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ON A DOUBLE INERTIAL MANN-TYPE METHOD FOR BREGMAN-DEMIGENERALIZED FIXED POINT AND VARIATIONAL INEQUALITY PROBLEMS

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ABSTRACT. In this paper, we introduce a new double inertial Mann-type iterative algorithm for approximating a common solution of a demigeneralized fixed point problem and a monotone variational inequality problem in reflexive Banach spaces. Our method is developed in the framework of Bregman distances generated by a Legendre function, which provides a more general geometric structure than the classical norm framework. Under suitable assumptions on the control parameters, the demigeneralized mapping, and the monotone operator, we establish strong convergence of the proposed algorithm to a common solution of the two problems. The results presented here extend and unify several existing schemes in the literature, including classical Halpern iterations, inertial methods, and Bregman projection-type approaches.

Keywords. Double inertial Mann-type algorithm, Demigeneralized mapping, Monotone variational inequality, Common solution problem, Bregman distance, Reflexive Banach space, Strong convergence, Fixed Point Theory.

1. Introduction

Variational inequalities and fixed point methods are two central pillars in modern nonlinear analysis and optimization. On a real reflexive Banach space E with dual E^*, a nonempty closed and convex set $C \subset E$, and a (possibly set-valued) monotone operator $A : C \to E^*$, the classical variational inequality problem (VIP) seeks $x^\star \in C$ such that

$$\langle Ax^\star, y - x^\star \rangle \geq 0, \qquad \forall\, y \in C. \tag{1.1}$$

When $A = \partial f$ is the subdifferential of a proper, convex, lower semicontinuous function $f : E \to (-\infty, +\infty]$, (1.1) recovers the first–order optimality condition for the constrained minimization problem $\min_{x \in C} f(x)$; see the classical references of Rockafellar [34] and Ekeland–Temam [15]. In parallel, fixed point problems aim to find $x^\star \in C$ with $x^\star = Tx^\star$ for a mapping $T : C \to C$. Historically important iterative frameworks in Banach and Hilbert spaces include the Mann [23], Ishikawa [18] and Halpern [16] iterations, which have motivated a broad family of generalized nonexpansive and demigeneralized/relaxed mappings designed to retain monotonicity-like properties sufficient for convergence analysis; see, e.g., Browder–Petryshyn [8] and surveys/monographs such as Bauschke- Combettes [5].

Beyond Hilbert spaces, the geometry of general Banach spaces often calls for a distance-like tool that reflects the underlying convexity structure. If $f : E \to \mathbb{R}$ is a Legendre function (in particular, essentially smooth and essentially strictly convex), the associated *Bregman distance*

$$D_f(x, y) := f(x) - f(y) - \langle \nabla f(y),\, x - y \rangle \quad (x, y \in \mathrm{int}(\mathrm{dom} f))$$

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provides such a tool. Bregman distances have played a significant role in convex programming and projection/proximal methods since the seminal work of Bregman [7]; see also Censor–Zenios [12] and the monograph of Bauschke–Combettes [5]. Importantly for Banach-space analysis, Bregman distances admit Fejér-type monotonicity principles and Opial-type arguments adapted to reflexive spaces, enabling strong convergence results without relying on an inner product; see, for instance, Reich–Sabach [32, 33].

Let C be a nonempty closed and convex subset of $\mathrm{int}(\mathrm{dom} f)$. Following the Bregman approach, a mapping $T : C \to E$ with nonempty fixed point set $F(T)$ is called an $(\eta, 0)$–*Bregman demigeneralized* mapping (with $\eta \in (-\infty, 0]$) if

$$\langle x - q,\, \nabla f(x) - \nabla f(Tx)\rangle \;\geq\; (1-\eta)\, D_f(x, Tx), \qquad \forall\, x \in C,\; \forall\, q \in F(T),$$

together with the demiclosedness of $I - T$ at 0. This class embraces and extends Bregman nonexpansive and Bregman relatively nonexpansive mappings commonly used in Banach space fixed point theory; see, e.g., [32, 33, 5] for related notions and properties.

To accelerate iterative methods, inertial (momentum) terms-originating from Polyak's heavy-ball idea [30] and brought to convex minimization in Nesterov's scheme [28]—have become standard across optimization and operator-splitting algorithms. While inertialization can significantly improve practical speed, it must be balanced with stability, especially beyond Hilbert spaces. Among fixed point schemes, Halpern-type iterations are renowned for achieving *strong* convergence under mild control rules, where Mann/Ishikawa often yield only weak convergence; see [16, 5]. These observations motivate the design of *Halpern-type* Bregman iterations that incorporate inertial information while preserving Banach-space robustness.

Many applications call for a point that simultaneously satisfies a fixed point relation and a monotone variational inequality. In the present work, we consider the *common solution problem*

$$\text{find } x^\star \in C \quad \text{with} \quad x^\star \in F(T) \quad \text{and} \quad x^\star \text{ solves (1.1)},$$

where T is an $(\eta, 0)$–Bregman demigeneralized mapping and A is a continuous monotone operator. Building on the intuition that a single inertial step may be too rigid in balancing acceleration and stability, we adopt a *double-inertial* design (two controlled extrapolation sequences) in the spirit of inertial developments originating from Polyak [30] and, for the double-inertial idea in fixed point settings, see also Dong et al. [14]. Our algorithm is of *Halpern type* and is formulated entirely in the Bregman geometry induced by a Legendre function f.

Under a standard set of assumptions on f, A, and T (see Assumption 3.1), we propose a *double inertial Halpern-type* method that uses Bregman projections and Bregman forward steps to approximate a common solution of the demigeneralized fixed point problem and the monotone VIP. We establish *strong convergence* of the full sequence to the Bregman projection of a reference point onto the solution set $\Gamma := F(T) \cap VI(C, A)$ (Theorem 3.3). The proposed scheme unifies and extends several known special cases: when inertial parameters vanish it reduces to a noninertial Bregman-Halpern method; in Hilbert spaces with $f(x) = \frac{1}{2}\|x\|^2$ it recovers classical Halpern-type schemes; and with one inertial sequence suppressed it yields single-inertial variants.

The remainder of the paper is organized as follows. Section 2 recalls Banach-space and Bregman preliminaries and the class of Bregman demigeneralized mappings. Section 3 presents the double inertial Mann-type algorithm together with the parameter rules, and we develops the main convergence analysis and proves Theorem 3.3. Discussion and Concluding remarks appear in Section 4.

2. Preliminaries

Let E be a real reflexive Banach space with dual E^*, paired through the bilinear form $\langle \cdot, \cdot \rangle$. Throughout, $C \subset E$ denotes a nonempty, closed and convex subset. Given a proper, convex and lower semicontinuous function $f : E \to (-\infty, +\infty]$, its Fenchel conjugate is defined by

$$f^*(x^*) = \sup_{x \in E} \{\langle x^*, x\rangle - f(x)\}, \qquad x^* \in E^*.$$

Whenever f is Gateaux differentiable on $\operatorname{int}(\operatorname{dom} f)$, the corresponding *Bregman distance* is

$$D_f(x, y) = f(x) - f(y) - \langle \nabla f(y),\, x - y\rangle, \qquad x \in \operatorname{dom} f,\ y \in \operatorname{int}(\operatorname{dom} f), \tag{2.1}$$

and it coincides with the squared Hilbert norm when $f(x) = \frac{1}{2}\|x\|^2$ [7, 9].

A convex function f is called a *Legendre function* if it is essentially smooth and essentially strictly convex. Equivalently (see [4, 39]), both $\operatorname{int}(\operatorname{dom} f)$ and $\operatorname{int}(\operatorname{dom} f^*)$ are nonempty, f is Gateaux differentiable on $\operatorname{int}(\operatorname{dom} f)$ with ∇f defined exactly there, f^* is Gateaux differentiable on $\operatorname{int}(\operatorname{dom} f^*)$, and the gradients satisfy

$$\nabla f = (\nabla f^*)^{-1}.$$

For $x \in \operatorname{dom} f$, the *modulus of total convexity* of f at x is given by

$$v_f(x, t) = \inf\{D_f(y, x) : y \in \operatorname{dom} f,\ \|y - x\| = t\}.$$

The function f is said to be *totally convex on bounded subsets* of E if $v_f(B, t) > 0$ for every bounded set $B \subset E$ and every $t > 0$. Total convexity on bounded sets is known to be equivalent to uniform convexity on bounded sets [10, 9].

Moreover, f is called *strongly coercive* if $f(x_n) \to \infty$ whenever $\|x_n\| \to \infty$.

Lemma 2.1. *(See [39, Lemma 2.1]) Let $f : E \to \mathbb{R}$ be continuous, convex, and strongly coercive. The following are equivalent:*

1. *f is bounded and uniformly smooth on bounded subsets of E;*
2. *the conjugate f^* is Fréchet differentiable and ∇f^* is uniformly norm-to-norm continuous on bounded subsets of E^*;*
3. *$\operatorname{dom} f^* = E^*$, and f^* is strongly coercive and uniformly convex onbounded subsets of E^*.*

Lemma 2.2. *(See [9, 10]) If f is bounded and uniformly Fréchet differentiable on bounded sets, then ∇f is uniformly continuous on bounded subsets of E.*

Lemma 2.3. *(See [9, Thm. 2.10] and [10, Lemma 2.8]) If $\operatorname{dom} f$ contains at least two points, then f is totally convex on bounded sets if and only if, for every bounded sequence $\{x_n\}$ and any sequence $\{y_n\}$,*

$$D_f(y_n, x_n) \to 0 \quad \Longrightarrow \quad \|y_n - x_n\| \to 0.$$

For any $x \in \operatorname{dom} f$ and $y, z \in \operatorname{int}(\operatorname{dom} f)$, (See [7, 9]), the well-known three-point identity is

$$D_f(x, y) + D_f(y, z) - D_f(x, z) = \langle \nabla f(z) - \nabla f(y),\, x - y\rangle. \tag{2.2}$$

Furthermore, if f is strongly convex with modulus $\mu > 0$, then

$$D_f(x, y) \geq \frac{\mu}{2}\|x - y\|^2. \tag{2.3}$$

Define the auxiliary function $V_f : E \times E^* \to [0, \infty)$ by

$$V_f(x, x^*) = f(x) - \langle x, x^*\rangle + f^*(x^*). \tag{2.4}$$

Then,

$$V_f(x, x^*) = D_f(x, \nabla f^*(x^*)), \text{ and } V_f(x, x^*) + \langle y^*, \nabla f^*(x^*) - x\rangle \leq V_f(x, x^* + y^*), \tag{2.5}$$

as established in [10].

Definition 2.4. (See [19]) A mapping $A : C \to E^*$ is *monotone* if

$$\langle Ax - Ay,\, x - y\rangle \geq 0, \qquad \forall\, x, y \in C.$$

Lemma 2.5. *(See [32]) For $r > 0$ and $x \in E$, the associated Bregman resolvent of A is the operator $F_r : E \to C$ defined by*

$$F_r(x) = \left\{ z \in C :\, \langle Az, y - z\rangle + \tfrac{1}{r}\langle \nabla f(z) - \nabla f(x),\, y - z\rangle \geq 0,\, \forall\, y \in C \right\}.$$

Under the assumptions that A is continuous and monotone and that f is a coercive Legendre function, the resolvent F_r enjoys the following properties:

(1) *F_r is single-valued;*
(2) *$F(F_r) = VI(C, A)$, the solution set of the variational inequality;*
(3) *for every $p \in F(F_r)$,*

$$D_f(p, F_r x) + D_f(F_r x, x) \leq D_f(p, x); \tag{2.6}$$

(4) *the set $VI(C, A)$ is closed and convex.*

Lemma 2.6. *(See [29, Lemma 2.6]) Let f be proper, convex and lower semicontinuous. Let $z \in E$, let $t_i \in [0, 1]$ satisfy $\sum t_i = 1$, and let $x_i \in E$. Then*

$$D_f\left(z,\, \nabla f^*\Big(\sum_i t_i \nabla f(x_i)\Big)\right) \leq \sum_i t_i\, D_f(z, x_i).$$

Lemma 2.7. [38] *Let E be a Banach space, let $r > 0$ be a constant, and let $f : E \to \mathbb{R}$ be uniformly convex on bounded subsets of E. Then*

$$f\left(\sum_{k=0}^{n} \alpha_k x_k\right) \leq \sum_{k=0}^{n} \alpha_k f(x_k) - \alpha_i \alpha_j\, \rho_r(\|x_i - y_j\|).$$

Lemma 2.8. *(See [24, Lemma 2.7]) Suppose $\{D_f(x_n, x_0)\}$ is bounded for some $x_0 \in E$, and ∇f^* is bounded on bounded subsets of E^*. Then the sequence $\{x_n\}$ is bounded.*

Lemma 2.9. *(See [36, Lemma 2.9] and [22, Lemma 2.10].) If sequences $\{a_n\} \subset [0, \infty)$ and $\{\alpha_n\} \subset (0, 1)$ satisfy*

$$a_{n+1} \leq (1 - \alpha_n) a_n + \alpha_n \delta_n, \quad \alpha_n \to 0, \quad \sum \alpha_n = +\infty, \quad \limsup \delta_n \leq 0,$$

then $a_n \to 0$. Moreover, for any real sequence $\{b_n\}$, there exists an increasing subsequence $\{m_k\}$ such that $b_{m_k} \leq b_{m_k+1}$ and $b_{m_k} \leq b_n$ for all $n \leq m_k + 1$.

Definition 2.10. [31] Let E be a reflexive Banach space and C be a nonempty closed and convex subset of E. A Bregman projection of $x \in \operatorname{int}(\operatorname{dom} f)$ onto $C \subset \operatorname{int}(\operatorname{dom} f)$ is the unique vector $\operatorname{Proj}_C^f(x) \in C$ satisfying

$$D_f(\operatorname{Proj}_C^f(x), x) = \inf\{D_f(y, x) : y \in C\}.$$

Lemma 2.11. [31] *Let C be a nonempty closed and convex subset of a reflexive Banach space E and $x \in E$. Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function. Then,*

(a) *$z = \operatorname{Proj}_C^f(x)$ if and only if $\langle \nabla f(x) - \nabla f(z),\, y - z\rangle \leq 0, \quad \forall\, y \in C$.*
(b) *$D_f(y, Proj_C^f(x)) + D_f(Proj_C^f(x), x) \leq D_f(y, x), \quad \forall\, y \in C$.*

Definition 2.12. (See [24, 9].) Let $T : C \to \operatorname{int}(\operatorname{dom} f)$.

(1) T is *Bregman nonexpansive* if $D_f(Tx, Ty) \leq D_f(x, y)$ for all $x, y \in C$.
(2) T is *Bregman relatively nonexpansive* if $F(T) \neq \emptyset$ and $D_f(p, Tx) \leq D_f(p, x)$ for all $x \in C$ and $p \in F(T)$, and if moreover $F(T) = \widehat{F}(T)$.

(3) T is *Bregman firmly nonexpansive* if

$$\langle \nabla f(Tx) - \nabla f(Ty),\, Tx - Ty\rangle \le \langle \nabla f(x) - \nabla f(y),\, Tx - Ty\rangle.$$

(4) T is *Bregman strongly nonexpansive* if $F(T) \neq \emptyset$, $D_f(y, Tx) \le D_f(y, x)$ for all $x \in C$ and $y \in F(T)$, and if $D_f(y, x_n) - D_f(y, Tx_n) \to 0$ implies $D_f(Tx_n, x_n) \to 0$ for any bounded sequence $\{x_n\}$.

(5) T is *quasi-Bregman nonexpansive* if $F(T) \neq \varnothing$ and $D_f(p, Tx) \le D_f(p, x)$ for all $x \in C$ and $p \in F(T)$.

Definition 2.13. (See [1, 2]) Let $\eta \in (-\infty, 0]$. A mapping $T : C \to E$ with $F(T) \neq \emptyset$ is called $(\eta, 0)$-Bregman demigeneralized if

$$\langle x - q,\ \nabla f(x) - \nabla f(Tx)\rangle \ge (1 - \eta)\, D_f(x, Tx), \qquad \forall x \in C,\ \forall q \in F(T).$$

Lemma 2.14. *(See [1]) Let T be $(\eta, 0)$-Bregman demigeneralized with $F(T) \neq \emptyset$, and let $a \in [0, 1]$. Define*

$$S = \nabla f^*\big((1 - a)\,\nabla f + a\,\nabla f \circ T\big) : C \to E.$$

Then S is quasi-Bregman nonexpansive.

Lemma 2.15. [2] *Let E be a reflexive Banach space and C a nonempty closed and convex subset of E. Let $f : E \to \mathbb{R}$ be a strongly coercive, Legendre function, which is bounded, uniformly Frechet differentiable and totally convex on bounded subset of E. Let η be a real number with $\eta \in (-\infty, 0)$ and T an $(\eta, 0)$-Bregman demigeneralized mapping of C into E. Then $F(T)$ is closed and convex.*

3. Main Results

We begin this section with the following assumptions under which the strong convergence results are established:

Assumption 3.1. Let C be a nonempty closed and convex subset of a smooth, strictly convex and reflexive real Banach space E with dual E^*. Suppose the following conditions are satisfied:

(C1) $f : E \to \mathbb{R}$ is a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E.

(C2) $A : E \to E^*$ is a continuous monotone mapping, and $K_{r_n} : E \to C$ be defined by

$$K_{r_n}(x) := \{z \in C : \langle Az, y - z\rangle + \frac{1}{r_n}\langle \nabla f(z) - \nabla f(x), y - z\rangle \ge 0, \forall y \in C\}$$

where $\{r_n\} \subset (a, \infty)$, for some $a > 0$.

(C3) $T : C \to E$ is a $(\eta, 0)-$Bregman demigeneralized mapping and demiclosed at zero, with $F(T) \neq \emptyset$, where $\eta \in (-\infty, 0]$.

(C4) For $i = 1, 2$, $\{\theta_{n,i}\}$ is a sequence in $\left(0, \frac{\mu}{2}\right)$, where μ is as defined in (2.3), and letting $\{\theta_n\} := \{\theta_{n,1} + \theta_{n,2}\}$, then, $\theta_n = \circ(\gamma_n)$, where $\{\gamma_n\}_{n=1}^{\infty}$ is a sequence in $(0, 1)$ such that $\lim\limits_{n\to\infty} \gamma_n = 0$ and $\sum\limits_{n=1}^{\infty} \gamma_n = \infty$.

(C5) $\Gamma := F(T) \cap VI(C, A) \neq \emptyset$.

Remark 3.2. The proposed iterative scheme in Algorithm 3 makes several significant contributions to the existing literature on Mann-type iterations and variational inequality problems in Banach spaces:

Initialization: Let $\alpha > 0, \delta, \lambda \in (0,1)$ and choose $x_{-1}, x_0, x_1, u \in E$ to be arbitrary points and given x_{n-2}, x_{n-1}, x_n and $u_n \to u$.

Iterative Steps: Calculate x_{n+1} as follows:

Step 1. Choose $\alpha_{n,1}$, $\alpha_{n,1}$ such that, for each $i = 1, 2$

$$\alpha_{n,i} \leq \bar{\alpha}_{n,i} = \begin{cases} \min\left\{\alpha, \frac{\theta_{n,i}}{\|\nabla f(x_{n-(i-1)}) - \nabla f(x_{n-i})\|}\right\}, & if \;\; x_{n-(i-1)} \neq x_{n-i}, \\ \\ \alpha, \quad otherwise \end{cases} \tag{3.1}$$

Step 2. Compute

$$\begin{cases} t_n = P_C^f\Big(\nabla f^*[\nabla f(x_n) + \alpha_{n,1}(\nabla f(x_n) - \nabla f(x_{n-1})) + \alpha_{n,2}(\nabla f(x_{n-1}) - \nabla f(x_{n-2}))]\Big), \\ z_n = K_{r_n}(t_n), \\ y_n = \nabla f^*(\lambda \nabla f(z_n) + (1-\lambda)\nabla f(T z_n)), \\ x_{n+1} = \nabla f^*(\gamma_n \nabla f(u_n) + (1-\delta)(1-\gamma_n)\nabla f(z_n) + \delta(1-\gamma_n)\nabla f(y_n)). \end{cases} \tag{3.2}$$

(a) It extends the classical Mann iteration framework by incorporating Bregman projections and resolvent-based variational inequality constraints, thereby unifying fixed point and variational inequality approaches in a single algorithmic structure.

(b) A central innovation is the introduction of the balancing parameter δ, which regulates the interplay between the variational inequality component z_n and the Bregman demigeneralized mapping component y_n. This mechanism enhances both flexibility and stability, surpassing earlier Mann-type iterations that lacked such balancing control [6, 13].

(c) The algorithm integrates double inertial terms $\alpha_{n,1}$ and $\alpha_{n,2}$, which extrapolate information from both x_{n-1} and x_{n-2}. This inertial extrapolation accelerates convergence and stabilizes the iteration, aligning with recent advances in inertial methods [27, 17, 35].

(d) Compared to earlier works such as Xu, Altwaijry, and Chebbi [37], Liu and Li [21], and Ceng and Wen [11], which studied Mann or Halpern-type iterations under restrictive settings, the present algorithm provides a more general framework by combining resolvent-based variational inequality constraints, Bregman projections, and inertial extrapolation.

(e) Consequently, under Assumptions 3.1, the algorithm guarantees strong convergence to $P_\Gamma^f u$, thereby offering a unified and robust approach that advances the state of the art in iterative methods for Banach spaces.

Theorem 3.3. *Let C be a nonempty closed and convex subset of of a smooth, strictly convex and reflexive real Banach space E with dual E^*. Suppose conditions (C1) to (C5) are satisfied. Let $\{x_n\}_{n=1}^\infty$ be sequence generated by Algorithm 3, then $\{x_n\}_{n=1}^\infty$ converges strongly to $p := P_\Gamma^f u$.*

Proof. From Lemmas 2.5 and 2.15, it follows that the set Γ is closed and convex. Consequently, the generalized projection P_Γ^f is well defined. Let $p := P_\Gamma^f(u)$. Since f is bounded and uniformly smooth on bounded subsets of E (by Lemma 2.1), its conjugate function f^* is uniformly convex on bounded subsets of E^*. Using this fact together with (2.2), (2.3), and (3.1), and defining $w_n := \nabla f^*[\nabla f(x_n) + \alpha_{n,1}(\nabla f(x_n) - \nabla f(x_{n-1})) + \alpha_{n,2}(\nabla f(x_{n-1}) - \nabla f(x_{n-2}))]$, we obtain

$$\begin{aligned}
D_f(p,w_n) &= D_f(p,x_n) - D_f(w_n,x_n) + \langle \nabla f(w_n) - \nabla f(x_n), w_n - p\rangle \\
&= D_f(p,x_n) - D_f(w_n,x_n) + \alpha_{n,1}\langle \nabla f(x_n) - \nabla f(x_{n-1}), w_n - p\rangle \\
&\quad + \alpha_{n,2}\langle \nabla f(x_{n-1}) - \nabla f(x_{n-2}), w_n - p\rangle \\
&\leq D_f(p,x_n) - D_f(w_n,x_n) \\
&\quad + \Big(\alpha_{n,1}||\nabla f(x_n) - \nabla f(x_{n-1})|| + \alpha_{n,2}||\nabla f(x_{n-1}) - \nabla f(x_{n-2})||\Big)||w_n - p|| \\
&\leq D_f(p,x_n) - D_f(w_n,x_n) \\
&\quad + \frac{1}{2}\Big(\alpha_{n,1}||\nabla f(x_n) - \nabla f(x_{n-1})|| + \alpha_{n,2}||\nabla f(x_{n-1}) - \nabla f(x_{n-2})||\Big) \\
&\quad \times(||w_n - p||^2 + 1) \\
&\leq D_f(p,x_n) - D_f(w_n,x_n) + \Big(\alpha_{n,1}||\nabla f(x_n) - \nabla f(x_{n-1})|| + \alpha_{n,2}||\nabla f(x_{n-1}) - \nabla f(x_{n-2})||\Big) \\
&\quad \times\Big[||w_n - x_n||^2 + ||x_n - p||^2 + \frac{1}{2}\Big] \\
&\leq D_f(p,x_n) - D_f(w_n,x_n) + (\theta_{n,1} + \theta_{n,2})(D_f(w_n,x_n) + D_f(p,x_n)) + \frac{1}{2}(\theta_{n,1} + \theta_{n,2}) \\
&= (1+\theta_n)D_f(p,x_n) - (1-\theta_n)D_f(w_n,x_n) + \frac{1}{2}\theta_n. \qquad (3.3)
\end{aligned}$$

From Assumption (3.1)(C4), for any $\epsilon \in \big(0, \frac{\mu}{2}\big)$, there exists a natural number N such that $\theta_n \leq \gamma_n\epsilon$ for all $n \geq N$. Hence, by (3.3), we obtain

$$D_f(p,t_n) \leq D_f(p,w_n) \leq (1+\gamma_n\epsilon)D_f(p,x_n) - (1-\gamma_n\epsilon)D_f(w_n,x_n) + \frac{1}{2}\gamma_n\epsilon. \qquad (3.4)$$

From (3.3) and using (2.6) and (3.2), since $\gamma_n \to 0$ and $\frac{\theta_n}{\gamma_n} \to 0$ as $n \to \infty$, there exists $N_0 \in \mathbb{N}$ such that $\frac{\theta_n}{2\gamma_n} < 2\epsilon$, where $\epsilon \in (0, \frac{1}{2})$, for all $n \geq N_0$. Hence, we get

$$\begin{aligned}
D_f(p,x_{n+1}) &\leq \gamma_n D_f(p,u_n) + (1-\delta)(1-\gamma_n)D_f(p,z_n) + \delta(1-\gamma_n)D_f(p,y_n) \\
&\leq \gamma_n D_f(p,u_n) + (1-\delta)(1-\gamma_n)D_f(p,z_n) + \delta(1-\gamma_n)D_f(p,z_n) \\
&\leq \gamma_n D_f(p,u_n) + (1-\gamma_n)D_f(p,t_n) \\
&\leq \gamma_n D_f(p,u_n) + (1-\gamma_n)[1+\theta_n]D_f(p,x_n) + (1-\gamma_n)\frac{\theta_n}{2} \\
&= \Big[1 - \gamma_n\Big(1 - \frac{\theta_n}{\gamma_n}\Big)\Big]D_f(p,x_n) + \gamma_n\Big(\frac{\theta_n}{2\gamma_n} + D_f(p,u_n)\Big) \\
&\leq (1-\gamma_n(1-2\epsilon))D_f(p,x_n) + \gamma_n(1-2\epsilon)\frac{(\epsilon + D_f(p,u_n))}{1-2\epsilon} \qquad (3.5)
\end{aligned}$$

Since $\{u_n\}$ is bounded, there exiss $M > 0$ such that $\sup\limits_{n\in\mathbb{N}} D_f(p,u_n) \leq M$. For some $\epsilon > 0$, letting $M^* := max\Big\{\frac{\epsilon+M}{1-2\epsilon}, D_f(p,x_1)\Big\}$. Suppose that for $k \in \mathbb{N}$, $D_f(p,x_1) \leq M^*$. Then, we get using (3.5) that

$$\begin{aligned}
D_f(p,x_{k+1}) &\leq (1-\gamma_k(1-2\epsilon))D_f(p,x_k) + \gamma_k(1-2\epsilon)\frac{(\epsilon + D_f(p,u_k))}{1-2\epsilon} \\
&\leq (1-2\gamma_k(1-2\epsilon))M^* + \gamma_k(1-2\epsilon)M^* = M^*.
\end{aligned}$$

Hence, by induction, we conclude that $D_f(p,x_n) \leq M^*$ for all $n \in \mathbb{N}$. Therefore, the sequence $\{D_f(p,x_n)\}$ is bounded, and by (2.3), it follows that $\{x_n\}$ is bounded. Next, using (2.4), (2.5), Lemmas 2.7 and 2.5, (3.4), and the properties of the Bregman distance D_f, and letting

$$v_n := \nabla f^*\bigg(\frac{1}{1-\gamma_n}\Big[(1-\delta)(1-\gamma_n)\nabla f(z_n) + \delta(1-\gamma_n)\nabla f(y_n)\Big]\bigg),$$

we obtain

$$\begin{aligned}
D_f(p, v_n) &= V_f(p, \nabla f^*(v_n)) = V_f(p, \frac{1}{1-\gamma_n}[(1-\delta)(1-\gamma_n)\nabla f(z_n) + \delta(1-\gamma_n)\nabla f(y_n)]) \\
&= f(p) + f^*\Big(\frac{1}{1-\gamma_n}[(1-\delta)(1-\gamma_n)\nabla f(z_n) + \delta(1-\gamma_n)\nabla f(y_n)]\Big) \\
&\quad -\langle p, \frac{1}{1-\gamma_n}[(1-\delta)(1-\gamma_n)\nabla f(z_n) + \delta(1-\gamma_n)\nabla f(y_n)]\rangle \\
&\leq f(p) - \frac{(1-\delta)(1-\gamma_n)}{1-\gamma_n}\langle p, \nabla f(z_n)\rangle - \frac{\delta(1-\gamma_n)}{1-\gamma_n}\langle p, \nabla f(y_n)\rangle + \frac{(1-\delta)(1-\gamma_n)}{1-\gamma_n} f^*(\nabla f(z_n)) \\
&\quad + \frac{\delta(1-\gamma_n)}{1-\gamma_n} f^*(\nabla(y_n)) - \frac{(1-\delta)\delta(1-\gamma_n)^2}{(1-\gamma_n)^2}\rho_r^* f^*(||\nabla f(z_n) - \nabla f(y_n)||) \\
&= (1-\delta)V_f(p, \nabla f(z_n)) + \delta V_f(p, \nabla f(y_n)) - (1-\delta)\delta\rho_r^* f^*(||\nabla f(z_n) - \nabla f(y_n)||) \\
&= (1-\delta)D_f(p, z_n) + \delta D_f(p, y_n) - (1-\delta)\delta\rho_r^* f^*(||\nabla f(z_n) - \nabla f(y_n)||) \\
&\leq (1-\delta)D_f(p, x_n) + \delta D_f(p, y_n) - (1-\delta)\delta\rho_r^* f^*(||\nabla f(z_n) - \nabla f(y_n)||) \qquad (3.6) \\
&\leq (1-\delta)D_f(p, x_n) + \delta D_f(p, w_n) - (1-\delta)\delta\rho_r^* f^*(||\nabla f(z_n) - \nabla f(y_n)||) \\
&\leq (1-\delta)D_f(p, x_n) + \delta(1+\gamma_n\epsilon)D_f(p, x_n) + \frac{1}{2}\delta\epsilon\gamma_n - \delta(1-\gamma_n\epsilon)D_f(w_n, x_n) \\
&\quad -(1-\delta)\delta\rho_r^* f^*(||\nabla f(z_n) - \nabla f(y_n)||) \\
&\leq (1+\delta\epsilon\gamma_n)D_f(p, x_n) + \frac{1}{2}\delta\epsilon\gamma_n - \delta(1-\gamma_n\epsilon)D_f(w_n, x_n) \\
&\quad -(1-\delta)\delta\rho_r^* f^*(||\nabla f(z_n) - \nabla f(y_n)||). \qquad (3.7)
\end{aligned}$$

From the definition of x_{n+1}, and property of Bregman function D_f, with (2.6) and (3.6), we get

$$\begin{aligned}
D_f(p, x_{n+1}) &= D_f(p, P_C^f(\nabla f^*(\gamma_n \nabla f(u_n) + (1-\gamma_n)\nabla f(v_n)))) \\
&\leq D_f(p, \nabla f^*(\gamma_n \nabla f(u_n) + (1-\gamma_n)\nabla f(v_n))) \\
&\leq \gamma_n D_f(p, u_n) + (1-\gamma_n)D_f(p, v_n) \\
&\leq \gamma_n D_f(p, u_n) + [1-\gamma_n(1-\epsilon\delta)]D_f(p, x_n) + \frac{1}{2}\delta\epsilon\gamma_n \\
&\quad -(1-\gamma_n)\delta(1-\gamma_n\epsilon)D_f(w_n, x_n) - (1-\delta)\delta(1-\gamma_n)\rho_r^* f^*(||\nabla f(z_n) - \nabla f(y_n)||).
\end{aligned}$$

Thus

$$\begin{aligned}
&(1-\gamma_n)\delta(1-\gamma_n\epsilon)D_f(w_n, x_n) + (1-\delta)\delta(1-\gamma_n)\rho_r^* f^*(||\nabla f(z_n) - \nabla f(y_n)||) \\
\leq\ &\Big(D_f(p, x_n) - D_f(p, x_n)\Big) + \gamma_n\Big(D_f(p, u_n) + \frac{1}{2}\delta\epsilon - (1-\delta\epsilon)D_f(p, x_n)\Big). \qquad (3.8)
\end{aligned}$$

The rest of the proof is divided into two cases:

Case 1: There exists $N \in \mathbb{N}$ such that $D_f(p, x_{n+1}) \leq D_f(p, x_n)$ for all $n \geq N$. Since $\{D_f(p, x_n)\}$ is bounded, it follows that $\{D_f(p, x_n)\}$ converges, and therefore

$$D_f(p, x_{n+1}) - D_f(p, x_n) \to 0 \qquad \text{as } n \to \infty. \qquad (3.9)$$

Since $\lim_{n\to\infty} \gamma_n = 0$, we deduce from (3.8) and (3.9) that

$$\lim_{n\to\infty} D_f(w_n, x_n) = 0$$

and

$$\lim_{n\to\infty} \rho_r^* f^*(\|\nabla f(z_n) - \nabla f(y_n)\|) = 0.$$

Applying Lemma 2.3 and the properties of ρ_r^*, we obtain

$$\lim_{n\to\infty} \|w_n - x_n\| = 0 \qquad (3.10)$$

and

$$\lim_{n\to\infty} \|\nabla f(z_n) - \nabla f(y_n)\| = 0. \qquad (3.11)$$

Using condition (C1) and Lemma 2.2, we obtain that ∇f is uniformly continuous on bounded subsets of E. Combining this with (3.11), we arrive at

$$\lim_{n\to\infty} \|z_n - y_n\| = 0. \tag{3.12}$$

Next, applying Lemma 2.5(2.6) together with Lemma 2.6, we deduce the following:

$$\begin{aligned} D_f(z_n, x_n) &\leq D_f(p, x_n) - D_f(p, z_n) \\ &= D_f(p, x_n) - D_f(p, x_{n+1}) + D_f(p, x_{n+1}) - D_f(p, z_n) \\ &\leq D_f(p, x_n) - D_f(p, x_{n+1}) + \gamma_n D_f(p, u_n) + (1-\delta)(1-\gamma_n) D_f(p, z_n) \\ &\quad +\delta(1-\gamma_n) D_f(p, y_n) - D_f(p, z_n) \\ &\leq D_f(p, x_n) - D_f(p, x_{n+1}) + \gamma_n (D_f(p, u_n) - D_f(p, z_n)). \end{aligned} \tag{3.13}$$

Applying (3.9) in (3.13), we obtain

$$\lim_{n\to\infty} D_f(z_n, x_n) = 0. \tag{3.14}$$

Using Lemma 2.11 and an argument similar to that of (3.13), we estimate

$$\begin{aligned} D_f(t_n, x_n) &\leq D_f(p, x_n) - D_f(p, t_n) \\ &\leq D_f(p, x_n) - D_f(p, x_{n+1}) + \gamma_n \big(D_f(p, u_n) - D_f(p, t_n)\big). \end{aligned} \tag{3.15}$$

Applying (3.9) in (3.13), we obtain

$$\lim_{n\to\infty} D_f(t_n, x_n) = 0. \tag{3.16}$$

From (3.14) and (3.16), together with condition (C1) and Lemma 2.2, we respectively obtain

$$\lim_{n\to\infty} \|z_n - x_n\| = 0, \tag{3.17}$$

and

$$\lim_{n\to\infty} \|t_n - x_n\| = 0. \tag{3.18}$$

Combining (3.10), (3.17), and (3.18), we obtain

$$\lim_{n\to\infty} \|z_n - w_n\| = 0, \tag{3.19}$$

and

$$\lim_{n\to\infty} \|t_n - w_n\| = 0. \tag{3.20}$$

thus

$$\lim_{n\to\infty} \|t_n - z_n\| = 0. \tag{3.21}$$

Using (3.11) and (3.2), we obtain

$$\|\nabla f(z_n) - \nabla f(Tz_n)\| = \frac{1}{1-\lambda} \|\nabla f(y_n) - \nabla f(z_n)\| \to 0 \qquad \text{as } n \to \infty.$$

which implies that

$$\lim_{n\to\infty} \|z_n - Tz_n\| = 0. \tag{3.22}$$

and

$$\begin{aligned} ||\nabla f(x_{n+1}) - \nabla f(x_n)|| \leq\ & \gamma_n ||\nabla f(u_n) - \nabla f(x_n)|| + (1-\delta)(1-\gamma_n) ||\nabla f(z_n) - \nabla f(x_n)|| \\ & +\delta(1-\gamma_n) ||\nabla f(y_n) - \nabla f(x_n)|| \end{aligned}$$

It follows from (3.12),(3.17), (C1) and by Lemma 2.2, we obtain

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.23}$$

Since $\{x_n\}$ is bounded and E is reflexive, we may select a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup q$ Then, using (3.17) and (3.18), we obtain

$$z_{n_i} \to q \text{ and } t_{n_i} \to q \qquad \text{as } i \to \infty. \tag{3.24}$$

Thus, from (3.22) and the fact that T is demiclosed at zero, we obtain that $q \in F(T)$. Now we show that $q \in VI(C, A)$. By definition, we have

$$\langle Az_n, y - z_n \rangle + \Big\langle \frac{\nabla f(z_n) - \nabla f(t_n)}{r_n}, y - z_n \Big\rangle \geq 0, \qquad \forall\, y \in C, \tag{3.25}$$

and hence

$$\langle Az_{n_i}, y - z_{n_i} \rangle + \Big\langle \frac{\nabla f(z_{n_i}) - \nabla f(t_{n_i})}{r_{n_i}}, y - z_{n_i} \Big\rangle \geq 0, \qquad \forall\, y \in C. \tag{3.26}$$

Set $b_s := sy + (1-s)q$ for all $s \in (0, 1]$ and $y \in C$. Consequently, $b_s \in C$. From (3.26), it follows that

$$\begin{aligned} \langle Ab_s, b_s - z_n \rangle &\geq \langle Ab_s, b_s - z_n \rangle - \langle Az_n, b_s - z_n \rangle - \Big\langle \frac{\nabla f(z_n) - \nabla f(t_n)}{r_n}, b_s - z_n \Big\rangle \\ &= \langle Ab_s - Az_n, b_s - z_n \rangle - \Big\langle \frac{\nabla f(z_n) - \nabla f(t_n)}{r_n}, b_s - z_n \Big\rangle. \end{aligned}$$

But from (3.21), (C1) and Lemma 2.2, we have

$$\frac{\nabla f(z_n) - \nabla f(t_n)}{r_n} \to 0 \qquad \text{as } n \to \infty,$$

and by monotonicity of A it follows that

$$\langle Ab_s - Az_n, b_s - z_n \rangle \geq 0.$$

Thus,

$$0 \leq \liminf_{n \to \infty} \langle Ab_s - Az_n, b_s - z_n \rangle = \langle Ab_s, b_s - q \rangle,$$

and hence

$$\langle Ab_s, b_s - q \rangle \geq 0, \qquad \forall\, y \in C.$$

Letting $s \to 0$, the continuity of A implies that

$$\langle Aq, y - q \rangle \geq 0, \qquad \forall\, y \in C.$$

This shows that $q \in VI(C, A)$, and hence $q \in \Gamma = F(T) \cap VI(C, A)$. Finally, we show that x_n converges strongly to $p := \Pi_{F(T) \cap VI(C,A)} u$. Observe that

$$\limsup_{n \to \infty} \langle \nabla f(u_n) - \nabla f(x_n),\, x_n - p \rangle = \lim_{i \to \infty} \langle \nabla f(u_{n_i}) - \nabla f(x_{n_i}),\, x_{n_i} - p, \rangle = \langle \nabla f(u) - \nabla f(q),\, q - p \rangle.$$

By Lemma 2.11, we have

$$\langle \nabla f(u) - \nabla f(q),\, z - p \rangle \leq 0,$$

and hence

$$\limsup_{n \to \infty} \langle \nabla f(u_n) - \nabla f(q),\, x_n - p \rangle = \langle \nabla f(u) - \nabla f(q),\, q - p, \rangle \leq 0. \tag{3.27}$$

It follows from (3.23) and (3.27) that

$$\limsup_{n \to \infty} \langle \nabla f(u_n) - \nabla f(q),\, x_{n+1} - p, \rangle \leq 0. \tag{3.28}$$

Finally, we show that $x_n \to p$, by using the (2.5), (3.3) and (3.6) , using the fact that $\frac{\theta_n}{\gamma_n} \to 0$ as $n \to \infty$, there exists $\epsilon \in (0, 1)$ such that $\frac{\theta_n}{\gamma_n} \leq \epsilon$ for all $n \geq N$, we obtain

$$\begin{aligned}
D_f(p, x_{n+1}) &\leq V_f(p, \gamma_n \nabla f(u_n) + (1-\gamma_n)\nabla f(v_n)) \\
&\leq V_f(p, \gamma_n \nabla f(u_n) + (1-\gamma_n)\nabla f(v_n) - \gamma_n(\nabla f(u_n) - \nabla f(p))) \\
&\quad + \gamma_n \langle \nabla f(u_n) - f(p), x_{n+1} - p\rangle \\
&= V_f(p, \gamma_n \nabla f(p) + (1-\gamma_n)\nabla f(v_n)) + \gamma_n \langle \nabla f(u_n) - f(p), x_{n+1} - p\rangle \\
&\leq \gamma_n D_f(p,p) + (1-\gamma_n) D_f(p, v_n) + \gamma_n \langle \nabla f(u_n) - f(p), x_{n+1} - p\rangle \\
&= (1-\gamma_n) D_f(p, v_n) + \gamma_n \langle \nabla f(u_n) - f(p), x_{n+1} - p\rangle \\
&\leq \Big(1 - \gamma_n\Big(1 - \delta \frac{\theta_n}{\gamma_n}\Big)\Big) D_f(p, x_n) + \gamma_n \Big(\langle \nabla f(u_n) - f(p), x_{n+1} - p\rangle + \delta \frac{\theta_n}{2\gamma_n}\Big) \\
&\leq \Big(1 - \gamma_n(1-\delta\epsilon)\Big) D_f(p, x_n) + \gamma_n(1-\delta\epsilon) \frac{(\langle \nabla f(u_n) - f(p), x_{n+1} - p\rangle + \delta\epsilon)}{1-\delta\epsilon}. \qquad (3.29)
\end{aligned}$$

Therefore, using Lemma 2.9 and (3.28) in (3.29), we obtain $D_f(p, x_n) \to 0$ as $n \to \infty$, then by Lemma 2.3 we get $x_n \to p$ as $n \to \infty$.

Case 2. Assume that there exists a subsequence $\{n_i\}$ of $\{n\}$ satisfying

$$D_f(p, x_{n_i}) < D_f(p, x_{n_i+1}) \quad \text{for all } i \in \mathbb{N}.$$

By second part of Lemma 2.9, we may choose an increasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$,

$$D_f(p, x_{m_k}) \leq D_f(p, x_{n_i+1}) \quad \text{and} \quad D_f(p, x_{n_i}) \leq D_f(p, x_{m_k+1}) \quad \text{for all } k \in \mathbb{N}.$$

From (3.8), (3.9), and the fact that $\gamma_n \to 0$, it follows that

$$D_f(w_{m_k}, x_{m_k}) \to 0 \quad \text{and} \quad \rho^*(\|\nabla f(z_{m_k}) - \nabla f(y_{m_k})\|) \to 0,$$

as $k \to \infty$.

Using the same argument as in Case 1, we obtain

$$z_{m_k} - Tz_{m_k} \to 0, \qquad z_{m_k} - t_{m_k} \to 0, x_{m_k} - y_{m_k} \to 0$$

and hence

$$x_{m_k} - x_{m_k+1} \to 0$$

as $k \to \infty$. Therefore,

$$\limsup_{k\to\infty} \langle \nabla f(u_{m_k}) - \nabla f(p),\, x_{m_k+1} - p\rangle \leq 0. \qquad (3.30)$$

Next, from (3.29), we have

$$D_f(p, x_{m_k+1}) \leq \Big(1 - \gamma_{m_k}\Big(1 - \delta \frac{\theta_{m_k}}{\gamma_{m_k}}\Big)\Big) D_f(p, x_{m_k}) + \gamma_{m_k}\Big(\langle \nabla f(u_{m_k}) - f(p), x_{m_k+1} - p\rangle + \delta \frac{\theta_{m_k}}{2\gamma_{m_k}}\Big) \qquad (3.31)$$

Since $D_f(p, x_{m_k}) \leq D_f(p, x_{n_i+1})$, inequality (3.31) yields

$$\begin{aligned}
\gamma_{m_k}\Big(1 - \delta \frac{\theta_{m_k}}{\gamma_{m_k}}\Big) D_f(p, x_{m_k}) &\leq D_f(p, x_{m_k}) - D_f(p, x_{m_k+1}) \\
&\quad + \gamma_{m_k}\Big(\langle \nabla f(u_{m_k}) - f(p), x_{m_k+1} - p\rangle + \delta \frac{\theta_{m_k}}{2\gamma_{m_k}}\Big).
\end{aligned}$$

In particular, because $\gamma_{m_k} > 0$, we deduce

$$D_f(p, x_{m_k}) \leq \frac{\langle \nabla f(u_{m_k}) - f(p), x_{m_k+1} - p\rangle + \delta \frac{\theta_{m_k}}{2\gamma_{m_k}}}{(1 - \delta \frac{\theta_{m_k}}{\gamma_{m_k}})}$$

From (3.30) it then follows that

$$D_f(p, x_{m_k}) \to 0 \quad \text{as } k \to \infty.$$

Combining this limit with (3.31) shows that

$$D_f(p, x_{n_i+1}) \to 0 \quad \text{as } i \to \infty.$$

Since $D_f(p, x_k) \leq D_f(p, x_{m_k+1})$ for all k, the two sequences converge to the same limit. Thus, we conclude that $\{x_k\}$ converges strongly to $p = P_\Gamma f(u)$, the Bregman projection of u onto Γ, which completes the proof for Case 2. □

Letting $T = I$, the identity mapping on E in Theorem 3.3, we obtain the following corollary.

Corollary 3.4. *Let C be a nonempty, closed, and convex subset of int(dom$f$). Suppose that conditions (C1), (C2) and (C4) in Assumption 3.1 and (3.1) holds and suppose that $V(C, A)$ is nonempty. For any $u, x_0, x_1 \in E$, consider the sequence $\{x_n\}$ defined by*

$$\begin{cases} t_n = P_C^f\Big(\nabla f^*[\nabla f(x_n) + \alpha_{n,1}(\nabla f(x_n) - \nabla f(x_{n-1})) + \alpha_{n,2}(\nabla f(x_{n-1}) - \nabla f(x_{n-2}))]\Big), \\ x_{n+1} = \nabla f^*(\gamma_n \nabla f(u_n) + (1-\gamma_n)\nabla f(K_{r_n}(t_n)). \end{cases} \tag{3.32}$$

Then the sequence $\{x_n\}$ converges strongly to $p = P^f_{V(C,A)}(u)$.

4. Discussion and Conclusion

Table 1. Comparison of Iterative Schemes

Scheme	**Update Rule**	**Features**	**Convergence Guarantee**
Standard Mann iteration	$x_{n+1} = (1-\gamma_n)x_n + \gamma_n T x_n$	Simple averaging with nonexpansive mapping	Weak convergence in Banach spaces
Single inertial Mann iteration	$x_{n+1} = (1-\gamma_n)x_n + \gamma_n T x_n + \alpha_n(x_n - x_{n-1})$	Adds one inertial term for acceleration	Strong convergence under restrictive assumptions
Double inertial Bregman-Mann scheme (Algorithm 3)	See Eq. (3.2) with $\alpha_{n,1}, \alpha_{n,2}$	Two inertial terms (x_{n-1}, x_{n-2}), Bregman projections, variational inequality constraints, balancing parameter δ	Strong convergence to $P^f_\Gamma u$ under Assumptions 3.1

Discussion: Table 1 highlights the progression from the classical Mann iteration to inertial and double inertial schemes. The standard Mann iteration provides only weak convergence in Banach spaces, while single inertial Mann iterations introduce acceleration but often require restrictive assumptions for strong convergence. In contrast, Algorithm 3 incorporates two inertial parameters $\alpha_{n,1}$ and $\alpha_{n,2}$, together with Bregman projections and the balancing parameter δ. This unified framework not only accelerates convergence but also stabilizes the iterative process by combining information from both the variational inequality component and the Bregman demigeneralized mapping. Recent works confirm the effectiveness of double inertial extrapolation: Muangchoo et al. [27] established strongly convergent Mann-type iterations with double inertial terms for bilevel pseudo-monotone variational inequalities in Banach spaces; Huang et al. [17] introduced a subgradient extragradient algorithm with double inertial steps for variational inequality and fixed point problems in Hilbert spaces; and Ugboh et al. [35] developed double inertial extrapolation methods for split generalized equilibrium and variational inequality problems. Consequently, the proposed scheme extends and strengthens existing inertial Mann-type methods, ensuring strong convergence to $P^f_\Gamma u$ under Assumptions 3.1.

Remark 4.1 (Applications).

(i) **Convex optimization in signal processing:** Problems such as compressed sensing, denoising, and MRI reconstruction can be modeled as convex optimization tasks with variational inequality constraints. The strong convergence of Algorithm 3 ensures reliable reconstruction from incomplete or noisy data, while the double inertial parameters $\alpha_{n,1}$ and $\alpha_{n,2}$ accelerate convergence. See Moradlou et al. [26] and Li et al. [20] for related inertial and compressed sensing approaches.

(ii) **Game theory and equilibrium problems:** Variational inequality formulations are central to computing Nash equilibria and analyzing market models. The proposed framework, combining Bregman projections with inertial Mann-type iterations, guarantees strong convergence in Banach spaces. The balancing parameter δ provides flexibility in handling both monotone operators and nonexpansive mappings. Related works include Matsuo et al. [25].

Concluding Remark

In this paper, we have introduced a new double inertial Bregman-Mann type iterative scheme for solving variational inequality problems and fixed point problems in Banach spaces. The proposed algorithm incorporates two inertial parameters $\alpha_{n,1}$ and $\alpha_{n,2}$, together with the balancing parameter δ, which unifies the contributions of the variational inequality component and the Bregman demigeneralized mapping. This framework extends classical Mann iterations and recent inertial variants, while ensuring strong convergence to $P_{\Gamma}^{f}u$ under suitable assumptions.

The novelty of our approach lies in the integration of double inertial extrapolation with Bregman projections, which accelerates convergence and enhances stability. The comparison with existing schemes demonstrates that our method generalizes both standard and single inertial Mann iterations, while offering stronger convergence guarantees. Furthermore, the applications to convex optimization in signal processing and to equilibrium problems in game theory illustrate the practical relevance of the result. Future research directions include extending the algorithm to stochastic settings, exploring adaptive choices of inertial parameters, and applying the framework to large-scale optimization problems in data science and economics. We believe that the proposed scheme provides a versatile foundation for further developments in iterative methods for variational inequalities and fixed point theory.

Statements and Declarations

The authors declare that they have no conflict of interest, and the manuscript has no associated data.

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