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A STEP SIZE RESOLVENT METHOD FOR MIXED VARIATIONAL INEQUALITIES AND HIERARCHICAL FIXED POINT

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Abstract. The objective of this study is to develop an approximation for the common solution to mixed variational inequalities governed by monotone and uniformly continuous operators and hierarchical fixed point in real Hilbert space. We propose a resolvent method with double inertial extrapolations. Our approach departs from traditional Armijo-type line searches by using a simple step-size adjustment rule that generates a non-monotonic sequence. We introduce a less restrictive condition that ensures the convergence of our inertial method without requiring knowledge of the Lipschitz constant. Finally, we present numerical results to evaluate the performance of the proposed algorithm.

Keywords. Mixed variational inequalities; resolvent operator; the hierarchical fixed point; strong convergence; nonexpansive mappings; uniformly continuous.

1. Introduction

Consider a real Hilbert space H, where $\langle \cdot, \cdot \rangle$ is the inner product and $\| \cdot \|$ is the associated norm. Let $\psi : H \to \mathbb{R}$ be continuous function and $G : H \to H$ be a nonlinear mapping. The mixed variational inequalities (MVI), finding $x \in H$ such that

$$\langle G(x), y - x \rangle + \psi(y) - \psi(x) \geq 0, \quad \forall y \in H. \tag{1.1}$$

We refer to the set of solutions of (1.1) as $MVI(G, \psi)$. The MVI provides a versatile framework that unifies diverse challenges from continuous optimization and variational analysis. It encompasses classical models such as minimization and linear complementarity problems, alongside a broad spectrum of other variational inequalities. These classes of variational inequalities are essential for modeling phenomena across a wide range of disciplines; see, for instance, [1, 10, 18, 20, 21, 22, 23, 24, 25, 26, 27, 32, 34, 36, 49, 57, 58].

The presence of a nonlinear term within the MVI framework introduces significant complexities, hindering the direct application of projection-based methods for developing efficient numerical algorithms. To overcome this, we adopt the resolvent operator approach, a technique pioneered by [12]. As established in the literature [2, 3, 4, 5, 8, 17, 28, 29, 37, 38, 39, 40, 41, 52, 57, 58], the resolvent operator approach serves as a powerful foundation for iterative methods, providing superior numerical efficiency for various classes of (1.1). A shared feature among these resolvent operator-based approaches is their fundamental reliance on line-search procedures. Establishing convergence for such iterative methods requires G to satisfy monotonicity and global Lipschitz continuity. Crucially, these algorithms rely on the Lipschitz constant of G, which is often difficult to estimate. Moreover, a recognized drawback of

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this approach is that the descent of the stepsize decay often leads to a significant slowdown the algorithm's convergence during the final stages of the process. The practical applicability of existing algorithms is often constrained by these factors. Our objective is to overcome these hurdles by proposing a simplified, more resilient framework that operates independently of the function's specific structural characteristics. Furthermore, let us investigate the fixed point problem for the operator T, according to its definition

$$\text{find} \quad x \in C \quad \text{such that} \quad T(x) = x, \tag{1.2}$$

where C is nonempty closed and convex. We denote by $F(T)$ the set of solutions of (1.2). Fixed point theory has gained substantial momentum in recent years, evolving into a highly active field of mathematical research due to its broad utility across various disciplines; see for example [6, 7, 9, 11, 13, 14, 16, 19, 43, 48].

The hierarchical fixed point problem (HFPP), as studied by Moudafi and Mainge [35], seeks a point x^* within the fixed point set of a nonexpansive mapping T relative to another nonexpansive mapping S. Specifically, find $x^* \in F(T)$ such that

$$\langle (I - S)x^*, x - x^* \rangle \geq 0, \quad \forall x \in F(T). \tag{1.3}$$

Let Λ represent the solution set of the HFPP (1.3). It is important to note that whenever Λ is nonempty, it constitutes a closed and convex set.

The HFPP is essential in fields ranging from optimization to differential equations. To solve this problem when contraction mappings are present, the viscosity scheme is frequently employed. This method was pioneered by Moudafi and Mainge [35], Xu [53] proved strong convergence toward a unique solution for HFPP involving both contraction and nonexpansive mappings. A key advantage of the viscosity scheme is its ability to find a solution within a fixed point set without requiring the direct calculation of a metric projection at any step. Furthermore, Yao et al. [55] explored singleton solutions of the HFPP as strong limits, expanding the scope of the problem by utilizing metric projections and countable families of nonexpansive mappings. Many researchers developed and analysed iterative algorithms for finding common element of solution sets of HFPP and other problems. For example, Kazmi et al. [30] introduced and analyzed a Krasnoselski-Mann type algorithm designed to identify a common element between the solution sets of the HFPP and the split mixed equilibrium problem. This approach was subsequently refined by Kim and Majee [31], who substituted the nonexpansive self-mapping T with an average of a finite family $\{T_i\}$ consisting of k_i-strictly pseudocontractive non-self mappings. Notably, their modification utilized a step size independent of the operator norm. Recently, to enhance the convergence rate of the method in [31], Chuasuk and Kaewcharoen [15] proposed an inertial Krasnoselski-Mann type algorithm. Their work addresses common solutions for HFPP involving k-strictly pseudocontractive non-self mappings and split generalized mixed equilibrium problems.

Synthesizing previous findings and an extensive review of the relevant literature, we propose a resolvent method with double inertial extrapolations to approximate the common solution to mixed variational inequalities (1.1) and the hierarchical fixed point (1.3) in the setting of real Hilbert spaces. Our approach departs from traditional Armijo-type line searches by using a simple step-size adjustment rule that generates a non-monotonic sequence. In the study of inertial methods, parameters are frequently constrained to be non-decreasing with a fixed upper bound. Furthermore, many existing frameworks require the inertia parameter to depend on the Lipschitz constant, which can be challenging to estimate in practice. To address these limitations, we introduce a less restrictive condition that ensures the convergence of our inertial method without requiring knowledge of the Lipschitz constant.

2. Preliminaries

We begin by establishing the fundamental definitions and preliminary results essential for the analysis of our proposed method.

Definition 2.1. The mapping $T : C \to H$ is said to be

(a) monotone if
$$\langle Tx - Ty, x - y\rangle \geq 0, \qquad \forall x, y \in C;$$
(b) pseudomonotone if
$$\langle Tx, y - x\rangle \geq 0 \Rightarrow \langle Ty, y - x\rangle \geq 0, \qquad \forall x, y \in C;$$
(c) nonexpansive if
$$\|Tx - Ty\| \leq \|x - y\|, \qquad \forall x, y \in C;$$
(d) L-Lipschitz continuous if there exists a constant $L > 0$ such that
$$\|Tx - Ty\| \leq L\|x - y\|, \qquad \forall x, y \in C;$$
(e) contraction if there exists a constant $0 \leq k < 1$ such that
$$\|Tx - Ty\| \leq k\|x - y\|, \qquad \forall x, y \in C;$$

Definition 2.2. [12] Let A be a maximal monotone operator, then the resolvent operator associated with A is defined as
$$J_A(u) = (I + \rho A)^{-1}(u), \quad \forall u \in H,$$
where $\rho > 0$ is a constant and I is the identity operator.

Remark 2.1. It is well known that the subdifferential $\partial\psi$ of a convex, proper and lower-semicontinuous function $\psi : H \longrightarrow \mathbb{R}$ is a maximal monotone with respect to the first argument, its resolvent is defined by
$$J_\psi = (I + \rho\partial\psi)^{-1}. \tag{2.1}$$

The resolvent operator $J_\psi$defined by (2.1) has the following characterization,

Lemma 2.1. [12]. *For a given $u \in H$, $z \in H$ satisfies the inequality*
$$\langle u - z, v - u\rangle + \rho\psi(v) - \rho\psi(u) \geq 0, \quad \forall v \in H, \tag{2.2}$$
if and only if
$$u = J_\psi[z],$$
where J_ψ is resolvent operator defined by (2.1).

Remark 2.2. It follows from Lemma 2.1 that
$$\langle J_\psi[z] - z, v - J_\psi[z]\rangle + \rho\psi(v) - \rho\psi(J_\psi[z]) \geq 0, \qquad \forall u, v, z \in H \tag{2.3}$$
which implies that
$$-2\langle v - J_\psi[z], J_\psi[z] - z\rangle \leq 2\rho\Big(\psi(v) - \psi(J_\psi[z])\Big), \forall v \in H.$$
Since
$$2\langle v - J_\psi[z], J_\psi[z] - z\rangle = \|v - z\|^2 - \|v - J_\psi[z]\|^2 - \|J_\psi[z] - z\|^2.$$
Then we obtain
$$\|v - J_\psi[z]\|^2 \leq \|v - z\|^2 - \|J_\psi[z] - z\|^2 + 2\rho\Big(\psi(v) - \psi(J_\psi[z])\Big), \forall v \in H. \tag{2.4}$$

Lemma 2.2. [42] *Every Hilbert space satisfies the Opial's condition, i.e., for any sequence $\{x_n\}$ in the space that converges weakly to a point x_0, the following property holds:*
$$\lim_{n\to\infty}\inf \|x_n - x_0\| < \lim_{n\to\infty}\inf \|x_n - y\| \tag{2.5}$$
for all points y in the space that are not equal to x_0.

Lemma 2.3. [51] *A function F defined on a convex domain is uniformly continuous, if and only if, for every $\epsilon > 0$, there exists a $K < \infty$ such that $\|F(a) - F(b)\| \leq K\|a - b\| + \epsilon$.*

Lemma 2.4. [47] *Let $\{c_n\}$ and $\{\omega_n\}$ be two nonnegative real sequences satisfy*

$$c_{n+1} \leq c_n + \omega_n, \quad \forall n \geq 1.$$

If $\sum_{n=0}^{\infty} \omega_n < \infty$, then $\lim\limits_{n\to\infty} c_n$ exists.

Lemma 2.5. [45] *Let $\{c_n\}$ be a positive real sequence, $\{\omega_n\}$ be a sequence in $(0,1)$ such that $\sum\limits_{n=1}^{\infty} \omega_n = \infty$ and ϑ_n is a sequence of real numbers. Suppose that*

$$c_{n+1} \leq (1 - \omega_n)c_n + \omega_n \vartheta_n, \quad \forall n \geq 1.$$

If $\limsup_{k\to\infty} \vartheta_{n_k} \leq 0$ for all subsequences $\{c_{n_k}\}$ of $\{c_n\}$ satisfying the condition $\liminf\limits_{k\to\infty}(c_{n_k+1} - c_{n_k}) \geq 0$. Then, $\lim\limits_{n\to\infty} c_n = 0$.

Finally, we require the following standard inequalities and equalities, which are valid in Hilbert spaces.

(i)
$$\|x + v\|^2 \leq \|x\|^2 + 2\langle v, x + v\rangle, \qquad \forall x, v \in H. \tag{2.6}$$

(ii) For each $v_1, \cdots, v_m \in H$ and $\delta_1, \cdots, \delta_m \in [0,1]$ with $\sum_{i=1}^{m} \delta_i = 1$, the following equality holds
$$\|\delta_1 v_1 + \cdots + \delta_m v_m\|^2 = \sum_{i=1}^{m} \delta_i \|v_i\|^2 - \sum_{1\leq i\leq j\leq m} \delta_i \delta_j \|v_i - v_j\|^2. \tag{2.7}$$

3. The Proposed Method and Some Properties

Let H be real Hilbert space. Let $\psi : H \to \mathbb{R}\bigcup\{+\infty\}$ be bifunction convex. Let $T, S : H \to H$ be nonexpansive mappings. Let $f : H \to H$ be a contraction mapping with constant $k \in [0,1)$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\delta_n\}$, $\{\mu_n\}$, $\{\epsilon_n\}$, $\{\xi_n\}$, $\{\tau_n\}$ and $\{\rho_n\}$ are nonnegative sequences satisfying the following conditions:

(a) $\alpha_n + \beta_n + \delta_n = 1$, and $\liminf_{n\to\infty} \beta_n \delta_n > 0$;

(b) $\lim\limits_{n\to\infty} \dfrac{\epsilon_n}{\alpha_n} = 0$ and $\lim\limits_{n\to\infty} \dfrac{\xi_n}{\alpha_n} = 0$;

(c) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$;

(d) $0 < \liminf\limits_{n\to\infty} \mu_n < \limsup\limits_{n\to\infty} \mu_n < 1$;

(e) $\sum\limits_{n=1}^{\infty} \tau_n < \infty$, $\lim\limits_{n\to\infty} r_n = 0$.

Assuming the usual conditions:

(C1) The operator $G : H \to H$ be monotone and uniformly continuous on H and satisfies the following property: whenever $\{x_n\} \in H, x_n \rightharpoonup x^*$, one has $\|G(x^*)\| \leq \liminf\limits_{n\to\infty} \|G(x_n)\|$.

(C2) The solution set $\Omega = F(S) \bigcap MVI(G, \psi) \bigcap \Lambda \neq \emptyset$.

To determine the common solutions of (1.1) and (1.3), we propose the following algorithm

Algorithm 3.1.

Step 0. The initial step:
Given $\chi \in (0,1), \gamma_n \in (0,1), \eta_n \in (0,1), \mu_n \in (0,1), \rho_1 > 0, \theta > 0, \varpi > 0$, and let $x_0, x_1 \in H$ be arbitrary.
Given x_{n-1}, x_n.

Step 1. Choose θ_n and ϖ_n such that

$$\theta_n := \begin{cases} \min\{\theta, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\} & \text{if } x_n \neq x_{n-1}, \\ \theta, & \text{otherwise.} \end{cases} \tag{3.1}$$

And

$$\varpi_n := \begin{cases} \min\{\varpi, \frac{\xi_n}{\|x_n - x_{n-1}\|}\} & \text{if } x_n \neq x_{n-1}, \\ \varpi, & \text{otherwise.} \end{cases} \tag{3.2}$$

Step 2. Set

$$a_n = \gamma_n x_n + (1-\gamma_n)\Big[x_n + \theta_n(x_n - x_{n-1})\Big],$$

$$b_n = \eta_n x_n + (1-\eta_n)\Big[x_n + \varpi_n(x_n - x_{n-1})\Big],$$

and compute

$$z_n = J_\psi\Big(a_n - \rho_n G(a_n)\Big),$$

if $z_n = a_n$ then stop, a_n is a solution of (1.1). Else, do Step 3.

Step 3. Compute

$$u_n = J_\psi\Big(a_n - \rho_n G(z_n)\Big).$$

Step 4. Compute

$$v_n = (1-\mu_n)u_n + \mu_n S(u_n)$$

and

$$x_{n+1} = \alpha_n h(b_n) + \beta_n v_n + \delta_n T(v_n).$$

Update

$$\rho_{n+1} := \begin{cases} \min\Big\{\frac{(r_n+\chi)(\|a_n - z_n\|^2 + \|u_n - z_n\|^2)}{2\langle G(a_n) - G(z_n), u_n - z_n\rangle}, \rho_n + \tau_n\Big\} & \text{if } \langle G(a_n) - G(z_n), u_n - z_n\rangle > 0, \\ \rho_n + \tau_n & \text{otherwise.} \end{cases} \tag{3.3}$$

Set $n := n+1$ and go to Step 1.

Given that ψ is an indicator function of a closed convex set C in H, its corresponding resolvent operator then J_ψ becomes the metric projection P_C onto C and consequently Algorithm 3.1 reduces to

Algorithm 3.2.

Step 0. The initial step:
Given $\chi \in (0,1), \gamma_n \in (0,1), \eta_n \in (0,1), \mu_n \in (0,1), \rho_1 > 0, \theta > 0, \varpi > 0$, and let $x_0, x_1 \in H$ be arbitrary.
Given x_{n-1}, x_n.

Step 1. Choose θ_n and ϖ_n such that

$$\theta_n := \begin{cases} \min\{\theta, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\} & \text{if } x_n \neq x_{n-1}, \\ \theta, & \text{otherwise.} \end{cases}$$

And

$$\varpi_n := \begin{cases} \min\{\varpi, \frac{\xi_n}{\|x_n - x_{n-1}\|}\} & \text{if } x_n \neq x_{n-1}, \\ \varpi, & \text{otherwise.} \end{cases}$$

Step 2. Set

$$a_n = \gamma_n x_n + (1-\gamma_n)\Big[x_n + \theta_n(x_n - x_{n-1})\Big],$$

$$b_n = \eta_n x_n + (1-\eta_n)\Big[x_n + \varpi_n(x_n - x_{n-1})\Big],$$

and compute

$$z_n = P_C\Big(a_n - \rho_n G(a_n)\Big),$$

if $z_n = a_n$ then stop, a_n is a solution of (1.1). Else, do Step 3.

Step 3. Compute

$$u_n = P_C\Big(a_n - \rho_n G(z_n)\Big).$$

Step 4. Compute

$$v_n = (1-\mu_n)u_n + \mu_n S(u_n)$$

and

$$x_{n+1} = \alpha_n h(b_n) + \beta_n v_n + \delta_n T(v_n).$$

Update

$$\rho_{n+1} := \begin{cases} \min\left\{\frac{(r_n+\chi)(\|a_n-z_n\|^2+\|u_n-z_n\|^2)}{2\langle G(a_n)-G(z_n),u_n-z_n\rangle}, \rho_n+\tau_n\right\} & \text{if } \langle G(a_n)-G(z_n), u_n-z_n\rangle > 0, \\ \rho_n+\tau_n & \text{otherwise.} \end{cases}$$

Set $n := n+1$ and go to Step 1.

Remark 3.1. By condition (b), from (3.1) we we obtain

$$\theta_n\|x_n - x_{n-1}\| \le \epsilon_n \quad \text{and} \quad \varpi_n\|x_n - x_{n-1}\| \le \xi_n.$$

then

$$\lim_{n\to\infty} \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| = 0 \tag{3.4}$$

and

$$\lim_{n\to\infty} \frac{\varpi_n}{\alpha_n}\|x_n - x_{n-1}\| = 0. \tag{3.5}$$

Thus, there exist $N_1 > 0$ and $N_2 > 0$ such that

$$\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \le N_1, \forall n \in \mathbf{N} \tag{3.6}$$

and

$$\frac{\varpi_n}{\alpha_n}\|x_n - x_{n-1}\| \le N_2, \forall n \in \mathbf{N}. \tag{3.7}$$

The proof of global convergence for our method relies on the foundational lemmas presented below.

Lemma 3.1. [11] *Let $\{\rho_n\}$ be a sequence defined by (3.3). Then, we have $\lim_{n\to\infty} \rho_n = \rho$, where $\rho \in \left[\min\left(\frac{\chi}{M}, \rho_1\right), \rho_1 + \sum_{n=1}^{\infty}\tau_n\right]$.*

Remark 3.2. It follows from Lemma 3.1 and condition (e) that

$$\lim_{n\to\infty}\left(1 - \frac{(r_n+\chi)\rho_n}{\rho_{n+1}}\right) = 1-\chi > 0, \tag{3.8}$$

there exists $n_0 > 0$ such that for all $n \ge n_0$, we have $1 - \frac{(r_n+\chi)\rho_n}{\rho_{n+1}} > \frac{1-\chi}{2} > 0$.

Lemma 3.2. *Given the sequence $\{x_n\}$ produced by Algorithm 3.1 and a point $x^* \in \Omega$, the following holds*

$$\|u_n - x^*\|^2 \leq \|a_n - x^*\|^2 - \Big(1 - \frac{(r_n+\chi)\rho_n}{\rho_{n+1}}\Big)\|u_n - z_n\|^2 - \Big(1 - \frac{(r_n+\chi)\rho_n}{\rho_{n+1}}\Big)\|z_n - a_n\|^2. \quad (3.9)$$

Proof. It follows from (2.4) that

$$\begin{aligned} \|u_n - x^*\|^2 &= \|J_\psi\Big(a_n - \rho_n G(z_n)\Big) - x^*\|^2 \\ &\leq \|a_n - \rho_n G(z_n) - x^*\|^2 - \|u_n - a_n + \rho_n G(z_n)\|^2 \\ &\quad +2\rho_n[\psi(x^*) - \psi(u_n)] \\ &= \|a_n - x^*\|^2 - 2\rho_n\langle u_n - x^*, G(z_n)\rangle - \|u_n - a_n\|^2 \\ &\quad +2\rho_n[\psi(x^*) - \psi(u_n)]. \end{aligned}$$

Since $x^* \in MVI(G, \psi)$, we have

$$\langle G(x^*), z_n - x^*\rangle + \psi(z_n) - \psi(x^*) \geq 0$$

and using the monotonicity of G, we get

$$\langle G(z_n), z_n - x^*\rangle + \psi(z_n) - \psi(x^*) \geq 0. \quad (3.10)$$

On the other hand, inequality (2.3) yields

$$\langle a_n - \rho_n G(a_n) - z_n, z_n - u_n\rangle + \rho_n\psi(u_n) - \rho_n\psi(z_n) \geq 0\,. \quad (3.11)$$

By virtue of (3.10) and (3.11), we obtain

$$\begin{aligned} \|u_n - x^*\|^2 &\leq \|a_n - x^*\|^2 - 2\rho_n\langle G(z_n), u_n - x^*\rangle - \|u_n - a_n\|^2 \\ &\quad +2\rho_n[\psi(x^*) - \psi(u_n)] + 2\rho_n\Big(\langle G(z_n), z_n - x^*\rangle + \psi(z_n) - \psi(x^*)\Big) \\ &\quad -2\langle a_n - \rho_n G(a_n) - z_n, u_n - z_n\rangle + 2\rho_n\Big(\psi(u_n) - \psi(z_n)\Big) \\ &= \|a_n - x^*\|^2 - 2\rho_n\langle G(z_n), u_n - z_n\rangle - \|u_n - z_n\|^2 - \|z_n - a_n\|^2 \\ &\quad -2\langle u_n - z_n, z_n - a_n\rangle \\ &\quad -2\langle a_n - \rho_n G(a_n) - z_n, u_n - z_n\rangle \\ &= \|a_n - x^*\|^2 - \|u_n - z_n\|^2 - \|z_n - a_n\|^2 + 2\rho_n\langle G(a_n) \\ &\quad -G(z_n), u_n - z_n\rangle. \end{aligned} \quad (3.12)$$

Invoking (3.3), we obtain the following

$$\|u_n - x^*\|^2 \leq \|a_n - x^*\|^2 - \Big(1 - \frac{(r_n+\chi)\rho_n}{\rho_{n+1}}\Big)\|u_n - z_n\|^2 - \Big(1 - \frac{(r_n+\chi)\rho_n}{\rho_{n+1}}\Big)\|z_n - a_n\|^2,$$

which leads to the desired assertion of this lemma. □

Lemma 3.3. *Let $\{x_n\}$ be the sequence generated by Algorithm 3.1. Then, for a given $x^* \in \Omega$, there exists an integer n_0 such that $\{x_n\}$ is bounded for all $n \geq n_0$.*

Proof. Observe that

$$\begin{aligned}
\|x_{n+1}-x^*\| &= \|\alpha_n h(b_n)+\beta_n v_n+\delta_n T(v_n)-x^*\| \\
&\le \alpha_n\|h(b_n)-x^*\|+\beta_n\|v_n-x^*\|+\delta_n\|T(v_n)-x^*\| \\
&\le \alpha_n\|h(b_n)-h(x^*)\|+\alpha_n\|h(x^*)-x^*\|+\beta_n\|v_n-x^*\| \\
&\quad +\delta_n\|T(v_n)-x^*\| \\
&\le \alpha_n k\|b_n-x^*\|+\alpha_n\|h(x^*)-x^*\|+\beta_n\|v_n-x^*\| \\
&\quad +\delta_n\|T(v_n)-T(x^*)\| \\
&\le \alpha_n k\|b_n-x^*\|+\alpha_n\|h(x^*)-x^*\|+(1-\alpha_n)\|v_n-x^*\|.
\end{aligned} \tag{3.13}$$

On the other hand, applying (2.7) in conjunction with Lemma 3.2, we obtain

$$\begin{aligned}
\|v_n-x^*\|^2 &= \Big\|(1-\mu_n)u_n+\mu_n S(u_n)-x^*\Big\|^2 \\
&= \Big\|(1-\mu_n)(u_n-x^*)+\mu_n\Big(S(u_n)-x^*\Big)\Big\|^2 \\
&\le (1-\mu_n)\|u_n-x^*\|^2+\mu_n\Big\|S(u_n)-x^*\Big\|^2 \\
&\quad -\mu_n(1-\mu_n)\Big\|S(u_n)-u_n\Big\|^2 \\
&= (1-\mu_n)\|u_n-x^*\|^2+\mu_n\Big\|S(u_n)-S(x^*)\Big\|^2 \\
&\quad -\mu_n(1-\mu_n)\Big\|S(u_n)-u_n\Big\|^2 \\
&\le (1-\mu_n)\|u_n-x^*\|^2+\mu_n\|u_n-x^*\|^2-\mu_n(1-\mu_n)\Big\|S(u_n)-u_n\Big\|^2 \\
&= \|u_n-x^*\|^2-\mu_n(1-\mu_n)\Big\|S(u_n)-u_n\Big\|^2 \\
&\le \|u_n-x^*\|^2 \\
&\le \|a_n-x^*\|^2.
\end{aligned} \tag{3.14}$$

In view of (3.6), we have

$$\begin{aligned}
\|a_n-x^*\| &= \|\gamma_n x_n+(1-\gamma_n)\Big[x_n+\theta_n(x_n-x_{n-1})\Big]-x^*\| \\
&\le \gamma_n\|x_n-x^*\|+(1-\gamma_n)\Big[\|x_n-x^*\|+\alpha_n\Big(\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|\Big)\Big] \\
&\le \|x_n-x^*\|+\alpha_n N_1.
\end{aligned} \tag{3.15}$$

Observing (3.7), we obtain

$$\begin{aligned}
\|b_n-x^*\| &= \|\eta_n x_n+(1-\eta_n)\Big[x_n+\varpi_n(x_n-x_{n-1})\Big]-x^*\| \\
&\le \eta_n\|x_n-x^*\|+(1-\eta_n)\Big[\|x_n-x^*\|+\alpha_n\Big(\frac{\varpi_n}{\alpha_n}\|x_n-x_{n-1}\|\Big)\Big] \\
&\le \|x_n-x^*\|+\alpha_n N_2.
\end{aligned} \tag{3.16}$$

Applying (3.14), (3.15) and (3.16) in (3.13), we obtain

$$\begin{aligned}
\|x_{n+1}-x^*\| &\leq \Big(1-(1-k)\alpha_n\Big)\|x_n-x^*\|+\alpha_n\|h(x^*)-x^*\|+\alpha_n\Big(N_1+k\alpha_n N_2\Big)\\
&\leq \Big(1-(1-k)\alpha_n\Big)\|x_n-x^*\|+\alpha_n\|h(x^*)-x^*\|+\alpha_n(N_1+N_2)\\
&= \Big(1-(1-k)\alpha_n\Big)\|x_n-x^*\|+\alpha_n(1-k)\Big(\frac{\|h(x^*)-x^*\|+(N_1+N_2)}{1-k}\Big)\\
&\leq \max\Big(\|x_n-x^*\|,\frac{\|h(x^*)-x^*\|+(N_1+N_2)}{1-k}\Big). \qquad (3.17)
\end{aligned}$$

By induction on n, we obtain

$$\|x_n-x^*\|\leq \max\Big(\|x_{n_0}-x^*\|,\frac{\|h(x^*)-x^*\|+(N_1+N_2)}{1-k}\Big),\forall n\geq n_0.$$

Hence $\{x_n\}$ is bounded and consequently, we deduce that $\{a_n\},\{b_n\},\{z_n\},\{v_n\},\{u_n\}$ and $\{h(b_n)\}$ are bounded. □

4. Convergence Analysis

We now demonstrate the strong convergence of the proposed method. Notably, our approach bypasses the traditional two-case analysis frequently utilized in the literature, offering a more direct and unified proof structure.

Theorem 4.1. *Algorithm 3.1 generates a sequence $\{x_n\}$ that converges strongly to a point $\tilde{x}\in\Omega$. This limit point is characterized as $\tilde{x}=P_\Omega[h(\tilde{x})]$.*

Proof. Let $\tilde{x}\in\Omega$ and $aa_n=x_n+\theta_n(x_n-x_{n-1})$, we have

$$\begin{aligned}
\|aa_n-\tilde{x}\|^2 &\leq \|x_n-\tilde{x}\|^2+\theta_n^2\|x_n-x_{n-1}\|^2+2\theta_n\|x_n-\tilde{x}\|\|x_n-x_{n-1}\|\\
&= \|x_n-\tilde{x}\|^2+\alpha_n\theta_n\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|^2+2\alpha_n\|x_n-\tilde{x}\|\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|.
\end{aligned}$$

In view of the definition of a_n, we have the following

$$\begin{aligned}
\|a_n-\tilde{x}\|^2 &\leq \gamma_n\|x_n-\tilde{x}\|^2+(1-\gamma_n)\|aa_n-\tilde{x}\|^2\\
&\leq \gamma_n\|x_n-\tilde{x}\|^2+(1-\gamma_n)\|x_n-\tilde{x}\|^2+\alpha_n\theta_n\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|^2\\
&\quad +2\alpha_n\|x_n-\tilde{x}\|\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|\\
&= \|x_n-\tilde{x}\|^2+\alpha_n qq_n \qquad (4.1)
\end{aligned}$$

where

$$qq_n=\theta_n\|x_n-x_{n-1}\|\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|+2\|x_n-\tilde{x}\|\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|. \qquad (4.2)$$

Let $bb_n=x_n+\varpi_n(x_n-x_{n-1})$, we get

$$\begin{aligned}
\|bb_n-\tilde{x}\|^2 &\leq \|x_n-\tilde{x}\|^2+\varpi_n^2\|x_n-x_{n-1}\|^2+2\varpi_n\|x_n-\tilde{x}\|\|x_n-x_{n-1}\|\\
&= \|x_n-\tilde{x}\|^2+\alpha_n\varpi_n\frac{\varpi_n}{\alpha_n}\|x_n-x_{n-1}\|^2+2\alpha_n\|x_n-\tilde{x}\|\frac{\varpi_n}{\alpha_n}\|x_n-x_{n-1}\|.
\end{aligned}$$

Applying the definition of b_n, we obtain

$$\begin{aligned}
\|b_n-\tilde{x}\|^2 &\leq \eta_n\|x_n-\tilde{x}\|^2+(1-\eta_n)\|bb_n-\tilde{x}\|^2\\
&\leq \|x_n-\tilde{x}\|^2+\alpha_n\varpi_n\frac{\varpi_n}{\alpha_n}\|x_n-x_{n-1}\|^2+2\alpha_n\|x_n-\tilde{x}\|\frac{\varpi_n}{\alpha_n}\|x_n-x_{n-1}\|\\
&= \|x_n-\tilde{x}\|^2+\alpha_n pp_n \qquad (4.3)
\end{aligned}$$

where

$$pp_n = \varpi_n\|x_n - x_{n-1}\|\frac{\varpi_n}{\alpha_n}\|x_n - x_{n-1}\| + 2\|x_n - \tilde{x}\|\frac{\varpi_n}{\alpha_n}\|x_n - x_{n-1}\|. \tag{4.4}$$

By combining (3.4) into (3.5), it can be shown that

$$\lim_{n\to\infty} qq_n = 0 \quad \text{and} \quad \lim_{n\to\infty} pp_n = 0. \tag{4.5}$$

In view of (2.7) and lemma 3.2, and taking into account (3.14), (4.1) and (4.3), it follows that

$$\begin{aligned}
\|x_{n+1} - \tilde{x}\|^2 &= \|\alpha_n h(b_n) + \beta_n v_n + \delta_n T(v_n) - \tilde{x}\|^2 \\
&\leq \alpha_n\|h(b_n) - \tilde{x}\|^2 + \beta_n\|v_n - \tilde{x}\|^2 + \delta_n\|T(v_n) - \tilde{x}\|^2 \\
&\quad -\beta_n\delta_n\|T(v_n) - v_n\|^2 \\
&\leq \alpha_n\Big(\|h(b_n) - h(\tilde{x})\| + \|h(\tilde{x}) - \tilde{x}\|\Big)^2 + \beta_n\|v_n - \tilde{x}\|^2 \\
&\quad +\delta_n\|T(v_n) - T(\tilde{x})\|^2 - \beta_n\delta_n\|T(v_n) - v_n\|^2 \\
&\leq \alpha_n\Big(k\|b_n - \tilde{x}\| + \|h(\tilde{x}) - \tilde{x}\|\Big)^2 + (1-\alpha_n)\|v_n - \tilde{x}\|^2 - \beta_n\delta_n\|T(v_n) - v_n\|^2 \\
&\leq \alpha_n\|b_n - \tilde{x}\|^2 + \alpha_n\Big(2\|b_n - \tilde{x}\|\|h(\tilde{x}) - \tilde{x}\| + \|h(\tilde{x}) - \tilde{x}\|^2\Big) \\
&\quad +(1-\alpha_n)\|u_n - \tilde{x}\|^2 \\
&\quad -\mu_n(1-\mu_n)(1-\alpha_n)\Big\|S(u_n) - u_n\Big\|^2 - \beta_n\delta_n\|T(v_n) - v_n\|^2 \\
&\leq \alpha_n\|b_n - \tilde{x}\|^2 + \alpha_n\Big(2\|b_n - \tilde{x}\|\|h(\tilde{x}) - \tilde{x}\| + \|h(\tilde{x}) - \tilde{x}\|^2\Big) \\
&\quad +(1-\alpha_n)\|a_n - \tilde{x}\|^2 \\
&\quad -(1-\alpha_n)\Big(1 - \frac{(r_n+\chi)\rho_n}{\rho_{n+1}}\Big)\|u_n - z_n\|^2 \\
&\quad -(1-\alpha_n)\Big(1 - \frac{(r_n+\chi)\rho_n}{\rho_{n+1}}\Big)\|z_n - a_n\|^2 \\
&\quad -\mu_n(1-\mu_n)(1-\alpha_n)\Big\|S(u_n) - u_n\Big\|^2 - \beta_n\delta_n\|T(v_n) - v_n\|^2 \\
&\leq \alpha_n\Big(\|x_n - \tilde{x}\|^2 + \alpha_n pp_n\Big) \\
&\quad +\alpha_n\Big(2\|b_n - \tilde{x}\|\|h(\tilde{x}) - \tilde{x}\| + \|h(\tilde{x}) - \tilde{x}\|^2\Big) + (1-\alpha_n)\Big(\|x_n - \tilde{x}\|^2 + \alpha_n qq_n\Big) \\
&\quad -(1-\alpha_n)\Big(1 - \frac{(r_n+\chi)\rho_n}{\rho_{n+1}}\Big)\|u_n - z_n\|^2 \\
&\quad -(1-\alpha_n)\Big(1 - \frac{(r_n+\chi)\rho_n}{\rho_{n+1}}\Big)\|z_n - a_n\|^2 - \mu_n(1-\mu_n)(1-\alpha_n)\Big\|S(u_n) - u_n\Big\|^2 \\
&\quad -\beta_n\delta_n\|T(v_n) - v_n\|^2 \\
&\leq \|x_n - \tilde{x}\|^2 + \alpha_n\Big(2\|b_n - \tilde{x}\|\|h(\tilde{x}) - \tilde{x}\| + \|h(\tilde{x}) - \tilde{x}\|^2 + qq_n + pp_n\Big) \\
&\quad -(1-\alpha_n)\Big(1 - \frac{(r_n+\chi)\rho_n}{\rho_{n+1}}\Big)\|u_n - z_n\|^2 \\
&\quad -(1-\alpha_n)\Big(1 - \frac{(r_n+\chi)\rho_n}{\rho_{n+1}}\Big)\|z_n - a_n\|^2 \\
&\quad -\mu_n(1-\mu_n)(1-\alpha_n)\Big\|S(u_n) - u_n\Big\|^2 - \beta_n\delta_n\|v_n - u_n\|^2.
\end{aligned} \tag{4.6}$$

Suppose that $\{\|x_{n_k}-\tilde{x}\|^2\}$ is a subsequence of $\{\|x_n-\tilde{x}\|^2\}$ satisfying

$$\liminf_{k\to\infty}\Big(\|x_{n_k+1}-\tilde{x}\|^2-\|x_{n_k}-\tilde{x}\|^2\Big)\geq 0. \tag{4.7}$$

Inequality (4.6) then yields

$$\begin{aligned}&(1-\alpha_{n_k})\Big(1-\frac{(r_{n_k}+\chi)\rho_{n_k}}{\rho_{n_k+1}}\Big)\|u_{n_k}-z_{n_k}\|^2+(1-\alpha_{n_k})\Big(1-\frac{(r_{n_k}+\chi)\rho_{n_k}}{\rho_{n_k+1}}\Big)\|z_{n_k}-a_{n_k}\|^2\\&+\mu_{n_k}(1-\mu_{n_k})(1-\alpha_{n_k})\Big\|S(u_{n_k})-u_{n_k}\Big\|^2+\beta_{n_k}\delta_{n_k}\|T(v_{n_k})-v_{n_k}\|^2\\\leq\;&\|x_{n_k}-\tilde{x}\|^2-\|x_{n_k+1}-\tilde{x}\|^2+\alpha_{n_k}\Big(2\|b_{n_k}-\tilde{x}\|\|h(\tilde{x})-\tilde{x}\|+\|h(\tilde{x})-\tilde{x}\|^2+qq_{n_k}+pp_{n_k}\Big).\end{aligned}$$

In light of the above inequality and (4.7), and noting that $\lim_{n\to\infty}\alpha_n=0$, it follows that

$$\begin{aligned}&\limsup_{k\to\infty}\Big((1-\alpha_{n_k})\Big(1-\frac{(r_{n_k}+\chi)\rho_{n_k}}{\rho_{n_k+1}}\Big)\|u_{n_k}-z_{n_k}\|^2\\&+(1-\alpha_{n_k})\Big(1-\frac{(r_{n_k}+\chi)\rho_{n_k}}{\rho_{n_k+1}}\Big)\|z_{n_k}-a_{n_k}\|^2\\&+\mu_{n_k}(1-\mu_{n_k})(1-\alpha_{n_k})\Big\|S(u_{n_k})-u_{n_k}\Big\|^2+\beta_{n_k}\delta_{n_k}\|T(v_{n_k})-v_{n_k}\|^2\Big)\\\leq\;&\limsup_{k\to\infty}\Big(\|x_{n_k}-\tilde{x}\|^2-\|x_{n_k+1}-\tilde{x}\|^2\\&+\alpha_{n_k}\Big(2\|b_{n_k}-\tilde{x}\|\|h(\tilde{x})-\tilde{x}\|+\|h(\tilde{x})-\tilde{x}\|^2+qq_{n_k}+pp_{n_k}\Big)\Big)\\=\;&-\liminf_{k\to\infty}\Big(\|x_{n_k+1}-\tilde{x}\|^2-\|x_{n_k}-\tilde{x}\|^2\Big)\\\leq\;&0.\end{aligned}$$

Recalling (3.8) alongside conditions (a), (c), and (d), we observe that

$$\lim_{k\to\infty}\|u_{n_k}-z_{n_k}\|=0,\ \lim_{k\to\infty}\|a_{n_k}-z_{n_k}\|=0,\ \lim_{k\to\infty}\|T(v_{n_k})-v_{n_k}\|=0,\ \lim_{k\to\infty}\Big\|S(u_{n_k})-u_{n_k}\Big\|=0. \tag{4.8}$$

By applying the triangle inequality to the first two limits in (4.8), we immediately obtain

$$\lim_{k\to\infty}\|a_{n_k}-u_{n_k}\|=0. \tag{4.9}$$

Furthermore, from (3.6), we observe that

$$\begin{aligned}\|a_{n_k}-x_{n_k}\|&=(1-\gamma_n)\|\theta_{n_k}(x_{n_k}-x_{n_k-1})\|\\&\leq\theta_{n_k}\|x_{n_k}-x_{n_k-1}\|\\&\leq\alpha_{n_k}N_1.\end{aligned}$$

Given that $\alpha_{n_k}\to 0$ as $k\to\infty$, we conclude

$$\lim_{k\to\infty}\|a_{n_k}-x_{n_k}\|=0. \tag{4.10}$$

Applying the triangle inequality, we have

$$\|u_{n_k}-x_{n_k}\|\leq\|u_{n_k}-a_{n_k}\|+\|a_{n_k}-x_{n_k}\|.$$

Based on (4.9) and (4.10), it follows that

$$\lim_{k\to\infty}\|u_{n_k}-x_{n_k}\|=0. \tag{4.11}$$

Next, considering the relationship between v_{n_k} and x_{n_k}

$$\|v_{n_k}-x_{n_k}\|\leq\|v_{n_k}-u_{n_k}\|+\|u_{n_k}-x_{n_k}\|\leq\|S(u_{n_k})-u_{n_k}\|+\|u_{n_k}-x_{n_k}\|.$$

In view of (4.8) and (4.11), we obtain

$$\lim_{k\to\infty}\|v_{n_k}-x_{n_k}\|=0. \tag{4.12}$$

For the term involving the operator T, we observe that

$$\|T(v_{n_k}) - u_{n_k}\| \leq \|T(v_{n_k}) - v_{n_k}\| + \|v_{n_k} - x_{n_k}\| + \|u_{n_k} - x_{n_k}\|.$$

By combining (4.8), (4.11) and (4.12), we conclude that

$$\lim_{k\to\infty} \|T(v_{n_k}) - u_{n_k}\| = 0. \tag{4.13}$$

From the definition of x_{n_k+1}, we have

$$\begin{aligned}\|x_{n_k+1} - x_{n_k}\| &\leq \alpha_{n_k}\|h(b_{n_k}) - x_{n_k}\| + \beta_{n_k}\|v_{n_k} - x_{n_k}\| + \delta_{n_k}\|T(v_{n_k}) - x_{n_k}\| \\ &\leq \alpha_{n_k}\|h(b_{n_k}) - x_{n_k}\| + (\beta_{n_k} + \delta_{n_k})\|v_{n_k} - x_{n_k}\| + \delta_{n_k}\|T(v_{n_k}) - v_{n_k}\|.\end{aligned}$$

Taking the limit as $k \to \infty$ and substituting the results from (4.8) and (4.12), we obtain

$$\lim_{k\to\infty} \|x_{n_k+1} - x_{n_k}\| = 0. \tag{4.14}$$

We now proceed to prove that $\omega_w(x_n) \subset \Omega$, where $\omega_w(x_n)$ denotes the set of weak cluster points of $\{x_n\}$, defined as

$$\omega_w(x_n) = \{x \in H : x_{n_i} \rightharpoonup x \text{ for some subsequence } \{x_{n_i}\} \text{ of } \{x_n\}\}.$$

Given that the sequence $\{x_n\}$ is bounded, the set $\omega_w(x_n)$ is nonempty. Let $\tilde{x} \in \omega_w(x_n)$; then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \tilde{x}$ as $k \to \infty$. Furthermore, since $\lim_{k\to\infty} \|a_{n_k} - x_{n_k}\| = 0$, it follows that a_{n_k} also converges weakly to $\tilde{x}$ as $k \to \infty$.

We shall now demonstrate that $\tilde{x} \in MVI(G, \psi)$. Recalling that $z_n = J_\psi(a_n - \rho_n G(a_n))$ and applying (2.2), we obtain

$$\psi(y) - \psi(z_n) + \langle G(a_n), y - z_n\rangle + \frac{1}{\rho_n}\langle y - z_n, z_n - a_n\rangle \geq 0, \quad \forall y \in H$$

and

$$\psi(y) - \psi(z_{n_k}) + \langle G(a_{n_k}), y - z_{n_k}\rangle + \langle y - z_{n_k}, \frac{z_{n_k} - a_{n_k}}{\rho_{n_k}}\rangle \geq 0, \quad \forall y \in H. \tag{4.15}$$

Setting $c_t = ty + (1-t)\tilde{x} \in H$ for $t \in (0,1]$ and $y \in H$, noting that $c_t \in H$. It then follows from (4.15) that

$$\begin{aligned}\langle G(c_t), c_t - z_{n_k}\rangle &\geq \psi(z_{n_k}) - \psi(c_t) + \langle G(c_t), c_t - z_{n_k}\rangle \\ &\quad -\langle G(a_{n_k}), c_t - z_{n_k}\rangle - \langle c_t - z_{n_k}, \frac{z_{n_k} - a_{n_k}}{\rho_{n_k}}\rangle) \\ &= \psi(z_{n_k}) - \psi(c_t) + \langle G(c_t) - G(z_{n_k}), c_t - z_{n_k}\rangle \\ &\quad +\langle G(z_{n_k}) - G(a_{n_k}), c_t - z_{n_k}\rangle - \langle c_t - z_{n_k}, \frac{z_{n_k} - a_{n_k}}{\rho_{n_k}}\rangle.\end{aligned} \tag{4.16}$$

Given that G is uniformly continuous on H and noting that $\lim_{k\to\infty} \|z_{n_k} - a_{n_k}\| = 0$ (see (4.8)), it follows that $\lim_{k\to\infty} \|G(z_{n_k}) - G(a_{n_k})\| = 0$. Utilizing the monotonicity of G, the weak lower semicontinuity of ψ, and the fact that $z_{n_k} \rightharpoonup \tilde{x}$, we deduce from (4.16) that

$$\langle G(c_t), c_t - \tilde{x}\rangle \geq \psi(\tilde{x}) - \psi(c_t). \tag{4.17}$$

Using the fact that ψ is convex, we obtain

$$\langle G(c_t), y - \tilde{x}\rangle \geq \psi(\tilde{x}) - \psi(y), \quad \forall y \in H.$$

Letting $t \to 0_+$ we have

$$\psi(y) - \psi(\tilde{x}) + \langle G(\tilde{x}), y - \tilde{x}\rangle \geq 0, \quad \forall y \in H,$$

which implies that $\tilde{x} \in MVI(G, \psi)$. Next, we show that $\tilde{x} \in F(S)$. Given that $\lim\limits_{k\to\infty} \|u_{n_k} - x_{n_k}\| = 0$ (see(4.11)), it follows that $u_{n_k} \rightharpoonup \tilde{x}$ as $k \to \infty$. We apply the triangle inequality and the properties of the operator

$$\|u_{n_k} - S(\tilde{x})\| \leq \|u_{n_k} - S(u_{n_k})\| + \|S(u_{n_k}) - S(\tilde{x})\| \leq \|u_{n_k} - S(u_{n_k})\| + \|u_{n_k} - \tilde{x}\|.$$

Taking the limit inferior on both sides and utilizing (4.8), we obtain

$$\liminf_{k\to\infty} \|u_{n_k} - S(\tilde{x})\| \leq \liminf_{k\to\infty} \|u_{n_k} - \tilde{x}\|.$$

By the Opial property of the Hilbert space H (see(Lemma 2.2)), this inequality implies that $S(\tilde{x}) = \tilde{x}$, thus confirming $\tilde{x} \in F(S)$.
Following a similar logic, since $\lim_{k\to\infty} \|v_{n_k} - x_{n_k}\| = 0$ (see(4.12)), we have $v_{n_k} \rightharpoonup \tilde{x}$ as $k \to \infty$. We then observe

$$\|v_{n_k} - T(\tilde{x})\| \le \|v_{n_k} - T(v_{n_k})\| + \|T(v_{n_k}) - T(\tilde{x})\| \le \|v_{n_k} - T(v_{n_k})\| + \|v_{n_k} - \tilde{x}\|.$$

Applying the limit inferior and (4.8) yields

$$\liminf_{k\to\infty} \|v_{n_k} - T(\tilde{x})\| \le \liminf_{k\to\infty} \|v_{n_k} - \tilde{x}\|.$$

Again, invoking the Opial property, we conclude that $T(\tilde{x}) = \tilde{x}$, which implies that $\tilde{x} \in F(T)$. Next, we show that $\tilde{x} \in \Lambda$. From the monotonicity of $I - S$ and $\forall z \in F(T)$, we have

$$\begin{aligned}
\langle \frac{T(v_{n_k}) - v_{n_k}}{\mu_{n_k}}, u_{n_k} - z\rangle &= \langle (I-S)(u_{n_k}) - (I-S)(z), u_{n_k} - z\rangle + \langle (I-S)(z), u_{n_k} - z\rangle \\
&\quad + \langle \frac{T(v_{n_k}) - u_{n_k}}{\mu_{n_k}}, u_{n_k} - z\rangle \\
&\ge \langle (I-S)(z), u_{n_k} - z\rangle + \langle \frac{T(v_{n_k}) - u_{n_k}}{\mu_{n_k}}, u_{n_k} - z\rangle
\end{aligned} \tag{4.18}$$

Since $\lim_{k\to\infty} \|u_{n_k} - x_{n_k}\| = 0$, we have that $u_{n_k} \rightharpoonup \tilde{x}$ as $k \to \infty$. It follows from (4.8), (4.13) and (4.18) that

$$\langle (I-S)(z), \tilde{x} - z\rangle \le 0 \quad \forall z \in F(T). \tag{4.19}$$

For any $0 < t \le 1$ and $z \in F(T)$, let $cc_t = tz + (1-t)\tilde{x}$, using the fact that $F(T)$ is convex, we have $cc_t \in F(T)$. Then from (4.19), we obtain

$$0 \ge \langle (I-S)(cc_t), \tilde{x} - cc_t\rangle = t\langle (I-S)(cc_t), \tilde{x} - z\rangle \quad \forall z \in F(T)$$

which yields

$$0 \ge \langle (I-S)(cc_t), \tilde{x} - z\rangle \quad \forall z \in F(T).$$

Letting $t \to 0_+$ we have

$$\langle (I-S)(\tilde{x}), \tilde{x} - z\rangle \le 0 \quad \forall z \in F(T),$$

which implies that $\tilde{x} \in \Lambda$. Since $\tilde{x} \in \omega_w(x_n)$, it follows that $\omega_w(x_n) \subset \Omega$. We next aim to show that

$$\limsup_{k\to\infty} \langle h(\tilde{x}) - \tilde{x}, x_{n_k+1} - \tilde{x}\rangle \le 0.$$

To this end, let $\{x_{n_{k_j}}\}$ be a subsequence of $\{x_{n_k}\}$ that converges weakly to some $\hat{x} \in \Omega$, such that

$$\lim_{j\to\infty} \langle h(\tilde{x}) - \tilde{x}, x_{n_{k_j}} - \tilde{x}\rangle = \limsup_{k\to\infty} \langle h(\tilde{x}) - \tilde{x}, x_{n_k} - \tilde{x}\rangle.$$

Given that $x_{n_{k_j}} \rightharpoonup \hat{x} \in \Omega$ and recalling the characterization of the metric projection $\tilde{x} = P_\Omega[h(\tilde{x})]$, we obtain

$$\limsup_{k\to\infty} \langle h(\tilde{x}) - \tilde{x}, x_{n_k+1} - \tilde{x}\rangle = \limsup_{k\to\infty} \langle h(\tilde{x}) - \tilde{x}, x_{n_k} - \tilde{x}\rangle = \langle h(\tilde{x}) - \tilde{x}, \hat{x} - \tilde{x}\rangle \le 0. \tag{4.20}$$

Furthermore, as $\tilde{x} \in \Omega$, it follows from (2.6) that

$$\begin{aligned}
\|x_{n_k+1} - \tilde{x}\|^2 &= \Big\|\alpha_{n_k}(h(b_{n_k}) - h(\tilde{x})) + \beta_{n_k}(v_{n_k} - \tilde{x}) + \delta_{n_k}(T(v_{n_k}) - \tilde{x}) \\
&\quad + \alpha_{n_k}(h(\tilde{x}) - \tilde{x})\Big\|^2 \\
&\le \Big\|\alpha_{n_k}(h(b_{n_k}) - h(\tilde{x})) + \beta_{n_k}(v_{n_k} - \tilde{x}) + \delta_{n_k}(T(v_{n_k}) - \tilde{x})\Big\|^2 \\
&\quad + 2\alpha_{n_k}\langle h(\tilde{x}) - \tilde{x}, x_{n_k+1} - \tilde{x}\rangle \\
&\le \alpha_{n_k}\|h(b_{n_k}) - h(\tilde{x})\|^2 + \beta_{n_k}\|v_{n_k} - \tilde{x}\|^2 + \delta_{n_k}\|v_{n_k} - \tilde{x}||^2 \\
&\quad + 2\alpha_{n_k}\langle h(\tilde{x}) - \tilde{x}, x_{n_k+1} - \tilde{x}\rangle \\
&\le \alpha_{n_k} k\|b_{n_k} - \tilde{x}\|^2 + (1 - \alpha_{n_k})\|v_{n_k} - \tilde{x}\|^2 + 2\alpha_{n_k}\langle h(\tilde{x}) - \tilde{x}, x_{n_k+1} - \tilde{x}\rangle.
\end{aligned} \tag{4.21}$$

Combining Lemma 3.2 with the results from (4.1) and (4.3), we can reduce (4.21) to

$$\begin{aligned}\|x_{n_k+1}-\tilde{x}\|^2 &\le \Big(1-(1-k)\alpha_{n_k}\Big)\|x_{n_k}-\tilde{x}\|^2+\alpha_{n_k}\Big(kpp_{n_k}+(1-\alpha_{n_k})qq_{n_k}\Big)\\ &\quad +2\alpha_{n_k}\langle h(\tilde{x})-\tilde{x},x_{n_k+1}-\tilde{x}\rangle\\ &\le \Big(1-(1-k)\alpha_{n_k}\Big)\|x_{n_k}-\tilde{x}\|^2+(1-k)\alpha_{n_k}\left(\frac{pp_{n_k}+qq_{n_k}}{1-k}+\frac{2\langle h(\tilde{x})-\tilde{x},x_{n_k+1}-\tilde{x}\rangle}{1-k}\right)\\ &= (1-\sigma_{n_k})\|x_{n_k}-\tilde{x}\|^2+\sigma_{n_k}\left(\frac{pp_{n_k}+qq_{n_k}}{1-k}+\frac{2\langle h(\tilde{x})-\tilde{x},x_{n_k+1}-\tilde{x}\rangle}{1-k}\right)\end{aligned}$$

where $\sigma_{n_k}=(1-k)\alpha_{n_k}$ and define

$$\phi_{n_k}=\frac{pp_{n_k}+qq_{n_k}}{1-k}+\frac{2\langle h(\tilde{x})-\tilde{x},x_{n_k+1}-\tilde{x}\rangle}{1-k}.$$

Given that $\sum_{n_k=1}^{\infty}\alpha_{n_k}=\infty$ and $\lim_{k\to\infty}\alpha_{n_k}=0$, it follows immediately that

$$\sum_{n_k=1}^{\infty}\sigma_{n_k}=\infty \quad \text{and} \quad \lim_{k\to\infty}\sigma_{n_k}=0.$$

Furthermore, invoking (4.5) and (4.20), we obtain

$$\limsup_{k\to\infty}\phi_{n_k}\le 0.$$

Condition (4.7) ensures that all hypotheses of Lemma 2.5 are satisfied. Consequently, we deduce that $\lim_{n\to\infty}\|x_n-\tilde{x}\|^2=0$, which implies $\lim_{n\to\infty}\|x_n-\tilde{x}\|=0$. This establishes the strong convergence of $\{x_n\}$ to $\tilde{x}$ and completes the proof. □

5. Numerical Examples

In this section, we provide numerical experiments to demonstrate the convergence properties of our proposed algorithm. We compare the performance of our method against several established approaches from the literature. Across all tests, the parameters are defined as follows:

- Proposed Method: We select $r_1=0.35,\chi=0.4,\theta=0.65,\varpi=0.22,\tau_n=\frac{1}{(n+1)^{1.1}},\epsilon_n=\xi_n=\frac{1}{(2n+1)^3},\rho_n=\frac{1}{(n+1)},\alpha_n=\frac{1}{2(n+3)},\beta_n=\frac{n}{2(n+3)},\delta_n=\frac{n+5}{2(n+3)},\mu=0.1,\eta_n=\gamma_n=\frac{1}{n},k=0.5,T(x)=\frac{x}{2},S(x)=\frac{x}{5}$, and $e_n=\|z_n-a_n\|$.
- Method of Yang [54]: $\lambda_1=0.8,\mu=0.9,\alpha_n=0.4$, and $e_n=\|w_n-y_n\|$.
- Method of Shehu et al. [46]: $\lambda_n=0.7,\mu=0.9,\alpha_n=0.4,\tau_1=0.8$, and $e_n=\|w_n-y_n\|$.
- Method of Thong et al. [50]: $\lambda_n=0.8,\mu=0.9,\nu_n=0.4,\tau_1=0.8$, and $e_n=\|w_n-y_n\|$.

Example 5.1: Let the mapping $G:\mathbb{R}^n\to\mathbb{R}^n$ be defined by $G(x)=Mx+q$, where $q\in\mathbb{R}^n$ and $M=N^TN+U+D$. Here, N is an $n\times n$ matrix, U is an $n\times n$ skew-symmetric matrix, and D is an $n\times n$ diagonal matrix with non-negative entries. Under these conditions, M is positive semi-definite, ensuring that G is both monotone and uniformly continuous. The convergence of the algorithms proposed in [50] and [54] requires G to be monotone and L-Lipschitz continuous. It is straightforward to show that G satisfies these requirements with a Lipschitz constant $L=\|M\|$. The feasible set is defined as:

$$C=\{x\in\mathbb{R}^n:-5\le x_i\le 5,\quad i=1,\dots,n\}.$$

For our numerical simulations, we set q as the zero vector and generate the entries of N,U, and D randomly. We initialize the algorithms with $x_1=\text{ones}(n,1)$ and $x_0=2x_1$. The iterations are terminated once the stopping criterion $e_n<10^{-4}$ is satisfied.

Table 5.1 summarizes the performance of the proposed method in comparison to those in [46], [50], and [54]. The results are categorized by problem dimension (n) and include both the number of iterations (No. It.) and the total computational time.

Table 1: Numerical results for Example 5.1

n	The method in [54]		The method in [46]		The method in [50]		The proposed method	
	No. It.	CPU(Sec.)	No. It.	CPU(Sec.)	No. It.	CPU(Sec.)	No. It.	CPU(Sec.)
$n = 10$	274	3.234	240	2.897	194	2.218	37	0.408
$n = 20$	768	9.338	582	7.089	313	3.704	34	0.377
$n = 30$	1222	15.235	1241	16.445	440	4.491	35	0.529
$n = 50$	1334	17.584	1320	15.954	1059	13.114	388	5.032

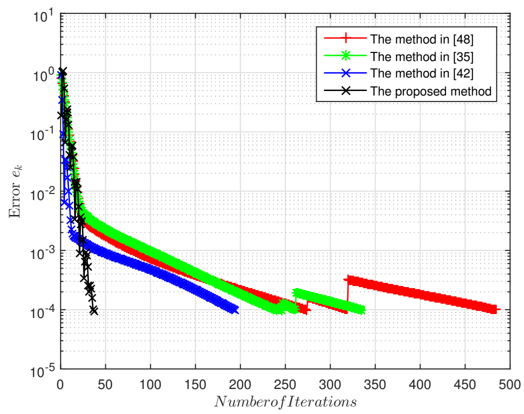

Figure 1: $n = 10$

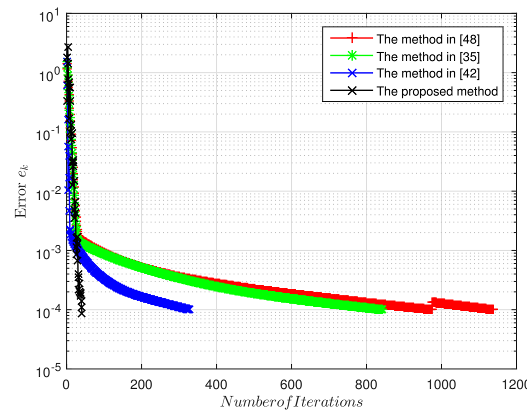

Figure 2: $n = 30$

Example 5.2: Consider the Hilbert space $H = L^2([0,1])$ equipped with the inner product $\langle x, y\rangle = \int_0^1 x(t)y(t)dt$ and the induced norm $\|x\| = \left(\int_0^1 |x(t)|^2 dt\right)^{1/2}$. Let the feasible set C be the unit ball, defined by $C = \{x \in L^2([0,1]) : \|x\| \leq 1\}$. We define the mapping $G : C \to H$ as follows:

$$G(x)(t) = \int_0^1 (x(t) - E(t,s)f(x(s)))\, ds + g(t),$$

where $E(t,s)$, $f(x)$, and $g(t)$ are given by:

$$E(t,s) = \frac{2tse^{(t+s)}}{e\sqrt{e^2-1}}, \quad f(x) = \cos(x), \quad g(t) = \frac{2te^t}{e\sqrt{e^2-1}}.$$

The mapping G is known to be monotone and L-Lipschitz continuous with $L = 1$. This specific test problem is adapted from [44]. For the numerical implementation, we initialize the algorithms with $x_1(t) = x_0(t) = \sin(t)$ and employ the stopping criterion $e_n < 10^{-4}$. The comparative results between our proposed method and the approaches described in [46], [50], and [54] are summarized in Table 2.

Table 2: Numerical results for Example 5.2

n	The method in [54]		The method in [46]		The method in [50]		The proposed method	
	No. It.	CPU(Sec.)	No. It.	CPU(Sec.)	No. It.	CPU(Sec.)	No. It.	CPU(Sec.)
$n = 50$	48	0.114	37	0.321	30	0.127	10	0.032
$n = 100$	42	3.589	36	3.116	31	2.958	10	0.451
$n = 150$	48	8.627	36	5.412	30	2.387	10	0.852
$n = 200$	48	14.167	36	9.512	30	7.624	10	1.873

Remark 5.1. Figures 1, 2, 3 and 4, along with Tables 1 and 2, illustrate the effectiveness and feasibility of the proposed method.

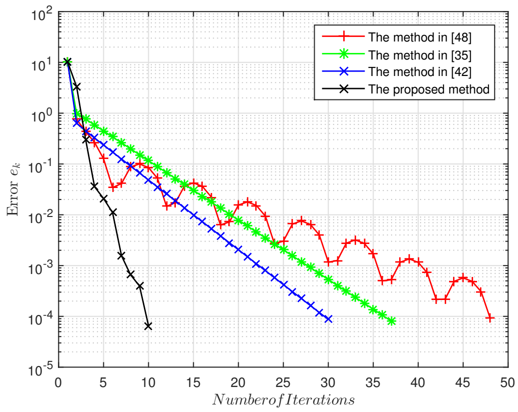

Figure 3: $n = 50$

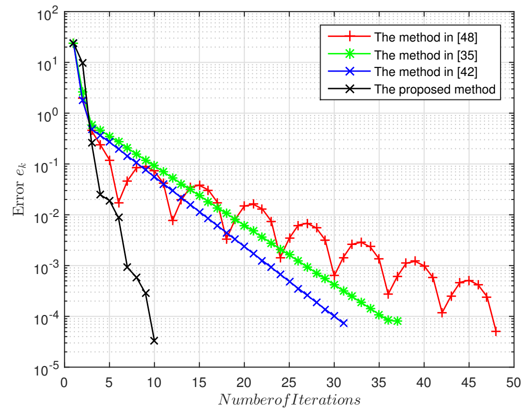

Figure 4: $n = 250$

6. Conclusions

This paper introduces a resolvent-based method designed to identify common solutions for problems (1.1) and (1.3) within the context of a real Hilbert space. Departing from traditional Armijo-type line search techniques, our algorithm employs a straightforward, self-adaptive step-size strategy. Under suitable conditions, we prove the strong convergence of the proposed algorithm. By generalizing and refining existing methods, our framework proves applicable to a wide array of challenges, most notably in the fields of variational inequalities and nonlinear complementarity.

Statements and Declarations

The author declares that has no conflict of interest, and the manuscript has no associated data.

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