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FENCHEL CONJUGATES ON THE PRODUCT OF A HADAMARD MANIFOLD AND ITS TANGENT BUNDLE

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Abstract. The interplay between convexity and duality is fundamental in optimization theory. Extending these ideas from Euclidean spaces to Riemannian manifolds, particularly Hadamard manifolds, has attracted significant attention due to applications in nonlinear geometric structures. In this paper, we study the theory of Fenchel conjugation and its generalizations on Hadamard manifolds for functions defined on the product of a manifold and its tangent bundle. We introduce the concepts of parametric conjugate, biconjugate, and establish their fundamental properties, including order-preserving behavior, translation invariance and scaling properties. These results extend basic properties of convex analysis to the manifold setting. The results obtained in this paper can be applied in duality theory and optimization on Hadamard manifolds.

Keywords. Monotone bifunction, monotone vector fields, Fenchel conjugate, biconjugate, Hadamard manifold, Riemannian manifold.

1. Introduction

The interplay between convexity and duality has long been a central theme in optimization theory and nonlinear analysis [2, 12]. Over the past decades, the extension of these ideas from Euclidean spaces to Riemannian manifolds has attracted increasing attention, driven by applications where data naturally live on nonlinear geometric structures [3, 14].

As complete and simply connected Riemannian manifolds with nonpositive curvature, Hadamard manifolds provide a natural framework for global convexity [1, 13, 6, 9]. On these manifolds, geodesics mimic the behavior of straight lines, making it possible to define convex functions and also monotone operators in a way that parallels the Euclidean case [11, 15]. A key concept in this context is the Fenchel conjugate, which establishes a duality between a function and its conjugate [2, 12]. Bergmann et al. in [3] recently proposed a new definition of the Fenchel conjugate on Riemannian manifolds based on the tangent bundle, resolving the point-dependence issue present in earlier formulations. This led to a version of the Fenchel-Moreau theorem for geodesically convex functions and a separation theorem for convex sets on Hadamard manifolds [14].

Bifunctions and their monotonicity properties also play a significant role, particularly in equilibrium problems [4, 8]. Concepts such as BO-maximal monotonicity and cyclic monotonicity for bifunctions have been developed in [8].

In this paper, we investigate Fenchel conjugation and its generalizations on Hadamard manifolds for functions defined on the product of a manifold and its tangent bundle. We introduce the Fenchel biconjugate, the θ-Fenchel conjugate and its biconjugate and study their fundamental properties.

The paper is structured as follows. Section 2 recalls some basic notions from convex analysis and Riemannian geometry [6, 10, 5, 13, 15].

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2. Preliminaries

Throughout this paper, the extended real numbers and the nonnegative real numbers are denoted by $\overline{\mathbb{R}}$ and $\mathbb{R}_+$, respectively, indeed $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ and $\mathbb{R}_+ := [0, +\infty)$ and we follow the convention that:

$$\infty + (-\infty) = (-\infty) + \infty = \infty, \quad 0 \cdot \infty = \infty \cdot 0 = \infty, \quad 0 \cdot (-\infty) = (-\infty) \cdot 0 = 0.$$

Let X and Y be two nonempty sets. A *set-valued mapping* $T : X \multimap Y$ assigns to each $x \in X$ a subset $T(x)$ of Y. Its *domain* and *graph* are defined by

$$\mathcal{D}_T := \{x \in X : T(x) \neq \emptyset\} \text{ and } \mathrm{Gr}(T) = \{(x, y) \in X \times Y : y \in T(x)\},$$

respectively. Given a Banach space X and its dual space X^*, a set-valued mapping $T : X \multimap X^*$ is said to be *monotone* if

$$\langle x^* - y^*, x - y\rangle \geq 0, \ \forall x, y \in \mathcal{D}_T, \ \forall x^* \in T(x), \ \forall y^* \in T(y).$$

A topological space $\mathcal{M}$ is an *n-dimensional topological manifold*, if it is Hausdorff, second countable and locally Euclidean of dimension n. Let $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ be open subsets. A function $f : U \to V$ is *smooth* (C^∞), if each of its component functions has continuous partial derivatives of all orders. A linear map $X : C^\infty(\mathcal{M}) \to \mathbb{R}$ is called a *derivation at p* if it satisfies

$$X(fg) = f(p)X(g) + g(p)X(f), \ \forall f, g \in C^\infty(\mathcal{M}).$$

The set of all derivations of $C^\infty(\mathcal{M})$ at p is a vector space called the *tangent space* to $\mathcal{M}$ at p and is denoted by $\mathrm{T}_p\mathcal{M}$. An element of $\mathrm{T}_p\mathcal{M}$ is called a *tangent vector* at p. For any smooth manifold $\mathcal{M}$, the *tangent bundle* of $\mathcal{M}$ is defined as the disjoint union $\mathrm{T}\mathcal{M} := \coprod_{p\in\mathcal{M}} \mathrm{T}_p\mathcal{M}$, and the *cotangent bundle* of $\mathcal{M}$ is its dual

$$\mathrm{T}^*\mathcal{M} := \coprod_{p\in\mathcal{M}} \mathrm{T}^*_p\mathcal{M}, \text{ where } \mathrm{T}^*_p\mathcal{M} := (\mathrm{T}_p\mathcal{M})^*.$$

For every smooth manifold $\mathcal{M}$, the tangent bundle $\mathrm{T}\mathcal{M}$ carries a natural topology and smooth structure that makes it into a smooth manifold [10]. Throughout this paper, we write $u_p \in \mathrm{T}\mathcal{M}$ (respectively $\xi_p \in \mathrm{T}^*\mathcal{M}$) to mean $u_p \in \mathrm{T}_p\mathcal{M}$ (respectively $\xi_p \in \mathrm{T}^*_p\mathcal{M}$) for some $p \in \mathcal{M}$. The natural duality between $\mathrm{T}\mathcal{M}$ and $\mathrm{T}^*\mathcal{M}$ is given by $\langle\langle \xi_p, u_p\rangle\rangle = \xi_p(u_p) \in \mathbb{R}$ for $u_p \in \mathrm{T}\mathcal{M}$ and $\xi_p \in \mathrm{T}^*\mathcal{M}$.

A *set-valued vector field* on the smooth manifold $\mathcal{M}$ is a continuous map $\mathsf{A} : \mathcal{M} \multimap \mathrm{T}\mathcal{M}$, that assigns to each point $p \in \mathcal{M}$ a set of tangent vectors $\mathsf{A}(p) \subseteq \mathrm{T}_p\mathcal{M}$. The collection of all set-valued vector fields on $\mathcal{M}$ is denoted by $\chi(\mathcal{M})$. A continuous map $\gamma : J \to \mathcal{M}$, where $J \subseteq \mathbb{R}$ is an interval, is called a *curve* in $\mathcal{M}$.

Let $\mathcal{M}$ be a smooth, connected, complete n-dimensional manifold, where $n \in \mathbb{N}$ satisfies $n \geq 2$. A *Riemannian metric* on $\mathcal{M}$ is given by an inner product on each tangent space $\mathrm{T}_p\mathcal{M}$ that depends smoothly on the base point p, denoted by $\langle ., .\rangle : \mathrm{T}_p\mathcal{M} \times \mathrm{T}_p\mathcal{M} \to \mathbb{R}$.
A *Riemannian manifold* is a differentiable manifold equipped with a Riemannian metric. A smooth curve $\gamma : I \to \mathcal{M}$ is called a *geodesic* if $\nabla_{\dot\gamma(t)}\dot\gamma(t) = 0$ for all $t \in I$, where ∇ is the Levi-Civita connection and $\dot\gamma(t) \in T_{\gamma(t)}\mathcal{M}$ is the tangent vector of γ at t. For any $p \in \mathcal{M}$ and $u \in T_p\mathcal{M}$, let $\gamma_u(\cdot, p)$ denote the unique geodesic with $\gamma_u(0, p) = p$ and $\dot\gamma_u(0, p) = u$.

A Riemannian manifold $\mathcal{M}$ is *complete* if for all $p \in \mathcal{M}$, the *exponential map* $\exp_p : \mathrm{T}_p\mathcal{M} \to \mathcal{M}$ given by $\exp_p u = \gamma_u(1, p)$, is defined for all $u \in \mathrm{T}_p\mathcal{M}$, i.e., if every geodesic starting from p is defined for all $t \in \mathbb{R}$.

A Riemannian manifold is called a *Hadamard manifold* if it is simply connected, complete, and has nonpositive sectional curvature everywhere. Every Riemannian metric introduces a natural isomorphism between the tangent and cotangent bundles, known as the *musical isomorphisms*. These are defined by

$$\begin{array}{rcl} \flat : T\mathcal{M} & \to & T^*\mathcal{M}, \\ u_p & \mapsto & u_p^\flat, \end{array} \qquad \begin{array}{rcl} \sharp : T_p^*\mathcal{M} & \to & T_p\mathcal{M}, \\ \xi_p & \mapsto & \xi_p^\sharp, \end{array}$$

which at each point $p \in \mathcal{M}$ satisfy

$$\langle\!\langle u_p^\flat, v_p \rangle\!\rangle = \langle u_p, v_p \rangle, \quad \langle \xi_p^\sharp, u_p \rangle = \xi_p(u_p), \quad (u_p, v_p \in \mathrm{T}\mathcal{M}, \xi_p \in \mathrm{T}^*\mathcal{M}).$$

Definition 2.1. [13] Let $f : \mathcal{M} \to \overline{\mathbb{R}}$ be an arbitrary function. Then

(i) f is *proper*, if

$$\mathrm{dom}(f) := \{p \in \mathcal{M} : f(p) < \infty\} \neq \emptyset \text{ and } f(p) > -\infty, \ \forall p \in \mathcal{M}.$$

(ii) The *epigraph* of f is defined as

$$\mathrm{epi}(f) := \{(p, r) \in \mathcal{M} \times \mathbb{R} : f(p) \leq r\}.$$

(iii) A function f is called *lower semicontinuous*, if $\mathrm{epi}(f)$ is closed.

Definition 2.2. [13, Definition 5.9] A real-valued function f on a complete Riemannian manifold $\mathcal{M}$ is said to be *geodesically convex*, if for every geodesic $\gamma : I \subseteq \mathbb{R} \to \mathcal{M}$ and any $a, b \in I$, the following inequality holds for all $t \in [0, 1]$:

$$f\big(\gamma(ta + (1-t)b)\big) \leq t f\big(\gamma(a)\big) + (1-t) f\big(\gamma(b)\big).$$

Definition 2.3. [11] Let $f : \mathcal{M} \to \overline{\mathbb{R}}$ be a proper, lower semicontinuous and geodesically convex function. The *subdifferential* of f at x is defined by

$$\partial f(x) = \{u_x \in \mathrm{T}\mathcal{M} : \langle u_x, \exp_x^{-1} y \rangle \leq f(y) - f(x), \ \forall y \in \mathcal{M}\}.$$

Definition 2.4. [5] A vector field $\mathsf{A} \in \chi(\mathcal{M})$ is said to be *monotone*, if

$$\langle u_x, \exp_x^{-1} y \rangle \leq \langle v_y, -\exp_y^{-1} x \rangle, \ \forall x, y \in \mathcal{D}_\mathsf{A}, \forall u_x \in \mathsf{A}(x), \forall v_y \in \mathsf{A}(y).$$

A is said to be *maximal monotone* if it is monotone and for each $x \in \mathcal{D}_\mathsf{A}$ and each $u_x \in \mathrm{T}\mathcal{M}$, the condition

$$\langle u_x, \exp_x^{-1} y \rangle \leq \langle v_y, -\exp_y^{-1} x \rangle, \ \forall y \in \mathcal{D}_\mathsf{A}, v_y \in \mathsf{A}(y),$$

implies $u_x \in \mathsf{A}(x)$.

Definition 2.5. A mapping $\mathbb{b} : \mathcal{M} \times \mathcal{M} \to \overline{\mathbb{R}}$ is called a *bifunction* and its *domain* is defined as

$$\mathrm{dom}(\mathbb{b}) = \{x \in \mathcal{M} : \mathbb{b}(x, y) > -\infty, \ \forall y \in \mathcal{M}\}.$$

The bifunction $\mathbb{b}$ is said to be *normal*, if

(i) $\mathbb{b} \not\equiv -\infty$,
(ii) for every $y \in \mathcal{M}$, one has $\mathbb{b}(x, y) = -\infty$ if and only if $x \notin \mathrm{dom}(\mathbb{b})$.

A normal bifunction $\mathbb{b}$ is called *monotone*, if

$$\mathbb{b}(x, y) \leq -\mathbb{b}(y, x), \ \forall x, y \in \mathcal{M}.$$

3. Fenchel Conjugate

In this section, we introduce an alternative form of the Fenchel conjugate on Hadamard manifolds, inspired by the approach in [3, Definition 3.1]. We then study its fundamental properties and applications.

Definition 3.1. Let $f : \mathcal{M} \to \overline{\mathbb{R}}$ be an arbitrary function. The *Fenchel conjugate* of f is the function $f^* : \mathrm{T}^*\mathcal{M} \to \overline{\mathbb{R}}$ defined by

$$f^*(\xi_p) := \sup_{x \in \mathcal{M}} \left\{ \langle\langle \xi_p, \exp_p^{-1} x \rangle\rangle - f(x) \right\}, \ (\xi_p \in \mathrm{T}^*\mathcal{M}), \tag{3.1}$$

where $\langle\langle ., . \rangle\rangle$ denotes the duality pairing between $\mathrm{T}_p^*\mathcal{M}$ and $\mathrm{T}_p\mathcal{M}$.

Proposition 3.2. For proper functions $f, g : \mathcal{M} \to \overline{\mathbb{R}}$ with $\alpha \in \mathbb{R}$ and $\lambda > 0$, we have:

(i) $f \leq g \Longrightarrow g^* \leq f^*$.
(ii) $g = f + \alpha \Longrightarrow g^* = f^* - \alpha$.
(iii) $g(p) = \lambda f(p) \Longrightarrow g^* = \lambda f^*\left(\frac{\cdot}{\lambda}\right)$.

Proof. Under the assumption that $C = \mathcal{M}$ in [3, Lemma 3.7], the proof is similar. □

Now we present the subdifferential on the Hadamard space as previously defined by Ferreira et al. in [7] and Udrişte [15].

Definition 3.3. Let $f : \mathcal{M} \to \overline{\mathbb{R}}$ be a proper geodesically convex function. The *subdifferential* of f at a point $p \in \mathrm{dom}(f)$ is the set $\partial^* f(p) \subseteq \mathrm{T}_p^*\mathcal{M}$ defined by

$$\partial^* f(p) = \{\xi_p \in \mathrm{T}_p^*\mathcal{M} : \langle\langle \xi_p, \exp_p^{-1} q \rangle\rangle + f(p) \leq f(q), \ \forall q \in \mathcal{M}\}.$$

Proposition 3.4. (Fenchel-Young inequality) Suppose that $f : \mathcal{M} \to \overline{\mathbb{R}}$ is a proper geodesically convex function and $p \in \mathrm{dom}(f)$. Then for every $\xi_p \in \mathrm{T}_p^*\mathcal{M}$ the following inequality holds:

$$\langle\langle \xi_p, \exp_p^{-1} q \rangle\rangle \leq f(q) + f^*(\xi_p), \ \forall q \in \mathcal{M}.$$

Moreover, for $p \in \mathrm{dom}(f)$ and $\xi_p \in \mathrm{T}_p^*\mathcal{M}$, the equality

$$f(p) + f^*(\xi_p) = 0,$$

holds if and only if $\xi_p \in \partial^* f(p)$.

Proof. The inequality follows directly from the definition of f^* since

$$f^*(\xi_p) \geq \langle\langle \xi_p, \exp_p^{-1} q \rangle\rangle - f(q), \ \forall q \in \mathcal{M}.$$

Now, assume that $\xi_p \in \partial^* f(p)$. Since $p \in \mathrm{dom}(f)$, so $f(p)$ is finite. By definition 3.3,

$$\langle\langle \xi_p, \exp_p^{-1} q \rangle\rangle + f(p) \leq f(q), \ \forall q \in \mathcal{M},$$

Equivalently,

$$\langle\langle \xi_p, \exp_p^{-1} q \rangle\rangle - f(q) \leq -f(p), \ \forall q \in \mathcal{M}.$$

Taking the supremum over $q \in M$ yields

$$f^*(\xi_p) = \sup_{q \in \mathcal{M}} \left\{ \langle\langle \xi_p, \exp_p^{-1} q \rangle\rangle - f(q) \right\} \leq -f(p). \tag{3.2}$$

Choosing $q = p$ in the definition of f^* gives

$$f^*(\xi_p) \geq \langle\langle \xi_p, \exp_p^{-1} p \rangle\rangle - f(p) = 0 - f(p) = -f(p). \tag{3.3}$$

From (3.2) and (3.3) we obtain $f^*(\xi_p) = -f(p)$, i.e. $f(p) + f^*(\xi_p) = 0$.

Conversely, let $f(p) + f^*(\xi_p) = 0$ with $p \in \operatorname{dom}(f)$. Then $f(p)$ is finite and $f^*(\xi_p) = -f(p) \in \mathbb{R}$. By the definition of the conjugate as a supremum,

$$\langle\!\langle \xi_p, \exp_p^{-1} q \rangle\!\rangle - f(q) \le f^*(\xi_p) = -f(p),\ \forall q \in \mathcal{M},$$

which is the same as

$$\langle\!\langle \xi_p, \exp_p^{-1} q \rangle\!\rangle + f(p) \le f(q),\ \forall q \in \mathcal{M}.$$

This is precisely the condition $\xi_p \in \partial^* f(p)$. $\square$

We now extend the Fenchel conjugate to functions defined on the product of the manifold and its tangent bundle.

Definition 3.5. Let $\mathsf{h} : \mathcal{M} \times \mathrm{T}\mathcal{M} \to \overline{\mathbb{R}}$ be an arbitrary function. The *Fenchel conjugate* of h is the function $\mathsf{h}^* : \mathrm{T}^*\mathcal{M} \times \mathcal{M} \to \overline{\mathbb{R}}$ defined as

$$\mathsf{h}^*(\xi_p, x) = \sup_{(y,v_q) \in \mathcal{M} \times \mathrm{T}\mathcal{M}} \big\{ \langle\!\langle \xi_p, \exp_p^{-1} y \rangle\!\rangle + \langle v_q, \exp_q^{-1} x \rangle - \mathsf{h}(y, v_q) \big\},\ \big((\xi_p, x) \in \mathrm{T}^*\mathcal{M} \times \mathcal{M} \big).$$

Proposition 3.6. Let $\mathsf{h}, \mathsf{k} : \mathcal{M} \times \mathrm{T}\mathcal{M} \to \overline{\mathbb{R}}$ be functions. Assume that h and k are homogeneous of degree 1 in the second variable, i.e., for every $x \in \mathcal{M}$, $u_p \in \mathrm{T}\mathcal{M}$ and every $\lambda > 0$,

$$\mathsf{h}(x, \lambda u_p) = \lambda \mathsf{h}(x, u_p),\ \mathsf{k}(x, \lambda u_p) = \lambda \mathsf{k}(x, u_p),$$

and nonnegative, i.e. for all $(x, u_p) \in \mathcal{M} \times \mathrm{T}\mathcal{M}$ we have $\mathsf{h}(x, u_p) \ge 0$ and $\mathsf{k}(x, u_p) \ge 0$. Then for every $(\xi_p, x) \in \mathrm{T}^*\mathcal{M} \times \mathcal{M}$, the following inequality holds:

$$(\mathsf{h} + \mathsf{k})^*(\xi_p, x) \le \mathsf{h}^*\big(\tfrac{\xi_p}{2}, x\big) + \mathsf{k}^*\big(\tfrac{\xi_p}{2}, x\big).$$

Proof. Assume that $(\xi_p, x) \in \mathrm{T}^*\mathcal{M} \times \mathcal{M}$. By Definition 3.5,

$$\begin{aligned}
(\mathsf{h} + \mathsf{k})^*(\xi_p, x) &= \sup_{(y,v_q) \in \mathcal{M} \times \mathrm{T}\mathcal{M}} \big\{ \langle\!\langle \xi_p, \exp_p^{-1} y \rangle\!\rangle + \langle v_q, \exp_q^{-1} x \rangle - (\mathsf{h} + \mathsf{k})(y, v_q) \big\} \\
&= \sup_{(y,v_q) \in \mathcal{M} \times \mathrm{T}\mathcal{M}} \big\{ \langle\!\langle \xi_p, \exp_p^{-1} y \rangle\!\rangle + \langle v_q, \exp_q^{-1} x \rangle - \mathsf{h}(y, v_q) - \mathsf{k}(y, v_q) \big\} \\
&= \sup_{(y,v_q) \in \mathcal{M} \times \mathrm{T}\mathcal{M}} \big\{ 2\big\langle\!\!\big\langle \tfrac{\xi_p}{2}, \exp_p^{-1} y \big\rangle\!\!\big\rangle + 2\big\langle \tfrac{v_q}{2}, \exp_q^{-1} x \big\rangle - \mathsf{h}(y, v_q) - \mathsf{k}(y, v_q) \big\}.
\end{aligned}$$

By the subadditivity of the supremum, we obtain

$$\begin{aligned}
(\mathsf{h} + \mathsf{k})^*(\xi_p, x) \le &\sup_{(y,v_q) \in \mathcal{M} \times \mathrm{T}\mathcal{M}} \big\{ \big\langle\!\!\big\langle \tfrac{\xi_p}{2}, \exp_p^{-1} y \big\rangle\!\!\big\rangle + \big\langle \tfrac{v_q}{2}, \exp_q^{-1} x \big\rangle - \mathsf{h}(y, v_q) \big\} \\
&+ \sup_{(y,v_q) \in \mathcal{M} \times \mathrm{T}\mathcal{M}} \big\{ \big\langle\!\!\big\langle \tfrac{\xi_p}{2}, \exp_p^{-1} y \big\rangle\!\!\big\rangle + \big\langle \tfrac{v_q}{2}, \exp_q^{-1} x \big\rangle - \mathsf{k}(y, v_q) \big\}.
\end{aligned}$$

Set $w_q := \frac{v_q}{2}$, then $v_q = 2w_q$. Substituting this and using the homogeneity of degree 1 of h and k in the second variable we obtain:

$$\begin{aligned}
(\mathsf{h} + \mathsf{k})^*(\xi_p, x) \le &\sup_{(y,w_q) \in \mathcal{M} \times \mathrm{T}\mathcal{M}} \big\{ \big\langle\!\!\big\langle \tfrac{\xi_p}{2}, \exp_p^{-1} y \big\rangle\!\!\big\rangle + \big\langle w_q, \exp_q^{-1} x \big\rangle - 2\mathsf{h}(y, w_q) \big\} \\
&+ \sup_{(y,w_q) \in \mathcal{M} \times \mathrm{T}\mathcal{M}} \big\{ \big\langle\!\!\big\langle \tfrac{\xi_p}{2}, \exp_p^{-1} y \big\rangle\!\!\big\rangle + \big\langle w_q, \exp_q^{-1} x \big\rangle - 2\mathsf{k}(y, w_q) \big\} \\
\le &\sup_{(y,w_q) \in \mathcal{M} \times \mathrm{T}\mathcal{M}} \big\{ \big\langle\!\!\big\langle \tfrac{\xi_p}{2}, \exp_p^{-1} y \big\rangle\!\!\big\rangle + \big\langle w_q, \exp_q^{-1} x \big\rangle - \mathsf{h}(y, w_q) \big\} \\
&+ \sup_{(y,w_q) \in \mathcal{M} \times \mathrm{T}\mathcal{M}} \big\{ \big\langle\!\!\big\langle \tfrac{\xi_p}{2}, \exp_p^{-1} y \big\rangle\!\!\big\rangle + \big\langle w_q, \exp_q^{-1} x \big\rangle - \mathsf{k}(y, w_q) \big\} \\
= &\mathsf{h}^*\big(\tfrac{\xi_p}{2}, x\big) + \mathsf{k}^*\big(\tfrac{\xi_p}{2}, x\big).
\end{aligned}$$

$\square$

Proposition 3.7. Suppose that $\mathsf{h}, \mathsf{k} : \mathcal{M} \times \mathrm{T}\mathcal{M} \to \overline{\mathbb{R}}$ are two functions and let $\alpha \in \mathbb{R}$ and $\lambda > 0$. Then the following statements hold for every $\xi_p \in \mathrm{T}^*\mathcal{M}$ and every $x \in \mathcal{M}$:

(i) If $\mathsf{h}(y, v_q) \leq \mathsf{k}(y, v_q)$ for all $(y, v_q) \in \mathcal{M} \times \mathrm{T}\mathcal{M}$, then

$$\mathsf{h}^*(\xi_p, x) \geq \mathsf{k}^*(\xi_p, x).$$

(ii) If $\mathsf{k}(y, v_q) = \mathsf{h}(y, v_q) + \alpha$ for all $(y, v_q) \in \mathcal{M} \times \mathrm{T}\mathcal{M}$ and $\alpha \in \mathbb{R}$, then

$$\mathsf{k}^*(\xi_p, x) = \mathsf{h}^*(\xi_p, x) - \alpha.$$

(iii) Fix $w_q \in \mathrm{T}\mathcal{M}$. If $\mathsf{k}(y, v_q) = \mathsf{h}(y, v_q + w_q)$ for all $(y, v_q) \in \mathcal{M} \times \mathrm{T}\mathcal{M}$, then

$$\mathsf{h}^*(\xi_p, x) = \mathsf{k}^*(\xi_p, x) + \langle w_q, \exp_q^{-1} x\rangle.$$

Proof. (i) The assumption $\mathsf{h}(y, v_q) \leq \mathsf{k}(y, v_q)$ implies that $-\mathsf{h}(y, v_q) \geq -\mathsf{k}(y, v_q)$. Therefore, for every $(\xi_p, x) \in \mathrm{T}^*\mathcal{M} \times \mathcal{M}$,

$$\begin{aligned}
\mathsf{h}^*(\xi_p, x) &= \sup_{(y,v_q)\in\mathcal{M}\times\mathrm{T}\mathcal{M}} \left\{\langle\!\langle \xi_p, \exp_p^{-1} y\rangle\!\rangle + \langle v_q, \exp_q^{-1} x\rangle - \mathsf{h}(y, v_q)\right\} \\
&\geq \sup_{(y,v_q)\in\mathcal{M}\times\mathrm{T}\mathcal{M}} \left\{\langle\!\langle \xi_p, \exp_p^{-1} y\rangle\!\rangle + \langle v_q, \exp_q^{-1} x\rangle - \mathsf{k}(y, v_q)\right\} \\
&= \mathsf{k}^*(\xi_p, x).
\end{aligned}$$

(ii) Let $\mathsf{k}(y, v_q) = \mathsf{h}(y, v_q) + \alpha$ for all $(y, v_q) \in \mathcal{M} \times \mathrm{T}\mathcal{M}$. Then for every $(\xi_p, x) \in \mathrm{T}^*\mathcal{M} \times \mathcal{M}$,

$$\begin{aligned}
\mathsf{k}^*(\xi_p, x) &= \sup_{(y,v_q)\in\mathcal{M}\times\mathrm{T}\mathcal{M}} \left\{\langle\!\langle \xi_p, \exp_p^{-1} y\rangle\!\rangle + \langle v_q, \exp_q^{-1} x\rangle - \mathsf{k}(y, v_q)\right\}, \\
&= \sup_{(y,v_q)\in\mathcal{M}\times\mathrm{T}\mathcal{M}} \left\{\langle\!\langle \xi_p, \exp_p^{-1} y\rangle\!\rangle + \langle v_q, \exp_q^{-1} x\rangle - \mathsf{h}(y, v_q) - \alpha\right\}, \\
&= \sup_{(y,v_q)\in\mathcal{M}\times\mathrm{T}\mathcal{M}} \left\{\langle\!\langle \xi_p, \exp_p^{-1} y\rangle\!\rangle + \langle v_q, \exp_q^{-1} x\rangle - \mathsf{h}(y, v_q)\right\} - \alpha, \\
&= \mathsf{h}^*(\xi_p, x) - \alpha.
\end{aligned}$$

(iii) Fix $w_q \in \mathrm{T}\mathcal{M}$. Define

$$\mathsf{k}(y, v_q) := \mathsf{h}(y, v_q + w_q),\ \forall (y, v_q) \in \mathcal{M} \times \mathrm{T}\mathcal{M}.$$

Take any $\xi_p \in \mathrm{T}^*\mathcal{M}$ and $x \in \mathcal{M}$. Then

$$\begin{aligned}
\mathsf{k}^*(\xi_p, x) &= \sup_{(y,v_q)\in\mathcal{M}\times\mathrm{T}\mathcal{M}} \left\{\langle\!\langle \xi_p, \exp_p^{-1} y\rangle\!\rangle + \langle v_q, \exp_q^{-1} x\rangle - \mathsf{k}(y, v_q)\right\}, \\
&= \sup_{(y,v_q)\in\mathcal{M}\times\mathrm{T}\mathcal{M}} \left\{\langle\!\langle \xi_p, \exp_p^{-1} y\rangle\!\rangle + \langle v_q, \exp_q^{-1} x\rangle - \mathsf{h}(y, v_q + w_q)\right\}.
\end{aligned}$$

Set $z_q := v_q + w_q$, then

$$\langle v_q, \exp_q^{-1} x\rangle = \langle z_q, \exp_q^{-1} x\rangle - \langle w_q, \exp_q^{-1} x\rangle.$$

Thus

$$\begin{aligned}
\mathsf{k}^*(\xi_p, x) &= \sup_{(y,z_q)\in\mathcal{M}\times\mathrm{T}\mathcal{M}} \left\{\langle\!\langle \xi_p, \exp_p^{-1} y\rangle\!\rangle + \langle z_q, \exp_q^{-1} x\rangle - \langle w_q, \exp_q^{-1} x\rangle - \mathsf{h}(y, z_q)\right\}, \\
&= \sup_{(y,z_q)\in\mathcal{M}\times\mathrm{T}\mathcal{M}} \left\{\langle\!\langle \xi_p, \exp_p^{-1} y\rangle\!\rangle + \langle z_q, \exp_q^{-1} x\rangle - \mathsf{h}(y, z_q)\right\} - \langle w_q, \exp_q^{-1} x\rangle, \\
&= \mathsf{h}^*(\xi_p, x) - \langle w_q, \exp_q^{-1} x\rangle.
\end{aligned}$$

□

Proposition 3.8. Let $\{\mathsf{h}_i\}_{i\in I} : \mathcal{M} \times \mathrm{T}\mathcal{M} \to \overline{\mathbb{R}}$ be a family of functions. Define

$$\mathsf{h}_{\inf}(x, u_p) := \inf_{i\in I} \mathsf{h}_i(x, u_p),\ \mathsf{h}_{\sup}(x, u_p) := \sup_{i\in I} \mathsf{h}_i(x, u_p),\ (x, u_p) \in \mathcal{M} \times \mathrm{T}\mathcal{M}.$$

Then for every $(\xi_p, x) \in \mathrm{T}^*\mathcal{M} \times \mathcal{M}$

(i) $\mathsf{h}^*_{\inf}(\xi_p, x) = \sup_{i\in I} \mathsf{h}^*_i(\xi_p, x)$.

(ii) $\mathsf{h}^*_{\sup}(\xi_p, x) \le \inf_{i\in I} \mathsf{h}^*_i(\xi_p, x)$.

Proof. (i) By using the definition of the conjugate for every $\xi_p \in T^*\mathcal{M}$ and $x \in \mathcal{M}$,

$$\begin{aligned}
\mathsf{h}^*_{\inf}(\xi_p, x) &= \sup_{(y,v_q)\in\mathcal{M}\times\mathrm{T}\mathcal{M}} \big\{\langle\langle \xi_p, \exp_p^{-1} y\rangle\rangle + \langle v_q, \exp_q^{-1} x\rangle - \mathsf{h}_{\inf}(y, v_q)\big\}\\
&= \sup_{(y,v_q)\in\mathcal{M}\times\mathrm{T}\mathcal{M}} \big\{\langle\langle \xi_p, \exp_p^{-1} y\rangle\rangle + \langle v_q, \exp_q^{-1} x\rangle - \inf_{i\in I} \mathsf{h}_i(y, v_q)\big\}\\
&= \sup_{(y,v_q)\in\mathcal{M}\times\mathrm{T}\mathcal{M}} \big\{\langle\langle \xi_p, \exp_p^{-1} y\rangle\rangle + \langle v_q, \exp_q^{-1} x\rangle + \sup_{i\in I} \big(-\mathsf{h}_i(y, v_q)\big)\big\}\\
&= \sup_{(y,v_q)\in\mathcal{M}\times\mathrm{T}\mathcal{M}} \Big[\sup_{i\in I} \big\{\langle\langle \xi_p, \exp_p^{-1} y\rangle\rangle + \langle v_q, \exp_q^{-1} x\rangle - \mathsf{h}_i(y, v_q)\big\}\Big]\\
&= \sup_{i\in I} \Big[\sup_{(y,v_q)\in\mathcal{M}\times\mathrm{T}\mathcal{M}} \big\{\langle\langle \xi_p, \exp_p^{-1} y\rangle\rangle + \langle v_q, \exp_q^{-1} x\rangle - \mathsf{h}_i(y, v_q)\big\}\Big]\\
&= \sup_{i\in I} \mathsf{h}^*_i(\xi_p, x)
\end{aligned}$$

(ii) $\mathsf{h}_i \le \mathsf{h}_{\sup}$ for each fixed i. Let $(\xi_p, x) \in \mathrm{T}^*\mathcal{M} \times \mathcal{M}$. By Proposition 3.7 part (i) we have

$$\mathsf{h}^*_{\sup}(\xi_p, x) \le \mathsf{h}^*_i(\xi_p, x).$$

Taking the infimum over i on the right-hand side yields $\mathsf{h}^*_{\sup}(\xi_p, x) \le \inf_{i\in I} \mathsf{h}^*_i(\xi_p, x)$ and hence the proof is completed. □

Definition 3.9. Let $\mathcal{M}$ be a Hadamard manifold and $\theta > 0$ be a positive real number. The *θ-Fenchel conjugate* of a function $\mathsf{h} : \mathcal{M} \times \mathrm{T}\mathcal{M} \to \overline{\mathbb{R}}$, denoted by h^*_θ, is defined as

$$\begin{aligned}
&\mathsf{h}^*_{(\theta)} : \mathrm{T}^*\mathcal{M} \times \mathcal{M} \to \overline{\mathbb{R}}\\
&\qquad (\xi_p, x) \mapsto \sup_{(y,v_q)\in\mathcal{M}\times\mathrm{T}\mathcal{M}} \big\{\langle\langle \xi_p, \exp_p^{-1} y\rangle\rangle + \theta\langle v_q, \exp_q^{-1} x\rangle - \mathsf{h}(y, v_q)\big\}.
\end{aligned}$$

Note that $\mathsf{h}^*_{(1)} = \mathsf{h}^*$ for $\theta = 1$.

Proposition 3.10. For an arbitrary function $\mathsf{h} : \mathcal{M} \times \mathrm{T}\mathcal{M} \to \overline{\mathbb{R}}$, every $\lambda > 0$ and every $x \in \mathcal{M}$ the following identities hold:

(i) $(\lambda\mathsf{h})^*_{(\theta)}(\xi_p, x) = \lambda\mathsf{h}^*_{(\frac{\theta}{\lambda})}\big(\frac{\xi_p}{\lambda}, x\big)$, where $\xi_p \in \mathrm{T}^*\mathcal{M}$.

(ii) Define $\mathsf{k}(x, u_p) := \lambda\mathsf{h}\big(x, \frac{u_p}{\lambda}\big)$, where $u_p \in \mathrm{T}\mathcal{M}$. Then

$$\mathsf{k}^*_{(\theta)}(\xi_p, x) = \lambda\mathsf{h}^*_{(\theta)}\big(\tfrac{\xi_p}{\lambda}, x\big),\ \forall \xi_p \in \mathrm{T}^*\mathcal{M}.$$

Proof. (i) For every $(\xi_p, x) \in \mathrm{T}^*\mathcal{M} \times \mathcal{M}$,

$$\begin{aligned}
(\lambda\mathsf{h})^*_{(\theta)}(\xi_p, x) &= \sup_{(y,v_q)\in\mathcal{M}\times\mathrm{T}\mathcal{M}} \big\{\langle\langle \xi_p, \exp_p^{-1} y\rangle\rangle + \theta\langle v_q, \exp_q^{-1} x\rangle - \lambda\mathsf{h}(y, v_q)\big\}\\
&= \lambda \sup_{(y,v_q)\in\mathcal{M}\times\mathrm{T}\mathcal{M}} \big\{\big\langle\!\big\langle \tfrac{\xi_p}{\lambda}, \exp_p^{-1} y\big\rangle\!\big\rangle + \tfrac{\theta}{\lambda}\langle v_q, \exp_q^{-1} x\rangle - \mathsf{h}(y, v_q)\big\}\\
&= \lambda\mathsf{h}^*_{(\frac{\theta}{\lambda})}\big(\tfrac{\xi_p}{\lambda}, x\big).
\end{aligned}$$

(ii) Assume that $x \in \mathcal{M}$ and $\xi_p \in \mathrm{T}^*\mathcal{M}$. By the definition of the θ-conjugate, we have

$$\begin{aligned}\mathsf{k}^*_{(\theta)}(\xi_p, x) &= \sup_{(y,v_q)\in\mathcal{M}\times\mathrm{T}\mathcal{M}} \left\{\langle\!\langle \xi_p, \exp_p^{-1} y\rangle\!\rangle + \theta\langle v_q, \exp_q^{-1} x\rangle - \mathsf{k}(y, v_q)\right\}\\ &= \sup_{(y,v_q)\in\mathcal{M}\times\mathrm{T}\mathcal{M}} \left\{\langle\!\langle \xi_p, \exp_p^{-1} y\rangle\!\rangle + \theta\langle v_q, \exp_q^{-1} x\rangle - \lambda\mathsf{h}\big(y, \tfrac{v_q}{\lambda}\big)\right\}.\end{aligned}$$

Set $w_q := \frac{v_q}{\lambda}$. Since $\langle v_q, \exp_q^{-1} x\rangle = \lambda\langle w_q, \exp_q^{-1} x\rangle$, we obtain

$$\begin{aligned}\mathsf{k}^*_{(\theta)}(\xi_p, x) &= \lambda \sup_{(y,w_q)\in\mathcal{M}\times\mathrm{T}\mathcal{M}} \left\{\left\langle\!\!\left\langle \tfrac{\xi_p}{\lambda}, \exp_p^{-1} y\right\rangle\!\!\right\rangle + \theta\langle w_q, \exp_q^{-1} x\rangle - \mathsf{h}(y, w_q)\right\}\\ &= \lambda\mathsf{h}^*_{(\theta)}\big(\tfrac{\xi_p}{\lambda}, x\big).\end{aligned}$$

□

4. Fenchel Biconjugate

Recently Bergmann et al. [3] and Louzeiro et al. [14] studied the concept of Fenchel duality on Hadamard manifolds using a tangent bundle approach that overcomes the point-dependence issues of earlier definitions. Based on these developments, we now introduce the Fenchel biconjugate for functions on $\mathcal{M} \times \mathrm{T}\mathcal{M}$. We establish its key properties and explore its relationship with the Fenchel conjugate, providing the foundation for a Fenchel-Moreau type theorem in our setting. A parametric generalization of the Fenchel biconjugate, called the θ-Fenchel biconjugate, is introduced in Definition 4.4.

Definition 4.1. Let $f : \mathcal{M} \to \overline{\mathbb{R}}$ be given. The *Fenchel biconjugate* of f is $f^{**} : \mathcal{M} \to \overline{\mathbb{R}}$ defined by

$$f^{**}(x) = \sup_{\xi_p\in\mathrm{T}^*\mathcal{M}} \{\langle\!\langle \xi_p, \exp_p^{-1} x\rangle\!\rangle - f^*(\xi_p)\},\ (x \in \mathcal{M}).$$

Proposition 4.2. Let $f : \mathcal{M} \to \overline{\mathbb{R}}$ be a function. Then $f^{**}(x) \le f(x)$ for all $x \in \mathcal{M}$.

Proof. For every $x \in \mathcal{M}$, by definition of f^{**}, we have

$$\begin{aligned}f^{**}(x) &= \sup_{\xi_p\in\mathrm{T}^*\mathcal{M}} \{\langle\!\langle \xi_p, \exp_p^{-1} x\rangle\!\rangle - f^*(\xi_p)\}\\ &= \sup_{\xi_p\in\mathrm{T}^*\mathcal{M}} \left\{\langle\!\langle \xi_p, \exp_p^{-1} x\rangle\!\rangle - \sup_{y\in\mathcal{M}}\{\langle\!\langle \xi_p, \exp_p^{-1} y\rangle\!\rangle - f(y)\}\right\}\\ &= \sup_{\xi_p\in\mathrm{T}^*\mathcal{M}} \inf_{y\in\mathcal{M}} \left\{\langle\!\langle \xi_p, \exp_p^{-1} x\rangle\!\rangle - \langle\!\langle \xi_p, \exp_p^{-1} y\rangle\!\rangle + f(y)\right\}\\ &\le \sup_{\xi_p\in\mathrm{T}^*\mathcal{M}} \left\{\langle\!\langle \xi_p, \exp_p^{-1} x\rangle\!\rangle - \langle\!\langle \xi_p, \exp_p^{-1} x\rangle\!\rangle + f(x)\right\}\\ &= f(x).\end{aligned}$$

□

Proposition 4.3. Suppose that $f, g : \mathcal{M} \to \overline{\mathbb{R}}$ are proper functions, $\alpha \in \mathbb{R}$ and $\lambda > 0$. Then for every $x \in \mathcal{M}$ the following statements hold:

(i) If $f \le g$, then $f^{**} \le g^{**}$.
(ii) If $g = f + \alpha$, then $g^{**} = f^{**} + \alpha$.
(iii) If $g = \lambda f$, then $g^{**} = \lambda f^{**}$.

Proof. (i) From $f \le g$ and by Proposition 3.2, we obtain $g^* \le f^*$. Taking the conjugate again and using Proposition 3.2 once more gives

$$f^{**} = (f^*)^* \le (g^*)^* = g^{**}.$$

(ii) By the assumption $g = f + \alpha$ and in view of Proposition 3.2(ii) and applying the biconjugate, the statement follows trivially.

(iii) Assume that $g = \lambda f$. Using Proposition 3.2(iii), for every $\xi_p \in \mathrm{T}^*\mathcal{M}$ we have

$$g^*(\xi_p) = \lambda f^*\big(\tfrac{\xi_p}{\lambda}\big). \tag{4.1}$$

Now by Definition 4.1 and (4.1) for every $x \in \mathcal{M}$, we can write

$$\begin{aligned} g^{**}(x) &= \sup_{\xi_p \in \mathrm{T}^*\mathcal{M}} \{\langle\!\langle \xi_p, \exp_p^{-1} x\rangle\!\rangle - g^*(\xi_p)\} \\ &= \sup_{\xi_p \in \mathrm{T}^*\mathcal{M}} \{\langle\!\langle \xi_p, \exp_p^{-1} x\rangle\!\rangle - \lambda f^*\big(\tfrac{\xi_p}{\lambda}\big)\}. \end{aligned}$$

Set $\eta_p := \frac{\xi_p}{\lambda} \in \mathrm{T}^*\mathcal{M}$, so that $\xi_p := \lambda\eta_p$. Then

$$\begin{aligned} g^{**}(x) &= \sup_{\eta_p \in \mathrm{T}^*\mathcal{M}} \{\langle\!\langle \lambda\eta_p, \exp_p^{-1} x\rangle\!\rangle - \lambda f^*(\eta_p)\} \\ &= \lambda \sup_{\eta_p \in \mathrm{T}^*\mathcal{M}} \{\langle\!\langle \eta_p, \exp_p^{-1} x\rangle\!\rangle - f^*(\eta_p)\} = \lambda f^{**}(x). \end{aligned}$$ ☐

Definition 4.4. Let $\mathcal{M}$ be a Hadamard manifold and let $\mathsf{h} : \mathcal{M} \times \mathrm{T}\mathcal{M} \to \overline{\mathbb{R}}$ be a proper function. For a given $\theta > 0$, the *θ-Fenchel biconjugate* of h is the function

$$\mathsf{h}^{**}_{(\theta)} : \mathcal{M} \times \mathrm{T}\mathcal{M} \to \overline{\mathbb{R}}$$

defined by

$$\mathsf{h}^{**}_{(\theta)}(y, v_q) = \sup_{(\xi_p, x) \in \mathrm{T}^*\mathcal{M} \times \mathcal{M}} \big\{\langle\!\langle \xi_p, \exp_p^{-1} y\rangle\!\rangle + \theta\langle v_q, \exp_q^{-1} x\rangle - \mathsf{h}^*_{(\theta)}(\xi_p, x)\big\}.$$

Proposition 4.5. Let $\mathsf{h}, \mathsf{k} : \mathcal{M} \times \mathrm{T}\mathcal{M} \to \overline{\mathbb{R}}$ be proper functions. If $\mathsf{h} \le \mathsf{k}$, then

$$\mathsf{h}^{**}_{(\theta)} \le \mathsf{k}^{**}_{(\theta)}.$$

Proof. This is an immediate consequence of Proposition 4.3(i), the assumption $\mathsf{h} \le \mathsf{k}$, and Definition 4.4. ☐

Remark 4.6. For the θ-Fenchel biconjugate, the translation property and a modified scaling property hold as follows:

(i) If $\mathsf{k} = \mathsf{h} + \alpha$ for some $\alpha \in \mathbb{R}$, then $\mathsf{k}^{**}_{(\theta)} = \mathsf{h}^{**}_{(\theta)} + \alpha$.
(ii) If $\mathsf{k} = \lambda\mathsf{h}$ for some $\lambda > 0$, then $\mathsf{k}^{**}_{(\theta)} = \lambda\, \mathsf{h}^{**}_{(\theta/\lambda)}$.

In particular, the identity $\mathsf{k}^{**}_{(\theta)} = \lambda\mathsf{h}^{**}_{(\theta)}$ does *not* hold in general (unless $\lambda = 1$).

Proposition 4.7. Let $\mathsf{h} : \mathcal{M} \times \mathrm{T}\mathcal{M} \to \overline{\mathbb{R}}$ be a proper function. Then

$$\mathsf{h}^{**}(y, v_q) \le \mathsf{h}(y, v_q),\ \forall (y, v_q) \in \mathcal{M} \times \mathrm{T}\mathcal{M}.$$

Proof. For any $(\xi_p, x) \in \mathrm{T}^*\mathcal{M} \times \mathcal{M}$, from Definition 3.5, we have

$$\mathsf{h}^*(\xi_p, x) \ge \langle\!\langle \xi_p, \exp_p^{-1} y\rangle\!\rangle + \langle v_q, \exp_q^{-1} x\rangle - \mathsf{h}(y, v_q).$$

Then

$$\langle\!\langle \xi_p, \exp_p^{-1} y\rangle\!\rangle + \langle v_q, \exp_q^{-1} x\rangle - \mathsf{h}^*(\xi_p, x) \le \mathsf{h}(y, v_q).$$

Taking the supremum over all (ξ_p, x) yields $\mathsf{h}^{**}(y, v_q) \le \mathsf{h}(y, v_q)$. ☐

5. Conclusion

We have extended the theory of Fenchel conjugation to functions defined on the product of a Hadamard manifold and its tangent bundle. By making use of the natural duality between the tangent and cotangent bundles, we introduced the notion of Fenchel conjugate, its parametric θ-version, and the corresponding biconjugates. We have established fundamental properties such as order preservation, behaviour under translation, and precise scaling relations. In particular, the scaling rule for the θ-Fenchel biconjugate was stated with the necessary modification of the parameter θ, clarifying a point that could otherwise cause confusion.

These results provide a solid foundation for developing a Fenchel-Moreau type duality theorem in this geometric framework and for studying duality-based algorithms on Hadamard manifolds. Future work may address applications to equilibrium problems involving monotone bifunctions, to primal-dual optimisation methods, and to further properties of maximally monotone vector fields on Riemannian manifolds.

Statements and Declarations

The authors declare that they have no conflict of interest, and the manuscript has no associated data.

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