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A SUPRAMETRIC FRAMEWORK FOR Θ-CONTRACTIONS WITH APPLICATION TO THE FOURTH-ORDER EULER-BERNOULLI BEAM EQUATION

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Abstract. This manuscript offers a comprehensive study of Θ-contractions, thereby demonstrating both the existence and uniqueness of fixed points results in the setting of complete suprametric spaces. We present an example that satisfies the proposed contraction conditions in a suprametric space, but not in a standard metric space. Moreover, the example illustrates the proposed contraction while demonstrating that the Banach contraction fails, highlighting the novelty of our approach. Then we employ our main result to a fourth-order Euler–Bernoulli beam equation with clamped–clamped boundary value problem. By transforming the original problem into an integral equation, which shows that the solution operator maintains the proposed contraction properties within a complete suprametric space. This study significantly expands fixed point theory beyond its usual application in metric spaces. This provides a strong framework for both theoretical analysis and practical problem-solving in nonlinear contexts.

Keywords. Fixed point, Θ-contractions, Euler–Bernoulli beam, Boundary value problems.
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1. Introduction

Jleli et al. [1] have introduced the concept of θ-contractions, which is additional to the classical notion of Banach contractions in [2]. Their contribution provided a flexible framework that allows researchers to examine contractive mappings across various types of metric spaces, while their research broadened the scope of fixed point theory. Building on this foundation, Jleli et al. [3] created a unified framework. This framework allowed researchers to study contraction mappings using both topological and metric space methods. As a result, the study developed better techniques for finding fixed points and expanded the application of contraction principles to more theoretical areas. Hussain et al. [4] proved that metric space completeness acts as a necessary condition for certain contractive mapping techniques because it enables proof of fixed point existence, which validates the core principles of fixed point theorems. Liu et al. [5] developed new theoretical grounds through their presentation of θ-type and θ-type Suzuki contractions which create fresh research pathways that enable scientists to discover fixed points across different metric space categories. Ahmad et al. [6] went through the work of generalized θ contractions to study fixed point results. These discoveries enabled them to develop new extensions of θ, a contraction that can be very effectively used both in generalized metric spaces and in solving nonlinear problems. Minak et al. [7] introduced the ordered θ contractions that not only continued the research on fixed points in partially ordered metric spaces but also offered new tools for analytical investigation.

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Zheng et al. [8] developed their work on contractive mappings through the introduction of their new function composition method, which combined the θ and ϕ functions because this method allowed them to derive fixed point theorems with fewer restrictions while extending their research to a broader range of nonlinear systems. In 2018, Imdad et al. [9] came up with the idea of weak θ contractions as a looser variation of complete θ contractions that provided more freedom to the researchers in establishing new fixed-point theorems, which, as a result, were applied to fractal theory studies, thus being a significant theoretical element of iterated function systems. Altun et al.[10] continued their research work in 2019 when they applied θ-contractions to Perov-type vector-valued mappings, which resulted in new fixed point results for semilinear operator systems because their methods enabled researchers to work with multi-dimensional systems. The researchers used auxiliary functions to create relational properties, which they used to establish fixed point theorems through the introduction of $(\theta, \psi)_{\mathcal{R}}$-weak contractions, which Imdad et al. [11] introduced in 2020 because this research connected theoretical concepts with real-world applications in integral equations. Ali et al. [12] used the Bianchini-Grandolfi gauge function together with parameter b to study how θ_b contractions change under different conditions. The current trend, which involves developing better mathematical models, depends on researchers who expand contraction concepts because these concepts provide essential support for current scientific studies.

Suprametric spaces develop into a new mathematical field that builds upon traditional metric spaces and becomes increasingly popular. Berzig [13] established fundamental research for his field in 2022 when he presented the formal definition of suprametric spaces together with their essential topological and analytical characteristics. The research established two major findings that described the structural framework of suprametric spaces while demonstrating their practical applications through initial research. Developing these ideas in 2023, Berzig [14] deeply explains such points in a paper that is entirely dedicated to nonlinear contractions on the b, suprametric spaces. Following this research, several delicate contractive behaviors were revealed, which are in fact very different from those of the classical metric and the b-metric spaces and these serve as pretty good illustrations of the richness and the versatility of the suprametric framework. Simultaneously, S. K. Panda et al. [15] proposed the concept of extended suprametric spaces and provided correlations with Stone-type results. In this work, the authors have firmly established the mathematical bases of suprametric spaces by linking them to well-known topological results, thus enhancing their mathematical utility and leading to abstract analysis. The progress of this study was continued by Berzig [16], who provided additional fixed-point results in generalized suprametric spaces. Those results were improvements on previous works and opened the way towards a more profound understanding of the existence and behavior when structural conditions are relaxed. Such results underscore the effectiveness of suprametric systems to address nonlinear problems that cannot be addressed through traditional measures of metrics. Ahmad et al. [17, 18, 19] examined the concept of multivalued and proximal contraction in suprametric contexts, where convergence and the closest proximity point were proved, and they were directly applied to nonlinear fractional differential equations and two-dimensional Volterra integral equations. Younis et al. [20] employ a fixed point approach to discuss the existence and uniqueness of the solutions of Chua's attractor model within the framework of suprametric spaces, incorporating the Atangana–Baleanu derivative and a two-step Lagrange polynomial approximation. Aldosari et al. [21] introduced a strong extended s-suprametric space as a new mathematical concept, which extends existing suprametric systems and provides a framework to study fixed points. This advancement enables rigorous treatment of nonlinear boundary value problems, including applications in chemical diffusion and satellite coupling systems, which show both theoretical and practical importance.

Motivated by the extensive literature on Θ-contractions and the inherent flexibility of suprametric spaces, this research aims to extend fixed point theory beyond the limitations of classical metric frameworks. Many real-world problems require distance measurements that do not comply with standard triangle inequality rules, although Θ-contractions work well in basic metric systems. Suprametric spaces allow generalizations of distance measurements, but researchers have yet to establish complete contraction mapping systems for these spaces. The research gap is addressed through our development of advanced contraction mappings, which include rational Θ-contractions and rational Θ-Gupta-Saxena contractions that function within complete suprametric spaces. The research develops two objectives that aim to create practical solutions to complex nonlinear problems through the development of nonlinear analysis methods.

2. Preliminaries

In this section, we recall several fundamental definitions from the existing literature that are essential for the development of this work. These basic concepts provide the necessary framework for suprametric spaces and generalized contraction mappings, and will be used throughout the paper to establish our main results.
Barzig [13] introduced the notion of a *suprametric space*, which is defined as follows:

Definition 2.1. [13] Consider $d : X \times X \to [0, \infty)$ as a mapping on a nonempty set X that adheres to the following conditions:

(1) $d(x, y) \geq 0$
(2) $d(x, y) = 0 \ \Leftrightarrow x = y$,
(3) $d(x, y) = d(y, x)$,
(4) $d(x, z) \leq d\,(x, y) + d\,(y, z) + \hbar d\,(x, y)\, d\,(y, z)$

for all $x, y, z \in X$ with $\hbar \geq 0$, then the pair (X, d) is referred to as suprametric space.

Every metric naturally satisfies the conditions of a suprametric, but one can also generate suprametrics from existing metrics by suitably altering or weakening the triangle inequality. Various constructions allow defining such functions that maintain the essential properties of a distance, while providing more flexibility. This approach facilitates the investigation of fixed point results in spaces more general than conventional metric spaces.

Example 2.2. Let (X, d) be a metric space, and let $\eta, \mu > 0$ be fixed positive numbers. Define the following two functions on X:

(1) $d_1^{\eta}(x, y) = d(x, y)\,(d(x, y) + \eta)$,
(2) $d_2^{\mu}(x, y) = \mu\,(e^{d(x,y)} - 1)$.

Then d_1^{η} and d_2^{μ} are suprametrics with constants $\hbar = \frac{2}{\eta}$ and $\hbar = \frac{1}{\mu}$, respectively.

For a specific instance, take $X = \mathbb{R}$ with $d(x, y) = |x - y|$. By choosing $d = d_1^1$ or $d = d_2^1$, these functions fail to satisfy the usual triangle inequality, as

$$d(0, 1) + d(1, 2) < d(0, 2).$$

Additionally, in this example, d is not a suprametric with constant $\hbar = \frac{1}{3}$, demonstrating that any suprametric with a given constant $\hbar$ is automatically a suprametric for all constants $\hbar' > \hbar$.

Definition 2.3. Let (X, d) be a suprametric space.

(i): A sequence $\{x_n\}$ is said to converge to $x \in X$ if, for any $\epsilon > 0$, there exists a positive integer N_ϵ such that $d(x_n, x) < \epsilon$ for all $n \geq N_\epsilon$. Symbolically,

$$\lim_{n\to\infty} x_n = x.$$

(ii): The sequence $\{x_n\}$ is called a Cauchy sequence if, for every $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$ for all $m, n \geq N_\epsilon$.
(iii): The space is said to be complete if every Cauchy sequence in X converges to a point in X.
(iv): If d is continuous, then each convergent sequence has a unique limit.

In 2014, Jleli et al. [1] introduced a significant generalization of the classical Banach contraction principle by proposing the concept of a Θ-contraction. This framework extends the standard contraction condition by employing an auxiliary function Θ, allowing a broader class of mappings to be analyzed within fixed point theory. The formal definition of the function class associated with Θ-contractions is given below.

Definition 2.4. Let Θ denote the family of all functions $\theta : (0, \infty) \to (1, \infty)$ satisfying the following conditions:

(Θ_1) Θ is nondecreasing;
(Θ_2) for each sequence $\{x_n\} \subseteq \mathbb{R}^+$,

$$\lim_{n\to\infty} \Theta(x_n) = 1 \iff \lim_{n\to\infty} x_n = 0;$$

(Θ_3) there exist constants $r \in (0, 1)$ and $\ell \in (0, \infty]$ such that

$$\lim_{t\to 0^+} \frac{\theta(t) - 1}{t^r} = \ell.$$

In 2017, Ahmad et al. [6] proposed a modification of the original Θ-contraction by replacing the condition (Θ_3) with a weaker continuity requirement, denoted as (Θ_3'), which is defined as follows:

Definition 2.5. Let $\Theta : (0, \infty) \to (1, \infty)$ be a function satisfying the following conditions:

(Θ_1) Θ is nondecreasing;
(Θ_2) for each sequence $\{x_n\} \subseteq \mathbb{R}^+$,

$$\lim_{n\to\infty} \Theta(x_n) = 1 \iff \lim_{n\to\infty} x_n = 0;$$

(Θ_3') Θ is continuous.

All functions Θ that satisfy the conditions (Θ_1), (Θ_2), (Θ_3') are considered to belong to the set Ω. Thus, we denote by Ω the collection of all such functions Θ.

These conditions ensure that the class Ω includes a wide variety of functions that are suitable for analyzing Θ-contractions. The nondecreasing property (Θ_1) preserves the order of distances, while the limit condition (Θ_2) guarantees that Θ accurately captures the behavior of sequences approaching zero. The continuity requirement (Θ_3') allows the application of standard analytical tools and ensures the stability of the contraction framework. Collectively, these properties make functions in Ω highly flexible and applicable in diverse fixed point problems. To illustrate, we provide several concrete examples of functions belonging to this class.

Example 2.6. Define the following functions for all $t > 0$:

(1) $\Theta_1(t) = e^{\sqrt{t}}$,
(2) $\Theta_2(t) = e^{\sqrt{te^t}}$,
(3) $\Theta_3(t) = e^t$,
(4) $\Theta_4(t) = \cosh t$,
(5) $\Theta_5(t) = 1 + \ln(1 + t)$,
(6) $\Theta_6(t) = e^{te^t}$.

Then the functions $\Theta_1, \Theta_2, \Theta_3, \Theta_4, \Theta_5, \Theta_6 \in \Omega$ satisfy the desired properties.

The manuscript is organized as follows.
In Section 1, we review the existing literature on Θ-contractions and suprametric spaces, and present the motivation for the present study.

Section 2 is devoted to recalling fundamental definitions and preliminary concepts from the literature that are essential for the development of the main results.

In Section 3, we establish several fixed point theorems in complete suprametric spaces, including Theorem 3.2 based on Θ-contractions (Definition 3.1), Theorem 3.4 based on rational Θ-contractions (Definition 3.3), and Theorem 3.7 based on rational Θ-Gupta-Saxena contractions (Definition 3.6). Theoretical result is supported by concrete example (Example 3.9).

In Section 4, we apply Theorem 3.2 to investigate the existence and uniqueness of solutions for the fourth-order Euler-Bernoulli beam equation 4.1 subject to clamped-clamped boundary conditions.

Finally, Section 5 concludes the paper with a discussion of the significance of the obtained results and presents several open problems for future research.

3. Main Results

In this section, we investigate the existence and uniqueness of fixed points in complete suprametric spaces by employing three distinct types of contractions: Θ-contraction, rational Θ-contraction, and rational Θ-Gupta-Saxena contraction. The theoretical results are further illustrated and validated through numerical simulations and graphical examples, demonstrating their practical applicability.

Definition 3.1. A mapping $F : X \to X$ is called a Θ-contraction, if there exist a function $\Theta \in \Omega$ and a constant $k \in (0, 1)$ such that for all $x, y \in X$ with $Fx \neq Fy$,

$$\Theta\big(d(Fx, Fy)\big) \leq \big[\Theta(d(x, y))\big]^k. \tag{3.1}$$

Here, Ω is the class of functions satisfying the standard conditions (Θ_1)–(Θ_3').

Theorem 3.2. *Let (X, d) be a complete suprametric space and $F : X \to X$ be a Θ-contraction. Then F has a unique fixed point $z \in X$, and for any $x_0 \in X$, the sequence $\{F^n x_0\}$ converges to z.*

Proof. Choose an arbitrary $x_0 \in X$ and define the sequence $\{x_n\}$ by $x_{n+1} = Fx_n$ for $n \geq 0$. If there exists n_0 such that $x_{n_0} = x_{n_0+1}$, then x_{n_0} is a fixed point. Otherwise, assume $x_n \neq x_{n+1}$ for all n. Since $Fx_{n-1} \neq Fx_n$, the contraction condition (3.1) of Definition 3.1 gives

$$\Theta\big(d(x_n, x_{n+1})\big) = \Theta\big(d(Fx_{n-1}, Fx_n)\big) \leq \big[\Theta\big(d(x_{n-1}, x_n)\big)\big]^k.$$

By induction,

$$1 < \Theta\big(d(x_n, x_{n+1})\big) \leq \big[\Theta\big(d(x_0, x_1)\big)\big]^{k^n} \quad \text{for all } n \geq 1.$$

Let $M = \Theta\big(d(x_0, x_1)\big) > 1$. Then $\big[M\big]^{k^n} \to 1$ as $n \to \infty$. Hence $\lim_{n\to\infty} \Theta\big(d(x_n, x_{n+1})\big) = 1$, and by (Θ_2),

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{3.2}$$

Now we show that $\{x_n\}$ is a Cauchy sequence. Suppose, for contradiction, that $\{x_n\}$ is not Cauchy. Then there exists $\epsilon > 0$ and subsequences $\{x_{p(n)}\}$, $\{x_{q(n)}\}$ with $p(n) > q(n) > n$ such that

$$d(x_{p(n)}, x_{q(n)}) \geq \epsilon, \quad d(x_{p(n)-1}, x_{q(n)}) < \epsilon \quad \text{for all } n. \tag{3.3}$$

Using the suprametric inequality,

$$d(x_{p(n)}, x_{q(n)}) \leq d(x_{p(n)}, x_{p(n)-1}) + d(x_{p(n)-1}, x_{q(n)}) + \hbar\, d(x_{p(n)}, x_{p(n)-1})\, d(x_{p(n)-1}, x_{q(n)}).$$

From (3.3), we have $d(x_{p(n)-1}, x_{q(n)}) < \epsilon$ and $d(x_{p(n)}, x_{q(n)}) \geq \epsilon$. Therefore,

$$\epsilon \leq d(x_{p(n)}, x_{q(n)}) \leq d(x_{p(n)}, x_{p(n)-1}) + \epsilon + \hbar\, d(x_{p(n)}, x_{p(n)-1})\, \epsilon. \tag{3.4}$$

Let $a_n = d(x_{p(n)}, x_{p(n)-1})$. From (3.2), we have $\lim_{n\to\infty} a_n = 0$. Taking the limit inferior and limit superior in (3.4):

$$\begin{aligned}\epsilon &\le \liminf_{n\to\infty} d(x_{p(n)}, x_{q(n)}) \\ &\le \limsup_{n\to\infty} d(x_{p(n)}, x_{q(n)}) \\ &\le \limsup_{n\to\infty} \big(a_n + \epsilon + \hbar a_n \epsilon\big) \\ &= \lim_{n\to\infty} a_n + \epsilon + \hbar\epsilon \lim_{n\to\infty} a_n \\ &= 0 + \epsilon + \hbar\epsilon \cdot 0 = \epsilon.\end{aligned}$$

Hence,

$$\liminf_{n\to\infty} d(x_{p(n)}, x_{q(n)}) = \limsup_{n\to\infty} d(x_{p(n)}, x_{q(n)}) = \epsilon,$$

which implies

$$\lim_{n\to\infty} d(x_{p(n)}, x_{q(n)}) = \epsilon. \tag{3.5}$$

From (3.2), we can choose $\delta > 0$ such that $2\delta + \hbar\delta^2 < \epsilon$. Then choose n_0 such that for all $n \ge n_0$,

$$d(x_{p(n)}, x_{p(n)+1}) < \delta \quad \text{and} \quad d(x_{q(n)}, x_{q(n)+1}) < \delta. \tag{3.6}$$

We claim that $d(x_{p(n)+1}, x_{q(n)+1}) > 0$ for all $n \ge n_0$. Suppose, to the contrary, that there exists $n \ge n_0$ such that $d(x_{p(n)+1}, x_{q(n)+1}) = 0$. Then using the suprametric inequality,

$$d(x_{p(n)}, x_{q(n)}) \le d(x_{p(n)}, x_{p(n)+1}) + d(x_{p(n)+1}, x_{q(n)}) + \hbar\, d(x_{p(n)}, x_{p(n)+1})\, d(x_{p(n)+1}, x_{q(n)}).$$

Since $d(x_{p(n)+1}, x_{q(n)+1}) = 0$, we have $x_{p(n)+1} = x_{q(n)+1}$, so

$$d(x_{p(n)+1}, x_{q(n)}) = d(x_{q(n)+1}, x_{q(n)}) = d(x_{q(n)}, x_{q(n)+1}).$$

Thus,

$$d(x_{p(n)}, x_{q(n)}) \le d(x_{p(n)}, x_{p(n)+1}) + d(x_{q(n)}, x_{q(n)+1}) + \hbar\, d(x_{p(n)}, x_{p(n)+1})\, d(x_{q(n)}, x_{q(n)+1}).$$

Using (3.6), we get

$$d(x_{p(n)}, x_{q(n)}) < \delta + \delta + \hbar\delta^2 = 2\delta + \hbar\delta^2 < \epsilon.$$

This contradicts (3.3) which states $d(x_{p(n)}, x_{q(n)}) \ge \epsilon$. Hence,

$$d(x_{p(n)+1}, x_{q(n)+1}) > 0 \quad \text{for all } n \ge n_0. \tag{3.7}$$

Since $d(x_{p(n)+1}, x_{q(n)+1}) > 0$, we have $Fx_{p(n)} \ne Fx_{q(n)}$, so the contraction condition (3.1) of Definition 3.1 applies:

$$\Theta\big(d(x_{p(n)+1}, x_{q(n)+1})\big) = \Theta\big(d(Fx_{p(n)}, Fx_{q(n)})\big) \le \big[\Theta\big(d(x_{p(n)}, x_{q(n)})\big)\big]^k.$$

Taking the limit as $n \to \infty$ and using the continuity of Θ (condition (Θ_3')) together with (3.5), we obtain

$$\Theta(\epsilon) \le [\Theta(\epsilon)]^k.$$

Since $\Theta(\epsilon) > 1$ and $k \in (0,1)$, this inequality implies $\Theta(\epsilon) \le 1$, a contradiction. Therefore, $\{x_n\}$ is a Cauchy sequence. By completeness of X, there exists $z \in X$ such that $\lim_{n\to\infty} x_n = z$.

Next, we show that z is a fixed point of F. If a subsequence $\{x_{n_k}\}$ satisfies $Fx_{n_k} = Fz$, then $x_{n_k+1} = Fz$, so taking limits gives $z = Fz$. Otherwise, for all sufficiently large n, we have $Fx_n \ne Fz$. Then the contraction condition (3.1) of Definition 3.1 yields

$$\Theta\big(d(Fx_n, Fz)\big) \le \big[\Theta\big(d(x_n, z)\big)\big]^k.$$

As $n \to \infty$, $d(x_n, z) \to 0$, so $\Theta\big(d(x_n, z)\big) \to 1$ by (Θ_2). Hence the right-hand side tends to 1, and consequently $\lim_{n\to\infty} \Theta\big(d(Fx_n, Fz)\big) = 1$. Again by (Θ_2), we get $\lim_{n\to\infty} d(Fx_n, Fz) = 0$. But $Fx_n = x_{n+1} \to z$, so $d(z, Fz) = 0$. Thus, $Fz = z$.
For **uniqueness**, suppose $u \in X$ is another fixed point with $u \neq z$. Then $d(z, u) > 0$ and $Fz \neq Fu$, utilizing the contraction condition (3.1) of Definition 3.1 we conclude

$$\Theta\big(d(z, u)\big) = \Theta\big(d(Fz, Fu)\big) \leq \big[\Theta\big(d(z, u)\big)\big]^k.$$

Since $\Theta\big(d(z, u)\big) > 1$ and $k \in (0, 1)$, this inequality implies $\Theta\big(d(z, u)\big) \leq 1$, a contradiction. Hence z is the unique fixed point of F. Moreover, for any $x_0 \in X$, the sequence $\{F^n x_0\} = \{x_n\}$ converges to z. This completes the proof. □

Definition 3.3. A mapping $F : X \to X$ is called Ćirić-Reich-Rus Θ-contraction if there exist a function $\Theta \in \Omega$ and a constant $k \in (0, 1)$ such that for all $x, y \in X$ with $Fx \neq Fy$,

$$\Theta\big(d(Fx, Fy)\big) \leq \big[\Theta(R(x, y))\big]^k, \tag{3.8}$$

where

$$R(x, y) = \max\big\{d(x, y), d(x, Fx), d(y, Fy)\big\}.$$

Theorem 3.4. *Let (X, d) be a complete suprametric space and $F : X \to X$ be a Ćirić-Reich-Rus Θ-contraction. Then F has a unique fixed point $z \in X$, and for any $x_0 \in X$, the sequence $\{F^n x_0\}$ converges to z.*

Proof. Choose an arbitrary $x_0 \in X$ and define the sequence $\{x_n\}$ by $x_{n+1} = Fx_n$ for all $n \geq 0$. If there exists n_0 such that $x_{n_0} = x_{n_0+1}$, then x_{n_0} is a fixed point and the proof is complete. Otherwise, assume $x_n \neq x_{n+1}$ for all n, so that $Fx_{n-1} \neq Fx_n$ and the contraction condition (3.8) of Definition 3.3 can be applied.
Let $d_n = d(x_n, x_{n+1})$ for $n \geq 0$. For $x = x_{n-1}$ and $y = x_n$, we have

$$R(x_{n-1}, x_n) = \max\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = \max\{d_{n-1}, d_n\}.$$

If we choose $\max\{d_{n-1}, d_n\} = d_n$ and applying the contraction condition we arrive at,

$$\Theta(d_n) \leq [\Theta(d_n)]^k. \tag{3.9}$$

which is a contraction. Therefore, $\max\{d_{n-1}, d_n\} = d_{n-1}$, and consequently $d_n < d_{n-1}$. Thus, the sequence $\{d_n\}$ is strictly decreasing and bounded below by 0. From (3.9), we obtain

$$\Theta(d_n) \leq [\Theta(d_{n-1})]^k.$$

By induction,

$$\Theta(d_n) \leq [\Theta(d_0)]^{k^n} \quad \text{for all } n \geq 1.$$

Since $\Theta(d_0) > 1$ and $k^n \to 0$ as $n \to \infty$, we have $[\Theta(d_0)]^{k^n} \to 1$. Hence, $\lim_{n\to\infty} \Theta(d_n) = 1$. By condition (Θ_2), it follows that

$$\lim_{n\to\infty} d_n = \lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{3.10}$$

Next, we demonstrate that the sequence $\{x_n\}$ is Cauchy. Suppose, on the contrary, that it is not. Then there exists $\epsilon > 0$ and subsequences $\{x_{p(n)}\}$, $\{x_{q(n)}\}$ with $p(n) > q(n) > n$. The subsequent steps follow exactly the same computations presented from equation (3.3) to equation (3.7) of Theorem 3.2. Therefore, we can apply the contraction condition (3.8) of Definition 3.3 to $x = x_{p(n)}$ and $y = x_{q(n)}$ for $n \geq n_0$:

$$\Theta\big(d(x_{p(n)+1}, x_{q(n)+1})\big) \leq \big[\Theta\big(R(x_{p(n)}, x_{q(n)})\big)\big]^k. \tag{3.11}$$

Now,

$$R(x_{p(n)}, x_{q(n)}) = \max\big\{d(x_{p(n)}, x_{q(n)}), d(x_{p(n)}, x_{p(n)+1}), d(x_{q(n)}, x_{q(n)+1})\big\}.$$

Since, we have $\lim_{n\to\infty} d(x_{p(n)}, x_{q(n)}) = \epsilon$, while $d(x_{p(n)}, x_{p(n)+1}) \to 0$ and $d(x_{q(n)}, x_{q(n)+1}) \to 0$ (as calculated in Theorem 3.2). Consequently, for sufficiently large n, the maximum is achieved by $d(x_{p(n)}, x_{q(n)})$, so

$$\lim_{n\to\infty} R(x_{p(n)}, x_{q(n)}) = \epsilon. \tag{3.12}$$

Moreover, using the suprametric inequality and (3.10), one can show that

$$\lim_{n\to\infty} d(x_{p(n)+1}, x_{q(n)+1}) = \epsilon. \tag{3.13}$$

Taking the limit as $n \to \infty$ in (3.11) and using the continuity of Θ together with (3.12) and (3.13), we obtain

$$\Theta(\epsilon) \leq [\Theta(\epsilon)]^k .$$

Since $\Theta(\epsilon) > 1$ and $k \in (0, 1)$, this inequality implies $\Theta(\epsilon) \leq 1$, a contradiction. Therefore, $\{x_n\}$ is a Cauchy sequence. By completeness of X, there exists $z \in X$ such that $\lim_{n\to\infty} x_n = z$. Now we prove that z is a fixed point of F. We consider two cases.
Case 1: There exists a subsequence $\{x_{n_k}\}$ such that $Fx_{n_k} = Fz$. Then $x_{n_k+1} = Fz$, and taking the limit as $k \to \infty$ yields $z = Fz$.
Case 2: For all sufficiently large n, we have $Fx_n \neq Fz$. Then we can apply the contraction condition (3.8) of Definition 3.3 to $x = x_n$ and $y = z$:

$$\Theta\big(d(Fx_n, Fz)\big) \leq \big[\Theta\big(R(x_n, z)\big)\big]^k , \tag{3.14}$$

where

$$R(x_n, z) = \max\{d(x_n, z), d(x_n, Fx_n), d(z, Fz)\} = \max\{d(x_n, z), d(x_n, x_{n+1}), d(z, Fz)\} .$$

As $n \to \infty$, we have $d(x_n, z) \to 0$, $d(x_n, x_{n+1}) \to 0$, and $d(z, Fz)$ remains fixed. Hence,

$$\lim_{n\to\infty} R(x_n, z) = d(z, Fz). \tag{3.15}$$

Also, $d(Fx_n, Fz) = d(x_{n+1}, Fz) \to d(z, Fz)$. Taking the limit in (3.14) and using the continuity of Θ, we get

$$\Theta\big(d(z, Fz)\big) \leq \big[\Theta\big(d(z, Fz)\big)\big]^k .$$

If $d(z, Fz) > 0$, then $\Theta(d(z, Fz)) > 1$, and since $k \in (0, 1)$, we have $[\Theta(d(z, Fz))]^k < \Theta(d(z, Fz))$, leading to a contradiction. Therefore, $d(z, Fz) = 0$, so $Fz = z$. Thus, in either case, z is a fixed point of F.
For **uniqueness**, suppose $u \in X$ is another fixed point of F with $u \neq z$. Then $d(z, u) > 0$ and $Fz \neq Fu$. Applying the contraction condition (3.8) of Definition 3.3 to $x = z$ and $y = u$,

$$\Theta\big(d(z, u)\big) = \Theta\big(d(Fz, Fu)\big) \leq \big[\Theta\big(R(z, u)\big)\big]^k ,$$

where

$$R(z, u) = \max\{d(z, u), d(z, Fz), d(u, Fu)\} = \max\{d(z, u), 0, 0\} = d(z, u).$$

Hence,

$$\Theta\big(d(z, u)\big) \leq \big[\Theta\big(d(z, u)\big)\big]^k .$$

Since $\Theta(d(z, u)) > 1$ and $k \in (0, 1)$, this implies $\Theta(d(z, u)) \leq 1$, a contradiction. Therefore, $u = z$. Thus, F has a unique fixed point $z \in X$, and for any $x_0 \in X$, the sequence $\{F^n x_0\} = \{x_n\}$ converges to z. This completes the proof. □

From Definition 3.3, it is evident that the Ćirić-Reich-Rus Θ-contraction property can be expressed in simplified forms by considering individual components of the rational term $R(x, y)$. Accordingly, we can state the following corollaries, which show that the contraction condition continues to hold when only selected terms of $R(x, y)$ are used.

Corollary 3.5. *We can obtain the following results from the Ćirić-Reich-Rus Θ-contraction property, if we choose*

(1)
$$R(x,y) = \max\big\{d(x,y), d(x,Fx)\big\},$$
(2)
$$R(x,y) = \max\big\{d(x,y), d(y,Fy)\big\},$$
showing that the contraction property holds even when considering only this component of the rational term.

Definition 3.6. A mapping $F : X \to X$ is called rational $\Theta-$Gupta-Saxena contraction, if there exist a function $\Theta \in \Omega$ and a constant $k \in (0,1)$ such that for all $x, y \in X$ with $Fx \neq Fy$,
$$\Theta\big(d(Fx,Fy)\big) \leq \big[\Theta(R(x,y))\big]^k, \tag{3.16}$$
where
$$R(x,y) = \max\left\{d(x,y), \frac{d(x,Fx)(1+d(y,Fy))}{1+d(x,y)}, \frac{d(y,Fy)(1+d(x,Fx))}{1+d(x,y)}\right\}.$$

Theorem 3.7. *Let $F : X \to X$ be a rational $\Theta-$Gupta-Saxena contraction on a complete suprametric space. Then F has a unique fixed point $z \in X$, and for any $x_0 \in X$, the sequence $\{F^n x_0\}$ converges to z.*

Proof. Let $x_0 \in X$ and define the sequence $\{x_n\}$ by $x_{n+1} = Fx_n$ for all $n \geq 0$. If there exists n_0 such that $x_{n_0} = x_{n_0+1}$, then x_{n_0} is a fixed point and the proof is complete. Otherwise, assume $x_n \neq x_{n+1}$ for all n, so that $Fx_{n-1} \neq Fx_n$ and the contraction condition can be applied. Let $d_n = d(x_n, x_{n+1})$ for $n \geq 0$. For $x = x_{n-1}$ and $y = x_n$ (with $n \geq 1$), we have $d(x_{n-1}, x_n) = d_{n-1} > 0$. From the contraction condition (3.16) of Definition 3.6, we obtain
$$\Theta(d_n) = \Theta\big(d(Fx_{n-1}, Fx_n)\big) \leq \big[\Theta\big(R(x_{n-1}, x_n)\big)\big]^k.$$
By definition, we can estimate the followings
$$R(x_{n-1}, x_n) = \max\left\{d_{n-1}, \frac{d_{n-1}(1+d_n)}{1+d_{n-1}}, d_n\right\}.$$
Consider the difference
$$\frac{d_{n-1}(1+d_n)}{1+d_{n-1}} - d_n = \frac{d_{n-1} - d_n}{1+d_{n-1}}.$$
Since $1 + d_{n-1} > 0$, the sign of this difference is determined by $d_{n-1} - d_n$. Consequently,
$$\frac{d_{n-1}(1+d_n)}{1+d_{n-1}} \leq d_n \quad \text{if and only if} \quad d_{n-1} \leq d_n,$$
and
$$\frac{d_{n-1}(1+d_n)}{1+d_{n-1}} \geq d_n \quad \text{if and only if} \quad d_{n-1} \geq d_n.$$
Therefore, the middle term never exceeds both d_{n-1} and d_n; it lies between them. Hence,
$$R(x_{n-1}, x_n) = \max\{d_{n-1}, d_n\}. \tag{3.17}$$
Suppose, for contradiction, that $\max\{d_{n-1}, d_n\} = d_n$. Then $R(x_{n-1}, x_n) = d_n$ and the contraction inequality yields
$$\Theta(d_n) \leq [\Theta(d_n)]^k.$$
Since $d_n > 0$, we have $\Theta(d_n) > 1$ by property (Θ_1). As $k \in (0,1)$, for any $t > 1$ we have $t^k < t$. Hence, $[\Theta(d_n)]^k < \Theta(d_n)$, contradicting (3). Therefore, we must have $\max\{d_{n-1}, d_n\} = d_{n-1}$, which implies $d_n < d_{n-1}$. Thus, $\{d_n\}$ is strictly decreasing. Moreover,
$$\Theta(d_n) \leq [\Theta(d_{n-1})]^k.$$

By induction, we arrive at

$$\Theta(d_n) \leq [\Theta(d_0)]^{k^n} \quad \text{for all } n \geq 1.$$

Since $\Theta(d_0) > 1$ and $k^n \to 0$ as $n \to \infty$, we have $[\Theta(d_0)]^{k^n} \to 1$. Hence, $\lim_{n\to\infty} \Theta(d_n) = 1$. By condition (Θ_2),

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{3.18}$$

Next, we demonstrate that the sequence $\{x_n\}$ is Cauchy. Suppose, on the contrary, that it is not. Then there exists $\epsilon > 0$ and subsequences $\{x_{p(n)}\}$, $\{x_{q(n)}\}$ with $p(n) > q(n) > n$. The subsequent steps follow exactly the same computations presented from equation (3.3) to equation (3.7) of Theorem 3.2. Since $d(x_{p(n)+1}, x_{q(n)+1}) > 0$, we have $Fx_{p(n)} \neq Fx_{q(n)}$, so the contraction condition applies:

$$\Theta\big(d(x_{p(n)+1}, x_{q(n)+1})\big) \leq \big[\Theta\big(R(x_{p(n)}, x_{q(n)})\big)\big]^k. \tag{3.19}$$

Now evaluate $R(x_{p(n)}, x_{q(n)})$:

$$\begin{aligned} A_n &= d(x_{p(n)}, x_{q(n)}), \\ B_n &= \frac{d(x_{p(n)}, x_{p(n)+1})\big(1 + d(x_{q(n)}, x_{q(n)+1})\big)}{1 + A_n}, \\ C_n &= \frac{d(x_{q(n)}, x_{q(n)+1})\big(1 + d(x_{p(n)}, x_{p(n)+1})\big)}{1 + A_n}. \end{aligned}$$

As $n \to \infty$, we have $A_n \to \epsilon$, where $d(x_{p(n)}, x_{p(n)+1}) \to 0$ and $d(x_{q(n)}, x_{q(n)+1}) \to 0$. Consequently,

$$B_n \to \frac{0 \cdot (1+0)}{1+\epsilon} = 0, \qquad C_n \to \frac{0 \cdot (1+0)}{1+\epsilon} = 0.$$

Thus,

$$\lim_{n\to\infty} R(x_{p(n)}, x_{q(n)}) = \lim_{n\to\infty} \max\{A_n, B_n, C_n\} = \epsilon. \tag{3.20}$$

Moreover, using the suprametric inequality and (3.18), one can show that

$$\lim_{n\to\infty} d(x_{p(n)+1}, x_{q(n)+1}) = \epsilon. \tag{3.21}$$

Taking the limit as $n \to \infty$ in (3.19) and using the continuity of Θ together with (3.20) and (3.21), we obtain

$$\Theta(\epsilon) \leq [\Theta(\epsilon)]^k.$$

Since $\Theta(\epsilon) > 1$ and $k \in (0,1)$, this inequality implies $\Theta(\epsilon) \leq 1$, a contradiction. Therefore, $\{x_n\}$ is a Cauchy sequence. By completeness of X, there exists $z \in X$ such that $\lim_{n\to\infty} x_n = z$. Now, we will show that z is a fixed point of F. We consider two cases.

Case 1: There exists a subsequence $\{x_{n_k}\}$ such that $Fx_{n_k} = Fz$. Then $x_{n_k+1} = Fz$, and taking the limit as $k \to \infty$ yields $z = Fz$.

Case 2: Assume that $Fx_n \neq Fz$ for all sufficiently large n. Then the contraction condition (3.16) of Definition 3.6 gives

$$\Theta(d(Fx_n, Fz)) \leq [\Theta(R(x_n, z))]^k. \tag{3.22}$$

Observe that

$$R(x_n, z) = \max\left\{ d(x_n, z), \frac{d(x_n, Fx_n)\big(1 + d(z, Fz)\big)}{1 + d(x_n, z)}, \frac{d(z, Fz)\big(1 + d(x_n, Fx_n)\big)}{1 + d(x_n, z)} \right\}.$$

As $n \to \infty$, we have $d(x_n, z) \to 0$ and $d(x_n, Fx_n) = d(x_n, x_{n+1}) \to 0$ by (3.18). Hence,

$$\lim_{n\to\infty} R(x_n, z) = d(z, Fz).$$

Also, $\lim_{n\to\infty} d(Fx_n, Fz) = d(z, Fz)$ because $Fx_n = x_{n+1} \to z$. Passing to the limit in (3.22) and using the continuity of Θ, we obtain

$$\Theta(d(z, Fz)) \le [\Theta(d(z, Fz))]^k.$$

Since $k \in (0, 1)$ and $\Theta(d(z, Fz)) > 1$ if $d(z, Fz) > 0$, this forces $d(z, Fz) = 0$. Hence, $Fz = z$.
For **uniqueness**, suppose $u \in X$ is another fixed point of F with $u \neq z$. Then $d(z, u) > 0$ and $Fz \neq Fu$. Applying the contraction condition,

$$\Theta(d(z, u)) = \Theta(d(Fz, Fu)) \le [\Theta(R(z, u))]^k.$$

Since z and u are fixed points,

$$R(z, u) = \max\left\{d(z, u), \frac{0\cdot(1+0)}{1+d(z, u)}, \frac{0\cdot(1+0)}{1+d(z, u)}\right\} = d(z, u).$$

Thus, $\Theta(d(z, u)) \le [\Theta(d(z, u))]^k$, which implies $\Theta(d(z, u)) \le 1$, contradicting $\Theta(d(z, u)) > 1$. Hence, $u = z$. Therefore, F has a unique fixed point $z \in X$, and for any $x_0 \in X$, the sequence $\{F^n x_0\} = \{x_n\}$ converges to z. □

From Definition 3.6, it is evident that the rational Θ-Gupta-Saxena contraction property can be obtained in simplified forms by considering individual components of the rational term $R(x, y)$. Consequently, we can state the following corollaries, which highlight that the contraction condition continues to hold when only selected terms of $R(x, y)$ are used.

Corollary 3.8. *We can obtain the following results from Θ-Gupta-Saxena contraction property, if we choose*

(1)

$$R(x, y) = \max\left\{d(x, y), \frac{d(x, Fx)(1+d(y, Fy))}{1+d(x, y)}\right\},$$

(2)

$$R(x, y) = \max\left\{d(x, y), \frac{d(y, Fy)(1+d(x, Fx))}{1+d(x, y)}\right\},$$

showing that the contraction property holds even when considering only this component of the rational term.

Example 3.9. Let $X = \{x_n : n \in \mathbb{N}\}$, $x_n = \frac{n(n+1)}{2}$ endowed with the distance

$$d(x, y) = |x-y|(|x-y|+1), \quad x, y \in X.$$

Consider $x_1 = 1, x_2 = 3, x_3 = 6$:

$$\begin{aligned} d(x_1, x_2) &= |1-3|(|1-3|+1) = 2\cdot 3 = 6,\\ d(x_2, x_3) &= |3-6|(|3-6|+1) = 3\cdot 4 = 12,\\ d(x_1, x_3) &= |1-6|(|1-6|+1) = 5\cdot 6 = 30. \end{aligned}$$

The triangle inequality fails since $d(x_1, x_3) = 30 > 6 + 12 = 18$, so (X, d) is not a metric space. However, X satisfies the suprametric inequality with $\hbar = \frac{2}{9}$ (for all $x, y \in X$ it holds for $\hbar = 1$)

$$d(x_1, x_3) = 30 \le 6 + 12 + \frac{2}{9}\cdot 6\cdot 12 = 34,$$

and similarly for other triples. Define the mapping $F : X \to X$ by

$$Fx_1 = x_1, \quad Fx_n = x_{n-1},\ n \ge 2.$$

Compute the ratio $\frac{d(Fx_1,Fx_m)}{d(x_1,x_m)}$, $m > 2$. First, we will calculate

$$x_m - x_1 = \frac{m(m+1)}{2} - 1 = \frac{m^2+m-2}{2},$$
$$d(x_1,x_m) = \frac{m^2+m-2}{2}\left(\frac{m^2+m-2}{2}+1\right) = \frac{(m^2+m-2)(m^2+m)}{4}.$$

Next, we will calculate

$$x_{m-1} - x_1 = \frac{(m-1)m}{2} - 1 = \frac{m^2-m-2}{2},$$
$$d(Fx_1,Fx_m) = \frac{(m^2-m-2)(m^2-m)}{4}.$$

After factorizing each term, we conclude

$$m^2-m-2 = (m-2)(m+1), \qquad m^2-m = m(m-1),$$
$$m^2+m-2 = (m-1)(m+2), \qquad m^2+m = m(m+1).$$

Thus,

$$\frac{d(Fx_1,Fx_m)}{d(x_1,x_m)} = \frac{(m-2)(m+1)\cdot m(m-1)}{(m-1)(m+2)\cdot m(m+1)} = \frac{m-2}{m+2} = \frac{1-\frac{2}{m}}{1+\frac{2}{m}} \to 1 \text{ as } m \to \infty.$$

Hence F is not a Banach contraction. Let

$$\Theta(t) = e^{\sqrt{te^t}},$$

it is easy to show that $\Theta \in \Omega$. After applying the Θ-contractive condition:

$$\theta(d(Fx,Fy)) \le [\theta(d(x,y))]^k.$$

Applying the natural logarithm and substituting $\theta(t)$ gives

$$\ln\theta(d(Fx,Fy)) \le k\ln\theta(d(x,y)),$$
$$\sqrt{d(Fx,Fy)e^{d(Fx,Fy)}} \le k\sqrt{d(x,y)e^{d(x,y)}}.$$

Squaring both sides yields

$$d(Fx,Fy)\,e^{d(Fx,Fy)} \le k^2\,d(x,y)\,e^{d(x,y)}.$$

Rewriting as a ratio form:

$$\frac{d(Fx,Fy)}{d(x,y)}\,e^{\,d(Fx,Fy)-d(x,y)} \le k^2.$$

Case 1: For $x = x_1, y = x_m,$ and $m > 2$, we have

$$d(Fx_1,Fx_m) = d(x_1,x_{m-1}) = \frac{(m^2-m-2)(m^2-m)}{4},$$
$$d(x_1,x_m) = \frac{(m^2+m-2)(m^2+m)}{4},$$
$$\frac{d(Fx_1,Fx_m)}{d(x_1,x_m)} = \frac{(m-2)(m+1)\cdot m(m-1)}{(m-1)(m+2)\cdot m(m+1)} = \frac{m-2}{m+2},$$
$$d(Fx_1,Fx_m) - d(x_1,x_m) = -m(m^2-1),$$
$$\frac{d(Fx_1,Fx_m)}{d(x_1,x_m)}e^{d(Fx_1,Fx_m)-d(x_1,x_m)} = \frac{m-2}{m+2}e^{-m(m^2-1)} \le k^2 = e^{-1}.$$

Case 2: For $x = x_n, y = x_m$, $m > n > 1$. We have $Fx_n = x_{n-1}$ and $Fx_m = x_{m-1}$. Now, we will compute distances

$$\begin{aligned} d(Fx_n, Fx_m) &= d(x_{n-1}, x_{m-1}) \\ &= \left|x_{m-1} - x_{n-1}\right|\Big(\left|x_{m-1} - x_{n-1}\right| + 1\Big) \\ &= \frac{(m-n)(m+n-1)}{2} \cdot \frac{(m-n)(m+n-1)+2}{2} \\ &= \frac{(m-n)(m+n-1)\big((m-n)(m+n-1)+2\big)}{4} \end{aligned}$$

$$\begin{aligned} d(x_n, x_m) &= |x_m - x_n|(|x_m - x_n| + 1) \\ &= \frac{(m-n)(m+n+1)}{2} \cdot \frac{(m-n)(m+n+1)+2}{2} \\ &= \frac{(m-n)(m+n+1)((m-n)(m+n+1)+2)}{4}. \end{aligned}$$

Now, we will compute their ratio for the inequality

$$\begin{aligned} \frac{d(Fx_n, Fx_m)}{d(x_n, x_m)} &= \frac{(m-n)(m+n-1)((m-n)(m+n-1)+2)}{(m-n)(m+n+1)((m-n)(m+n+1)+2)} \\ &= \frac{(m+n-1)((m-n)(m+n-1)+2)}{(m+n+1)((m-n)(m+n+1)+2)}. \end{aligned}$$

After that, We need to estimate the difference for the exponential power term

$$\begin{aligned} & d(Fx_n, Fx_m) - d(x_n, x_m) \\ &= \frac{(m-n)(m+n-1)((m-n)(m+n-1)+2)}{4} \\ &\quad - \frac{(m-n)(m+n+1)((m-n)(m+n+1)+2)}{4} \\ &= \frac{m-n}{4}\Big[(m+n-1)((m-n)(m+n-1)+2) \\ &\quad - (m+n+1)((m-n)(m+n+1)+2)\Big] \\ &= \frac{m-n}{4}\Big[(m-n)\big((m+n-1)^2 - (m+n+1)^2\big) \\ &\quad + 2\big((m+n-1) - (m+n+1)\big)\Big] \\ &= \frac{m-n}{4}\Big[(m-n)(-4(m+n)) + 2(-2)\Big] \\ &= -(m-n)(m+n) - 1. \end{aligned}$$

Substituting all the values, we get

$$\frac{d(Fx_n, Fx_m)}{d(x_n, x_m)}\, e^{d(Fx_n, Fx_m) - d(x_n, x_m)} = \frac{(m+n-1)((m-n)(m+n-1)+2)}{(m+n+1)((m-n)(m+n+1)+2)}\, e^{-(m-n)(m+n)-1}.$$

For all $m > n > 1$, the exponential decay ensures that

$$\frac{d(Fx_n, Fx_m)}{d(x_n, x_m)}\, e^{d(Fx_n, Fx_m) - d(x_n, x_m)} \leq k^2 = e^{-1},$$

so the Θ-contractive condition holds, which demonstrates F is a Θ-contraction but not a Banach contraction. Therefore, all the conditions of Theorem 3.2 are satisfied and F has a unique fixed point $x_1 = 1$.

4. Application to Fourth Order Euler-Bernoulli Beam Equation

The mathematical modeling of physical and engineering systems that demonstrate bending and elastic behavior and exhibit advanced deformation properties results in the development of fourth-order ordinary differential equations (ODEs). The field of structural mechanics uses beam and plate models together with elastic foundations and fluid-structure interaction problems, and advanced mechanical systems that depend on curvature and stiffness. The presence of a fourth-order derivative in the system models demonstrates that these systems require both displacement and slope, curvature, and higher geometric constraints for their accurate modeling of structural and dynamic behavior.

An essential component in the formulation of fourth–order boundary value problems is the specification of appropriate boundary conditions. Among the most important are the clamped–clamped (or built–in) boundary conditions, given by

$$x(0) = x'(0) = x(1) = x'(1) = 0.$$

The system is unmoving at its boundary points as it has fixed endpoints that do not allow movement or rotation there. In the engineering field, these kinds of constraints are very often referred to, e.g., when talking about structures that are connected to very rigid support points. The idea of clamped boundary conditions in mathematics is that the allowed solutions are very limited, but at the same time, these conditions bring stability and describe the form of the Green function that is going to fulfill these boundary conditions.

In a particularly important special case, the general fourth–order operator reduces to the simple form

$$x^{(4)}(t) = f(t), \tag{4.1}$$

subject to the same clamped–clamped boundary conditions as shown in the Figure 1. This model represents the canonical non–dimensional form of the Euler–Bernoulli beam equation and serves as a fundamental prototype in both theoretical analysis and practical applications.

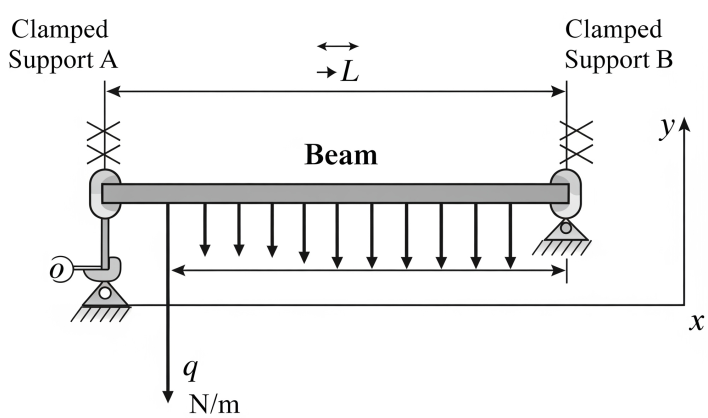

Figure 1. Euler–Bernoulli beam model

In its dimensional form, the Euler–Bernoulli beam equation is written as

$$EI\, x^{(4)}(t) = q(t),$$

where E denotes Young's modulus of the material, I is the second moment of area of the beam cross–section, and $q(t)$ represents the applied transverse load. By normalizing the beam length to unity and scaling the physical parameters so that $EI = 1$, one arrives at the simplified equation (4.1), where $f(t)$ denotes the corresponding non–dimensional load distribution. Physically, the clamped–clamped configuration models a beam that is fully restrained at both ends, leading to high stiffness, reduced deflection under loading, and characteristic bending patterns that differ markedly from those of simply supported or free beams.

This less complicated Euler, Bernoulli theory has been a staple in the toolkits of scientists and engineers of various disciplines over the years. Illustrative examples would be bridges, flooring of buildings, and structural members capable of carrying the load in civil and structural engineering, where it is used to simulate the bending behavior. Similarly, the idea is that the bending of wing spars and fuselage frames due to the aerodynamic forces can be described by the same form of the equation. Besides shafts, machine components, and robotic arms, mechanical engineering applications are also areas where this PDE can be effectively used. The balance maintained between the external load and the internal elastic rest spaces in all these scenarios is the physical basis of the equation, and through it, one can obtain predicted values of deflections, bending moments, and stresses that are necessary for safety and performance assessment.

The Green function method provides a complete and effective solution system to solve boundary value problems that require its application. The Green function $G(t, s)$ describes the deflection at a point t through the operation of a unit point load which acts from point s while it automatically meets the established boundary conditions. For the problem

$$x^{(4)}(t) = f(t), \qquad 0 \le t \le 1,$$

with clamped–clamped ends, the Green function $G(t, s)$ is a piecewise cubic polynomial in t, defined separately for $t < s$ and $t > s$. It is continuous together with its first and second derivatives at $t = s$, while its third derivative exhibits a unit jump at that point, reflecting the presence of a concentrated load.

The solution x belongs to $X = C[0, 1]$ and can be represented in integral form as

$$x(t) = \int_0^1 G(t, s)\, f\Big(s, x, x', x'', x''', x^{(4)}\Big)\; ds. \tag{4.2}$$

The corresponding Green function associated with (4.1) is given explicitly by

$$G(t, s) = \begin{cases} t^3\left(\dfrac{-2s^3 + 3s^2 - 1}{6}\right) + t^2\left(\dfrac{s^3 - 2s^2 + s}{2}\right), & 0 < t < s, \\ s^3\left(\dfrac{-2t^3 + 3t^2 - 1}{6}\right) + s^2\left(\dfrac{t^3 - 2t^2 + t}{2}\right), & s < t < 1. \end{cases} \tag{4.3}$$

This Green function allows the solution corresponding to any admissible load $f(t)$ to be obtained directly through integration, without the need to repeatedly solve the differential equation for different loading scenarios. The combination of clear physical interpretation and rigorous mathematical structure makes the Green function approach an invaluable tool, not only in theoretical analysis but also in practical engineering design and computation. Let $X = C[0, 1]$ with distance function

$$d(x, y) = \sup_{t\in[0,1]} |x(t) - y(t)|\big(|x(t) - y(t)| + \hbar\big).$$

Then (X, d) is a complete suprametric space. Define the operator $F : X \to X$ by

$$(Fx)(t) = \int_0^1 G(t, s) f(s, x(s))\, ds,$$

where $G(t,s)$ is the Green's function constructed above for the normalized ladder network model with Neumann boundary conditions. Then x is a unique solution of 4.2, iff it is fixed point of T. Next theorem provides us a sufficient condition for the existence and uniqueness of solution of problem 4.1.

Theorem 4.1. *Consider the Euler-Bernoulli beam equation with boundary conditions given in* (4.1). *Define the operator $F : X \to X$ as above, and suppose that for all $x, y \in X$ and $t \in [0,1]$,*

$$|f(t,x(t)) - f(t,y(t))| \leq |x(t) - y(t)|.$$

Then the Euler-Bernoulli beam equation admits a unique solution $u \in C[0,1]$.

Proof. Let $x, y \in X$ be arbitrary. By the definition of the operator T and the integral representation of the solution, we obtain

$$\begin{aligned} |Fx(t) - Fy(t)|\big(|Fx(t) - Fy(t)| + \hbar\big) &= \left| \int_0^1 G(t,s)\big[f(s,x(s)) - f(s,y(s))\big]\, ds \right| \\ &\quad \times \left(\left| \int_0^1 G(t,s)\big[f(s,x(s)) - f(s,y(s))\big]\, ds \right| + \hbar \right). \end{aligned}$$

Using the assumed condition on f together with the triangle inequality, we estimate the above expression as

$$\begin{aligned} |Fx(t) - Fy(t)|\big(|Fx(t) - Fy(t)| + \hbar\big) &\leq \left(|x(t) - y(t)| \int_0^1 |G(t,s)|\, ds \right) \\ &\quad \times \left(|x(t) - y(t)| \int_0^1 |G(t,s)|\, ds + \hbar \right). \end{aligned}$$

Since the Green function $G(t,s)$ is continuous on the compact set $[0,1] \times [0,1]$, it is bounded. Hence, there exists a constant $0 \leq \alpha < 1$ such that

$$|G(t,s)| \leq \alpha \quad \text{for all } t, s \in [0,1].$$

As a consequence, we have

$$\int_0^1 |G(t,s)|\, ds \leq \alpha \quad \text{for all } t \in [0,1].$$

Substituting this bound into the previous inequality yields

$$|Fx(t) - Fy(t)|\big(|Fx(t) - Fy(t)| + \hbar\big) \leq \alpha |x(t) - y(t)|\big(\alpha |x(t) - y(t)| + \hbar\big).$$

By suitably rearranging and factoring the terms on the right-hand side, the inequality can be rewritten in the form

$$|Fx(t) - Fy(t)|\big(|Fx(t) - Fy(t)| + \hbar\big) \leq \alpha^2 |x(t) - y(t)| \left(|x(t) - y(t)| + \frac{\hbar}{\alpha} \right).$$

Now, let us define $k = \alpha^2 \in (0,1)$ and introduce the modified constant $\hbar^* = \dfrac{\hbar}{\alpha} \geq 0$. With these notations, the above inequality becomes

$$|Fx(t) - Fy(t)|\big(|Fx(t) - Fy(t)| + \hbar\big) \leq k\, |x(t) - y(t)|\big(|x(t) - y(t)| + \hbar^*\big).$$

Taking the supremum over $t \in [0,1]$ on both sides, we obtain

$$\sup_{t \in [0,1]} |Fx(t) - Fy(t)|\big(|Fx(t) - Fy(t)| + \hbar\big) \leq k \sup_{t \in [0,1]} |x(t) - y(t)|\big(|x(t) - y(t)| + \hbar^*\big).$$

In terms of the metric d defined on X, this inequality is equivalently expressed as

$$d(Fx, Fy) \leq k\, d(x,y).$$

Equivalently, by applying the increasing mapping $\Theta(t) = e^t$ to both sides of the above inequality, we obtain the Θ-contraction formulation

$$e^{\left[d(Fx,Fy)\right]} \leq e^{\left[k\, d(x,y)\right]}.$$

This expression can be written in the equivalent form

$$e^{\left[d(Fx,Fy)\right]} \leq e^{k\left[d(x,y)\right]},$$

which, in terms of the mapping Θ, yields

$$\Theta\big(d(Fx,Fy)\big) \leq \big[\Theta(d(x,y))\big]^k.$$

Therefore, the operator F satisfies a Θ-contraction condition on the complete metric space (X,d). By the corresponding fixed point Theorem 3.2, F admits a unique fixed point in X, which corresponds to the unique solution of the Euler-Bernoulli beam boundary value problem. □

5. Conclusion

In this manuscript, we developed comprehensive study of fixed point theory, which studies complete suprametric spaces as its primary focus, while extending traditional metric space results to this wider and more adaptable environment. We established three different types of contractions, which include Θ-contractions, rational Θ-contractions, and rational Θ-Gupta-Saxena contractions. Each contraction class was formulated precisely, and the associated fixed point theorems were proven using iterative methods that are compatible with the suprametric structure.

The primary achievement of this research establishes a new generalization for traditional fixed point theorems. The suprametric approach enables us to use its triangle inequality to demonstrate fixed point theory applicability beyond traditional metric space limitations. This broader framework allows the analysis of a wider variety of distance-like structures, which often arise in both theoretical and applied contexts. Furthermore, the introduction of rational Θ-contractions and rational Θ-Gupta-Saxena contractions provides a more refined control over contraction behavior, incorporating additional interactions between points and their images. This significantly enhances the classical Θ-contraction paradigm and opens new avenues for applications. For each contraction type, we demonstrated that iterative sequences converge to the unique fixed point, offering a constructive method for approximating solutions. The theoretical results showed their real-world value through the demonstration of different examples and the use of graphical evidence, which established the convergence analysis. In addition, carefully constructed examples show how specific mappings in suprametric spaces satisfy the proposed contraction conditions, validating the theoretical framework in concrete settings.

To further demonstrate the applicability of our results, we applied the main fixed point theorem to a fourth-order boundary value problem from structural mechanics, namely the Euler–Bernoulli beam equation under clamped–clamped boundary conditions. By reformulating the problem as an integral equation and equipping the function space with an appropriate suprametric, we showed that the solution operator satisfies a Θ-contraction. The established abstraction theorem shows that our framework successfully solves non-linear engineering problems and scientific real-world problems through its ability to demonstrate the existence of one continuous solution.

This research expands fixed point theory boundaries through its demonstration of fixed point applications in metric spaces. The combination of generalized Θ-contractions with suprametric spaces allows us to establish existence and uniqueness results which offer both theoretical and practical applications. This research establishes a solid foundation for future research, which will investigate nonlinear analysis through fractional differential equations, integral equations, and other mathematical models that use suprametric structures.

Open Problems:

(1) Is it possible to extend Theorems 3.2, 3.4 and 3.7 to the setting of multivalued contractions $F : X \to CB(X)$, and establish corresponding fixed point or best proximity point results?

(2) Can Theorems 3.2, 3.4 and 3.7 be effectively employed to investigate the existence and uniqueness of solutions for fractional-order dynamical systems, such as the fractional-order Lorenz model [22], the fractional-order King cobra model [23], and other fractional-order systems [24]?

Statements and Declarations

The authors declare that they have no conflict of interest, and the manuscript has no associated data.

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