APPLICABLE NONLINEAR ANALYSIS
Volume 3 (2026), No. 3, 164–175
DOI: https://doi.org/10.69829/apna-026-0303-ta02
Tulipa Opera Scholarum

FIXED POINT THEOREMS FOR GENERALIZED (α, β)-S-NONEXPANSIVE MAPPINGS WITHOUT DEMI-CLOSEDNESS PRINCIPLE IN BANACH SPACES

ESKANDAR NARAGHIRAD[1,*]

[1] *Department of Mathematics, Faculty of Sciences, Yasouj University, Yasouj 75918, Iran*

ABSTRACT. We introduce a new class of nonlinear mappings called generalized (α, β)-S-nonexpansive mappings and show that this class of nonlinear mappings can be considered as an extension of of nonexpansive, firmly-nonexpansive and (α, β)-nonexpansive mappings. Beyond the continuity assumption on the mappings and without using the classical demiclosedness principle, we will verify new existence and convergence results for these mappings in Banach spaces. To illustrate the results, we include some examples in the paper.

Keywords. Nonlinear mapping, Condition (C), Generalized (α, β)-S-nonexpansive mapping, Banach space, Opial property.

1. INTRODUCTION

In the whole article $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{R}$ the set of real numbers. We suppose a Banach space $(E, \|\cdot\|)$ has a dual space E^* and we indicate the duality pairing by $\langle u, u^* \rangle$, $\forall u \in E$ and $u^* \in E^*$. The strong convergence and the weak convergence of a sequence $\{u_n\}_{n\in\mathbb{N}} \subset E$ to $u \in E$ are indicated by $u_n \to u$ and $u_n \rightharpoonup u$, respectively. Let D be a subset of E and $T : D \longrightarrow D$ an operator. An element $v \in E$ is called a fixed point of T if $T(v) = v$ and we set $F(T) = \{v \in E : T(v) = v\}$. The operator $T : D \longrightarrow D$ is called a nonexpansive if $\|T(u) - T(v)\| \leq \|u - v\|$ for all $u, v \in D$ and quasinonexpansive [5], if $\|T(u) - v\| \leq \|u - v\|$ for all $u \in D$ and $v \in F(T)$. The first existence results of fixed points of these mappings were studied by Browder [3, 4], Gӧhde [9] and Kirk [12] in Hilbert and Banach spaces (see also [8] for detailed exposition). Many researchers have extended and generalized these kind of mappings in different ways. In 2008, Suzuki [25] introduced (as generalization) a class of operators known as mappings satisfying Condition (C) and investigated their fixed point results.

We adopt the following notations in the whole paper. Let Y be a nonempty subset of a Banach space E and $\mathcal{P}(Y)$ be the set of all nonempty subsets of Y. We denote respectively by

- $\mathcal{N}(Y) := \{W \in \mathcal{P}(Y) : W \text{ is nonempty}\}$,
- $\mathcal{CV}(Y) := \{W \in \mathcal{P}(Y) : W \text{ is nonempty, convex}\}$,
- $\mathcal{C}(Y) := \{W \in \mathcal{P}(Y) : W \text{ is nonempty, closed}\}$,
- $\mathcal{K}(Y) := \{W \in \mathcal{P}(Y) : W \text{ is nonempty, compact}\}$,
- $\mathcal{C}b(Y) := \{W \in \mathcal{P}(Y) : W \text{ is nonempty, closed and bounded}\}$,
- $\mathcal{C}v(Y) := \{W \in \mathcal{P}(Y) : W \text{ is nonempty, closed and convex}\}$.

Definition 1.1. [25] Let D be a subset of a Banach space E. A mapping $T : D \longrightarrow D$ is said to satisfy Condition (C) if for all $u, v \in D$, the condition $\frac{1}{2}\|u - T(u)\| \leq \|u - v\|$ implies $\|T(u) - T(v)\| \leq \|u - v\|$.

---

[*]Corresponding author.
E-mail address: eskandarrad@gmail.com, esnaraghirad@yu.ac.ir (E. Naraghirad)
2020 Mathematics Subject Classification: 47H10, 54H25.
Accepted: March 30, 2026.

In order to extend the Suzuki type operators, Aoyama and Kohsaka [1] introduced another class of (not necessarily continuous) operators namely α-nonexpansive mappings and investigated some fixed point results related to these mappings.

Definition 1.2. [1] Let E be a Banach space and $D \subset E$ be a nonempty set. A mapping $T : D \longrightarrow D$ is called α-nonexpansive if for any $z, w \in D$ and some $\alpha < 1$

$$\|T(z) - T(w)\|^2 \leq \alpha\|T(z) - w\|^2 + \alpha\|z - T(w)\|^2 + (1 - \alpha)\|z - w\|^2.$$

In order to prove weak and strong convergence theorems for single-valued and multivalued nonlinear mappings, the concept of demiclosedness principle was introduced by Opial [19]. It is remarkable to mention that the researchers have been deeply investigated the fixed point theorems by employing the demiclosedness principle during past forty (40) years or so (see, e.g, [15] and the references therein). Although some interesting results have been obtained, it is worth mentioning that, in some cases, the mapping T of the class of nonexpansive mappings defined in the setting of a Hilbert space H does not necessarily satisfy the demiclosedness principle due to Opial [19] (see [15], Example 2.1 for details). Consequently, it is natural to ask:
Question 1: Is there any way to obtain strong convergence theorems of Halpern's type for such mappings that fail to satisfy the original demiclosedness principle due to Opial in the setting of Banach spaces?

Naraghirad [15] provided an affirmative answer to the above question by introducing the idea of jointly demiclosedness principle (Recall that if $C \in \mathcal{N}(E)$, then a pair $S, T : C \to C$ satisfies jointly demiclosedness principle if $\{x_n\}_{n\in\mathbb{N}} \subset C$ converges weakly to a point $z \in C$ and $\lim_{n\to\infty} \|S(x_n) - T(x_n)\| = 0$, then $Sz = z$ and $Tz = z$; that is, $S - T$ is jointly demiclosed at zero). More precisely, the author proved the following theorem:

Theorem 1.3. [15] *Let E be a Banach space and $D \in \mathcal{C}v(E)$ and $v \in D$ be fixed. Let $S, T : D \to D$ be two quasi-nonexpansive self mappings such that $F = F(S) \cap F(T) \neq \emptyset$ is closed and convex. Let (S, T) satisfy jointly demiclosedness principle on D and $\{u_n\}_{n\in\mathbb{N}}$ be the sequence defined by*

$$\begin{cases} u_1 \in D \\ u_{n+1} = \gamma_n v + (1 - \gamma_n)y_n \\ y_n = \zeta_n S u_n + (1 - \zeta_n)T(u_n), \ \forall n \in \mathbb{N}, \end{cases} \tag{1.1}$$

where $\{\gamma_n\}_{n\in\mathbb{N}}$ and $\{\zeta_n\}_{n\in\mathbb{N}}$ are sequences of real numbers in $(0, 1)$. If the following conditions hold:
(c1) $\lim_{n\to\infty} \gamma_n = 0$*;*
(c2) $\sum_{n=1}^{\infty} \gamma_n = \infty$*;*
(c3) $0 < \liminf_{n\to\infty} \zeta_n(1 - \zeta_n) \leq \limsup_{n\to\infty} \zeta_n(1 - \zeta_n) < 1$.
Then, the sequence $\{u_n\}_{n\in\mathbb{N}}$ defined (1.1) converges strongly to $Q_F u$, where Q_F is the sunny nonexpansive retraction from E onto F.

Remark 1.4. It is worth mentioning that if $S = I$, where I is the identity mapping on E, then $I - T$ is demiclosed at zero (see, [15]). If S and T satisfy the demiclosedness principle according to Opial [19], then (S, T) satisfies the jointly demiclosedness. Unfortunately, the converse is not in general true as could be seen in Example 1 discussed in [15].

Now, we pose the following question:
Question 2: Is there any way to extend α-nonexpansive mappings to a more general class of mappings with the existence and convergence results in general Banach spaces?

This paper is organized as follows. In Section 2, preliminaries and a clear problem statement are provided. In Section 3, we first introduce such a new class of mappings namely, generalized (α, β)-S-nonexpansive mappings in general Banach spaces. In Section 4, beyond the continuity assumption on the mappings and without using the classical demiclosedness principle, we will verify new existence and convergence results for these mappings in Banach spaces and provide an affirmative answer to the above Question 2. To illustrate the facts, we include some useful examples. Our results unify and generalize corresponding ones discussed in [2, 10, 16, 20, 25]. Finally, the conclusion is in Section 5.

2. Preliminaries

Let E be a Banach space with the dual space E^*. We define the normalized duality mapping $J : E \longrightarrow 2^{E^*}$ as follows:

$$J(u) := \{f \in E^* : \langle u, f\rangle = \|u\|^2 = \|f\|^2\}, \ u \in E.$$

We say that E is a smooth Banach space if the limit

$$\lim_{n \longrightarrow \infty} \frac{\|u + tv\| - \|u\|}{t} \tag{2.1}$$

exists for each $u, v \in S_E := \{u \in E : \|u\| = 1\}$. In this case, the norm of E is called Gâteaux differentiable. It is known that J is single valued if E is smooth [8]. If for each $x \in S_E$, the limit (2.1) exists and is attained uniformly for $y \in S_E$. For more details on the Fréchet derivative, we refer the reader to see [8]. A Banach space E is called uniformly convex if for each $\varepsilon \in (0, 2]$ there is a $\delta > 0$ such that $\left\|\frac{u+v}{2}\right\| \le 1 - \delta, \forall u, v \in E$ with $\|u\| = \|v\| = 1$ and $\|u - v\| > \varepsilon$. A Banach space E enjoys the Opial property [19], if for every sequence $\{z_n\}_{n\in\mathbb{N}} \subset E$ with $z_n \rightharpoonup z$,

$$\liminf_{n \longrightarrow \infty} \|u_n - u\| < \liminf_{n \longrightarrow \infty} \|u_n - w\|$$

for every $w \in E$ ($w \neq u$). All Hilbert spaces and $\ell^p(1 < p < \infty)$ have the Opial property but, the Banach spaces $L_p[0, 2\pi](p \neq 2)$ do not enjoy the Opial property [8].

Let D be a nonempty subset of Banach space E and $\{u_n\}_{n\in\mathbb{N}}$ a bounded sequence in E. For each $x \in E$, define:

- : (i) asymptotic radius of $\{u_n\}_{n\in\mathbb{N}}$ at u by $r(u, \{u_n\}_{n\in\mathbb{N}}) := \limsup_{n\longrightarrow\infty} \|u_n - u\|$;
- : (ii) asymptotic radius of $\{u_n\}_{n\in\mathbb{N}}$ relative to D by $r(D, \{u_n\}_{n\in\mathbb{N}}) := \inf\{r(u, \{u_n\}_{n\in\mathbb{N}}) : u \in D\}$;
- : (iii) asymptotic center of $\{u_n\}_{n\in\mathbb{N}}$ relative to D by

$$A\left(D, \{u_n\}_{n\in\mathbb{N}}\right) := \{u \in D : r(u, \{u_n\}_{n\in\mathbb{N}}) = r(D, \{u_n\}_{n\in\mathbb{N}})\}.$$

It is worth mentioning that $A(D, \{u_n\}_{n\in\mathbb{N}})$ is nonempty. If E is uniformly convex then $A(D, \{u_n\}_{n\in\mathbb{N}})$ is singleton [8].

Definition 2.1. [24] Let $D \in \mathcal{N}(E)$ and $S, T : D \longrightarrow D$ be two mappings. The mappings $S, T : D \longrightarrow D$ are said to satisfy Jointly Condition (I) if there exists a nondecreasing function $h : [0, \infty) \longrightarrow [0, \infty)$ satisfying $h(0) = 0$, $\min\{\|u - T(u)\|, \|v - S(v)\|\} \ge h(\max\{d(u, F(T)), d(v, F(S))\})$ for all $u \in D$ and $h(c) > 0$ for all $c \in (0, \infty)$, where $d(u, A) := \inf\{\|u - p\| : p \in A\}$.

Lemma 2.2. [23] *Let E be a uniformly convex Banach space and $0 < c \le \tau_n \le d < 1$ for all $n \in \mathbb{N}$. Let $\{u_n\}_{n\in\mathbb{N}}$ and $\{v_n\}_{n\in\mathbb{N}}$ be such that $\limsup_{n\longrightarrow\infty} \|u_n\| \le s$, $\limsup_{n\longrightarrow\infty} \|v_n\| \le s$ and $\lim_{n\longrightarrow\infty} \|\tau_n u_n + (1 - \tau_n)v_n\| = s$ hold for some $s \ge 0$. Then $\lim_{n\longrightarrow\infty} \|u_n - v_n\| = 0$.*

3. Generalized (α, β)-S-Nonexpansive Mappings

We first introduce the following new class of mappings in Banach spaces.

Definition 3.1. Let $D \in \mathcal{N}(E)$ be fixed.
(i) A mapping $T : D \longrightarrow D$ is called a generalized (α,β)-nonexpansive mapping if there exist $\alpha,\beta \in [0,1)$ such that

$$\frac{1}{2}\|z-T(w)\| \le \|z-w\| \Longrightarrow \|T(z)-T(w)\| \le \alpha\|T(z)-w\| + \alpha\|z-T(w)\| + \beta\|z-Tz\| + \beta\|w-Tw\| + (1-2\alpha-2\beta)\|z-w\| \tag{3.1}$$

for all $z, w \in D$.
(ii) Let $S, T : D \longrightarrow D$ be two mappings. The mapping $T : D \longrightarrow D$ is called a generalized (α,β)-S-nonexpansive mapping if there exist $\alpha,\beta \in [0,1)$ such that

$$\frac{1}{2}\|z-T(z)\| \le \|z-w\| \Longrightarrow \|T(z)-S(w)\| \le \alpha\|T(z)-w\| + \alpha\|z-S(w)\| + \beta\|z-Tz\| + \beta\|w-Sw\| + (1-2\alpha-2\beta)\|z-w\| \tag{3.2}$$

for all $z, w \in D$.

In the special case of $S = T$, these two class of mappings are the same.
Now we include some essential properties of generalized (α,β)-S-nonexpansive mappings.

Remark 3.2. If $T : D \to D$ satisfies the Condition (C), then T is a generalized (α,β)-S-nonexpansive mapping. However, the converse does not hold as the following example indicates. When $\alpha = \beta = 0$ and $S = I$, the identity mapping on E, a generalized (α,β)-S-nonexpansive mapping reduces to a mapping satisfying Condition (C), however the reverse is not valid as the following example indicates.

Example 3.3. Let $D = [0,4] \subset \mathbb{R}$ be endowed with the usual norm. If we define $T : D \longrightarrow D$ by

$$T(z) = \begin{cases} 0, & \text{if } u \ne 4, \\ 2, & \text{if } z = 4 \end{cases}$$

Then for $z \in (2, 8/3]$ and $w = 4$,

$$\frac{1}{2}\|z-T(z)\| \le \|z-w\| \text{ and } \|T(z)-T(w)\| = 2 > \|z-w\|,$$

and T does not satisfy Condition (C). For $z \in (2,3]$ and $w = 4$, we also have

$$\frac{1}{2}\|w-T(w)\| \le \|z-w\| \text{ and } \|T(z)-T(w)\| > \|z-w\|,$$

and T does not enjoy Condition (C) but, T is an α-nonexpansive where $\alpha \ge \frac{1}{2}$ and a generalized (α,β)-S-nonexpansive mapping where $\alpha \ge \frac{1}{3}$ and $\beta = 0$. Also, the mapping T is not an (α,β)-nonexpansive mapping in the sense of [20]. Since, otherwise there are real numbers $\alpha,\beta \in [0,1]$ such that

$$|T(z)-T(w)|^2 \le \alpha|T(z)-w|^2 + \alpha|z-T(w)|^2 + \beta|z-T(z)|^2 + \beta|w-T(w)|^2 + (1-2\alpha-2\beta)|z-w|^2 \tag{3.3}$$

for all $z, w \in D$. Now, for $z = 2$ and $w = 4$ we get that

$$|T(2)-T(4)|^2 \le \alpha|T(2)-4|^2 + \alpha|2-T(4)|^2 + \beta|2-T(2)|^2 + \beta\|4-T(4)|^2 + (1-2\alpha-2\beta)|2-4|^2$$

which implies that $4 \le 4-4\alpha$ and hence $\alpha = 0$. Thus we have $\beta = 1$. If we select $z = 1$ and $w = 4$, then by (3.3) we obtain

$$|T(1)-T(4)|^2 \le \alpha|T(1)-4|^2 + \alpha|1-T(4)|^2 + \beta|1-T(1)|^2 + \beta\|4-T(4)|^2 + (1-2\alpha-2\beta)|1-4|^2$$

and hence $4 \le -4$ which is a contradiction.

Proposition 3.4. *Let $D \in \mathcal{N}(E)$ and $T : D \longrightarrow D$ be a generalized (α, β)-S-nonexpansive mapping with $F(T) \cap F(S) \neq \emptyset$. Then T is quasi-nonexpansive.*

Proof. Let $q \in F(T) \cap F(S)$ and $w \in D$. By the relation $\frac{1}{2}\|q - T(q)\| = 0 \leq \|w - q\|$ and in view of (3.1) we obtain

$$\begin{aligned} \|T(w) - q\| &= \|T(w) - S(q)\| \\ &\leq \alpha\|T(w) - q\| + \alpha\|w - S(q)\| + \beta\|T(w) - w\| + \beta\|q - S(q)\| \\ &\quad +(1 - 2\alpha - 2\beta)\|w - q\| \\ &\leq \alpha\|T(w) - q\| + \alpha\|w - q\| + \beta\|T(w) - q\| + \beta\|q - w\| + (1 - 2\alpha - 2\beta)\|w - q\|. \end{aligned}$$

This amounts to

$$(1 - \alpha - \beta)\|T(w) - q\| \leq (1 - \alpha - \beta)\|w - q\|.$$

Since $(1 - \alpha - \beta) > 0$, we arrive at $\|T(w) - q\| \leq \|w - q\|$ which finishes the proof. $\square$

In the following we present a result concerning the structure of fixed point set for a generalized (α, β)-S-nonexpansive mapping.

Lemma 3.5. *Let $D \in \mathcal{N}(E)$ be fixed. Let $T : D \longrightarrow D$ be a generalized (α, β)-S-nonexpansive mapping with $S : D \longrightarrow D$. Then $F(T) \cap F(S)$ is closed. If, in addition, E is strictly convex and D is convex, then $F(T) \cap F(S)$ is convex.*

Proof. Let $\{z_n\}_{n\in\mathbb{N}} \subset F(T) \cap F(S)$ be such that $z_n \to u \in D$ as $n \to \infty$. From $\frac{1}{2}\|z_n - S(z_n)\| = 0 \leq \|z_n - z\|$ and considering (3.2), we are led to

$$\begin{aligned} \|z_n - T(z)\| &= \|T(z_n) - T(z)\| \\ &\leq \|T(z_n) - S(z_n)\| + \|T(z) - S(z_n)\| \\ &\leq \alpha\|T(z) - z_n\| + \alpha\|z - S(z_n)\| + \beta\|z - T(z)\| + \beta\|z_n - S(z_n)\| \\ &\quad +(1 - 2\alpha - 2\beta)\|z_n - z\| \\ &\leq \alpha\|T(z) - z_n\| + \alpha\|z_n - z\| + \beta\|z - z_n\| + (1 - 2\alpha - 2\beta)\|z_n - z\|. \end{aligned}$$

This amounts to

$$(1 - \alpha - \beta) \lim_{n\longrightarrow\infty} \|z_n - T(z)\| \leq (1 - \alpha - \beta) \lim_{n\longrightarrow\infty} \|z_n - z\|.$$

Since $1 - \alpha - \beta > 0$, we obtain

$$\lim_{n\longrightarrow\infty} \|z_n - T(z)\| \leq \lim_{n\longrightarrow\infty} \|z_n - z\|,$$

and $T(z) = z$ and hence $F(T)$ is closed. Similarly, we can prove that $F(S)$ is closed.

If E is strict convexity and D is convex, then we select $\eta \in (0, 1)$ and $z, w \in F(T) \cap F(S)$ with $z \neq w$. Setting $y = \eta z + (1 - \eta)w \in D$, by employing $\frac{1}{2}\|z - T(z)\| = 0 \leq \|z - y\|$, Proposition 3.4 and (3.2), we arrive at

$$\|T(z) - T(y)\| \leq \|z - y\|.$$

Similarly, we get

$$\|T(v) - T(y)\| \leq \|v - y\|.$$

Thus we get

$$\begin{aligned} \|z - w\| &\leq \|z - T(y)\| + \|T(y) - w\| = \|T(z) - T(y)\| + \|T(y) - T(w)\| \\ &\leq \|z - y\| + \|v - y\| = (1 - \eta)\|z - w\| + \eta\|z - w\| = \|z - w\|. \end{aligned}$$

The strict convexity of E ensures the existence of an element $\mu \in [0,1]$ with $T(w) = \mu z + (1-\mu)w$. Now

$$(1-\mu)\|z-w\| = \|T(z) - T(y)\| \le \|z-y\| = (1-\eta)\|z-w\|$$

and

$$\mu\|z-w\| = \|T(w) - T(y)\| \le \|w-y\| = \eta\|z-w\|.$$

By the above two inequalities $1-\mu \le 1-\eta$ and $\mu \le \eta$. This implies $\eta = \mu$ and hence $w \in F(T)$. □

4. Weak and Strong Convergence Theorems

In this section, we provide the convergence theorems for generalized (α, β)-S-nonexpansive mappings in Banach spaces with $S : D \longrightarrow D$. We first introduce the following iterative scheme related to generalized (α, β)-S-nonexpansive mappings:
Let $S, T : D \longrightarrow D$ be two mappings.

$$\begin{cases} z_1 \in D \\ z_{n+1} = (1-\theta_n)T(z_n) + \theta_n S(w_n) \\ w_n = (1-\lambda_n)z_n + \lambda_n T(z_n), \ n \in \mathbb{N}, \end{cases} \tag{4.1}$$

where $\{\theta_n\}_{n\in\mathbb{N}}$ and $\{\lambda_n\}_{n\in\mathbb{N}}$ are sequences in $(0,1)$. For the particular case of $S = T$, the iterative scheme (4.1) becomes the corresponding algorithm proposed by Agarwal et al. [2]. We refer the reader to some other references [10, 13, 18] in relation to the algorithm (4.1).

Let us verify the following lemmas which will be essential for proving our main results. The results of this section provide an affirmative answer to Question 2.

Lemma 4.1. *Let $D \in \mathcal{N}(E)$ and $T : D \longrightarrow D$ be a generalized (α, β)-S-nonexpansive mapping with $S : D \longrightarrow D$. Then, $\forall z, w \in D$, we have:*
(i) $\|T(z) - S(Tz)\| \le \|z - T(z)\|$.
(ii) Either $\frac{1}{2}\|z - T(z)\| \le \|z - w\|$ *or* $\frac{1}{2}\|T(z) - S(T(z))\| \le \|T(z) - w\|$.
(iii) Either $\|T(z) - S(w)\| \le \alpha\|T(z) - w\| + \alpha\|z - S(w)\| + \beta\|z - T(z)\| + \beta\|w - S(w)\| + (1 - 2\alpha - 2\beta)\|z - w\|$
or $\|S(T(z)) - S(w)\| \le \alpha\|T(z) - S(w)\| + \alpha\|S(T(z)) - w\| + \beta\|z - T(z)\| + \beta\|w - S(w)\| + (1 - 2\alpha - 2\beta)\|T(z) - w\|$.

Proof. Since $\frac{1}{2}\|z - T(z)\| \le \|z - T(z)\|$, in view of (3.2) we conclude that

$$\|T(z) - S(T(z))\| \le \|z - T(z)\|.$$

To verify (ii), by contradiction, we assume that

$$\frac{1}{2}\|z - T(z)\| > \|z - v\| \text{ and } \frac{1}{2}\|T(z) - S(T(z))\| > \|T(z) - w\|.$$

Employing (i), we get

$$\|z - T(z)\| \le \|z - w\| + \|T(z) - w\| < \frac{1}{2}\|z - T(z)\| + \frac{1}{2}\|T(z) - S(T(z))\| \le \|z - T(z)\|,$$

which is a contradiction. This verifies the condition (ii) and the condition (iii) is deduced by (ii). □

Lemma 4.2. *Let $D \in \mathcal{N}(E)$ and $T : D \longrightarrow D$ be a generalized (α, β)-S-nonexpansive mapping such that $S : D \longrightarrow D$. Then for all $z, w \in D$,*

$$\|z - S(w)\| \le \frac{3+\alpha-\beta}{1-\alpha-\beta}\|z - T(z)\| + \|z - w\|, \ \|T(z) - w\| \le \frac{3+\alpha-\beta}{1-\alpha-\beta}\|w - S(w)\| + \|z - w\|.$$

Proof. In view of Lemma 4.1, we obtain for any $z, w \in D$ either

$$\begin{array}{rcl} \|T(z)-S(w)\| & \leq & \alpha\|T(z)-w\|+\alpha\|z-S(w)\|+\beta\|z-Tz\|+\beta\|w-Sw\| \\ & & +(1-2\alpha-2\beta)\|z-w\| \end{array}$$

or

$$\begin{array}{rcl} \|S(T(z))-S(w)\| & \leq & \alpha\|T(z)-S(w)\|+\alpha\|S(T(z))-w\| \\ & & +\beta\|z-Tz\|+\beta\|w-Sw\|+(1-2\alpha-2\beta)\|T(z)-w\|. \end{array}$$

In the first case, we get

$$\begin{array}{rcl} \|z-T(w)\| & \leq & \|z-S(z)\|+\|T(w)-S(z)\| \\ & \leq & \|z-S(z)\|+\alpha\|T(w)-z\|+\alpha\|w-S(z)\| \\ & & +\beta\|w-T(w)\|+\beta\|z-S(z)\|+(1-2\alpha-2\beta)\|z-w\| \\ & \leq & \|z-S(z)\|+\alpha\|T(w)-z\|+\alpha\|w-z\|+\alpha\|z-S(z)\| \\ & & +\beta\|z-T(w)\|+\beta\|z-w\|+\beta\|z-S(z)\| \\ & & +(1-2\alpha-2\beta)\|z-w\| \\ & \leq & (1+\alpha+\beta)\|z-S(z)\|+(\alpha+\beta)\|T(w)-z\| \\ & & +(1-\alpha-\beta)\|z-w\|. \end{array}$$

This implies that

$$\begin{array}{rcl} \|z-T(w)\| & \leq & \dfrac{1+\alpha+\beta}{1-\alpha-\beta}\|T(z)-z\|+\|z-w\| \\ & \leq & \dfrac{(3+\alpha+\beta)}{(1-\alpha-\beta)}\|z-T(z)\|+\|z-w\|. \end{array}$$

Similarly, we obtain

$$\begin{array}{rcl} \|w-T(z)\| & \leq & \|w-S(w)\|+\|S(w)-T(z)\| \\ & \leq & \|w-S(w)\|+\alpha\|T(z)-w\|+\alpha\|z-S(w)\| \\ & & +\beta\|z-T(z)\|+\beta\|w-S(w)\|+(1-2\alpha-2\beta)\|z-w\| \\ & \leq & \|w-S(w)\|+\alpha\|T(z)-v\|+\alpha\|v-z\|+\alpha\|w-S(w)\| \\ & & +\beta\|w-T(z)\|+\beta\|z-w\|+\beta\|w-S(w)\| \\ & & +(1-2\alpha-2\beta)\|z-w\| \\ & \leq & (1+\alpha+\beta)\|w-S(w)\|+(\alpha+\beta)\|w-T(z)\| \\ & & +(1-\alpha-\beta)\|z-w\|. \end{array}$$

This implies

$$\begin{array}{rcl} \|w-T(z)\| & \leq & \dfrac{(1+\alpha+\beta)}{(1-\alpha-\beta)}\|w-S(w)\|+\|z-w\| \\ & \leq & \dfrac{(3+\alpha+\beta)}{(1-\alpha-\beta)}\|w-S(w)\|+\|z-w\|. \end{array}$$

In the other case, we obtain

$$\begin{aligned}
\|z-S(w)\| &\leq \|z-T(z)\|+\|T(z)-S(T(z))\|+\|S(T(z))-S(w)\| \\
&\leq 2\|z-T(z)\|+\alpha\|T(z)-S(w)\|+\alpha\|S(T(z))-w\| \\
&\quad +\beta\|S(T(z))-T(z)\|+\beta\|w-S(w)\|+(1-2\alpha-2\beta)\|T(z)-w\| \\
&\leq 2\|z-T(z)\|+\alpha\|T(z)-z\|+\alpha\|z-S(w)\|+\alpha\|S(T(z))-T(z)\| \\
&\quad +\alpha\|T(z)-w\| \\
&\quad +\beta\|T(z)-z\|+\beta\|w-z\|+\beta\|z-S(w)\|+(1-2\alpha-2\beta)\|T(z)-w\| \\
&\leq 2\|z-T(z)\|+\alpha\|T(z)-z\|+\alpha\|z-S(w)\| \\
&\quad +\alpha\|S(T(z))-T(z)\|+\alpha\|T(z)-z\|+\alpha\|z-w\| \\
&\quad +\beta\|T(z)-z\|+\beta\|w-z\|+\beta\|z-S(w)\| \\
&\quad +(1-2\alpha-2\beta)\|T(z)-z\|+(1-2\alpha-2\beta)\|z-w\| \\
&\leq (2+\alpha+\beta)\|z-T(z)\|+\alpha\|S(w)-z\|+\alpha\|z-T(z)\| \\
&\quad +\beta\|z-T(z)\|+(1-\alpha-\beta)\|T(z)-z\|+(1-2\alpha-2\beta)\|z-w\|.
\end{aligned}$$

This implies

$$\|z-S(w)\| \leq \frac{(3+\alpha+\beta)}{(1-\alpha-\beta)}\|z-T(z)\|+\|z-w\|.$$

Therefore in both the cases, we get the conclusion that

$$\max\{\|T(z)-w\|,\|z-S(w)\|\} \leq \frac{3+\alpha+\beta}{1-\alpha-\beta}\max\{\|z-T(z)\|,\|w-S(w)\|\}+\|z-w\|.$$

Therefore in both the cases, we get the conclusion. □

Proposition 4.3. *(Jointly Demiclosedness Principle). Let E be a Banach space with the Opial property, $D \in \mathcal{C}(E)$ and $T : D \longrightarrow D$ a generalized (α,β)-S-nonexpansive mapping with $S : D \longrightarrow D$. If $\{z_n\}_{n\in\mathbb{N}} \subset D$ with $z_n \rightharpoonup z \in E$, $\lim_{n\longrightarrow\infty}\|z_n-S(z_n)\|=0$ and $\lim_{n\longrightarrow\infty}\|T(z_n)-S(z_n)\|=0$, then $T(z)=z$ and $S(z)=z$. That is, (T,S) satisfies the jointly demiclosedness principle.*

Proof. By the assumption, we have $\lim_{n\longrightarrow\infty}\|z_n-S(z_n)\|=0$ and $\lim_{n\longrightarrow\infty}\|T(z_n)-S(z_n)\|=0$, which imply that $\lim_{n\longrightarrow\infty}\|z_n-T(z_n)\|=0$. In the light of Lemma 4.2, for all $n\in\mathbb{N}$, we obtain

$$\|z_n-T(z)\| \leq \frac{(3+\alpha+\beta)}{(1-\alpha-\beta)}\|z_n-S(z_n)\|+\|z_n-z\|.$$

and

$$\|z_n-S(z)\| \leq \frac{(3+\alpha+\beta)}{(1-\alpha-\beta)}\|z_n-T(z_n)\|+\|z_n-z\|.$$

This implies

$$\liminf_{n\longrightarrow\infty}\|z_n-T(z)\| \leq \liminf_{n\longrightarrow\infty}\|z_n-z\|,\ \liminf_{n\longrightarrow\infty}\|z_n-S(z)\| \leq \liminf_{n\longrightarrow\infty}\|z_n-z\|.$$

By the Opial property, we get $T(z)=S(z)=z$. □

Lemma 4.4. *Let $D \in \mathcal{C}v(E)$ and $T : D \longrightarrow D$ be a generalized (α,β)-S-nonexpansive mapping such that $S : D \longrightarrow D$. Let $\{z_n\}_{n\in\mathbb{N}}$ be the sequence defined by (4.1) and $z \in F(T)$. Then the following statements are satisfied:*
(1) $\max\{\|z_{n+1}-z\|,\|w_n-z\|\} \leq \|z_n-z\|, \forall n\in\mathbb{N}$;
(2) $\lim_{n\longrightarrow\infty}\|z_n-z\|$ *exists.*

Proof. In view of (4.1) and Proposition 3.4, we deduce that

$$\begin{aligned} \|w_n - z\| &= \|(1-\lambda_n)z_n + \lambda_n T(z_n) - z\| \leq (1-\lambda_n)\|z_n - z\| + \lambda_n \|T(z_n) - z\| \\ &\leq (1-\lambda_n)\|z_n - z\| + \lambda_n \|z_n - z\| = \|z_n - z\| \end{aligned}$$

Employing (4.1) and Proposition 3.4, we obtain

$$\begin{aligned} \|z_{n+1} - z\| &= \|(1-\theta_n)T(z_n) + \theta_n S(w_n) - z\| \leq (1-\theta_n)\|T(z_n) - z\| + \theta_n \|S(w_n) - z\| \\ &\leq (1-\theta_n)\|z_n - z\| + \theta_n \|w_n - z\| \leq (1-\theta_n)\|z_n - z\| + \theta_n \|z_n - z\| = \|z_n - z\|. \end{aligned}$$

This verifies that $\{\|z_n - z\|\}_{n\in\mathbb{N}}$ is bounded and nonincreasing and hence $\lim_{n\longrightarrow\infty} \|z_n - z\|$ exists. □

Now, we provide the following essential an important result which will be employed in the sequel.

Theorem 4.5. *Let E be a uniformly convex Banach space, $D \in \mathcal{C}(E)$ and $T : D \longrightarrow D$ a generalized (α, β)-S-nonexpansive mapping with $S : D \longrightarrow D$. Let $\{z_n\}_{n\in\mathbb{N}}$ be a sequence with $u_1 \in D$ defined by (4.1). Then the following assertions are equivalent:*
(i) $\{z_n\}_{n\in\mathbb{N}}$ is bounded, $\lim_{n\longrightarrow\infty} \|T(z_n) - z_n\| = 0$ and $\lim_{n\longrightarrow\infty} \|S(w_n) - v_n\| = 0$;
(ii) $F(T) \cap F(S) \neq \emptyset$.

Proof. $(i) \Longrightarrow (ii)$. Let $\{z_n\}_{n\in\mathbb{N}}$ be bounded, $\lim_{n\longrightarrow\infty} \|T(z_n) - z_n\| = 0$ and $\lim_{n\longrightarrow\infty} \|S(w_n) - w_n\| = 0$. Since E is uniformly convex, $A(D, \{z_n\}_{n\in\mathbb{N}}) \neq \emptyset$. Taking $u \in A(D, \{z_n\}_{n\in\mathbb{N}})$, we deduce that

$$r(T(z), \{z_n\}_{n\in\mathbb{N}}) = \limsup_{n\longrightarrow\infty} \|z_n - T(u)\|.$$

Using Lemma 4.2, one obtains

$$\begin{aligned} r(T(u), \{z_n\}_{n\in\mathbb{N}}) &= \limsup_{n\longrightarrow\infty} \|z_n - T(u)\| \\ &\leq \frac{(3+\alpha+\beta)}{(1-\alpha-\beta)} \limsup_{n\longrightarrow\infty} \|S(w_n) - w_n\| + \limsup_{n\longrightarrow\infty} \|z_n - u\| = r(z, \{z_n\}_{n\in\mathbb{N}}). \end{aligned}$$

Since the asymptotic center of $\{z_n\}_{n\in\mathbb{N}}$ is uniquely determined, we arrive at $T(u) = u$. On the other hand, by the assumption $\lim_{n\longrightarrow\infty} \|S(w_n) - w_n\| = 0$, we conclude that $\lim_{n\longrightarrow\infty} \|z_n - w_n\| = 0$ and hence $A(D, \{z_n\}_{n\in\mathbb{N}}) = A(D, \{w_n\}_{n\in\mathbb{N}})$ and $r(T(u), \{z_n\}_{n\in\mathbb{N}}) = r(S(u), \{w_n\}_{n\in\mathbb{N}})$. This ensures that

$$\begin{aligned} r(S(u), \{w_n\}_{n\in\mathbb{N}}) &= \limsup_{n\longrightarrow\infty} \|w_n - S(u)\| \\ &\leq \frac{(3+\alpha+\beta)}{(1-\alpha-\beta)} \limsup_{n\longrightarrow\infty} \|T(z_n) - z_n\| + \limsup_{n\longrightarrow\infty} \|w_n - u\| = r(u, \{w_n\}_{n\in\mathbb{N}}). \end{aligned}$$

$(ii) \Longrightarrow (i)$. Let $F(T) \cap F(S) \neq \emptyset$ and $v \in F(T) \cap F(S)$ be fixed. In the light of Lemma 4.4 we see that $\lim_{n\longrightarrow\infty} \|z_n - v\|$ exists. Setting $\lim_{n\longrightarrow\infty} \|z_n - v\| = s$, in view of Proposition 3.4, we obtain $\limsup_{n\longrightarrow\infty} \|T(z_n) - w\| \leq s$. Using Lemma 4.4, we deduce that $\limsup_{n\longrightarrow\infty} \|w_n - v\| \leq s$. Employing Proposition 3.4, we are led to $\limsup_{n\longrightarrow\infty} \|S(w_n) - v\| \leq s$ and hence

$$r = \lim_{n\longrightarrow\infty} \|z_{n+1} - v\| = \lim_{n\longrightarrow\infty} \|(1-\theta_n)(T(z_n) - v) + \theta_n(T(w_n) - v)\|. \tag{4.2}$$

In view of (4.2) and Lemma 2.2, we get $\lim_{n\longrightarrow\infty} \|T(w_n) - T(z_n)\| = 0$ and therefore

$$\|z_{n+1} - T(z_n)\| = \|(1-\theta_n)T(z_n) + \theta_n S(w_n) - T(z_n)\| = \theta_n \|T(w_n) - T(z_n)\|.$$

Letting $n \longrightarrow \infty$ we get $\lim_{n\longrightarrow\infty} \|z_{n+1} - T(z_n)\| = 0$. Also, we have

$$\|z_{n+1} - S(w_n)\| \leq \|z_{n+1} - T(z_n)\| + \|S(w_n) - T(z_n)\|.$$

Thus we get $\lim_{n\longrightarrow\infty}\|z_{n+1}-S(w_n)\|=0$. Now, we observe that

$$\|z_{n+1}-v\|\leq\|z_{n+1}-S(w_n)\|+\|S(w_n)-w\|\leq\|z_{n+1}-S(w_n)\|+\|w_n-v\|,$$

which yields $r\leq\liminf_{n\longrightarrow\infty}\|w_n-v\|$. From (4.2), we get

$$\begin{aligned} r=\lim_{n\longrightarrow\infty}\|w_n-v\| &= \lim_{n\longrightarrow\infty}\|(1-\lambda_n)z_n+\lambda_nT(z_n)-v\| \\ &= \lim_{n\longrightarrow\infty}\|(1-\lambda_n)(z_n-v)+\lambda_n(T(z_n)-v)\|. \end{aligned} \tag{4.3}$$

Finally, from (4.3) and Lemma 2.2, we conclude that $\lim_{n\longrightarrow\infty}\|T(z_n)-z_n\|=0$. □

The following lemma is essential for the weak convergence result in Banach spaces.

Lemma 4.6. *Suppose that the conditions of Theorem 4.5 are satisfied. Then* $\lim_{n\longrightarrow\infty}\langle z_n,J(p_1-p_2)\rangle$ *exists for any* $p_1,p_2\in F(T)\cap F(S)$*; in particular* $\langle z-w,J(p_1-p_2)\rangle=0$ *for all* $u,v\in\omega_w(z_n)$*, where* $\omega_w(z_n)$ *is the set of all weak limit points of* $\{z_n\}_{n\in\mathbb{N}}$*.*

Proof. By similar discussion to that of Lemma 2.3 in [11] we arrive at the desired conclusion. □

Theorem 4.7. *Let E be a ufniformly convex Banach space and D,T,S and $\{z_n\}_{n\in\mathbb{N}}$ satisfy the conditions of Theorem 4.5. Suppose that either of the following condition holds:*
(i) E satisfies the Opial's property;
(ii) E has a Fréchet differential norm and (T,S) satisfies the jointly demiclosedness principle.
If $F(T)\cap F(S)\neq\emptyset$, then there exists p in $F(T)$ with $z_n\rightharpoonup p$ as $n\to\infty$.

Proof. Employing Theorem 4.5, we deduce that $\{z_n\}_{n\in\mathbb{N}}$ is bounded and $\lim_{n\longrightarrow\infty}\|T(z_n)-z_n\|=0$. Since E is reflexive, there exists a subsequence $\{z_{n_j}\}_{j\in\mathbb{N}}$ of $\{z_n\}_{n\in\mathbb{N}}$ such that $\{z_{n_j}\}_{j\in\mathbb{N}}$ converges weakly to some $p\in D$. First, we assume that (i) is true. From Theorem 4.5, it follows that $p\in F(T)\cap F(S)$. Arguing by contradiction $\{z_n\}_{n\in\mathbb{N}}$ does not converge weakly to p. Then there exist a subsequence $\{z_{n_k}\}_{k\in\mathbb{N}}$ of $\{z_n\}_{n\in\mathbb{N}}$ and $q\in D$ such that $\{z_{n_k}\}_{k\in\mathbb{N}}$ converges weakly to q and $p\neq q$. By Theorem 4.5, we have $q\in F(T)$. It follows from Lemma 5.4 that $\lim_{n\longrightarrow\infty}\|z_n-u\|$ exists for all $u\in F(T)\cap F(S)$. The Opial property assures that

$$\begin{aligned} \lim_{n\longrightarrow\infty}\|z_n-p\| &= \lim_{j\longrightarrow\infty}\|z_{n_j}-p\|<\lim_{j\longrightarrow\infty}\|z_{n_j}-q\|=\lim_{n\longrightarrow\infty}\|z_n-q\|=\lim_{k\longrightarrow\infty}\|z_{n_k}-q\| \\ &< \lim_{k\longrightarrow\infty}\|z_{n_k}-p\|=\lim_{n\longrightarrow\infty}\|z_n-p\|, \end{aligned}$$

which is absurd and therefore $z_n\rightharpoonup p\in F(T)$. Since $\|z_n-w_n\|\to 0$ as $n\to\infty$, we find that $w_n\rightharpoonup p\in F(S)$. Thus $p\in F(T)\cap F(S)$.

If (ii) is satisfied, then from Lemma 4.6, we get $\langle z-w,J(p-q)\rangle=0$ for all $z,w\in\omega_w(z_n)$. By the jointly demiclosedness of (S,T), we discover that $p,q\in F(T)\cap F(S)$ and hence $\|p-q\|^2=\langle p-q,J(p-q)\rangle=0$ and hence $p=q$. □

Theorem 4.8. *Let $D\in\mathcal{C}(E)$ and $T:D\longrightarrow D$ be a generalized (α,β)-S-nonexpansive mapping with $S:D\longrightarrow D$ such that $F(T)\cap F(S)\neq\emptyset$. Let $\{z_n\}_{n\in\mathbb{N}}$ be a sequence with $z_1\in D$ defined by (4.1). If* $\liminf_{n\longrightarrow\infty}\{\max\{d(z_n,F(T)),d(w_n,F(S))\}\}=0$*, then the sequence $\{z_n\}_{n\in\mathbb{N}}$ converges strongly to a fixed point of T, where $d(z,A):=\inf\{\|z-p\|:p\in A\}$.*

Proof. If $\liminf_{n\longrightarrow\infty}\{\max\{d(z_n,F(T)),d(w_n,F(S))\}\}=0$ then we get $\liminf_{n\longrightarrow\infty}d(z_n,F(T))=0$. Hence we can select a subsequence $\{u_n\}_{n\in\mathbb{N}}$ of $\{z_n\}_{n\in\mathbb{N}}$ with

$$\lim_{n\longrightarrow\infty}d(u_n,F(T))=0. \tag{4.4}$$

The relation (4.4) assures the existence of a subsequence $\{u_{n_i}\}_{i\in\mathbb{N}}$ of $\{u_n\}_{n\in\mathbb{N}}$ with $\|u_{n_i}-p_i\|\leq\frac{1}{2^i}$ for all $i\in\mathbb{N}$, where $\{p_i\}_{i\in\mathbb{N}}\subset F(T)\cap F(S)$. In view of Lemma 4.4, we are led to

$$\|u_{n_{i+1}}-p_i\|\leq\|u_{n_i}-p_i\|\leq\frac{1}{2^i}. \tag{4.5}$$

Using (4.5), we conclude that

$$\|p_{i+1} - p_i\| \leq \|p_{i+1} - u_{n_{i+1}}\| + \|u_{n_{i+1}} - p_i\| < \frac{1}{2^{i+1}} + \frac{1}{2^i} < \frac{1}{2^{i-1}}.$$

This implies that $\{p_i\}_{i\in\mathbb{N}}$ is a Cauchy sequence in $F(T)$. Employing Lemma 3.5, we conclude that $F(T)$ is closed and hence there exists $z \in F(T)$ such that $p_i \to z$. Since $\|z_n - w_n\| \to 0$ as $n \to \infty$, we find that $p_i \to z \in F(S)$ as $n \to \infty$. Employing $\|u_{n_i} - p\| \leq \|u_{n_i} - p_i\| + \|p_i - p\|$ we are led to $z_{n_i} \to z$ as $i \to \infty$. Finally, Lemma 4.4 ensures the existence of $\lim_{n\longrightarrow\infty} \|z_n - p\|$ and therefore $u_n \to p$ as $n \to \infty$. □

5. Conclusion

We have derived fixed point results for generalized (α, β)-S-nonexpansive mappings in Banach spaces. We have also presented new fixed point theorems for these mappings defined on closed and convex subsets of Banach spaces. In will be interesting to work on Bregman distance counterpart of the presented results of this paper for some future research study. Also, in future works we will employ our results to best proximity point, nonlinear variational inequality, split equilibrium problems and Halpern type iterations in the setting of Banach spaces (see, for some related articles, [6, 7, 14, 15, 17, 21, 22, 26]).

Statements and Declarations

The authors declare that they have no conflict of interest, and the manuscript has no associated data.

Acknowledgments

The authors would like to give their deep gratitude to anonymous referees for valuable suggestions and helpful comments.

References

[1] K. Aoyama and F. Kohsaka. Fixed point theorem for α-nonexpansive mappings in Banach spaces. *Nonlinear Analysis: Theory, Methods & Applications*, 74:4387-4391, 2011.
[2] R. P. Agarwal, D. O'Regan, and D. R. Sahu. Iterative construction of fixed points of nearly asymptotically nonexpansive mappings. *Journal of Nonlinear and Convex Analysis*, 8(1):61-79, 2007.
[3] F. E. Browder. Fixed-point theorems for noncompact mappings in Hilbert space. *Proceedings of the National Academy of Sciences of the United States of America*, 53:1272-1276, 1965.
[4] F. E. Browder. Nonexpansive nonlinear operators in Banach spaces. *Proceedings of the National Academy of Sciences of the United States of America*, 54:1041-1044, 1965.
[5] J. B. Diaz and F. T. Metcalf. On the structure of the set subsequential limit points of successive approximations. *Bulletin of the American Mathematical Society*, 73:516-519, 1967.
[6] Q.-L. Dong, L. Liu, X. Qin, and J.-C. Yao. An alternated inertial general splitting method with linearization for the split feasibility problem. *Optimization*, 72:2585-2607, 2023.
[7] E.-C. Godwin, T.-O. Alakoya, O.-T. Mewomo, and J.-C. Yao. Relaxed inertial Tseng extragradient method for variational inequality and fixed point problems. *Applicable Analysis*, 102:4253-4278, 2023.
[8] K. Goebel and W. A. Kirk. *Topics in Metric Fixed Point Theory*. Cambridge University Press, Cambridge, 1990.
[9] D. Göhde. Zum Prinzip def Kontraktiven Abbilding. *Mathematische Nachrichten*, 30:251-258, 1965.
[10] S. Ishikawa. Fixed points and iteration of a nonexpansive mapping in a Banach space. *Proceedings of the American Mathematical Society*, 59:65-71, 1976.
[11] S. H. Khan and J. K. Kim. Common fixed points of two nonexpansive mappings by a modified faster iteration scheme. *Bulletin of the Korean Mathematical Society*, 47(5):973-985, 2010.
[12] W. A. Kirk. A fixed point theorem for mappings which do not increase distances. *American Mathematical Monthly*, 72:1004-1006, 1965.
[13] W. R. Mann. Mean value methods in iteration. *Proceedings of the American Mathematical Society*, 4:506-510, 1953.
[14] E. Naraghirad. Halpern's iteration for Bregman relatively nonexpansive mappings in Banach spaces. *Numerical Functional Analysis and Optimization*, 34(10):1129-1155, 2013.

[15] E. Naraghirad. Approximation of common fixed points of nonlinear mappings satisfying jointly demi-closedness principle in Banach spaces. *Mediterranean Journal of Mathematics*, 14:Article ID 162, 2017.

[16] E. Naraghirad, N. C. Wong, and J. C. Yao. Approximating fixed point of α-nonexpansive mappings in uniformly convex and CAT(0) spaces. *Fixed Point Theory and Applications*, 2013:Article ID 57, 2013.

[17] E. Naraghirad and J.-C. Yao. Bregman weak relatively nonexpansive mappings in Banach spaces. *Fixed Point Theory and Applications*, 2013:Article ID 141, 2013.

[18] M. A. Noor. New approximation schemes for general variational inequalities. *Journal of Mathematical Analysis and Applications*, 251(1):217-229, 2000.

[19] Z. Opial. Weak convergence of the sequence of the succesive approximations for nonexpansive mappings. *Bulletin of the American Mathematical Society*, 73:591-597, 1967.

[20] R. Pant and R. Shukla. Approximating fixed points of generalized α-nonexpansive mappings in Banach spaces. *Numerical Functional Analysis and Optimization*, 38(2):248-266, 2017.

[21] A. Petrusel and G. Petrusel. On Reich's strict fixed point theorem for multi-valued operators in complete metric spaces. *Journal of Nonlinear and Variational Analysis*, 2:103-112, 2018.

[22] A. Petrusel, G. Petrusel, and J.-C. Yao. Variational analysis concepts in the theory of multivalued coincidence problems. *Journal of Nonlinear and Convex Analysis*, 19:935-958, 2018.

[23] J. Schu. Weak and strong convergence to fixed points of asymptotically nonexpansive mappings. *Bulletin of the Australian Mathematical Society*, 43(1):153-159, 1991.

[24] H. F. Senter and W. G. Dotson. Approximating fixed points of nonexpansive mappings. *Proceedings of the American Mathematical Society*, 44:375-380, 1974.

[25] T. Suzuki. Fixed point theorems and convergence theorems for some generalized nonexpansive mappings. *Journal of Mathematical Analysis and Applications*, 340:1088-1095, 2008.

[26] L.-J. Zhu and J.-C. Yao. Approximating solutions of a split fixed point problem of demicontractive operators. *Carpathian Journal of Mathematics*, 40(1):195-206, 2024.