APPLICABLE NONLINEAR ANALYSIS
Volume 3 (2026), No. 3, 148–163
DOI: https://doi.org/10.69829/apna-026-0303-ta01
Tulipa Opera Scholarum

SOLUTIONS FOR LINEAR NON-HOMOGENEOUS SET-VALUED DIFFERENTIAL EQUATIONS WITH MEMORY UNDER GENERALIZED DERIVATIVES

YONGLIN LIU[1],*

[1] *School of Mathematics and Statistics, Changsha University of Science and Technology, Changsha, China*

ABSTRACT. We study linear non-homogeneous set-valued differential equations in $\mathbb{R}^n$ with memory effects under generalized derivatives. The memory term is described by a convolution-type radius function, which leads to equivalent set-valued Volterra integral equations. Under suitable structural assumptions on the coefficient matrix, we derive explicit solution representations for the integral equation associated with the Hukuhara derivative, which yield globally defined solutions with monotone non-decreasing diameters. In contrast, for the Plotnikov–Skripnik derivative and the Bede–Gal derivative, we obtain ball-valued solutions that are typically only locally defined in time and have monotone decreasing diameters. Several examples are included to illustrate qualitative differences in the solution behavior induced by different notions of generalized differentiability.

Keywords. Set-valued differential equations, memory effects, generalized derivatives, non- homogeneous systems.

1. Introduction

Set-valued differential equations arise naturally in the mathematical modeling of dynamical systems involving uncertainty, imprecision, or incomplete information. Since the pioneering work of de Blasi and Iervolino [7], such equations have attracted considerable attention and have found applications in a wide range of fields, including control theory, population dynamics, and systems with fuzzy or interval uncertainty. Within this framework, a fundamental analytical issue concerns the choice of an appropriate notion of differentiability for set-valued mappings.

The classical approach is based on the Hukuhara derivative [11], which provides a natural extension of the standard derivative to the space of nonempty compact convex sets. However, a well-known limitation of the Hukuhara derivative is that the diameter of any solution obtained under this notion is necessarily non-decreasing. This restriction significantly limits its applicability in situations where the uncertainty of the system is expected to decrease over time.

To overcome this drawback, several generalized notions of differentiability have been introduced. In particular, the concepts of *Bede–Gal differentiability* [3, 31] and *Plotnikov–Skripnik differentiability* [27] allow the diameter of set-valued solutions to either increase or decrease, thereby providing a more flexible framework for modeling realistic dynamical phenomena. As a consequence, differential equations involving these generalized derivatives have been extensively investigated in the literature; see, for example, [16, 17, 30, 29, 28, 25, 9] and the references therein.

In recent years, systematic studies of linear homogeneous and non-homogeneous set-valued differential equations under generalized derivatives have been carried out; see, for instance, [16, 17, 15]. It has been shown that, even in the linear setting, the existence and explicit representation of solutions

*Corresponding author.
E-mail address: lyl14539@163.com (Y. Liu)
2020 Mathematics Subject Classification: 34G20, 26E25.
Accepted: March 19, 2026.

may strongly depend on structural properties of the coefficient matrix, such as the multiplicity of its singular values. These results reveal a delicate interplay between the algebraic features of the linear part and the qualitative behavior of set-valued solutions.

Motivated by these developments, the present paper is concerned with linear non-homogeneous set-valued differential equations incorporating memory effects. More precisely, we consider models in which the non-homogeneous term depends on the past history of the system through a convolution-type radius function. This formulation naturally leads to memory-dependent set-valued Volterra integral equations, whose solvability and qualitative properties depend both on the chosen notion of generalized differentiability and on the structure of the coefficient matrix.

Building upon solution representations available in the instantaneous setting, we analyze how the presence of memory influences the existence interval, representation, and diameter evolution of solutions. Under suitable structural assumptions on the coefficient matrix, including the case in which all singular values coincide, we show that explicit representations derived in the non-memory case can be extended to the memory-dependent framework. Moreover, we demonstrate that the qualitative distinction between different generalized derivatives persists: solutions associated with the Hukuhara derivative are globally defined and exhibit monotone non-decreasing diameters, whereas solutions corresponding to the Plotnikov–Skripnik and Bede–Gal derivatives are, in general, only locally defined and display monotone decreasing diameters.

The paper is organized as follows. In Section 2, basic notions and preliminary results concerning set-valued functions, generalized differentiability, and set-valued integration are recalled. Section 3 is devoted to the formulation of linear non-homogeneous problems with memory and to the derivation of solution representations under different generalized derivatives. Several illustrative examples are presented in Section 4.

2. Preliminaries

Let $K_c(\mathbb{R}^n)$ denote the family of all nonempty compact convex subsets of the Euclidean space $\mathbb{R}^n$, endowed with the Hausdorff metric

$$h(A,B) = \min\{r \geq 0 : \ A \subset B + B_r(0),\ B \subset A + B_r(0)\},$$

where $A, B \in K_c(\mathbb{R}^n)$ and

$$B_r(0) = \{x \in \mathbb{R}^n : \ \|x\| \leq r\}$$

denotes the closed ball of radius $r \geq 0$ centered at the origin.

For any $A, B \in K_c(\mathbb{R}^n)$ and $\lambda \in \mathbb{R}$, the Minkowski sum and scalar multiplication are defined by

$$A + B = \{a + b : \ a \in A,\ b \in B\}, \qquad \lambda A = \{\lambda a : \ a \in A\}.$$

The following properties of these operations, which will be used throughout the paper, are well known (see [26, 22]).

Proposition 2.1. *For all $A, B, C, D \in K_c(\mathbb{R}^n)$, $\lambda, \mu \in \mathbb{R}_+$, and $\beta \in \mathbb{R}$, the following statements hold.*

(i) $A + \{0\} = A$;
(ii) $A + B = B + A$;
(iii) $1A = A$ *and* $0A = \{0\}$;
(iv) $(\lambda + \mu)A = \lambda A + \mu A$;
(v) $\beta(A + B) = \beta A + \beta B$;
(vi) $h(A + C, B + C) = h(A, B)$;
(vii) $h(\beta A, \beta B) = |\beta|\, h(A, B)$.

The metric space $(K_c(\mathbb{R}^n), h)$ is complete. However, in general, the equality $A + (-1)A = \{0\}$ does not hold for arbitrary sets $A \in K_c(\mathbb{R}^n)$. To address this difficulty, the notion of the Hukuhara difference was introduced in [11].

Definition 2.2. [11] Let $A, B \in K_c(\mathbb{R}^n)$. If there exists a set $C \in K_c(\mathbb{R}^n)$ such that $A = B + C$, then the Hukuhara difference of A and B is said to exist and is denoted by $A \ominus_h B$, that is,

$$A \ominus_h B = C \quad \Longleftrightarrow \quad A = B + C.$$

Proposition 2.3. [30] *Let $A, B, C \in K_c(\mathbb{R}^n)$ and $\lambda \in \mathbb{R}_+$.*

(i) If the Hukuhara difference $A \ominus_h B$ exists, then it is unique.
(ii) $A \ominus_h A = \{0\}$.
(iii) If $A \ominus_h B$ exists, then $\lambda A \ominus_h \lambda B$ exists and $\lambda(A \ominus_h B) = \lambda A \ominus_h \lambda B$.
(iv) $(A + B) \ominus_h B = A$.
(v) If $A \ominus_h B$ exists, then there exists $\alpha_0 \geq 1$ such that $A \ominus_h \alpha B$ exists for all $\alpha \in (0, \alpha_0]$.
(vi) If $A \ominus_h (B + C)$ exists, then

$$A \ominus_h (B + C) = (A \ominus_h B) \ominus_h C = (A \ominus_h C) \ominus_h B.$$

(vii) If both $A \ominus_h B$ and $A \ominus_h C$ exist, then there exists $\alpha > 0$ such that $A \ominus_h \alpha(B + C)$ exists.

Definition 2.4. [4] A set $M \subset \mathbb{R}^n$ is said to be centrally symmetric with respect to a point $q \in \mathbb{R}^n$ if, for every $x \in M$, there exists $x' \in M$ such that $x + x' = 2q$.

The following lemma collects several properties of the Hukuhara difference and of balls in $K_c(\mathbb{R}^n)$ that will be used repeatedly in the sequel.

Lemma 2.5. [15, 22] *Let $A, B \in K_c(\mathbb{R}^n)$.*

(i) If A and B are centrally symmetric and $A \ominus_h B$ exists, then the Hukuhara differences

$$A \ominus_h (-1)A, \quad B \ominus_h (-1)B, \quad (-1)A \ominus_h B, \quad A \ominus_h (-1)B, \quad (-1)A \ominus_h (-1)B$$

exist.
(ii) If A and B are not centrally symmetric and $A \ominus_h B$ exists, then $(-1)A \ominus_h (-1)B$ exists, whereas the other corresponding differences do not.
(iii) If $A \ominus_h B$ exists, then either both A and B are centrally symmetric or neither is centrally symmetric.
(iv) If $X + Y = B_1(0)$ for some $X, Y \in K_c(\mathbb{R}^n)$, then there exist $\mu, \lambda \geq 0$ and $z_1, z_2 \in \mathbb{R}^n$ such that $X = B_\mu(z_1)$, $Y = B_\lambda(z_2)$, $\mu + \lambda = 1$, and $z_1 + z_2 = 0$.
(v) $B_{r_1}(z_1) + B_{r_2}(z_2) = B_{r_1+r_2}(z_1 + z_2)$.
(vi) $\lambda B_r(0) = B_{|\lambda| r}(0)$.

Let $\mathcal{A} \in L(\mathbb{R}^n)$ denote a linear operator on $\mathbb{R}^n$. Its action can be naturally extended to the space $K_c(\mathbb{R}^n)$ by setting

$$\mathcal{A}X = \{\mathcal{A}x : \ x \in X\}, \qquad X \in K_c(\mathbb{R}^n). \tag{2.1}$$

Proposition 2.6. [22] *Let $F, G \in K_c(\mathbb{R}^n)$ and let $\mathcal{A}, \mathcal{B} \in L(\mathbb{R}^n)$. Then the following relations hold.*

(i) $\mathcal{A}(F + G) = \mathcal{A}F + \mathcal{A}G$;
(ii) $\mathcal{A}(\mathcal{B}F) = (\mathcal{A}\mathcal{B})F$.

Remark 2.7. In general, for $\mathcal{A}, \mathcal{B} \in L(\mathbb{R}^n)$ and $F \in K_c(\mathbb{R}^n)$, the equality

$$(\mathcal{A} + \mathcal{B})F = \mathcal{A}F + \mathcal{B}F$$

does not necessarily hold.

Remark 2.8. Alternatively, for $X \in K_c(\mathbb{R}^n)$ and an $n \times n$ matrix A, the action of A on X may be defined by

$$AX = \{Ax : \; x \in X\}.$$

In particular, if $X = B_1(0)$ is centrally symmetric, then $AB_1(0)$ is an r-dimensional ellipsoid whose semi-axes are determined by the singular values of A, where $r = \operatorname{rank}(A)$(see [8]). Moreover, if A is an orthogonal matrix, then $AB_1(0) = B_1(0)$.

Definition 2.9. [11] Let $X : [0,T] \to K_c(\mathbb{R}^n)$ be a set-valued mapping. The Hukuhara derivative $D_H X(t) \in K_c(\mathbb{R}^n)$ at $t \in (0,T)$ is said to exist if, for all sufficiently small $h > 0$, the corresponding Hukuhara differences exist and satisfy

$$\lim_{h \to 0^+} \frac{X(t+h) \ominus_h X(t)}{h} = \lim_{h \to 0^+} \frac{X(t) \ominus_h X(t-h)}{h} = D_H X(t).$$

Remark 2.10. If a set-valued mapping $X(\cdot)$ is Hukuhara differentiable on $[0,T]$, then the diameter function $\operatorname{diam}(X(\cdot))$ is non-decreasing on $[0,T]$. The converse implication does not hold in general.

To overcome the restrictions of the Hukuhara derivative, several generalized notions of differentiability for set-valued mappings have been introduced. Among the available approaches, the Bede–Gal derivative and the Plotnikov–Skripnik derivative provide a flexible and effective framework for the formulation of set-valued differential equations.

Definition 2.11. [31, 2] Let $X : [0,T] \to K_c(\mathbb{R}^n)$. The Bede–Gal derivative $D_{bg}X(t) \in K_c(\mathbb{R}^n)$ at $t \in (0,T)$ is defined if, for sufficiently small $h > 0$, the corresponding Hukuhara differences exist and at least one of the following conditions holds.

(i)
$$\lim_{h \to 0} \frac{X(t+h) \ominus_h X(t)}{h} = \lim_{h \to 0} \frac{X(t) \ominus_h X(t-h)}{h};$$

(ii)
$$\lim_{h \to 0} \frac{X(t) \ominus_h X(t+h)}{-h} = \lim_{h \to 0} \frac{X(t-h) \ominus_h X(t)}{-h};$$

(iii)
$$\lim_{h \to 0} \frac{X(t+h) \ominus_h X(t)}{h} = \lim_{h \to 0} \frac{X(t-h) \ominus_h X(t)}{-h};$$

(iv)
$$\lim_{h \to 0} \frac{X(t) \ominus_h X(t+h)}{-h} = \lim_{h \to 0} \frac{X(t) \ominus_h X(t-h)}{h}.$$

The common value of the above limits is denoted by $D_{bg}X(t)$.

Definition 2.12. [27, 17] Let $X : [0,T] \to K_c(\mathbb{R}^n)$. The Plotnikov–Skripnik derivative $D_{ps}X(t) \in K_c(\mathbb{R}^n)$ at $t \in (0,T)$ is defined if, for sufficiently small $h > 0$, the corresponding Hukuhara differences exist and at least one of the following conditions holds.

(i)
$$\lim_{h \to 0} \frac{X(t+h) \ominus_h X(t)}{h} = \lim_{h \to 0} \frac{X(t) \ominus_h X(t-h)}{h};$$

(ii)
$$\lim_{h \to 0} \frac{X(t) \ominus_h X(t+h)}{h} = \lim_{h \to 0} \frac{X(t-h) \ominus_h X(t)}{h};$$

(iii)
$$\lim_{h \to 0} \frac{X(t+h) \ominus_h X(t)}{h} = \lim_{h \to 0} \frac{X(t-h) \ominus_h X(t)}{h};$$

(iv)
$$\lim_{h\to 0}\frac{X(t)\ominus_h X(t+h)}{h}=\lim_{h\to 0}\frac{X(t)\ominus_h X(t-h)}{h}.$$
The common value of the above limits is denoted by $D_{ps}X(t)$.

Remark 2.13. [13] If a set-valued mapping $X(\cdot)$ is Hukuhara differentiable on $[0,T]$, then it is both Bede–Gal differentiable and Plotnikov–Skripnik differentiable on $[0,T]$, and
$$D_H X(t)=D_{bg}X(t)=D_{ps}X(t),\qquad t\in[0,T].$$
There exist set-valued mappings that are Bede–Gal differentiable and Plotnikov–Skripnik differentiable but not Hukuhara differentiable. Moreover, there exist mappings for which $D_{bg}X(t)\neq D_{ps}X(t)$ for all t.

Theorem 2.14. [11, 27, 1] *Suppose that $X:[0,T]\to K_c(\mathbb{R}^n)$ is a set-valued mapping for which one of the derivatives $D_HX(\cdot)$, $D_{bg}X(\cdot)$, or $D_{ps}X(\cdot)$ exists on $(0,T]$.*

(i) If $\operatorname{diam}(X(t))$ is non-decreasing on $[0,T]$, then
$$X(t)=X(0)+\int_0^t DX(s)\,ds,\qquad t\in[0,T],$$
where $DX(t)$ denotes any of the above derivatives.

(ii) If $\operatorname{diam}(X(t))$ is decreasing on $[0,T]$, then
$$X(t)=X(0)\ominus_h\int_0^t D_{ps}X(s)\,ds\quad \text{or}\quad X(t)=X(0)\ominus_h(-1)\int_0^t D_{bg}X(s)\,ds.$$

For completeness, we recall the following classical result for linear set-valued differential equations under the Hukuhara derivative.

Theorem 2.15. *Consider the linear set-valued differential equation*
$$D_HX(t)=\lambda(t)X(t)+F(t),\qquad X(0)=X_0,\tag{2.2}$$
where $\lambda:\mathbb{R}\to\mathbb{R}_+$ is summable, $F:\mathbb{R}\to K_c(\mathbb{R}^n)$ is measurable, and there exists a summable function $k:\mathbb{R}\to\mathbb{R}_+$ such that $h(F(t),0)\leq k(t)$. Then the mapping $X(\cdot)$ defined by
$$X(t)=e^{\int_0^t\lambda(s)\,ds}X_0+e^{\int_0^t\lambda(s)\,ds}\int_0^t e^{-\int_0^s\lambda(\tau)\,d\tau}F(s)\,ds$$
is a solution of (2.2).

For generalized derivatives such as the PS and BG derivatives, the corresponding initial value problems may exhibit substantially different solvability and solution behavior compared with the Hukuhara case, even for linear equations. In particular, obtaining explicit and geometrically interpretable solution representations often requires additional structural assumptions on matrix coefficients. These considerations motivate the assumptions adopted in the subsequent analysis, where we focus on non-homogeneous models with memory effects leading to set-valued Volterra-type integral equations.

3. Linear Non-Homogeneous Set-Valued Differential Equations

Consider the set-valued linear non-homogeneous differential equation with memory perturbation of the form
$$DX(t)=AX(t)+B_{r(t)}(0)+\int_0^t k(t-s)\,B_{q(s)}(0)\,ds,\qquad X(0)=B_1(0),\tag{3.1}$$
where $DX(t)$ denotes one of the generalized derivatives $D_HX(t)$, $D_{ps}X(t)$, or $D_{bg}X(t)$ of the set-valued mapping $X(\cdot):[0,T]\to K_c(\mathbb{R}^n)$, A is an $n\times n$ matrix, $r,q:[0,T]\to\mathbb{R}_+=[0,\infty)$ are

bounded continuous functions, and the kernel $k : [0,T] \to \mathbb{R}$ satisfies $k \in L^1(0,T)$ and $k(t) \geq 0$ almost everywhere on $[0,T]$.

Define

$$\rho(t) := r(t) + \int_0^t k(t-s)q(s)\,ds, \qquad t \in [0,T]. \tag{3.2}$$

Since $k(t-s) \geq 0$ and $q(s) \geq 0$, by Lemma 2.5(vi) we have

$$k(t-s)B_{q(s)}(0) = B_{k(t-s)q(s)}(0),$$

and hence

$$B_{r(t)}(0) + \int_0^t k(t-s)\,B_{q(s)}(0)\,ds = B_{\rho(t)}(0).$$

Therefore, equation (3.1) is equivalent to

$$DX(t) = AX(t) + B_{\rho(t)}(0), \qquad X(0) = B_1(0). \tag{3.3}$$

The set-valued integral equations locally equivalent to (3.3) depend on the type of generalized derivative involved. For $t \in (0,T)$, we distinguish the following cases.

(i) If $DX(t) = D_H X(t)$, then the corresponding SVIE is

$$X(t) = B_1(0) + A\int_0^t X(s)\,ds + B_{R(t)}(0). \tag{3.4}$$

(ii) If $DX(t) = D_{ps}X(t)$, corresponding to case (ii) in Definition 2.12, then the SVIE is

$$X(t) + A\int_0^t X(s)\,ds + B_{R(t)}(0) = B_1(0). \tag{3.5}$$

(iii) If $DX(t) = D_{bg}X(t)$, corresponding to case (ii) in Definition 2.11, then the SVIE is

$$X(t) + (-1)A\int_0^t X(s)\,ds + B_{R(t)}(0) = B_1(0). \tag{3.6}$$

Here,

$$B_{R(t)}(0) = \{x \in \mathbb{R}^n :\ \|x\| \leq R(t)\}, \qquad R(t) = \int_0^t \rho(s)\,ds.$$

In all three cases, the integral is understood in the sense of Hukuhara.

Definition 3.1. A set-valued mapping $X : [0,T] \to K_c(\mathbb{R}^n)$ is called a solution of the integral equation (3.4), (3.5), or (3.6) if it is continuous and satisfies the corresponding equation on $[0,t_0]$, for some $t_0 < T$, depending on the type of derivative and the evolution of the solution diameter.

We also consider the set-valued linear non-homogeneous differential equation with memory perturbation of the form

$$DX(t) = (-1)AX(t) + B_{r(t)}(0) + \int_0^t k(t-s)\,B_{q(s)}(0)\,ds, \qquad X(0) = B_1(0), \tag{3.7}$$

where $DX(t)$ denotes one of the generalized derivatives $D_H X(t)$, $D_{ps}X(t)$, or $D_{bg}X(t)$.

Define

$$\rho(t) := r(t) + \int_0^t k(t-s)q(s)\,ds, \qquad t \in [0,T]. \tag{3.8}$$

Since $k(t-s) \geq 0$ and $q(s) \geq 0$, we obtain

$$B_{r(t)}(0) + \int_0^t k(t-s)\,B_{q(s)}(0)\,ds = B_{\rho(t)}(0),$$

and hence equation (3.7) is equivalent to

$$DX(t) = (-1)AX(t) + B_{\rho(t)}(0), \qquad X(0) = B_1(0). \tag{3.9}$$

As in the previous case, the set-valued integral equations locally equivalent to (3.9) depend on the type of generalized derivative involved. For $t \in (0, T)$, we distinguish the following cases.

(i) If $DX(t) = D_H X(t)$, then

$$X(t) = B_1(0) + (-1)A \int_0^t X(s)\, ds + B_{R(t)}(0), \tag{3.10}$$

where

$$R(t) = \int_0^t \rho(s)\, ds.$$

(ii) If $DX(t) = D_{ps}X(t)$, corresponding to case (ii) in Definition 2.12, then

$$X(t) + (-1)A \int_0^t X(s)\, ds + B_{R(t)}(0) = B_1(0). \tag{3.11}$$

(iii) If $DX(t) = D_{bg}X(t)$, corresponding to case (ii) in Definition 2.11, then

$$X(t) + A \int_0^t X(s)\, ds + B_{R(t)}(0) = B_1(0). \tag{3.12}$$

In all three cases, the integral is understood in the sense of Hukuhara.

Remark 3.2. Since the initial set $X(0) = B_1(0)$ and the forcing term

$$B_{r(t)}(0) + \int_0^t k(t-s) B_{q(s)}(0)\, ds$$

are centrally symmetric, the solutions of the integral equations associated with the PS-derivative coincide with those associated with the BG-derivative in the corresponding paired cases, whenever both are well defined. Although the corresponding integral equations may admit different formal expressions, in the ball-valued setting they reduce to the same scalar radius equation.

Theorem 3.3. *Assume that $\rho : [0, T] \to \mathbb{R}_+$ is bounded and continuous, and let*

$$R(t) = \int_0^t \rho(s)\, ds.$$

Define $X : [0, T] \to K_c(\mathbb{R}^n)$ by

$$X(t) = \sum_{m=0}^{\infty} \frac{t^m}{m!} A^m B_1(0) + \sum_{m=0}^{\infty} \frac{1}{m!} \left(\int_0^t (t-s)^m \rho(s)\, ds \right) A^m B_1(0). \tag{3.13}$$

Then $X(t)$ satisfies the integral equation (3.4) on $[0, T]$.

Proof. Since ρ is bounded on $[0, T]$ and $A \in \mathbb{R}^{n\times n}$, the matrix power series $\sum_{m=0}^{\infty} \frac{t^m}{m!} A^m$ converges absolutely and uniformly on $[0, T]$. Moreover, for all $t \in [0, T]$,

$$\int_0^t (t-s)^m \rho(s)\, ds \le \|\rho\|_\infty \int_0^t (t-s)^m\, ds = \frac{\|\rho\|_\infty}{m+1} t^{m+1}.$$

Hence, the second series in (3.13) also converges uniformly on $[0, T]$.

In particular, $X(t)$ is well defined in $K_c(\mathbb{R}^n)$ for all $t \in [0, T]$, and the integrals below are understood in the sense of Hukuhara and may be computed term by term.

Step 1: Homogeneous part. Set

$$X_1(t) := \sum_{m=0}^{\infty} \frac{t^m}{m!} A^m B_1(0).$$

Then

$$\int_0^t X_1(s)\, ds = \sum_{m=0}^{\infty} \frac{t^{m+1}}{(m+1)!} A^m B_1(0),$$

and hence

$$A \int_0^t X_1(s)\, ds = \sum_{m=1}^{\infty} \frac{t^m}{m!} A^m B_1(0) = X_1(t) - B_1(0).$$

Step 2: Forcing part. Set

$$X_2(t) := \sum_{m=0}^{\infty} \frac{1}{m!} \left(\int_0^t (t-s)^m \rho(s)\, ds \right) A^m B_1(0).$$

Then

$$\int_0^t X_2(\tau)\, d\tau = \sum_{m=0}^{\infty} \frac{1}{m!} \int_0^t \left(\int_0^\tau (\tau - s)^m \rho(s)\, ds \right) d\tau\, A^m B_1(0).$$

By Fubini's theorem,

$$\int_0^t \left(\int_0^\tau (\tau - s)^m \rho(s)\, ds \right) d\tau = \int_0^t \frac{(t-s)^{m+1}}{m+1} \rho(s)\, ds.$$

Therefore,

$$A \int_0^t X_2(\tau)\, d\tau = X_2(t) - \left(\int_0^t \rho(s)\, ds \right) B_1(0).$$

Since $R(t) \geq 0$, by Lemma 2.5(vi) we have

$$R(t) B_1(0) = B_{R(t)}(0).$$

Thus,

$$X_2(t) = A \int_0^t X_2(\tau)\, d\tau + B_{R(t)}(0).$$

Step 3: Combination. Let $X(t) = X_1(t) + X_2(t)$. Using the identities obtained in Steps 1 and 2, we obtain

$$B_1(0) + A \int_0^t X(\tau)\, d\tau + B_{R(t)}(0) = X(t),$$

which shows that $X(t)$ satisfies (3.4). □

Assume that $A \in \mathbb{R}^{n \times n}$ is symmetric. By the spectral theorem, there exist an orthogonal matrix Q and a diagonal matrix

$$\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$$

such that

$$A = Q \Lambda Q^T.$$

Set

$$|\Lambda| = \mathrm{diag}(|\lambda_1|, \ldots, |\lambda_n|), \qquad S = \mathrm{diag}(\mathrm{sgn}(\lambda_1), \ldots, \mathrm{sgn}(\lambda_n)).$$

For $k \geq 0$, define $D_k := S^k$. Then

$$\Lambda^k = |\Lambda|^k D_k,$$

and hence

$$A^k = Q \Lambda^k Q^T = Q |\Lambda|^k D_k Q^T, \qquad k \geq 0.$$

Moreover, since Q is orthogonal, we have $Q^T B_1(0) = B_1(0)$.

Theorem 3.4. *Assume that $A \in \mathbb{R}^{n\times n}$ is symmetric and nondegenerate, and let $\rho : [0,T] \to \mathbb{R}_+$ be bounded and continuous. Let $X(t)$ be defined by the series representation (3.13). Then $X(t)$ admits the exponential representation*

$$X(t) = Q\left(e^{t|\Lambda|}B_1(0) + \int_0^t e^{(t-s)|\Lambda|}\rho(s)\,ds\,B_1(0)\right). \tag{3.14}$$

Proof. Since A is symmetric and nondegenerate, we have $A = Q\Lambda Q^T$ with Q orthogonal and Λ diagonal with no zero entries. Using the identity $\Lambda^m = |\Lambda|^m D_m$, it follows that

$$A^m = Q\Lambda^m Q^T = Q|\Lambda|^m D_m Q^T, \qquad m \geq 0.$$

Because A is nondegenerate, the matrix D_m is orthogonal, and hence

$$D_m B_1(0) = B_1(0).$$

Moreover, since Q is orthogonal, we also have $Q^T B_1(0) = B_1(0)$. Therefore,

$$A^m B_1(0) = Q|\Lambda|^m D_m Q^T B_1(0) = Q|\Lambda|^m B_1(0), \qquad m \geq 0.$$

Substituting this identity into the series representation (3.13) and factoring out the orthogonal matrix Q yields (3.14). □

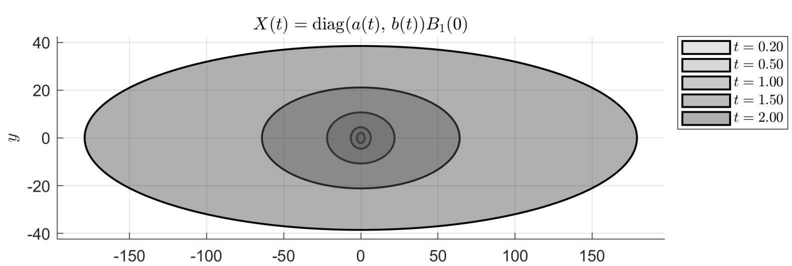

FIGURE 1. Evolution of the solution set $X(t) = \mathrm{diag}(a(t), b(t))B_1(0)$ in Example 3.5, visualized as a family of filled ellipses for several time instants t

Example 3.5. Consider the set-valued differential equation

$$DX(t) = AX(t) + B_{r(t)}(0) + \int_0^t k(t-s)B_{q(s)}(0)\,ds, \qquad X(0) = B_1(0).$$

Let

$$A = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}, \qquad r(t) \equiv 4, \qquad q(t) \equiv 2, \qquad k(t) = e^{-t}.$$

Then

$$\rho(t) = 4 + 2(1 - e^{-t}) = 6 - 2e^{-t}, \qquad R(t) = \int_0^t \rho(s)\,ds = 6t - 2(1 - e^{-t}).$$

Since A is symmetric and nondegenerate, we may take

$$Q = I, \qquad \Lambda = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}, \qquad |\Lambda| = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

Consequently,

$$e^{t|\Lambda|} = \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{t} \end{pmatrix}.$$

By Theorem 3.4, the solution admits the representation

$$X(t) = \left(e^{t|\Lambda|} + \int_0^t e^{(t-s)|\Lambda|} \rho(s)\, ds \right) B_1(0). \tag{3.15}$$

A direct computation yields

$$\int_0^t e^{(t-s)|\Lambda|} \rho(s)\, ds = \begin{pmatrix} 3(e^{2t}-1) - \frac{2}{3}(e^{2t} - e^{-t}) & 0 \\ 0 & 6(e^{t}-1) - (e^{t} - e^{-t}) \end{pmatrix}.$$

Therefore,

$$X(t) = \begin{pmatrix} \dfrac{10}{3} e^{2t} + \dfrac{2}{3} e^{-t} - 3 & 0 \\ 0 & 6e^{t} + e^{-t} - 6 \end{pmatrix} B_1(0).$$

The time evolution of the solution set $X(t)$ is illustrated in Fig. 1, where the expanding family of filled ellipses highlights the anisotropic growth induced by the spectrum of A together with the cumulative effect of the memory term.

Remark 3.6. If $A \in \mathbb{R}^{n\times n}$ is symmetric positive definite, then all eigenvalues of A are positive. Consequently, for the spectral decomposition $A = Q\Lambda Q^T$ one has

$$|\Lambda| = \Lambda \qquad \text{and} \qquad D_k = I \quad \text{for all } k \geq 0.$$

Therefore, the exponential representation (3.14) reduces to

$$X(t) = Q \left(e^{t\Lambda} B_1(0) + \int_0^t e^{(t-s)\Lambda} \rho(s)\, ds\, B_1(0) \right),$$

which provides a simpler expression for the solution of (3.4) in the positive-definite case.

As a particular structural case of the previous results, we now consider the situation where all singular values of the matrix A coincide.

Let $A = U\Sigma V^T$ be a singular value decomposition of A, where U and V are orthogonal matrices and $\Sigma = \mathrm{diag}(\sigma_1, \dots, \sigma_n)$ with $\sigma_i \geq 0$. Since orthogonal transformations preserve the Euclidean norm, we have

$$UB_r(0) = B_r(0), \qquad V^T B_r(0) = B_r(0), \qquad r \geq 0.$$

In particular, if all singular values of A are equal, that is, $\sigma_1 = \cdots = \sigma_n = \sigma > 0$, then $\Sigma = \sigma I$, and hence

$$AB_r(0) = U(\sigma I)V^T B_r(0) = \sigma\, U\big(V^T B_r(0)\big) = \sigma B_r(0), \qquad r \geq 0.$$

Theorem 3.7. *Assume that all singular values of $A \in \mathbb{R}^{n\times n}$ are equal to $\sigma > 0$. Let $\rho : [0,T] \to \mathbb{R}_+$ be bounded and continuous. Define $X : [0,T] \to K_c(\mathbb{R}^n)$ by*

$$X(t) = \big(e^{\sigma t} + J(t)\big) B_1(0), \qquad J(t) := \int_0^t e^{\sigma(t-s)} \rho(s)\, ds. \tag{3.16}$$

Then $X(t)$ satisfies the integral equation (3.4) on $[0,T]$.

Proof. Since all singular values of A are equal to $\sigma > 0$, it follows that

$$AB_r(0) = \sigma B_r(0), \qquad r \geq 0.$$

We now verify that the function

$$X(t) = \big(e^{\sigma t} + J(t)\big)B_1(0), \qquad J(t) := \int_0^t e^{\sigma(t-s)}\rho(s)\,ds,$$

satisfies the integral equation (3.4).

For $\alpha, \beta \geq 0$, by Lemma 2.5(v),

$$\alpha B_1(0) + \beta B_1(0) = (\alpha + \beta)B_1(0).$$

Moreover,

$$\int_0^t c(s)B_1(0)\,ds = \left(\int_0^t c(s)\,ds\right) B_1(0).$$

Since $c(\cdot)$ is nonnegative and continuous, we obtain

$$\int_0^t X(s)\,ds = \left(\int_0^t e^{\sigma s}\,ds + \int_0^t J(s)\,ds\right) B_1(0).$$

Therefore,

$$A\int_0^t X(s)\,ds = \sigma\left(\int_0^t e^{\sigma s}\,ds + \int_0^t J(s)\,ds\right) B_1(0).$$

A direct computation yields

$$\sigma\int_0^t e^{\sigma s}\,ds = e^{\sigma t} - 1,$$

and, by Fubini's theorem,

$$\int_0^t J(s)\,ds = \int_0^t \left(\int_0^s e^{\sigma(s-u)}\rho(u)\,du\right) ds = \frac{1}{\sigma}J(t) - \frac{1}{\sigma}R(t).$$

Consequently,

$$A\int_0^t X(s)\,ds = \big(e^{\sigma t} - 1 + J(t) - R(t)\big)B_1(0).$$

Finally, since $B_{R(t)}(0) = R(t)B_1(0)$, we obtain

$$B_1(0) + A\int_0^t X(s)\,ds + B_{R(t)}(0) = \big(e^{\sigma t} + J(t)\big)B_1(0) = X(t),$$

which proves that $X(t)$ satisfies (3.4). □

Remark 3.8. A fundamental difference between the Hukuhara derivative and the PS/BG derivatives lies in the evolution of the solution diameter. In the Hukuhara case, the diameter of a solution is monotone non-decreasing, and thus ball-valued solutions do not break down due to a loss of nonnegativity of the radius. In contrast, PS and BG derivatives allow decreasing diameters, and ball-valued solutions may cease to exist in finite time when the associated radius reaches zero. As a consequence, solutions in the PS/BG setting are naturally considered on a finite interval $[0, t_0]$, on which the radius function remains nonnegative.

Theorem 3.9. *Assume that all singular values of $A \in \mathbb{R}^{n\times n}$ are equal to $\sigma > 0$. Let $\rho : [0,T] \to \mathbb{R}_+$ be bounded and continuous, and set*

$$R(t) = \int_0^t \rho(s)\,ds, \qquad M := \|\rho\|_\infty = \sup_{t\in[0,T]} \rho(t).$$

Define $u : [0,T] \to \mathbb{R}$ by

$$u(t) = e^{-\sigma t} - \int_0^t e^{-\sigma(t-s)}\rho(s)\,ds, \tag{3.17}$$

and let

$$t_0 = \frac{1}{\sigma} \ln\Big(1 + \frac{\sigma}{M}\Big). \tag{3.18}$$

Define $X : [0, t_0] \to K_c(\mathbb{R}^n)$ *by* $X(t) = u(t)B_1(0)$. *Then* $X(t)$ *satisfies the integral equation* (3.5) *on* $[0, t_0]$.

Proof. Since all singular values of A are equal to $\sigma > 0$, we have

$$AB_r(0) = \sigma B_r(0) = B_{\sigma r}(0), \qquad r \geq 0.$$

We show that $X(t) = u(t)B_1(0)$ with $u(t)$ given by (3.17) satisfies (3.5) on $[0, t_0]$. First observe that u is continuously differentiable and solves the scalar linear ODE

$$u'(t) = -\sigma u(t) - \rho(t), \qquad u(0) = 1,$$

whose unique solution is exactly (3.17).

Step 1: Nonnegativity of $u(t)$ on $[0, t_0]$.

Since $0 \leq \rho(t) \leq M$, from (3.17) we have, for all $t \in [0, T]$,

$$u(t) \geq e^{-\sigma t} - M \int_0^t e^{-\sigma(t-s)}\, ds = e^{-\sigma t} - \frac{M}{\sigma}\big(1 - e^{-\sigma t}\big).$$

Thus $u(t) \geq 0$ whenever $e^{-\sigma t} \geq \frac{M}{\sigma + M}$, which is equivalent to

$$t \leq \frac{1}{\sigma} \ln\Big(1 + \frac{\sigma}{M}\Big) = t_0.$$

Hence $u(t) \geq 0$ on $[0, t_0]$, and therefore $X(t) = u(t)B_1(0) = B_{u(t)}(0)$ is well defined on $[0, t_0]$.

Step 2: Verification of (3.5).

Define

$$\Phi(t) := \int_0^t u(s)\, ds, \qquad t \in [0, t_0],$$

and set

$$f(t) := u(t) + \sigma\Phi(t) + R(t).$$

Then

$$f'(t) = u'(t) + \sigma u(t) + \rho(t) = 0.$$

Since $f(0) = u(0) = 1$, we obtain

$$u(t) + \sigma \int_0^t u(s)\, ds + R(t) = 1, \qquad t \in [0, t_0].$$

Substituting $X(t) = u(t)B_1(0)$ into the left-hand side of (3.5), we obtain

$$\int_0^t X(s)\, ds = \Phi(t)B_1(0),$$

and, since $\Phi(t) \geq 0$ on $[0, t_0]$,

$$A \int_0^t X(s)\, ds = \sigma\Phi(t)B_1(0) = B_{\sigma\Phi(t)}(0).$$

Therefore, by Lemma 2.5(v),

$$X(t) + A \int_0^t X(s)\, ds + B_{R(t)}(0) = B_{u(t)+\sigma\Phi(t)+R(t)}(0).$$

Using $u(t) + \sigma\Phi(t) + R(t) = 1$, we conclude that

$$X(t) + A \int_0^t X(s)\, ds + B_{R(t)}(0) = B_1(0), \qquad t \in [0, t_0],$$

that is, $X(t)$ satisfies (3.5) on $[0, t_0]$. □

Remark 3.10. The bound (3.18) is not optimal. Any $t_0 \in (0, T]$ such that $u(t) \geq 0$ on $[0, t_0]$ yields a valid existence interval. In particular, a sufficient condition is

$$u(t_0) \geq 0, \quad \text{i.e.,} \quad e^{-\sigma t_0} \geq \int_0^{t_0} e^{-\sigma(t_0 - s)} \rho(s)\, ds.$$

Remark 3.11. Since $B_1(0)$ and $B_{R(t)}(0)$ are centrally symmetric, we have $B_{R(t)}(0) = -B_{R(t)}(0)$. Hence, in the ball-valued setting considered above, the PS-type integral equation (3.5) and the BG-type integral equation (3.6) admit the same solution on $[0, t_0]$.

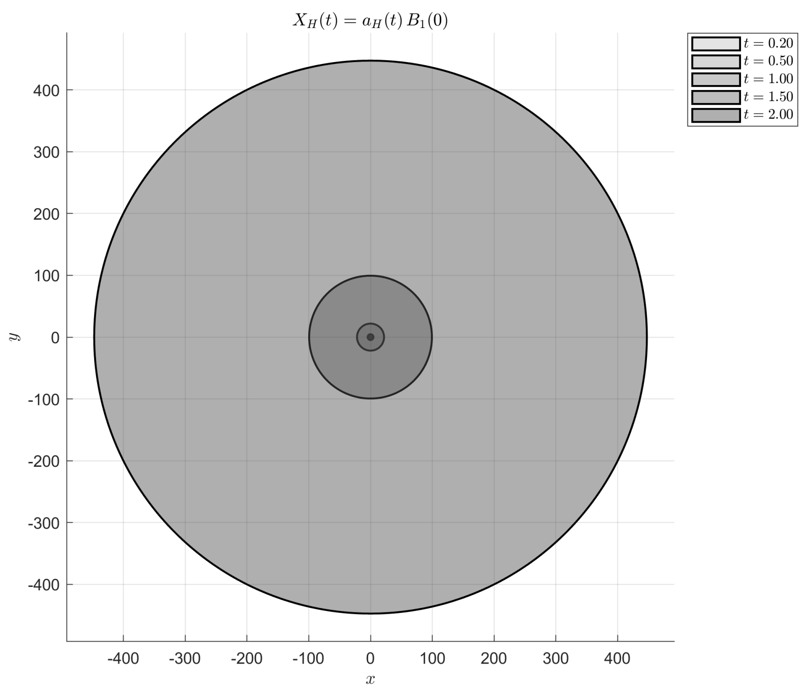

FIGURE 2. Hukuhara case in Example 3.12: evolution of $X_H(t) = a_H(t)B_1(0)$ for several time instants

Example 3.12. Let

$$A = \begin{pmatrix} \sqrt{5} & 2 \\ -2 & \sqrt{5} \end{pmatrix}.$$

A direct computation shows that all singular values of A are equal to $\sigma = 3$. In this example we take

$$\rho(t) = t, \qquad R(t) = \int_0^t \rho(s)\, ds = \frac{t^2}{2}.$$

(i) Hukuhara case. For the Hukuhara-type SVIE (3.4), Theorem 3.7 yields

$$X_H(t) = \left(e^{3t} + \int_0^t e^{3(t-s)}\, s\, ds \right) B_1(0), \qquad t \geq 0.$$

Moreover,

$$\int_0^t e^{3(t-s)}\, s\, ds = e^{3t} \int_0^t s e^{-3s}\, ds = \frac{1}{9} e^{3t} - \frac{t}{3} - \frac{1}{9}.$$

Hence,

$$X_H(t) = \left(\frac{10}{9}e^{3t} - \frac{t}{3} - \frac{1}{9}\right) B_1(0), \qquad t \geq 0. \tag{3.19}$$

In particular, $\mathrm{diam}(X_H(t))$ is increasing on $[0, +\infty)$. The corresponding evolution of the solution sets is illustrated in Fig. 2.

(ii) PS and BG cases. For the PS-type SVIE (3.5), Theorem 3.9 gives

$$X_{PS}(t) = u(t)B_1(0), \qquad u(t) = e^{-3t} - \int_0^t e^{-3(t-s)}\, s\, ds, \quad 0 \leq t \leq t_0,$$

where $t_0 > 0$ is chosen so that $u(t) \geq 0$ on $[0, t_0]$. A direct computation yields

$$\int_0^t e^{-3(t-s)}\, s\, ds = e^{-3t} \int_0^t se^{3s}\, ds = \frac{t}{3} - \frac{1}{9} + \frac{1}{9}e^{-3t}.$$

Therefore,

$$X_{PS}(t) = \left(\frac{8}{9}e^{-3t} - \frac{t}{3} + \frac{1}{9}\right) B_1(0), \qquad 0 \leq t \leq t_0, \tag{3.20}$$

where t_0 is the unique solution of

$$\frac{8}{9}e^{-3t} - \frac{t}{3} + \frac{1}{9} = 0 \quad \Longleftrightarrow \quad 8e^{-3t} - 3t + 1 = 0.$$

Moreover, $\mathrm{diam}(X_{PS}(t))$ is decreasing on $[0, t_0]$. Finally, by Remark 3.11, the BG-type SVIE (3.6) admits the same solution on $[0, t_0]$. The time evolution of the PS/BG solutions is shown in Fig. 3.

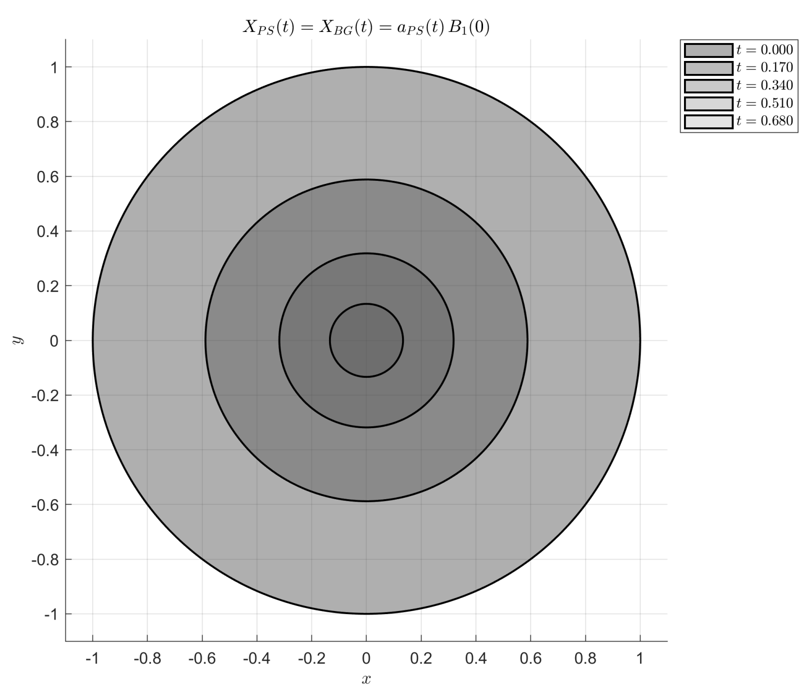

FIGURE 3. PS/BG case in Example 3.12: evolution of $X_{PS}(t) = X_{BG}(t) = a_{PS}(t)B_1(0)$ for $t \in [0, t_0]$, where $t_0 \approx 0.680$

4. Conclusions and Future Works

In this paper, we have investigated linear non-homogeneous set-valued differential equations in $\mathbb{R}^n$ with memory effects under generalized derivatives. By introducing a convolution-type radius function, the original problems were reformulated as memory-perturbed set-valued Volterra integral equations.

Under suitable structural assumptions on the coefficient matrix, we showed that solution representations known from the instantaneous case can be extended to the memory-dependent setting. Our analysis demonstrates that the qualitative distinction between different notions of generalized differentiability persists in the presence of memory effects: solutions associated with the Hukuhara derivative admit representations with monotone non-decreasing diameters, whereas solutions corresponding to the PS and BG derivatives are, in general, only locally defined and exhibit monotone decreasing diameters.

Possible directions for future research include the study of more general memory kernels and the investigation of set-valued differential equations with memory under weaker structural assumptions on the coefficient matrix. Another natural extension concerns fractional-order or impulsive set-valued systems with memory effects, which may provide further insight into the interplay between generalized differentiability and hereditary dynamics.

Statements and Declarations

The author declares that there is no conflict of interest, and the manuscript has no associated data.

References

[1] S. E. Amrahov, A. Khastan, N. A. Gasilov, and A. G. Fatullayev. Relationship between Bede–Gal differentiable set-valued functions and their associated support functions. *Fuzzy Sets and Systems*, 295:57–71, 2016.

[2] B. Bede and S. G. Gal. Almost periodic fuzzy-number-valued functions. *Fuzzy Sets and Systems*, 147(3):385–403, 2004.

[3] B. Bede and S. G. Gal. Generalizations of the differentiability of fuzzy-number-valued functions with applications to fuzzy differential equations. *Fuzzy Sets and Systems*, 151(3):581–599, 2005.

[4] V. G. Boltyanski and J. J. Castro. Centrally symmetric convex sets. *Journal of Convex Analysis*, 14(2):345–351, 2007.

[5] T. Bridgland. Trajectory integrals of set-valued functions. *Pacific Journal of Mathematics*, 33(1):43–68, 1970.

[6] Y. Chalco-Cano, H. Román-Flores, and M.-D. Jiménez-Gamero. Generalized derivative and π-derivative for set-valued functions. *Information Sciences*, 181(11):2177–2188, 2011.

[7] F. de Blasi and F. Iervolino. Equazioni differenziali con soluzioni a valore compatto convesso. *Bollettino dell'Unione Matematica Italiana*, 2(4-5):491–501, 1969.

[8] G. E. Forsythe and C. B. Moler. *Computer Solution of Linear Algebraic Systems*. Prentice-Hall, Englewood Cliffs, 1967.

[9] N. A. Gasilov. Solving a system of linear differential equations with interval coefficients. *Discrete and Continuous Dynamical Systems - Series B*, 26(5):2739–2747, 2021.

[10] L. T. Gomes, L. C. de Barros, and B. Bede. *Fuzzy Differential Equations in Various Approaches*. Springer, Berlin, 2015.

[11] M. Hukuhara. Integration des applications mesurables dont la valeur est un compact convexe. *Funkcialaj Ekvacioj*, 10(3):205–223, 1967.

[12] T. A. Komleva and A. V. Plotnikov. Differential inclusions with hukuhara derivative. *Nonlinear Oscillations*, 10(2):229–245, 2007.

[13] T. A. Komleva, A. V. Plotnikov, L. I. Plotnikova, and N. V. Skripnik. Conditions for the existence of basic solutions of linear multivalued differential equations. *Ukrainian Mathematical Journal*, 73(5):758–783, 2021.

[14] T. A. Komleva, A. V. Plotnikov, and N. V. Skripnik. Differential equations with set-valued solutions. *Ukrainian Mathematical Journal*, 60(10):1540–1556, 2008.

[15] T. A. Komleva, A. V. Plotnikov, and N. V. Skripnik. Existence of solutions of linear set-valued integral equations and their properties. *Journal of Mathematical Sciences*, 277(2):268–280, 2023.

[16] T. A. Komleva, A. V. Plotnikov, and N. V. Skripnik. Some properties of the solutions of linear set-valued differential equations in the space $\mathrm{conv}(\mathbb{R}^2)$. *Journal of Mathematical Sciences*, 279(3):365–377, 2024.

[17] T. A. Komleva, L. I. Plotnikova, N. V. Skripnik, and A. V. Plotnikov. Some remarks on linear set-valued differential equations. *Studia Universitatis Babeș-Bolyai Mathematica*, 65(3):411–427, 2020.

[18] V. Lakshmikantham, T. G. Bhaskar, and J. V. Devi. *Theory of Set Differential Equations in Metric Spaces*. Cambridge Scientific Publishers, Cambridge, 2006.

[19] V. Lakshmikantham and R. N. Mohapatra. *Theory of Fuzzy Differential Equations and Inclusions*. Taylor & Francis Ltd., London, 2003.

[20] M. T. Malinowski. On set differential equations in banach spaces–a second type hukuhara differentiability approach. *Applied Mathematics and Computation*, 219(1):289–305, 2012.

[21] A. A. Martynyuk. *Qualitative Analysis of Set-Valued Differential Equations*. Springer, Cham, 2019.

[22] N. A. Perestyuk, V. A. Plotnikov, A. M. Samoilenko, and N. V. Skripnik. *Differential Equations with Impulse Effects: Multivalued Right-Hand Sides with Discontinuities*, volume 40. Walter de Gruyter, Berlin, 2011.

[23] N. A. Perestyuk and N. V. Skripnik. Averaging of set-valued impulsive systems. *Ukrainian Mathematical Journal*, 65(1):140–157, 2013.

[24] A. V. Plotnikov, T. A. Komleva, and I. V. Molchanyuk. Existence and uniqueness theorem for set-valued volterra–hammerstein integral equations. *Asian-European Journal of Mathematics*, 11(3):Article ID 1850036, 2018.

[25] A. V. Plotnikov, T. A. Komleva, and N. V. Skripnik. Existence of basic solutions of first-order linear homogeneous set-valued differential equations. *Matematychni Studii*, 61(1):61–78, 2024.

[26] A. V. Plotnikov and N. V. Skripnik. *Differential Equations with "Clear" and Fuzzy Multivalued Right-Hand Side*. AstroPrint, Odessa, 2009.

[27] A. V. Plotnikov and N. V. Skripnik. Set-valued differential equations with generalized derivative. *Journal of Advanced Research in Pure Mathematics*, 3:144–160, 2011.

[28] A. V. Plotnikov and N. V. Skripnik. Existence and uniqueness theorems for generalized set differential equations. *International Journal of Control Science and Engineering*, 2(1):1–6, 2012.

[29] A. V. Plotnikov and N. V. Skripnik. An existence and uniqueness theorem to the cauchy problem for generalised set differential equations. *Dynamic Systems and Applications*, 20(4):433–445, 2013.

[30] A. V. Plotnikov and N. V. Skripnik. Conditions for the existence of local solutions of set-valued differential equations with generalized derivative. *Ukrainian Mathematical Journal*, 65(10):1498–1513, 2014.

[31] L. Stefanini and B. Bede. Generalized hukuhara differentiability of interval-valued functions and interval differential equations. *Nonlinear Analysis: Theory, Methods & Applications*, 71(3–4):1311–1328, 2009.