A}} = \{\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_l\}$ with $\mathcal{A}_t \in \mathbb{R}^{[m,n]}$ for all $t \in [l]$, let $\boldsymbol{x}$ be a solution of the $\mathrm{EVTCP}(\hat{\mathcal{A}}, \hat{\boldsymbol{q}})$. If $\boldsymbol{q}_t \geq 0$ for all $t \in [l]$, then*

(i) $\|\boldsymbol{x}\|_\infty^{l(m-1)} \leq \dfrac{\|\boldsymbol{q}_1 \circ \boldsymbol{q}_2 \circ \cdots \circ \boldsymbol{q}_l\|_\infty}{\alpha(T_{\hat{\mathcal{A}}})}$,

(ii) $\|\boldsymbol{x}\|_\infty^{l(m-1)} \leq \dfrac{\|\boldsymbol{q}_1 \circ \boldsymbol{q}_2 \circ \cdots \circ \boldsymbol{q}_l\|_\infty}{\beta(\hat{\mathcal{A}})}$,

(iii) $\|\boldsymbol{x}\|_\infty^{l} \leq \dfrac{\|\boldsymbol{q}_1^{\frac{1}{m-1}} \circ \boldsymbol{q}_2^{\frac{1}{m-1}} \circ \cdots \circ \boldsymbol{q}_l^{\frac{1}{m-1}}\|_\infty}{\alpha(F_{\hat{\mathcal{A}}})}$, *if m is even.*

Proof. If $\boldsymbol{q}_t = \boldsymbol{0}$ for all $t \in [l]$, then the $\mathrm{EVTCP}(\hat{\mathcal{A}}, \hat{\boldsymbol{q}})$ amounts to solving the homogeneous problem

$$(\mathcal{A}_1\boldsymbol{x}^{m-1}) \wedge (\mathcal{A}_2\boldsymbol{x}^{m-1}) \wedge \cdots \wedge (\mathcal{A}_l\boldsymbol{x}^{m-1}) = \boldsymbol{0}.$$

Since $\hat{\mathcal{A}}$ is an EVP-tensor tuple, the problem only has a zero solution $\boldsymbol{x} = \boldsymbol{0}$. Then, the conclusion holds obviously.

Assume that there exists $\boldsymbol{q}_t \in \mathbb{R}^n$. Let $\boldsymbol{x} \in \mathbb{R}^n$ be a solution. Since $\hat{\mathcal{A}}$ is an EVP-tensor tuple, it follows from Lemma 2.8 that for any nonzero $\boldsymbol{x}$, there exists an index $k \in [n]$ such that all $(\mathcal{A}_t\boldsymbol{x}^{m-1})_k$ for $t \in [l]$ have the same strict sign. Because $\boldsymbol{x}$ is a solution, at this index k, we must have

$$\min_{t\in[l]}(\boldsymbol{q}_t + \mathcal{A}_t\boldsymbol{x}^{m-1})_k = 0.$$

First, there exists at least one $t_0 \in [l]$ satisfying $(\boldsymbol{q}_{t_0} + \mathcal{A}_{t_0}\boldsymbol{x}^{m-1})_k = 0$, which yields $(\mathcal{A}_{t_0}\boldsymbol{x}^{m-1})_k = -(\boldsymbol{q}_{t_0})_k \leq 0$ (since $\boldsymbol{q}_{t_0} \geq \boldsymbol{0}$). Second, for all other $t \in [l]$, the terms involved in the minimum must be non-negative, yielding $(\boldsymbol{q}_t + \mathcal{A}_t\boldsymbol{x}^{m-1})_k \geq 0$, which means $(\mathcal{A}_t\boldsymbol{x}^{m-1})_k \geq -(\boldsymbol{q}_t)_k$. Because all terms share the same strict sign at index k, and the term associated with t_0 is less than or equal to zero, we must have $(\mathcal{A}_t\boldsymbol{x}^{m-1})_k < 0$ for all $t \in [l]$. By taking the absolute value, we obtain the strictly positive bounds:

$$0 < |(\mathcal{A}_t\boldsymbol{x}^{m-1})_k| \leq (\boldsymbol{q}_t)_k, \text{ for all } t \in [l].$$

Since all absolute values are strictly positive, multiplying them across all $t \in [l]$ perfectly preserves the inequality direction. We get

$$\prod_{t=1}^{l} |(\mathcal{A}_t\boldsymbol{x}^{m-1})_k| \leq \prod_{t=1}^{l}(\boldsymbol{q}_t)_k.$$

In addition, since $\boldsymbol{x} \mapsto \boldsymbol{x}^{\frac{1}{m-1}}$ is monotonically increasing and sign-preserving for even m, we obtain

$$\prod_{t=1}^{l} |(\mathcal{A}_t\boldsymbol{x}^{m-1})_k|^{\frac{1}{m-1}} \leq \prod_{t=1}^{l}(\boldsymbol{q}_t)_k^{\frac{1}{m-1}}, \quad \text{if } m \text{ is even.}$$

Therefore, by taking the maximum over all $i \in [n]$, we globally constrain the maximum product:

$$\max_{i\in[n]} \prod_{t=1}^{l} |(\mathcal{A}_t\boldsymbol{x}^{m-1})_i| \leq \max_{i\in[n]} \prod_{t=1}^{l}(\boldsymbol{q}_t)_i = \|\boldsymbol{q}_1 \circ \boldsymbol{q}_2 \circ \cdots \circ \boldsymbol{q}_l\|_\infty, \quad \textit{if } m \textit{ is even.}$$

Now, extracting the norm of $\boldsymbol{x}$ using the homogeneity of the constants, we obtain that

$$\begin{aligned}
\|\boldsymbol{x}\|_\infty^{l(m-1)}\beta(\hat{\mathcal{A}}) &= \|\boldsymbol{x}\|_\infty^{l(m-1)} \min_{\|\boldsymbol{y}\|_\infty=1} \max_{i\in[n]} \prod_{t=1}^{l} \left|(\mathcal{A}_t\boldsymbol{y}^{m-1})_i\right| \\
&\leq \max_{i\in[n]} \prod_{t=1}^{l} \left| \left(\mathcal{A}_t \left(\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|_\infty} \right)^{m-1} \right)_i \right| \cdot \|\boldsymbol{x}\|_\infty^{l(m-1)} \\
&= \max_{i\in[n]} \prod_{t=1}^{l} |(\mathcal{A}_t\boldsymbol{x}^{m-1})_i| \\
&\leq \|\boldsymbol{q}_1 \circ \boldsymbol{q}_2 \circ \cdots \circ \boldsymbol{q}_l\|_\infty,
\end{aligned}$$

which directly yields (ii).

Similarly, for the operator $T_{\mathcal{A}}$ where $(T_{\mathcal{A}}\boldsymbol{x}) = \|\boldsymbol{x}\|_2^{2-m}\mathcal{A}\boldsymbol{x}^{m-1}$, we have

$$\begin{aligned}
\|\boldsymbol{x}\|_\infty^{l}\alpha(T_{\hat{\mathcal{A}}}) &= \|\boldsymbol{x}\|_\infty^{l} \min_{\|\boldsymbol{y}\|_\infty=1} \max_{i\in[n]} \prod_{t=1}^{l} |(T_{\mathcal{A}_t}\boldsymbol{y})_i| \\
&\leq \max_{i\in[n]} \prod_{t=1}^{l} \left| \left(T_{\mathcal{A}_t} \left(\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|_\infty} \right) \right)_i \right| \cdot \|\boldsymbol{x}\|_\infty^{l} \\
&= \max_{i\in[n]} \prod_{t=1}^{l} |(T_{\mathcal{A}_t}\boldsymbol{x})_i| \\
&= \|\boldsymbol{x}\|_2^{l(2-m)} \max_{i\in[n]} \prod_{t=1}^{l} |(\mathcal{A}_t\boldsymbol{x}^{m-1})_i| \\
&\leq \|\boldsymbol{x}\|_2^{l(2-m)} \|\boldsymbol{q}_1 \circ \boldsymbol{q}_2 \circ \cdots \circ \boldsymbol{q}_l\|_\infty.
\end{aligned}$$

Since $m \geq 2$, we have $2 - m \leq 0$, so $\|\boldsymbol{x}\|_2^{l(2-m)} \leq \|\boldsymbol{x}\|_\infty^{l(2-m)}$. We obtain that

$$\|\boldsymbol{x}\|_\infty^{l(m-1)} \leq \frac{\|\boldsymbol{q}_1 \circ \boldsymbol{q}_2 \circ \cdots \circ \boldsymbol{q}_l\|_\infty}{\alpha(T_{\hat{\mathcal{A}}})},$$

which establish item (i) .

Finally, for the operator $F_{\mathcal{A}}$ when m is even, since $F_{\mathcal{A}}\left(\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|_\infty}\right) = \|\boldsymbol{x}\|_\infty^{-1}F_{\mathcal{A}}\boldsymbol{x}$, we obtain that

$$\begin{aligned}
\|\boldsymbol{x}\|_\infty^{l}\alpha(F_{\hat{\mathcal{A}}}) &= \|\boldsymbol{x}\|_\infty^{l} \min_{\|\boldsymbol{y}\|_\infty=1} \max_{i\in[n]} \prod_{t=1}^{l} |(F_{\mathcal{A}_t}\boldsymbol{y})_i| \\
&\leq \max_{i\in[n]} \prod_{t=1}^{l} \left| \left(F_{\mathcal{A}_t} \left(\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|_\infty} \right) \right)_i \right| \cdot \|\boldsymbol{x}\|_\infty^{l} \\
&= \max_{i\in[n]} \prod_{t=1}^{l} |(F_{\mathcal{A}_t}\boldsymbol{x})_i| \\
&= \max_{i\in[n]} \prod_{t=1}^{l} |(\mathcal{A}_t\boldsymbol{x}^{m-1})_i|^{\frac{1}{m-1}} \\
&\leq \left\| \boldsymbol{q}_1^{\frac{1}{m-1}} \circ \boldsymbol{q}_2^{\frac{1}{m-1}} \circ \cdots \circ \boldsymbol{q}_l^{\frac{1}{m-1}} \right\|_\infty, \quad \text{if } m \text{ is even.}
\end{aligned}$$

Dividing both sides by $\alpha(F_{\hat{\mathcal{A}}})$, we get item (iii). This completes the proof. □

By using Lemma 3.2, we obtain the comparison of these upper bounds in Theorem 5.1. The upper bound of (ii) is sharper than the result of (i).

Next, we give an upper bound for an EVR_0-tensor tuple.

Theorem 5.2. *Given an EVR_0-tensor tuple $\hat{\mathcal{A}} = \{\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_l\}$ with $\mathcal{A}_t \in \mathbb{R}^{[m,n]}$. Let $\boldsymbol{x}$ be a solution of the* $\mathrm{EVTCP}(\hat{\mathcal{A}}, \hat{\boldsymbol{q}})$. *If $\boldsymbol{q}_t \geq \mathbf{0}$ for all $t \in [l]$, then*

$$\|\boldsymbol{x}\|_\infty^{m-1} \leq \frac{\max_{t\in[l]} \|\boldsymbol{q}_t\|_\infty}{\gamma(\hat{\mathcal{A}})}.$$

Proof. If $\boldsymbol{x} = \mathbf{0}$, the inequality trivially holds since the right-hand side is nonnegative.

Assume $\boldsymbol{x} \neq \mathbf{0}$. Since $\boldsymbol{x}$ is a solution of the $\mathrm{EVTCP}(\hat{\mathcal{A}}, \hat{\boldsymbol{q}})$, it satisfies

$$(\boldsymbol{q}_1 + \mathcal{A}_1\boldsymbol{x}^{m-1}) \wedge (\boldsymbol{q}_2 + \mathcal{A}_2\boldsymbol{x}^{m-1}) \wedge \cdots \wedge (\boldsymbol{q}_l + \mathcal{A}_l\boldsymbol{x}^{m-1}) = \mathbf{0}.$$

This implies that for any dimension index $i \in [n]$, the minimum of these l components is exactly zero:

$$\min_{t\in[l]} (\boldsymbol{q}_t + \mathcal{A}_t\boldsymbol{x}^{m-1})_i = 0.$$

From this, we deduce two facts:

First, there exists at least one index $t_0 \in [l]$ such that $(\boldsymbol{q}_{t_0} + \mathcal{A}_{t_0}\boldsymbol{x}^{m-1})_i = 0$. Since $\boldsymbol{q}_{t_0} \geq \mathbf{0}$, we have $(\mathcal{A}_{t_0}\boldsymbol{x}^{m-1})_i = -(\boldsymbol{q}_{t_0})_i \leq 0$. Because the minimum over all $t \in [l]$ cannot exceed the value at t_0, we obtain

$$\min_{t\in[l]} (\mathcal{A}_t\boldsymbol{x}^{m-1})_i \leq (\mathcal{A}_{t_0}\boldsymbol{x}^{m-1})_i \leq 0.$$

Since this holds for every $i \in [n]$, taking the maximum over all components yields

$$\max_{i\in[n]} \min_{t\in[l]} (\mathcal{A}_t\boldsymbol{x}^{m-1})_i \leq 0.$$

Second, for all $t \in [l]$, the terms must be nonnegative, meaning $(\boldsymbol{q}_t + \mathcal{A}_t\boldsymbol{x}^{m-1})_i \geq 0$. This gives $(-\mathcal{A}_t\boldsymbol{x}^{m-1})_i \leq (\boldsymbol{q}_t)_i$. Applying the positive part operator $(\cdot)_+$ on both sides and noting that $\boldsymbol{q}_t \geq \mathbf{0}$, we get

$$([-\mathcal{A}_t\boldsymbol{x}^{m-1}]_+)_i \leq (\boldsymbol{q}_t)_i,$$
$$\max_{t\in[l]} \|[-\mathcal{A}_t\boldsymbol{x}^{m-1}]_+\|_\infty \leq \max_{t\in[l]} \|\boldsymbol{q}_t\|_\infty.$$

Now, using the definition of the generalized constant $\gamma(\hat{\mathcal{A}})$ and the $(m-1)$-th order homogeneity of the tensor product, we can extract the infinity norm of $\boldsymbol{x}$:

$$\begin{aligned}
\|\boldsymbol{x}\|_\infty^{m-1}\gamma(\hat{\mathcal{A}}) &= \|\boldsymbol{x}\|_\infty^{m-1} \min_{\|\boldsymbol{y}\|_\infty=1} \max\left\{\max_{i\in[n]} \min_{t\in[l]} (\mathcal{A}_t\boldsymbol{y}^{m-1})_i,\ \max_{t\in[l]} \|[-\mathcal{A}_t\boldsymbol{y}^{m-1}]_+\|_\infty\right\} \\
&\leq \max\left\{\max_{i\in[n]} \min_{t\in[l]} \left(\mathcal{A}_t\left(\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|_\infty}\right)^{m-1}\right)_i,\ \max_{t\in[l]} \left\|\left[-\mathcal{A}_t\left(\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|_\infty}\right)^{m-1}\right]_+\right\|_\infty\right\} \cdot \|\boldsymbol{x}\|_\infty^{m-1} \\
&= \max\left\{\max_{i\in[n]} \min_{t\in[l]} (\mathcal{A}_t\boldsymbol{x}^{m-1})_i,\ \max_{t\in[l]} \|[-\mathcal{A}_t\boldsymbol{x}^{m-1}]_+\|_\infty\right\}.
\end{aligned}$$

Substituting the upper bounds established earlier for the two terms inside the maximum, we obtain

$$\|\boldsymbol{x}\|_\infty^{m-1}\gamma(\hat{\mathcal{A}}) \leq \max\left\{0,\ \max_{t\in[l]} \|\boldsymbol{q}_t\|_\infty\right\}.$$

Since $\max_{t\in[l]} \|\boldsymbol{q}_t\|_\infty \geq 0$, the inequality simplifies to

$$\|\boldsymbol{x}\|_\infty^{m-1}\gamma(\hat{\mathcal{A}}) \leq \max_{t\in[l]} \|\boldsymbol{q}_t\|_\infty.$$

Because $\hat{\mathcal{A}}$ is an EVR_0-tensor tuple, we know that $\gamma(\hat{\mathcal{A}}) > 0$ (see Lemma 3.3). Dividing both sides by $\gamma(\hat{\mathcal{A}})$ and multiplying by $\|x\|_\infty^{m-1}$ yields the desired inequality

$$\|x\|_\infty^{m-1} \leq \frac{\max_{t\in[l]} \|q_t\|_\infty}{\gamma(\hat{\mathcal{A}})}.$$

This completes the proof. □

Since an EVP-tensor tuple is an EVR_0-tensor tuple by Proposition 1 in [19], the above conclusion is also true for an EVP-tensor tuple $\hat{\mathcal{A}}$.

6. Conclusions

In this paper, we extend the theoretical framework of solution bounds from the classical two-tensor VTCP to the multi-tensor EVTCP ($l \geq 2$). By defining two generalized structural quantities equipped with absolute value products, we successfully overcome the sign parity collapse inherent in odd-tensor systems. Consequently, we obtain elegant upper bounds for the solution set with an EVP-tensor tuple that preserve strict algebraic homogeneity. Furthermore, we derive the lower bounds by revealing that such estimations fundamentally rely on norm properties rather than the strict EVR_0 topological structure. Exploring tighter bounds or designing efficient numerical algorithms for the EVTCP are meaningful questions that deserve further consideration in future work.

Statements and Declarations

The authors declare no competing interests. All authors contributed to the study conception and design. Material preparation and data analysis were performed jointly by all authors. All authors read and approved the final manuscript.

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