APPLICABLE NONLINEAR ANALYSIS
Volume 4 (2027), No. 1, 10–28
DOI: https://doi.org/10.69829/apna-027-0401-ta02
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FLEXIBLE PROBABILISTIC MODELS FOR CAPTURING EXTREME RISK IN CRYPTOCURRENCY RETURNS

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Abstract. This study presents a comparative investigation of probabilistic mixture models for capturing extreme risk in cryptocurrency return distributions. Specifically, we evaluate the performance of the Gaussian Mixture Model (GMM), the Student-t Mixture Model (TMM), and the Laplace Mixture Model (LMM) using standardized daily return series from ten leading cryptocurrencies over five years. The primary objective is to assess each model's ability to accurately represent heavy tails and extreme return behavior, which are critical for reliable financial risk estimation. Model performance is evaluated using tail-sensitive risk measures, including Value-at-Risk (VaR) and Expected Shortfall (ES) at the 5% confidence level, alongside log-likelihood-based goodness-of-fit diagnostics. Empirical findings indicate that while the GMM effectively captures the central distribution, it systematically underestimates tail probabilities. The LMM provides moderate improvement in flexibility but does not consistently outperform alternative specifications in extreme regions. In contrast, the TMM demonstrates superior tail fit and delivers more accurate VaR and ES estimates across assets. These results underscore the importance of heavy-tailed mixture frameworks for robust risk modeling in high-volatility digital asset markets.

Keywords. Cryptocurrency, Mixture models, Tail risk, Value-at-Risk, Expected Shortfall, Heavy-tailed distributions.

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1. Introduction

Cryptocurrencies represent a transformative class of digital financial instruments that operate independently of centralized authorities. These assets rely on distributed ledger technologies, primarily blockchain, to enable peer-to-peer transactions, ensure transparency, and enforce cryptographic security [13]. Unlike traditional financial assets, cryptocurrencies are highly decentralized, often have fixed or algorithmically controlled supplies, and exhibit speculative trading behavior. Their market behavior is marked by extreme price volatility, heavy-tailed return distributions, and pronounced sensitivity to macroeconomic news, regulatory shifts, and investor sentiment [2, 3, 4]. As their adoption grows across global markets, the need for robust statistical models that can accurately capture the underlying risk dynamics of cryptocurrency returns becomes increasingly critical [14, 15].

A growing body of empirical research has highlighted that cryptocurrency returns exhibit heavy tails, volatility clustering, and regime shifts [6, 7, 14]. These characteristics challenge the assumptions of classical financial models that rely on normally distributed returns. In this context, robust tail risk modeling becomes essential for effective portfolio construction, hedging, and risk management in crypto-asset markets [7, 9, 15].

Mixture models provide a flexible framework for modeling such non-linear and non-normal behavior. They can capture multimodal distributions, skewness, and excess kurtosis by combining multiple

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2020 Mathematics Subject Classification: 91G70, 91G60, 62E17.
Accepted: May 26, 2026.

parametric components [11, 12]. In financial applications, Gaussian Mixture Models (GMMs) have been used to model return heterogeneity [8], while extensions like the Student-t Mixture Model (TMM) [16] and the Laplace Mixture Model (LMM) [10] offer better fits in the presence of extreme events and heavy tails. These models are especially relevant for cryptocurrency markets, where tail risk plays a dominant role in price dynamics [5, 8].

Recent studies have demonstrated the superior performance of mixture-based volatility models over traditional GARCH-type specifications when applied to Bitcoin and other digital assets [5, 7]. For example, Chi and Hao [5] show that mixture models better capture the risk profile of cryptocurrencies and are more suitable for options pricing and VaR estimation. Similarly, empirical findings from Haas et al. [8] and Alexander and Lazar [1] suggest that mixture frameworks outperform single-distribution models in terms of log-likelihood, tail fit, and risk forecasting.

Given the limitations of Gaussian assumptions in modeling crypto returns, this study applies and compares three parametric mixture models–GMM, TMM, and LMM–on standardized daily return series of ten leading cryptocurrencies over a five-year period. The models are evaluated based on their ability to capture tail behavior using metrics such as log-likelihood, Value-at-Risk (VaR), and Expected Shortfall (ES) at the 5% level. Special emphasis is placed on the comparative performance of these models in quantifying downside and upside tail risk, which is crucial for risk-aware decision-making in digital asset investment.

Our findings reveal that while GMM provides a good approximation of central tendencies, it underestimates tail probabilities. LMM improves upon this but lacks consistency across assets. The TMM emerges as the most robust model, offering superior tail fit and more reliable risk estimates, thereby reinforcing the importance of heavy-tailed mixture distributions in financial modeling of cryptocurrencies.

A systematic review of prior literature on extreme return modeling and distributional fitting is presented in Table 1 and Table 2. This table synthesizes key studies by summarizing their methodological frameworks, empirical datasets, and central insights. The reviewed works span a variety of approaches–including mixture distributions, copula models, extreme value theory, and volatility-based methods–applied across cryptocurrency, equity, and hybrid asset classes. This synthesis highlights the growing scholarly attention toward modeling heavy tails and volatility clustering, particularly in the context of cryptocurrencies. The reviewed evidence sets a strong foundation for our comparative modeling framework, which focuses on capturing tail risk using flexible mixture models tailored to the nonlinear, non-Gaussian behavior of digital asset returns.

2. Data Description

2.1. **Data collection and asset selection.** This study focuses on ten major cryptocurrencies selected based on market capitalization, liquidity, and trading consistency. The assets include Bitcoin (BTC), Ethereum (ETH), Tether (USDT), USD Coin (USDC), Binance Coin (BNB), Solana (SOL), Ripple (XRP), Cardano (ADA), Dogecoin (DOGE), and Avalanche (AVAX). Daily closing prices for each asset were collected over a five year spanning from August 2020 to July 2025 using the Yahoo Finance API.

2.2. **Return transformation and standardization.** Cryptocurrencies trade continuously across global markets without fixed opening or closing hours. However, for consistency and comparability, the daily price P_t is defined as the value recorded at 00:00 UTC each day. This timestamped price serves as a standardized daily reference point for computing returns.

Daily log returns are computed as:

$$R_t = \log\left(\frac{P_t}{P_{t-1}}\right) \tag{2.1}$$

TABLE 1. Systematic Review of Extreme Return Modeling (Part 1)

Study	Methodological Framework and Empirical Insights
Normal Mixture GARCH(1,1): Applications to Exchange Rate Modelling. Alexander and Lazar (2006).	**Data Description:** Exchange rate returns, 5 years (daily). **Techniques Used:** Mixture-GARCH modeling. **Key Contribution:** Foundational work on time-varying volatility using mixture distributions.
Financial Crises and Stock Market Contagion in a Multivariate Time-Varying Asymmetric Framework. Kenourgios, Samitas, and Paltalidis (2011).	**Data Description:** BRIC + US/UK markets, 1995–2006. **Techniques Used:** AG-DCC and copula regime-switching models. **Key Contribution:** Establishes contagion modeling under asymmetric volatility dynamics.
Macroeconomic Risk Factors and Stock Returns: The Arbitrage Pricing Approach. Torbira and Agbam (2017).	**Data Description:** Monthly equity returns, 10 years. **Techniques Used:** Arbitrage Pricing Theory (APT). **Key Contribution:** Examines macroeconomic factor risk in emerging markets.
On the Return Distributions of a Basket of Cryptocurrencies and Subsequent Implications. Boerner et al. (2021).	**Data Description:** Cryptocurrency returns, 1 year (daily). **Techniques Used:** Distributional modeling. **Key Contribution:** Empirical investigation of fat-tailed crypto return distributions.
Modeling Tail-Dependence of Crypto Assets with Extreme Value Theory. Opala, Fischer, and Svoboda (2022).	**Data Description:** Crypto daily returns, 6 months. **Techniques Used:** Extreme Value Theory (EVT). **Key Contribution:** Highlights crypto tail co-movement risks for financial institutions.
The Exponentiated Half-Logistic Skew-T Distribution with GARCH-Type Volatility Models. Adubisi et al. (2022).	**Data Description:** Financial returns, 5 years. **Techniques Used:** Skew-t distribution with GARCH and MLE. **Key Contribution:** Introduces flexible heavy-tailed skewed volatility modeling framework.
Speculative Bubbles in Bitcoin Markets? An Empirical Investigation. Cheah and Fry (2015).	**Data Description:** BTC weekly prices, 3 years. **Techniques Used:** Empirical bubble modeling. **Key Contribution:** Detects and models speculative bubbles in Bitcoin markets.
GEV Approach to Comparing the Riskiness of Bitcoin/USD and ZAR/USD Returns. Chikobvu and Ndlovu (2023).	**Data Description:** BTC/USD and ZAR/USD returns, 2 years. **Techniques Used:** Generalized Extreme Value (GEV) modeling. **Key Contribution:** Compares currency tail risk using EVT-based measures.
Bitcoin in Portfolio Selection: A Multivariate Distribution Approach. Contreras-Valdez et al. (2022).	**Data Description:** Crypto and traditional assets, 1 year. **Techniques Used:** Copula and multivariate distribution modeling. **Key Contribution:** Demonstrates Bitcoin's role in portfolio allocation.
Exploring the Dynamic Relationships Between Cryptocurrencies and Other Financial Assets. Corbet et al. (2018).	**Data Description:** Crypto and equities/bonds, 5 years. **Techniques Used:** VAR and correlation analysis. **Key Contribution:** Evidence of dynamic interlinkages across asset classes.

This transformation ensures time-additivity and helps stabilize variance, which is desirable for modeling financial time series.

To ensure comparability across assets and improve convergence of the mixture model estimators, all return series are standardized using z-scores:

$$Z_t = \frac{R_t - \mu}{\sigma} \tag{2.2}$$

where μ and σ denote the sample mean and standard deviation of R_t, respectively. The standardized returns exhibit zero mean and unit variance and are used throughout the empirical analysis.

2.3. **Distributional properties and tail thresholds.** After preprocessing, each asset yielded approximately 1,300 valid daily return observations. Table 3 presents summary statistics including skewness, kurtosis, and empirical percentile-based thresholds for extreme events. The 5th and 95th percentiles serve as lower and upper cutoffs for identifying tail events. All return series were standardized prior to computation to enable fair comparison of distributional characteristics.

TABLE 2. Systematic Review of Extreme Return Modeling (Part 2)

Study	Methodological Framework and Empirical Insights
Bitcoin's Return Behaviour: What Do We Know So Far?. Fajardo (2019).	**Data Description:** BTC prices, 4 years (daily). **Techniques Used:** Descriptive statistical analysis. **Key Contribution:** Summarizes statistical behavior of Bitcoin returns.
Mixed Normal Conditional Heteroskedasticity. Haas, Mittnik, and Paolella (2004).	**Data Description:** Financial returns, 10 years. **Techniques Used:** Mixed normal + GARCH. **Key Contribution:** Early model capturing fat tails and volatility clustering.
Value at Risk: The New Benchmark for Managing Financial Risk. Jorion (2001).	**Data Description:** Multi-asset risk datasets. **Techniques Used:** VaR theory. **Key Contribution:** Authoritative framework for capital risk modeling.
Diversification Evidence of Bitcoin and Gold from Wavelet Analysis. Bhuiyan, Husain, and Zhang (2023).	**Data Description:** BTC and gold returns, 2 years. **Techniques Used:** Wavelet correlation. **Key Contribution:** Analyzes Bitcoin-gold co-movement across time-frequency scales.
Modeling Volatility of Indian Banking Sector. Goyal and Kumar (2019).	**Data Description:** Indian bank index, 5 years. **Techniques Used:** GARCH modeling. **Key Contribution:** Examines sector-specific volatility dynamics.
Portfolio Selection. Markowitz (1952).	**Data Description:** Theoretical. **Techniques Used:** Modern Portfolio Theory. **Key Contribution:** Introduces mean-variance optimization framework.
Estimation of Portfolio Return and Value at Risk Using Gaussian Mixture Distributions. Tan and Chu (2012).	**Data Description:** Portfolio returns, 3 years. **Techniques Used:** Gaussian mixtures and VaR. **Key Contribution:** Enhances portfolio risk estimation using mixture distributions.
The Inefficiency of Bitcoin. Urquhart (2016).	**Data Description:** BTC daily prices, 4 years. **Techniques Used:** Random walk and ADF tests. **Key Contribution:** Provides early evidence of Bitcoin market inefficiency.
Volatility Models for Cryptocurrencies and Applications in the Options Market. Chi and Hao (2021).	**Data Description:** Crypto spot and options data, 2 years. **Techniques Used:** GARCH and implied volatility modeling. **Key Contribution:** Applies heavy-tailed modeling for crypto options pricing.
Robust Mixture Modelling Using the t Distribution. Peel and McLachlan (2000).	**Data Description:** Synthetic and empirical returns. **Techniques Used:** Student-t mixture model with EM algorithm. **Key Contribution:** Foundational framework for robust heavy-tailed mixture modeling.

TABLE 3. Skewness, Kurtosis, and Tail Thresholds of Standardized Log Returns

Asset	Skewness	Kurtosis	5th Percentile	95th Percentile
BTC-USD	-0.274	7.45	-1.5619	1.6050
ETH-USD	-0.139	9.47	-1.4774	1.5116
USDT-USD	-6.904	213.42	-0.9644	1.0063
USDC-USD	-0.277	56.18	-1.0185	1.0222
BNB-USD	0.548	19.18	-1.3574	1.4619
SOL-USD	-0.023	9.17	-1.3659	1.5943
XRP-USD	1.098	22.51	-1.3574	1.3430
ADA-USD	0.508	7.05	-1.4508	1.6051
DOGE-USD	5.134	85.44	-1.1348	1.2683
AVAX-USD	0.351	10.60	-1.4414	1.5328

The statistics reveal significant departures from normality across all assets. Elevated kurtosis values indicate fat tails, while skewness values, both positive and negative, indicate asymmetric return behavior. Stablecoins such as USDT and USDC exhibit extremely peaked distributions, whereas assets such as DOGE and XRP display pronounced right-tail risk.

2.4. **Visualization of extreme observations.** To illustrate the frequency and magnitude of extreme returns, Figure 1 presents histograms of the standardized return series. Blue bars denote observations within the 5th to 95th percentile range, while red bars highlight extreme left and right tail observations.

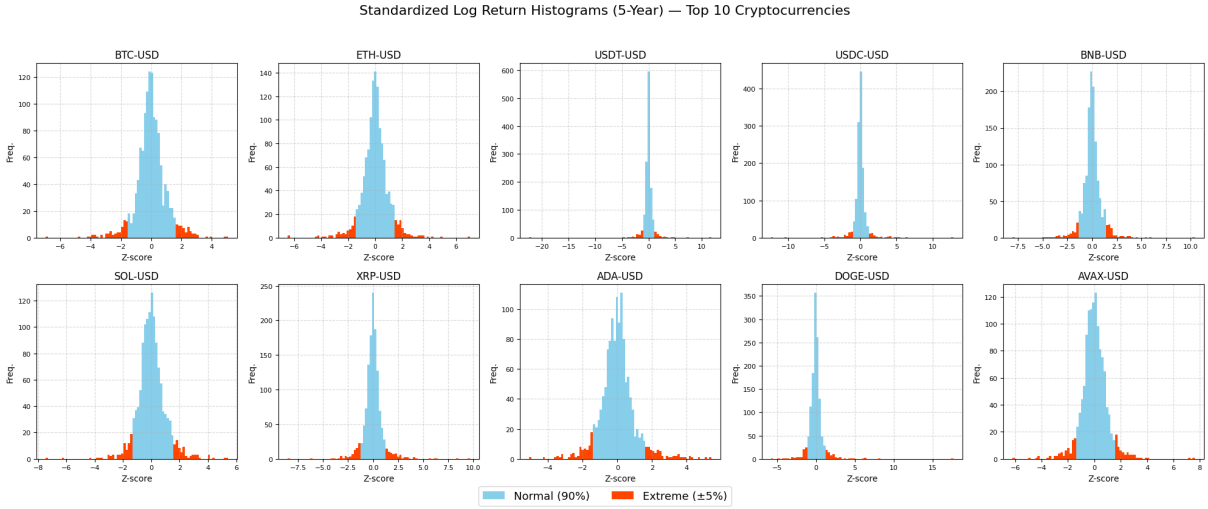


FIGURE 1. Standardized log return histograms for ten major cryptocurrencies

These empirical characteristics, including leptokurtosis, asymmetry, and heavy tails, motivate the use of flexible probabilistic models capable of capturing extreme observations. The subsequent section develops and evaluates such models using a mixture distribution framework.

3. METHODOLOGY

3.1. **Overview and motivation.** This section presents the probabilistic modeling framework designed to capture tail behavior in cryptocurrency return distributions. Empirical studies consistently demonstrate that standardized returns exhibit excess kurtosis and asymmetry, challenging the assumptions of classical single-distribution models such as the Gaussian or Student-t distribution.

To address this limitation, we employ a class of finite mixture models that flexibly account for multimodality, heavy tails, and latent regime-switching behavior in financial markets. The models considered in this study include the Gaussian Mixture Model (GMM), the Laplace Mixture Model (LMM), and the Student-t Mixture Model (TMM). Each model is fitted to the standardized return series using tailored Expectation–Maximization (EM) or EM-like estimation procedures. Model selection is guided by log-likelihood values, Akaike Information Criterion (AIC), and tail-specific risk metrics such as Value-at-Risk and Expected Shortfall.

3.2. Gaussian mixture model (GMM).

3.2.1. *Mathematical formulation.* The Gaussian Mixture Model (GMM) represents the return distribution as a weighted sum of K normal components, each characterized by a mean μ_k, variance σ_k^2, and mixing weight w_k. The probability density function (PDF) is defined as:

$$f_{\text{GMM}}(x) = \sum_{k=1}^{K} w_k \cdot \mathcal{N}(x \mid \mu_k, \sigma_k^2), \tag{3.1}$$

where

$$\mathcal{N}(x \mid \mu_k, \sigma_k^2) = \frac{1}{\sqrt{2\pi\sigma_k^2}} \exp\left(-\frac{(x-\mu_k)^2}{2\sigma_k^2}\right). \tag{3.2}$$

Each component represents a latent regime governing return dynamics.

3.2.2. *EM-based estimation.* The parameter set $\{w_k, \mu_k, \sigma_k^2\}$ is estimated via the Expectation –Maximization (EM) algorithm. The procedure alternates between computing soft assignments of observations to mixture components (E-step) and maximizing the expected complete-data log-likelihood (M-step).

Algorithm EM Algorithm for GMM

Input: Standardized returns $X = \{x_1, \ldots, x_N\}$, number of components K
Output: Estimated parameters w_k, μ_k, σ_k^2

Initialize $w_k = 1/K$
Initialize μ_k via percentiles
Initialize $\sigma_k = \mathrm{std}(X)/2$
while not converged **do**
  for each data point x_n **do**
    Compute posterior probability

$$\gamma_{nk} = \frac{w_k \cdot \mathcal{N}(x_n \mid \mu_k, \sigma_k^2)}{\sum_j w_j \cdot \mathcal{N}(x_n \mid \mu_j, \sigma_j^2)}$$

  end for
  for each component k **do**
    $N_k = \sum_n \gamma_{nk}$
    $w_k = N_k/N$
    $\mu_k = \frac{\sum_n \gamma_{nk} x_n}{N_k}$
    $\sigma_k^2 = \frac{\sum_n \gamma_{nk}(x_n - \mu_k)^2}{N_k}$
  end for
end while

3.2.3. *Model selection via AIC.* To determine the optimal number of mixture components, the Akaike Information Criterion (AIC) is computed for $K = 2$ to 10. For each fitted model:

$$\mathrm{AIC} = 2p - 2\log \mathcal{L}_{\max}, \qquad p = 3K - 1, \tag{3.3}$$

where p denotes the total number of estimated parameters and $\mathcal{L}_{\max}$ represents the maximized log-likelihood. The selected model corresponds to the specification yielding the minimum AIC value.

Algorithm GMM Model Selection via AIC

Input: Data X

for each $K = 2$ to 10 **do**
  Fit GMM with K components
  Compute maximized log-likelihood
  Compute AIC
end for
Return model with lowest AIC

3.3. **Laplace mixture model (LMM).**

3.3.1. *Mathematical formulation.* The Laplace Mixture Model (LMM) represents the return distribution as a weighted sum of K Laplace components. Each component is parameterized by a location parameter μ_k, a scale parameter b_k, and a mixing weight w_k. The mixture density is defined as:

$$f_{\text{LMM}}(x) = \sum_{k=1}^{K} w_k \cdot \text{Laplace}(x \mid \mu_k, b_k), \tag{3.4}$$

where the univariate Laplace density is given by

$$\text{Laplace}(x \mid \mu, b) = \frac{1}{2b} \exp\left(-\frac{|x-\mu|}{b}\right). \tag{3.5}$$

Compared to the Gaussian distribution, the Laplace distribution exhibits heavier tails and a sharper central peak, making it particularly suitable for modeling leptokurtic financial return distributions.

3.3.2. *EM-like estimation.* Parameter estimation for the LMM is performed using an EM-like iterative procedure. The E-step computes posterior responsibilities (soft assignments), and the M-step updates the parameters using weighted median and absolute deviation-based scale estimation.

Algorithm EM-like Algorithm for LMM

Input: Standardized returns $X = \{x_1, \ldots, x_N\}$, number of components K
Output: Estimated parameters w_k, μ_k, b_k

Initialize $w_k = 1/K$
Initialize μ_k via percentiles
Initialize $b_k = \text{std}(X)/2$
while not converged **do**
  for each data point x_n **do**
    Compute posterior probability

$$\gamma_{nk} = \frac{w_k \cdot \text{Laplace}(x_n \mid \mu_k, b_k)}{\sum_j w_j \cdot \text{Laplace}(x_n \mid \mu_j, b_j)}$$

  end for
  for each component k **do**
    $N_k = \sum_n \gamma_{nk}$
    $w_k = N_k/N$
    Update μ_k as weighted median of X using γ_{nk}
    $b_k = \frac{1}{N_k} \sum_n \gamma_{nk} \cdot |x_n - \mu_k|$
  end for
end while

3.3.3. *Model selection via AIC.* As in the GMM framework, the optimal number of components K is selected by minimizing the Akaike Information Criterion (AIC). The total number of parameters is given by $p = 3K - 1$, accounting for K location parameters, K scale parameters, and $K - 1$ independent mixing weights.

$$\text{AIC} = 2p - 2\log \mathcal{L}_{\max}, \qquad p = 3K - 1. \tag{3.6}$$

3.4. **Student-t mixture model (TMM).**

Algorithm LMM Model Selection via AIC

Input: Data X

1: **for** each $K = 2$ to 10 **do**
2: Fit LMM with K components using EM-like algorithm
3: Compute maximized log-likelihood
4: Compute AIC
5: **end for**
6: Return model with lowest AIC

3.4.1. *Mathematical formulation.* The Student-t Mixture Model (TMM) represents the return distribution as a weighted sum of K univariate t-distributions. Each component is characterized by a location parameter μ_k, scale parameter σ_k, degrees of freedom ν_k, and mixing weight w_k. The mixture density is given by:

$$f_{\text{TMM}}(x) = \sum_{k=1}^{K} w_k \cdot t(x \mid \nu_k, \mu_k, \sigma_k), \tag{3.7}$$

where $t(x \mid \nu, \mu, \sigma)$ denotes the Student-t density with ν degrees of freedom, centered at μ and scaled by σ. The heavier tails of the t-distribution make this specification particularly suitable for modeling extreme return behavior.

3.4.2. *EM-based estimation with latent scaling.* Parameter estimation for the TMM proceeds via an EM-like algorithm incorporating latent variables u_{nk}, which serve as observation-specific precision weights. These latent quantities downweight extreme observations during parameter updates, thereby enhancing robustness in the presence of heavy-tailed data.

3.4.3. *Model selection via AIC.* The number of mixture components K is selected by minimizing either the Akaike Information Criterion (AIC) or the Bayesian Information Criterion (BIC). Since the TMM involves four parameters per component (location, scale, degrees of freedom, and weight), the total parameter count is $p = 4K - 1$. The criteria are computed as:

$$\text{AIC} = 2p - 2\log \mathcal{L}_{\max}, \qquad \text{BIC} = \log(n)\, p - 2\log \mathcal{L}_{\max}. \tag{3.8}$$

3.5. **Extreme tail modeling rationale.** Cryptocurrency returns exhibit volatility clustering and fat tails, making classical parametric models inadequate for capturing extreme events. To address this, we adopt flexible mixture models—GMM, LMM, and TMM—that accommodate multimodality and heavy tails.

Model performance is assessed using both overall and tail-specific metrics. The full log-likelihood is defined as:

$$\log \mathcal{L}_{\text{full}} = \sum_{i=1}^{N} \log f(x_i), \tag{3.9}$$

where $f(x)$ is the fitted density and x_i are standardized returns. Tail performance is evaluated via:

$$\log \mathcal{L}_{\text{tail}}^{\text{left}} = \sum_{x_i < q_{0.05}} \log f(x_i), \qquad \log \mathcal{L}_{\text{tail}}^{\text{right}} = \sum_{x_i > q_{0.95}} \log f(x_i). \tag{3.10}$$

Risk sensitivity is further analyzed using Value-at-Risk (VaR) and Expected Shortfall (ES) at the 5% level:

Algorithm EM Algorithm for TMM

Input: Standardized returns $X = \{x_1, \ldots, x_N\}$, number of components K
Output: Estimated parameters $w_k, \mu_k, \sigma_k, \nu_k$

Initialize $w_k = 1/K$
Initialize μ_k via percentiles
Initialize $\sigma_k = \mathrm{std}(X)/2$
Initialize $\nu_k = 10$
while not converged **do**
  for each data point x_n **do**
    Compute responsibility

$$\gamma_{nk} = \frac{w_k \cdot t(x_n \mid \nu_k, \mu_k, \sigma_k)}{\sum_j w_j \cdot t(x_n \mid \nu_j, \mu_j, \sigma_j)}$$

  end for
  for each component k **do**
    $N_k = \sum_n \gamma_{nk}$
    $\delta_{nk} = \frac{(x_n - \mu_k)^2}{\sigma_k^2}$
    $u_{nk} = \frac{\nu_k + 1}{\nu_k + \delta_{nk}} \cdot \gamma_{nk}$
    $w_k = N_k / N$
    $\mu_k = \frac{\sum_n u_{nk} x_n}{\sum_n u_{nk}}$

$$\sigma_k^2 = \frac{\sum_n u_{nk}(x_n - \mu_k)^2}{N_k}$$

    for iteration = 1 to 10 **do**
      Compute gradient and Hessian of log-likelihood
      $\nu_k \leftarrow \nu_k - \frac{\nabla \log L}{\nabla^2 \log L}$
      if change $<$ tolerance **then**
        break
      end if
    end for
  end for
end while

Algorithm TMM Model Selection via Information Criteria

Input: Data X

for each $K = 2$ to 10 **do**
  Fit TMM with K components
  Compute maximized log-likelihood
  Compute AIC and BIC
end for
Return model with lowest AIC (or BIC)

$$\mathrm{VaR}_\alpha = \inf\{x : F(x) \geq \alpha\}, \qquad \mathrm{ES}_\alpha = \mathbb{E}[X \mid X < \mathrm{VaR}_\alpha]. \tag{3.11}$$

Visual diagnostics complement these metrics. Q-Q plots compare empirical and model quantiles. Survival function plots on a log scale reveal upper-tail behavior via $1 - F(x)$. Histogram overlays

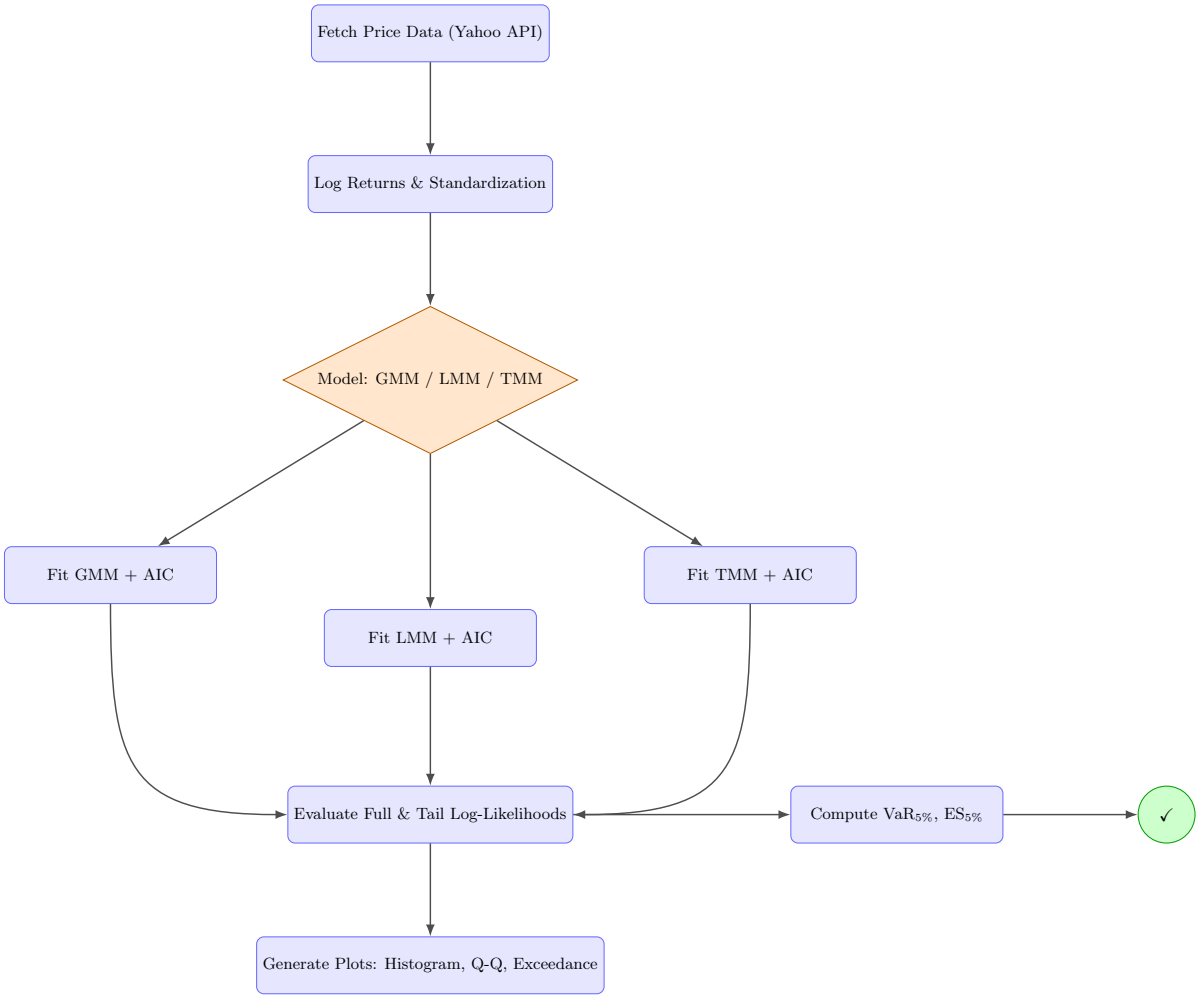


FIGURE 2. Workflow for Mixture Model Estimation and Tail Risk Evaluation

show empirical versus model-fitted densities. These tools aid qualitative interpretation of fit, especially in the tails.

3.6. **Computational framework.** The modeling pipeline integrates return transformation, mixture model estimation, and tail risk analysis into a reproducible structure. Daily log returns are first computed and standardized to enable comparability. Each asset's standardized returns are then fitted using Gaussian (GMM), Laplace (LMM), and Student-t (TMM) mixture models. The number of components $k \in [2, 10]$ is selected using AIC. Tail regions are identified using the empirical 5^{th} and 95^{th} percentiles.

Each model's fit is evaluated via full and tail-specific log-likelihoods, while tail risk is quantified using Value-at-Risk (VaR) and Expected Shortfall (ES) at the 5% level. Diagnostic plots—including Q-Q plots, histograms with PDF overlays, and exceedance curves—support visual evaluation of distributional fit and tail behavior.

Figure 2 and Algorithm provide schematic and procedural summaries of the complete computational process.

Algorithm Mixture Modeling and Tail Risk Evaluation

Input: Daily price data $\{P_t\}$
Output: VaR, ES, diagnostics for GMM, LMM, TMM
Compute log returns $R_t = \log(P_t/P_{t-1})$
Standardize: $Z_t = \frac{R_t - \mu}{\sigma}$
for each asset and model type **do**
    Fit model via EM, select $k \in [2, 10]$ by AIC
    Evaluate log-likelihoods (full, left, right tail)
    Compute $\text{VaR}_{5\%}$, $\text{ES}_{5\%}$
    Plot: histogram + PDF, Q-Q, exceedance
end for

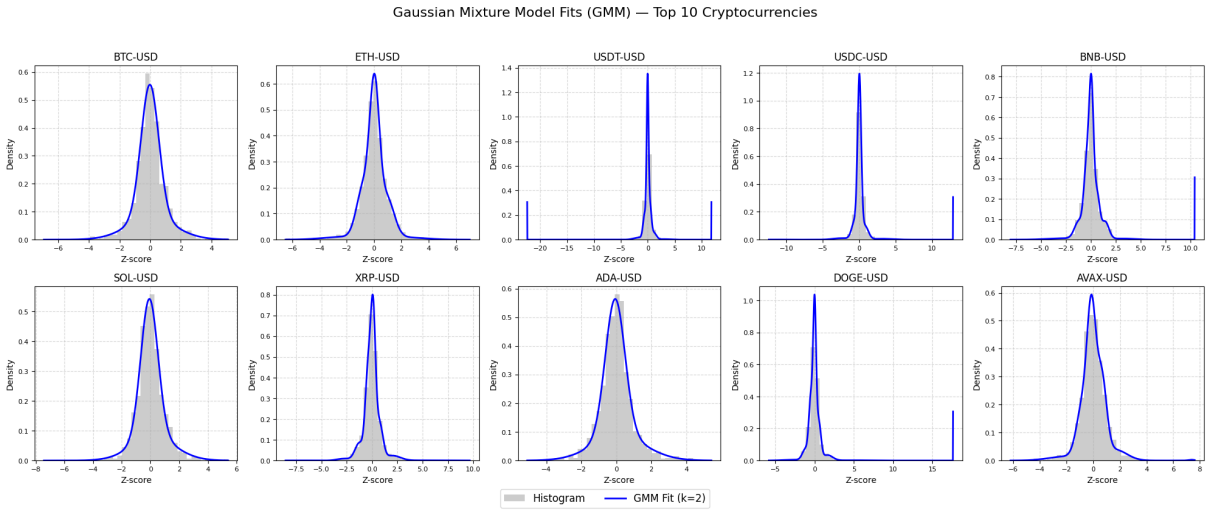


FIGURE 3. Gaussian Mixture Model Fits (GMM) across 10 cryptocurrencies

4. Empirical Results

4.1. **Fitting of mixture models.** Figure 3, Figure 4, and Figure 5 display the fitted densities from Gaussian Mixture Models (GMM), Laplace Mixture Models (LMM), and Student-t Mixture Models (TMM), respectively, overlaid on standardized return histograms for ten cryptocurrencies.

Visual inspection reveals that all three models generally capture the heavy tails and multimodal patterns present in return distributions, albeit with varying effectiveness. GMMs adapt well to the central mass of the distribution but often underfit extreme tails. LMMs tend to accommodate sharper peaks, particularly for stablecoins such as USDT and USDC. TMMs demonstrate greater flexibility in modeling both body and tails, especially for volatile assets including BTC, AVAX, and SOL.

Table 4 reports the best-performing number of components k^* selected via AIC for each model–asset pair. GMMs frequently favor higher component counts (e.g., DOGE, XRP), reflecting the need for multiple Gaussian components to approximate heavy-tailed behavior. TMMs achieve comparable or superior fit using fewer components by leveraging the inherent heavy-tailed structure of the Student-t distribution. In contrast, LMMs generally select smaller k, consistent with their peaked symmetric structure that suits assets exhibiting sharp kurtosis (e.g., USDC, ADA).

4.2. **Goodness-of-fit evaluation.** To assess how well each fitted distribution captures the empirical return structure, we present Q-Q plots comparing theoretical quantiles against observed returns.

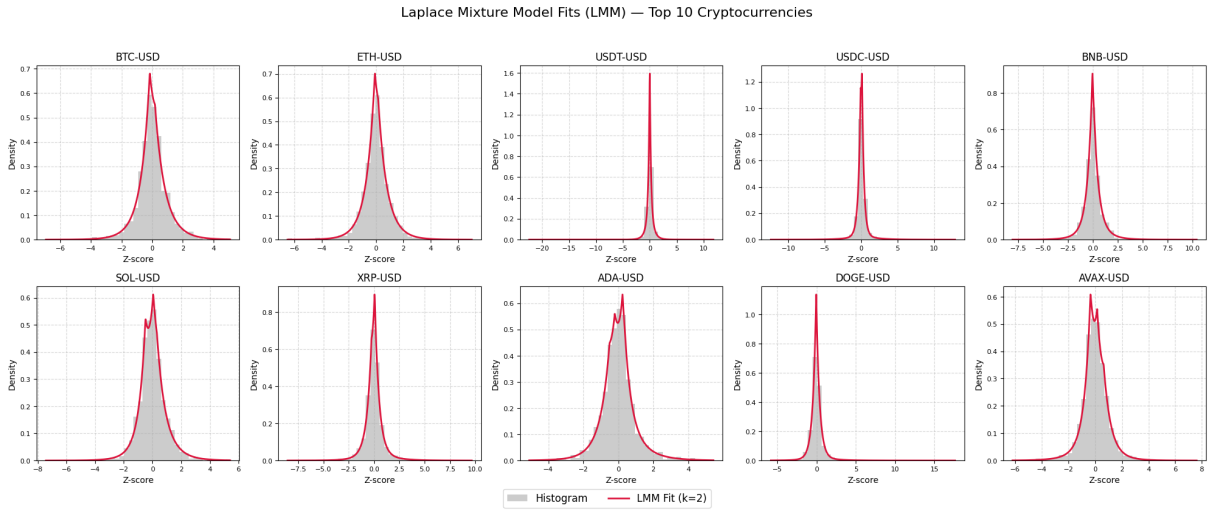


FIGURE 4. Laplace Mixture Model Fits (LMM) across 10 cryptocurrencies

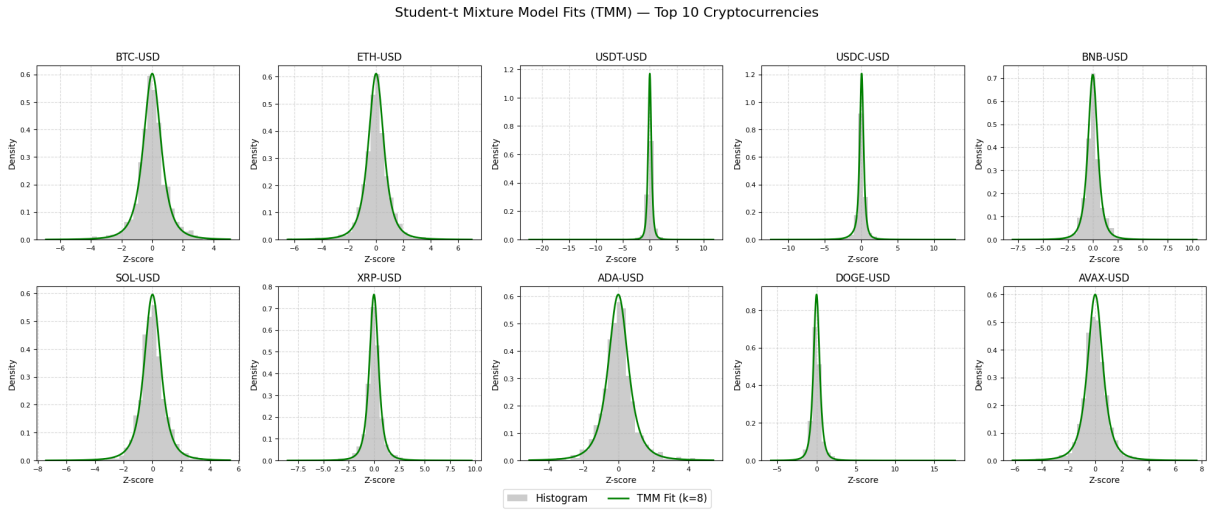


FIGURE 5. Student-t Mixture Model Fits (TMM) across 10 cryptocurrencies

Figures 6–8 show that the GMM tends to deviate in the tails, particularly underestimating extreme quantiles. LMM improves upon this with better alignment in the central region but still exhibits curvature in the tails. TMM shows the closest adherence to the 45-degree reference line across most assets, indicating the strongest overall fit in both the center and tails.

In terms of overall likelihood, Table 5 reports full-sample log-likelihoods across all ten assets. TMM achieves the highest values for most series, confirming its ability to accommodate heavy-tailed behavior more effectively than GMM and LMM.

4.3. **Tail risk metrics.** Tail risk analysis focuses on a model's ability to capture extreme losses and gains. We evaluate this via left and right tail log-likelihoods, Value-at-Risk (VaR), and Expected Shortfall (ES) at the 5% level for each asset–model pair. Tables 6–8 present these metrics for GMM, LMM, and TMM, respectively.

TABLE 4. Best number of mixture components (k^*) selected via AIC for each asset and model

Asset	GMM (k)	LMM (k)	TMM (k)
BTC-USD	2	2	8
ETH-USD	5	2	2
USDT-USD	9	3	3
USDC-USD	7	3	8
BNB-USD	8	2	6
SOL-USD	2	3	7
XRP-USD	8	3	2
ADA-USD	2	4	5
DOGE-USD	8	2	3
AVAX-USD	6	4	3

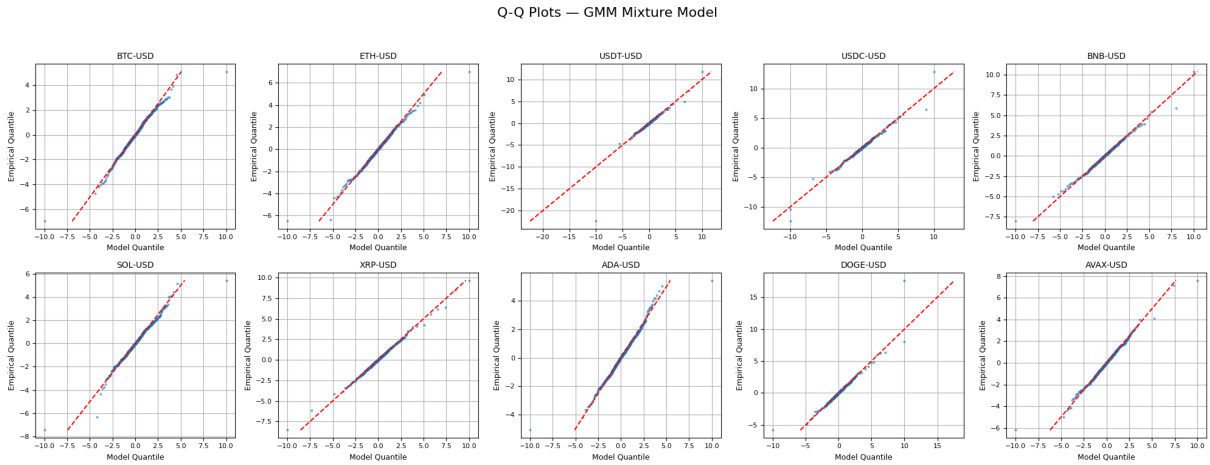


FIGURE 6. Q-Q Plots: GMM vs Empirical Returns

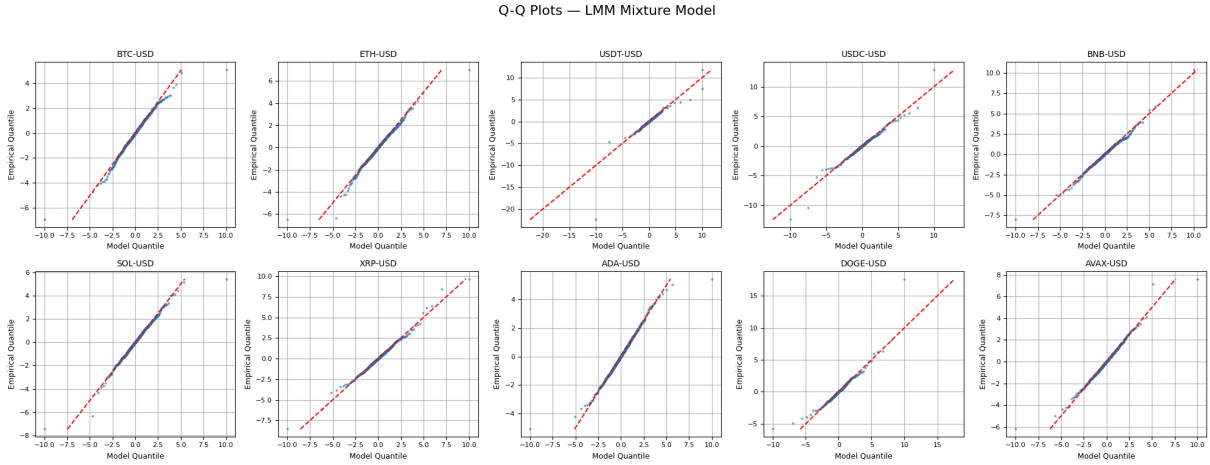


FIGURE 7. Q-Q Plots: LMM vs Empirical Returns

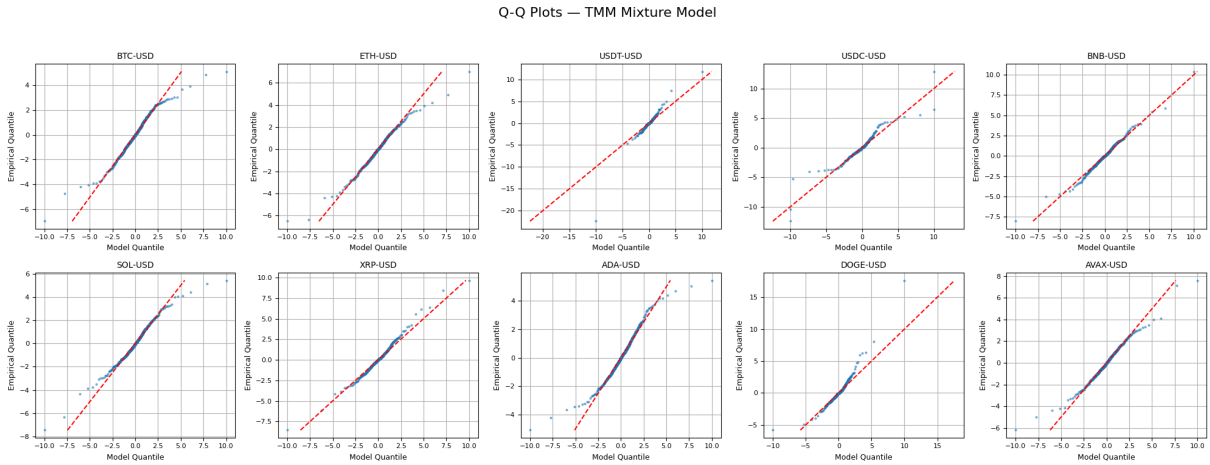


FIGURE 8. Q-Q Plots: TMM vs Empirical Returns

TABLE 5. Full-sample log-likelihoods for GMM, LMM, and TMM across all assets. Higher values indicate better overall fit

Asset	GMM LogLik	LMM LogLik	TMM LogLik
BTC-USD	-1738.04	-1730.57	-1738.49
ETH-USD	-1713.88	-1713.88	-1716.06
USDT-USD	-1011.09	-1030.69	-1069.97
USDC-USD	-1092.29	-1106.48	-1122.22
BNB-USD	-1563.30	-1576.05	-1586.48
SOL-USD	-1736.19	-1721.57	-1733.23
XRP-USD	-1504.00	-1506.26	-1515.37
ADA-USD	-1725.65	-1717.75	-1729.43
DOGE-USD	-1301.35	-1314.88	-1336.85
AVAX-USD	-1658.92	-1656.67	-1663.90

GMM results: As shown in Table 6, Gaussian mixtures perform reasonably well for moderate deviations but tend to underestimate the severity of tail events, particularly in assets with heavier tails such as ETH and SOL.

LMM results: Table 7 shows that Laplace mixtures yield improved tail likelihoods over GMM, particularly for stablecoins such as USDT and USDC. The sharper peak of the Laplace distribution aligns well with assets exhibiting lower volatility but still fails to fully capture heavy tails in highly volatile assets.

TMM results: As evident in Table 8, Student-t mixtures consistently provide the strongest fit for tail behavior, achieving the highest tail log-likelihoods and the most conservative VaR and ES estimates. This reflects the model's ability to accommodate heavy-tailed behavior, especially in assets such as DOGE, AVAX, and XRP.

4.4. **Exceedance probability plots.** To evaluate how well each model captures extreme events, we examine the empirical versus model-predicted exceedance probabilities in both tails. These plots are displayed on a logarithmic scale to emphasize rare but impactful events. Figures 9–14 compare left and right tail exceedance behavior for GMM, LMM, and TMM across all assets.

Table 6. Tail Risk Metrics for GMM

Asset	Left LL	Right LL	VaR (5%)	ES (5%)
BTC-USD	-255.99	-239.12	-0.7755	-1.5319
ETH-USD	-257.64	-246.87	-0.8182	-1.5180
USDT-USD	-241.01	-236.60	-0.3567	-0.9477
USDC-USD	-269.06	-261.40	-0.3529	-1.0596
BNB-USD	-256.21	-236.62	-0.6413	-1.3486
SOL-USD	-245.26	-242.91	-0.8503	-1.4900
XRP-USD	-242.80	-276.54	-0.6028	-1.2822
ADA-USD	-237.37	-254.73	-0.8104	-1.4557
DOGE-USD	-238.83	-267.48	-0.5779	-1.1251
AVAX-USD	-249.34	-239.73	-0.8082	-1.4899

Table 7. Tail Risk Metrics for LMM

Asset	Left LL	Right LL	VaR (5%)	ES (5%)
BTC-USD	-255.77	-234.96	-0.7566	-1.5049
ETH-USD	-249.58	-235.41	-0.7888	-1.4795
USDT-USD	-243.54	-252.97	-0.3667	-0.9626
USDC-USD	-275.88	-277.58	-0.3646	-1.0842
BNB-USD	-255.16	-250.88	-0.6414	-1.3486
SOL-USD	-242.06	-244.62	-0.8133	-1.4620
XRP-USD	-251.15	-272.99	-0.6079	-1.2917
ADA-USD	-235.56	-262.03	-0.8044	-1.4524
DOGE-USD	-242.18	-283.20	-0.5593	-1.0944
AVAX-USD	-237.91	-244.53	-0.7730	-1.4604

Table 8. Tail Risk Metrics for TMM

Asset	Left LL	Right LL	VaR (5%)	ES (5%)
BTC-USD	-260.21	-255.27	-0.7513	-1.4974
ETH-USD	-254.96	-251.64	-0.7828	-1.4694
USDT-USD	-280.62	-292.27	-0.3936	-0.9967
USDC-USD	-266.43	-302.46	-0.3452	-1.0392
BNB-USD	-271.80	-273.99	-0.6493	-1.3585
SOL-USD	-236.64	-260.11	-0.7821	-1.4318
XRP-USD	-268.79	-294.22	-0.6000	-1.2790
ADA-USD	-242.54	-275.31	-0.7871	-1.4207
DOGE-USD	-255.99	-308.57	-0.5517	-1.0871
AVAX-USD	-239.84	-246.10	-0.7710	-1.4532

GMM performance: As seen in Figures 9 and 10, Gaussian mixtures consistently underestimate tail exceedance, particularly in the extreme left tail, demonstrating poor alignment with empirical probabilities.

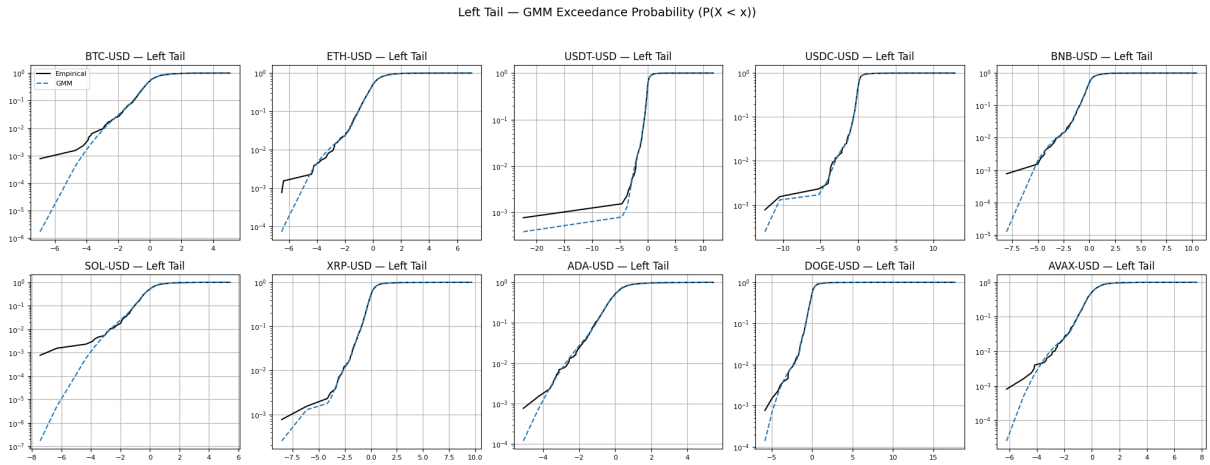


FIGURE 9. Left Tail Exceedance: GMM vs Empirical (log scale)

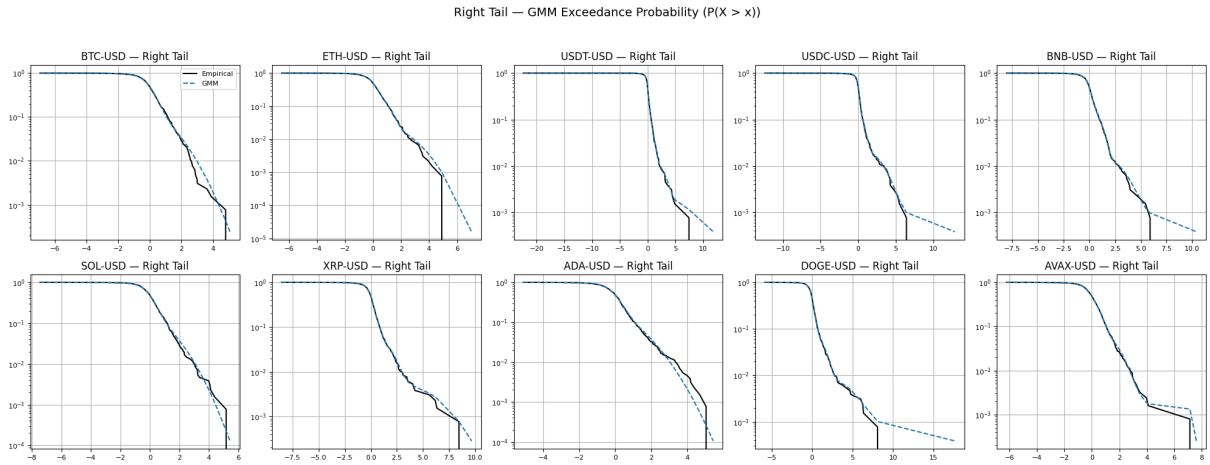


FIGURE 10. Right Tail Exceedance: GMM vs Empirical (log scale)

LMM performance: Figures 11 and 12 show modest improvements in tail approximation over GMM, especially in the right tail. However, LMM still falls short for heavy-tailed return distributions, such as those observed in highly volatile assets.

TMM performance: The Student-t mixture model (Figures 13 and 14) most closely matches empirical exceedance in both tails, capturing fat-tailed behavior effectively. This reinforces the findings from tail likelihoods and VaR/ES metrics, further validating TMM as the most suitable model for tail-sensitive applications.

5. DISCUSSION

This study conducted a comparative evaluation of three mixture model frameworks—Gaussian Mixture Models (GMM), Laplace Mixture Models (LMM), and Student-t Mixture Models (TMM)—for modeling the return distributions and tail risks of ten major cryptocurrencies. The empirical evidence clearly demonstrates distinct differences in model performance across dimensions of fit, risk estimation, and complexity.

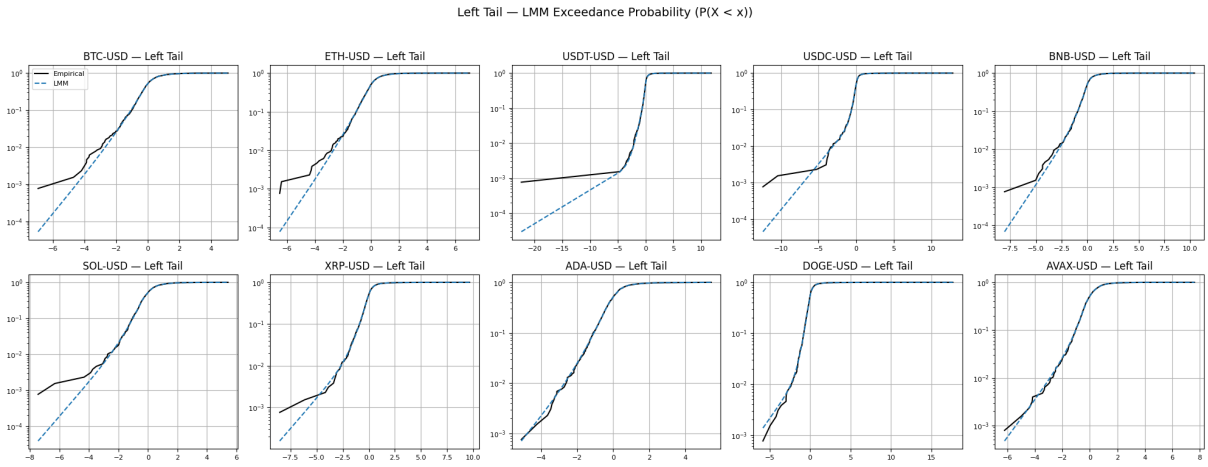


FIGURE 11. Left Tail Exceedance: LMM vs Empirical (log scale)

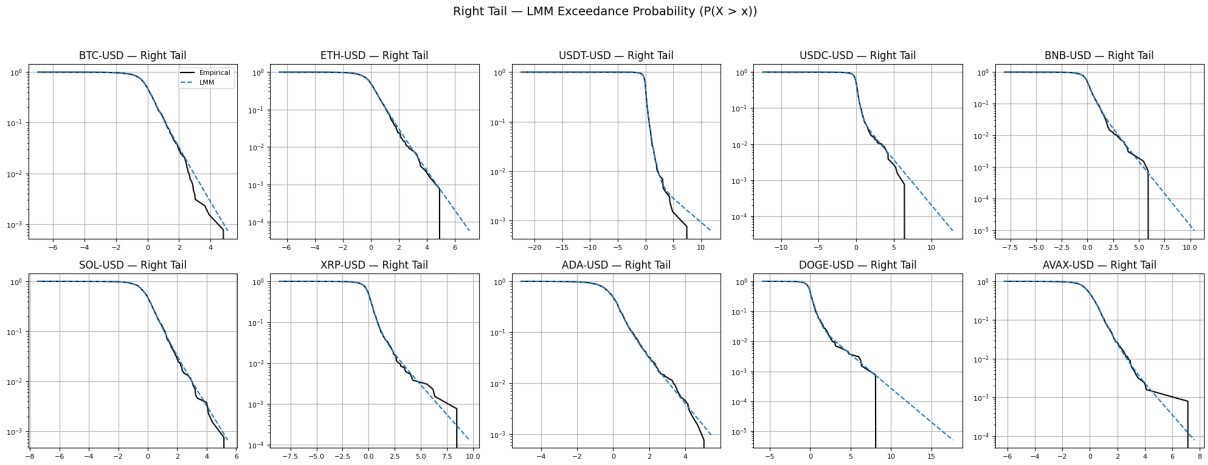


FIGURE 12. Right Tail Exceedance: LMM vs Empirical (log scale)

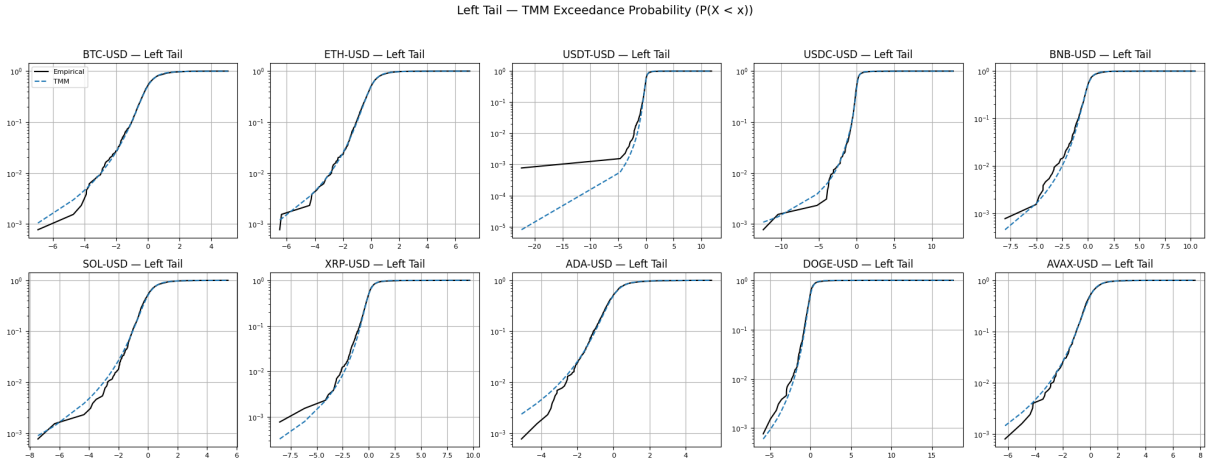


FIGURE 13. Left Tail Exceedance: TMM vs Empirical (log scale)

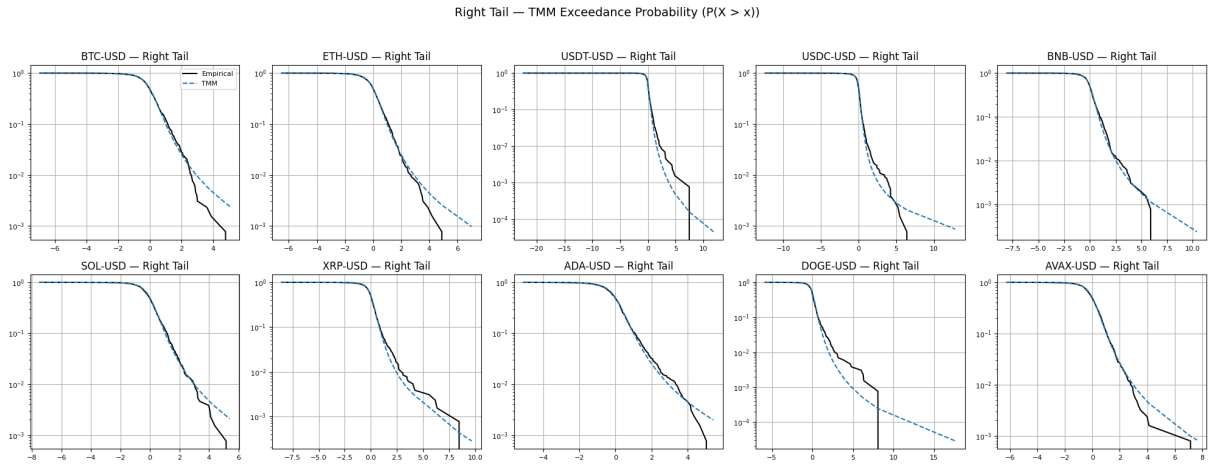


FIGURE 14. Right Tail Exceedance: TMM vs Empirical (log scale)

Our findings reveal that TMM consistently outperforms both GMM and LMM in capturing extreme return behavior. The Q-Q plots, tail log-likelihoods, and risk metrics (VaR and ES) collectively confirm TMM's superiority, particularly for high-volatility assets such as BTC, DOGE, and AVAX. TMM's fat-tailed nature enables it to provide more accurate approximations of empirical exceedance probabilities in both left and right tails, as confirmed by the log-scale exceedance plots. Moreover, TMM achieves these improvements with relatively fewer components—typically 2–3 for assets such as ETH and USDT—demonstrating efficiency without sacrificing flexibility.

In contrast, GMM exhibits the weakest tail fit, often underestimating tail probabilities and requiring more components to approximate the data. For instance, GMM selected $k = 8$ or more for DOGE, XRP, and USDT, yet still failed to capture extreme behavior adequately. This over-reliance on component count manifests in artifacts such as sharp, narrow spikes due to overfitting rare observations, a known limitation of GMMs in heavy-tailed settings.

LMM offers a middle ground—its peakedness improves fit over GMM, particularly for relatively stable assets such as USDT and USDC. However, its tail flexibility remains limited when compared to TMM, which more faithfully tracks empirical risk across the distribution's extremities.

From a complexity perspective, model performance is not simply a function of component count. While GMM tends to inflate k to compensate for its lack of tail flexibility, LMM and TMM achieve better or comparable fits using fewer components, owing to their distributional structure. This reinforces the notion that *distributional flexibility trumps model complexity* when modeling financial return data—especially in the context of cryptocurrencies.

Overall, the results advocate the use of heavy-tailed mixture models, particularly TMM, in scenarios requiring robust tail risk estimation. Such models not only enhance risk sensitivity but also mitigate overfitting, making them better suited for turbulent and heavy-tailed asset classes such as cryptocurrencies.

6. CONCLUSION

This study compares GMM, LMM, and TMM models for capturing the return distributions and tail risks of major cryptocurrencies. Using daily returns from ten assets, we evaluated model fit, complexity, and risk sensitivity through log-likelihoods, tail metrics, and exceedance plots.

The results show that GMM underperforms in the tails and often requires more components to approximate heavy-tailed behavior, occasionally producing unrealistic spikes due to overfitting. LMM improves fit for stablecoins but remains limited in extreme-risk scenarios. TMM consistently delivers

the strongest tail performance, with accurate Value-at-Risk and Expected Shortfall estimates and close alignment with empirical exceedance probabilities.

These findings underscore the importance of heavy-tailed models in cryptocurrency risk analysis. We recommend Student-t mixtures as a robust and parsimonious framework for tail-aware modeling and risk management in volatile asset classes.

Statements and Declarations

The authors declare that no supplementary material accompanies this article. The data used in this study are publicly available from Yahoo Finance. The computational procedures and scripts used in this study are available from the corresponding author upon reasonable request.

Acknowledgments

The authors are thankful to the Editor and the anonymous reviewers for their constructive comments and valuable suggestions, which significantly improved the quality and clarity of this manuscript.

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