APPLICABLE NONLINEAR ANALYSIS
Volume 4 (2027), No. 1, 29–53
DOI: https://doi.org/10.69829/apna-027-0401-ta03
Tulipa Opera Scholarum

SYSTEM OF NEUTRAL FUNCTIONAL INTEGRO-DIFFERENTIAL EVOLUTION EQUATIONS WITH INSTANTANEOUS AND NON-INSTANTANEOUS IMPULSES

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Abstract. In this paper, we investigate the existence of solutions for a system of impulsive functional integro-differential evolution equations with state-dependent delay on infinite intervals via resolvent operators in the sense of Grimmer. Our analysis is based on fixed point theorem with measures of noncompactness combined with a vector approach that uses matrices that converge to zero which was given in generalized Banach space. Moreover, an illustrative example is considered to support our new results.

Keywords. Fixed point theorem, infinite delay, impulsive integro-differential system, measures of noncompactness, mild solution, resolvent operator, state-dependent delay, generalized Banach space.
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1. Introduction

Impulsive events, which frequently occur in both nature and human activity, are the results of a sudden change in a system's state brought on by external disturbance. The phenomena of this type are split into two groups depending on the duration of the change. One is that this alteration only lasts for a brief period of time in comparison to the entire process, known as the instantaneous impulse. And when the effects are continuous, which means they can begin at any fixed point and last for a set amount of time. It is known as a non-instantaneous impulse. The theory of instantaneous impulsive differential equations has undergone rapid development over the years and played a very important role in modern applied mathematical models of real processes rising in phenomena studied in physics, population dynamics, chemical technology, biotechnology and economics, there are current results for impulsive evolution equations in [1, 4, 15] and its reference. But sometimes instantaneous impulses cannot describe the dynamics of certain evolution processes. For example, when one considers the hemodynamic equilibrium of a person, the introduction of the drugs in the bloodstream and the consequent absorption for the body are a gradual and continuous process. also a widely used example of noninstantaneous impulses is the introduction of insulin in the bloodstream which is abrupt change and the consequent absorption which is a gradual process as it remains active for a finite interval of time. Many researchers have investigated non-instantaneous impulses; for an example, see [2, 5, 9, 28].

On the other hand, due to their frequent appearance in applications as equation models, functional-differential equations with delays have drawn a lot of interest in recent years. see for instance [1, 3, 8, 11, 16, 26] and the references therein.

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2020 Mathematics Subject Classification: 47H10, 45J05, 47H08, 34K45.
Accepted: July 09, 2026.

Recently, many authors have investigated the existence of a solution for a system of impulsive differential equations using fixed point theorems (see, [7, 10, 12, 20, 23, 19, 24, 25]). The approach that uses matrices that converge to zero is more general in terms of the conditions that are imposed (we will give comments before the example).

Motivated by the above works, in this paper, we first discuss the existence of mild solutions for the following abstract model of impulsive functional integro-differential system with state-dependent delays:

$$\begin{cases} \frac{d}{dt}\left(\vartheta(t)-\Lambda_1(t,\vartheta_{\rho_1(t,\vartheta_t)},\varpi_{\rho_2(t,\varpi_t)})\right)=A_1(t)\left(\vartheta(t)-\Lambda_1(t,\vartheta_{\rho_1(t,\vartheta_t)},\varpi_{\rho_2(t,\varpi_t)})\right)\\ \qquad\qquad +\int_0^t B_1(t,s)\left(\vartheta(s)-\Lambda_1(s,\vartheta_{\rho_1(s,\vartheta_s)},\varpi_{\rho_2(s,\varpi_s)})\right)ds\\ \qquad\qquad +f_1\left(t,\vartheta_{\rho_1(t,\vartheta_t)},\varpi_{\rho_2(t,\varpi_t)},H_1(\vartheta(t),\varpi(t))\right);\ \text{for } t\in\widetilde{\Omega},\\ \frac{d}{dt}\left(\varpi(t)-\Lambda_2(t,\vartheta_{\rho_1(t,\vartheta_t)},\varpi_{\rho_2(t,\varpi_t)})\right)=A_2(t)\left(\varpi(t)-\Lambda_2(t,\vartheta_{\rho_1(t,\vartheta_t)},\varpi_{\rho_2(t,\varpi_t)})\right)\\ \qquad\qquad +\int_0^t B_2(t,s)\left(\varpi(s)-\Lambda_2(s,\vartheta_{\rho_1(s,\vartheta_s)},\varpi_{\rho_2(s,\varpi_s)})\right)ds\\ \qquad\qquad +f_2\left(t,\vartheta_{\rho_1(t,\vartheta_t)},\varpi_{\rho_2(t,\varpi_t)},H_2(\vartheta(t),\varpi(t))\right);\ \text{ for } t\in\widetilde{\Omega},\\ \vartheta(t_k^+)-\vartheta(t_k^-)=\Delta_k(\vartheta(t_k^-),\varpi(t_k^-)),\ k=1,\ldots,m,\\ \varpi(t_k^+)-\varpi(t_k^-)=\widetilde{\Delta}_k(\vartheta(t_k^-),\varpi(t_k^-)),\ k=1,\ldots,m,\\ (\vartheta(t),\vartheta(t))=(\Phi_1(t),\Phi_2(t));\ \text{if } t\in\mathbb{R}_-, \end{cases} \tag{1.1}$$

where $\Omega=\mathbb{R}^+$, $\widetilde{\Omega}=\Omega\backslash\widehat{\Omega}_k$, $\widehat{\Omega}_k=\{t_1,\ldots,t_k,\ldots\}$, with $0=t_0<t_1<t_2<\ldots<t_k<t_{k+1}\to+\infty,(k\to+\infty)$.

Next, we discuss the existence of mild solutions for the following system of integro-differential equations with state-dependent delays and noninstantaneous impulsive:

$$\begin{cases} \frac{d}{dt}\left(\vartheta(t)-\Lambda_1(t,\vartheta_{\rho_1(t,\vartheta_t)},\varpi_{\rho_2(t,\varpi_t)})\right)=A_1(t)\left(\vartheta(t)-\Lambda_1(t,\vartheta_{\rho_1(t,\vartheta_t)},\varpi_{\rho_2(t,\varpi_t)})\right)\\ \qquad\qquad +\int_0^t B_2(t,s)\left(\vartheta(s)-\Lambda_1(s,\vartheta_{\rho_1(s,\vartheta_s)},\varpi_{\rho_2(s,\varpi_s)})\right)ds\\ \qquad\qquad +f_1\left(t,\vartheta_{\rho_1(t,\vartheta_t)},\varpi_{\rho_2(t,\varpi_t)},H_1(\vartheta(t),\varpi(t))\right);\ \text{if } t\in I_k,\ k\in\mathbb{N}_0,\\ \frac{d}{dt}\left(\varpi(t)-\Lambda_2(t,\vartheta_{\rho_1(t,\vartheta_t)},\varpi_{\rho_2(t,\varpi_t)})\right)=A_2(t)\left(\varpi(t)-\Lambda_2(t,\vartheta_{\rho_1(t,\vartheta_t)},\varpi_{\rho_2(t,\varpi_t)})\right)\\ \qquad\qquad +\int_0^t B_2(t,s)\left(\varpi(s)-\Lambda_2(s,\vartheta_{\rho_1(s,\vartheta_s)},\varpi_{\rho_2(s,\varpi_s)})\right)ds\\ \qquad\qquad +f_2\left(t,\vartheta_{\rho_1(t,\vartheta_t)},\varpi_{\rho_2(t,\varpi_t)},H_2(\vartheta(t),\varpi(t))\right);\ \text{if } t\in I_k,\ k\in\mathbb{N}_0,\\ \vartheta(t)=\Theta_k\left(t,\vartheta\left(t_k^-\right),\varpi\left(t_k^-\right)\right);\ \text{if } t\in\Omega_k,\ k\in\mathbb{N},\\ \vartheta(t)=\Xi_k\left(t,\vartheta\left(t_k^-\right),\varpi\left(t_k^-\right)\right);\ \text{if } t\in\Omega_k,\ k\in\mathbb{N},\\ (\vartheta(t),\vartheta(t))=(\Phi_1(t),\Phi_2(t));\ \text{if } t\in\mathbb{R}_-, \end{cases} \tag{1.2}$$

where $I_0=[0,t_1]$, $I_k=(s_k,t_{k+1}]$, $\Omega_k=(t_k,s_k]$, $\mathbb{N}=\{1,2,\ldots\}$ and $\mathbb{N}_0=\mathbb{N}\cup\{0\}$ with $0=s_0<t_1\le s_1\le t_2<\ldots<s_{m-1}\le t_m\le s_m\le t_{m+1}\to+\infty,(k\to+\infty)$.

In systems (1.1) and (1.2) and for $i=1,2$, $A_i(t):D(A_i(t))\subset\mho\to\mho$, $B_i(t,s)$ are closed linear operators on $\mho$, with dense domains $D(A_i(t))$, which are independent of t, and $D(A_i(s))\subset D(B_i(t,s))$,

the operators H_i are defined by

$$H_i(\vartheta,\varpi)(t)=\int_0^a h_i(t,s,\vartheta(s),\varpi(s))ds,\ a>0.$$

The nonlinear terms $f_i:\Omega\times\S\times\S\times\mho\to\mho,\ \Lambda_i:\Omega\times\S\times\S\to\mho,\ \Theta_k,\ \Xi_k:\Omega_k\times\mho\times\mho\to\mho,$ $I_k,\ \bar{I}_k:\mho\times\mho\to\mho,\ k\in\mathbb{N},\ \Phi_i:\mathbb{R}_-\to\mho,\rho_i:\Omega\times\S\to(-\infty,\infty)$ are a given functions.

This work is organized of as follows. In Section 2, we recall some preliminary results and definitions which will be utilized throughout this manuscript In Section 3 and 4, we give the existence results of mild solutions to the problems (1.1) and (1.2) respectively. An example is presented in Section 5 to illustrate the application of our results.

2. Preliminaries

Let $BC(\Omega,\mho)$ be the Banach space of all bounded and continuous functions ϑ mapping $\Omega:=[0,+\infty)$ into $\mho$, with

$$\|\vartheta\|_\infty=\sup_{t\in\Omega}\|\vartheta(t)\|.$$

And $L^1([0,+\infty),\mho)$ be the Banach space of measurable functions $\vartheta:[0,+\infty)\to\mho$ which are Bochner integrable, with the norm

$$\|\vartheta\|_{L^1}=\int_0^{+\infty}\|\vartheta(t)\|dt.$$

We introduce the space

$$PC(\mathbb{R},\mho)=\Big\{\vartheta:\mathbb{R}\to\mho:\vartheta|_{\mathbb{R}^-}\in\S,\vartheta|_{\Omega_k}=Z_k;\ k\in\mathbb{N},\vartheta|_{I_k};\ k\in\mathbb{N}_0,\text{ is continuous,}$$
$$\vartheta\left(s_k^+\right),\vartheta\left(t_k^-\right)\text{ and }\vartheta\left(t_k^+\right)\text{ exists with }\vartheta(s_k^+)=\vartheta(s_k)\text{ and }\vartheta(t_k^-)=\vartheta(t_k)\Big\}.$$

Let $\widetilde{\Omega}_0=[0,t_1],\widetilde{\Omega}_k=(t_k,t_{k+1}],\bar{\Omega}_k=[t_k,t_{k+1}]$ for $k\in\mathbb{N},\ \vartheta(t^+)=\lim_{t\to t^+}\vartheta(t)$. We define the space of piecewise continuous functions:

$$\widetilde{PC}(\Omega,\mho)=\Big\{\vartheta:\mathbb{R}\to\mho:\vartheta|_{\mathbb{R}^-}\in\widetilde{\S},\ \vartheta|_{\widetilde{\Omega}_k}\in C\left(\widetilde{\Omega}_k,\mho\right),\text{ such that }\vartheta\left(t_k^-\right)\text{ and }\vartheta\left(t_k^+\right)\text{ exist and}$$
$$\text{satisfy }\vartheta\left(t_k^-\right)=\vartheta\left(t_k\right),\text{ for }k\in\mathbb{N}\Big\}.$$

2.1. **Generalized Banach space.**

Definition 2.1. Let X be a vector metric space on $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$. A map $\|\cdot\|:X\to\mathbb{R}_+^n$ is called a norm on X if it satisfies the following properties:

- $\|\vartheta\|=0$ then $\vartheta=(0,...,0)$;
- $\|\lambda\vartheta\|=|\lambda|\|\vartheta\|$ for $\vartheta\in X,\lambda\in\mathbb{K}$;
- $\|\vartheta+v\|\le\|\vartheta\|+\|v\|$ for every $\vartheta,v\in X$.

Remark 2.2. The pair $(X,\|\cdot\|_X)$ is called a generalized normed space. If the generalized metric generated by $\|\cdot\|_X$ $(i.e\ d(\vartheta,v)=\|\vartheta-v\|_X)$ is complete then the space $(X,\|\cdot\|_X)$ is called a generalized Banach space, where

$$\|\vartheta-v\|_X=\begin{pmatrix}\|\vartheta-v\|_1\\ \vdots\\ \|\vartheta-v\|_n\end{pmatrix}.$$

Let $X \times X = PC(\Omega, \mho) \times PC(\Omega, \mho)$ be endowed with the vector norm $\|\cdot\|_{X\times X}$ defined by $\|v\|_{X\times X} = (\|u_1\|_{PC}, \|u_2\|_{PC})$ for $v = (u_1, u_2)$. It is clear that $(PC(\Omega, \mho)\times\ PC(\Omega, \mho), \|\cdot\|_{X\times X})$ is a generalized Banach space.

In the case of generalized Banach spaces in the sense of Perov, the notations of convergent sequence, Cauchy sequence, completeness, open and closed subset are similar to those for usual metric spaces.

Theorem 2.3 ([27]). *Let $M = (a_{i,j})$ and $\widetilde{M} = (b_{i,j})$ be two $m \times r$ matrices. Then, $M \geq \widetilde{M}$ $(> \widetilde{M})$ if $a_{i,j} \geq b_{i,j}$ $(> b_{i,j})$, for all $1 \leq i \leq m,\ 1 \leq j \leq r$. If O is the null matrix and $A \geq O$ (> 0), we say that A is non-negative (positive) matrix. Finally, if $B = b_{i,j}$ is an arbitrary complex $m \times r$ matrix, then $|B|$ denotes the matrix with entries $|b_{i,j}|$.*

Definition 2.4. A square matrix M of real numbers is said to be convergent to zero if and only if, $M^m \to 0$ as $m \to \infty$.

Theorem 2.5 ([27]). *A square matrix M of real numbers is said to be convergent to zero if and only if its spectral radius $\rho(M)$ is strictly less than 1. In other words, this means that all the eigenvalues of M are in the open unit disc i.e. $|\lambda| < 1$, for every $\lambda \in \mathrm{C}$ with $\det(M - \lambda I) = 0$, where I denote the unit matrix of $\mathcal{M}_{m\times m}(\mathbb{R})$.*

Lemma 2.6 ([27]). *Let $M \in \mathcal{M}_{m\times m}(\mathbb{R}_+)$. Then, the following assertions are equivalent:*

- *M is convergent towards zero,*
- *$M^k \to 0$ as $k \to \infty$,*
- *The matrix $(I - M)$ is nonsingular and*

$$(I - M)^{-1} = I + M + M^2 + \ldots + M^k + \ldots$$

- *The matrix $(I - M)$ is nonsingular and $(I - M)^{-1}$ has nonnegative elements.*

Theorem 2.7 ([27]). *Let A and B be two $m \times m$ matrices with $O \leq |B| \leq A$. Then $\rho(B) \leq \rho(A)$.*

Lemma 2.8 ([6]). *If $A \in \mathcal{M}_{m\times m}(\mathbb{R}_+)$ is a matrix with $\rho(A) < 1$, then $\rho(A + B) < 1$ for every matrix $B \in \mathcal{M}_{m\times m}(\mathbb{R}_+)$ whose elements are small enough.*

Definition 2.9. Let $Q \in \mathcal{M}_{2\times 2}(\mathbb{R})$ is said to be order preserving (or positive) if

$$p_1 \leq p_0,\ q_1 \leq q_0,$$

imply

$$Q\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} \geq Q\begin{pmatrix} p_1 \\ q_1 \end{pmatrix}.$$

in the sense of components.

Lemma 2.10 ([22]). *Let*

$$Q = \begin{pmatrix} \alpha & -\beta \\ -\gamma & \delta \end{pmatrix},$$

where $\alpha, \beta, \gamma, \delta \geq 0$ and $\det Q > 0$. Then Q^{-1} is order preserving.

Let the state space $(\S, \|\cdot\|_\S)$ is a seminormed linear space of functions mapping $(-\infty, 0]$ into $\mathbb{R}$, and verifying (see [14] for more details):

(A_1) If $\vartheta \in PC$ and $\vartheta_0 \in \S$, then for every $t \in \Omega$, the following hold:
- (i) $\vartheta_t \in \S$,
- (ii) There exists $\wp > 0$ such that $|\vartheta(t)| \leq \wp \|\vartheta_t\|_\S$,
- (iii) There exist two functions $L(\cdot)$ and $M(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ independent of ϑ with L continuous and bounded and M locally bounded such that :

$$\|\vartheta_t\|_\S \leq L(t) \sup\{|\vartheta(s)| : 0 \leq s \leq t\} + M(t) \|\vartheta_0\|_\S .$$

(A_2) For the function ϑ in (A_1), ϑ_t is a $\S$ - valued continuous function on $\mathbb{R}^+ \setminus \Omega_k$.
(A_3) The space $\S$ is complete. Denote $L_* = \sup\{L(t) : t \in \Omega\}$, $M_* = \sup\{M(t) : t \in \Omega\}$, $\aleph = \max\{L_*, M_*\}$.

Now, let $p \in \mathbb{N}_0$ and $(\theta_k)_{k\in\mathbb{N}}$ be a sequence defined by

$$\theta_k = \begin{cases} t_{p+1} - t & , \text{ if } k = 2p+1,\ t \in \mathbb{R}^-, \\ s_p - t & , \text{ if } k = 2p,\ t \in \mathbb{R}^-. \end{cases}$$

Then, for $I_\theta = \mathbb{R}^- \setminus \{\theta_k : k = 2p,\ p \in \mathbb{N}\}$, consider

$$PC_\theta(\mathbb{R}^-, \mho) = \{\vartheta : \mathbb{R}^- \to \mho : \vartheta|_{I_\theta} \text{ is continuous, } \vartheta(\theta_k^-),\ \vartheta(\theta_k^+) \text{ exist with } \vartheta(\theta_k^-) = \vartheta(\theta_k)\},$$

and the space

$$C_\theta := \{\phi \in PC_\theta(\mathbb{R}^-, \mho) : \lim_{\tau\to-\infty} \phi(\tau) \text{ exist in } \mho\},$$

endowed with the norm

$$\|\phi\|_\theta = \sup\{|\phi(\tau)| : \tau \le 0\}.$$

And for $\widetilde{I}_\theta = \mathbb{R}^- \setminus \{\theta \in \mathbb{R}^- : \text{ if } t \in \Omega, \text{ then } (\theta + t) \in \widehat{\Omega}_k\}$, we define the space

$$\widetilde{PC}_\theta(\mathbb{R}^-, \mho) = \{\vartheta : \mathbb{R}^- \to \mho : \vartheta|_{\widetilde{I}_\theta} \text{ is continuous, } \vartheta(\theta_k^-),\ \vartheta(\theta_k^+) \text{ exist with } \vartheta(\theta_k^-) = \vartheta(\theta_k)\},$$

and the space

$$\widetilde{C}_\theta := \{\phi \in \widetilde{PC}_\theta(\mathbb{R}^-, \mho) : \lim_{\tau\to-\infty} \phi(\tau) \text{ exist in } \mho\}.$$

Then, the axioms $(A_1) - (A_3)$ are satisfied in the spaces C_θ, $\widetilde{C}_\theta$. So in all what follows, we consider the phase space $\S = \{C_\theta, \text{ if } u \in PC(\mathbb{R}, \mho)\}$, and $\S = \{\widetilde{C}_\theta, \text{ if } u \in \widetilde{PC}(\mathbb{R}, \mho)\}$, and let

$$BPC(\mathbb{R}, \mho) = \{\vartheta \in PC(\mathbb{R}, \mho) : \vartheta \text{ is bounded on } \mathbb{R}^+ \text{ with respect to the norm } \|\cdot\|_{BPC}\},$$

$$\mathcal{X} = \{\vartheta \in \widetilde{PC}(\mathbb{R}, \mho) : \vartheta \text{ is bounded on } \mathbb{R}^+ \text{ with respect to the norm } \|\cdot\|_{\mathcal{X}}\},$$

where

$$\|\vartheta\|_{BPC} = \sup_{t\in\mathbb{R}} \{\|\vartheta(t)\|\}, \text{ and } \|\vartheta\|_{\mathcal{X}} = \sup_{t\in\mathbb{R}} \{\|\vartheta(t)\|\}.$$

We consider the following Cauchy problem

$$\begin{cases} \vartheta'(t) = A(t)\vartheta(t) + \int_0^t B(t,s)\vartheta(s)ds; \text{ for } t \ge 0, \\ \vartheta(0) = \vartheta_0 \in \mho. \end{cases} \tag{2.1}$$

The existence and properties of a resolvent operator has been discussed in [13]. We shall make the following assumptions

(I) $A(t)$ generates a uniformly continuous semigroup of evolution operators in $\mho$,
(II) Suppose Y is the Banach space formed from $D(A(t))$ with the graph norm. $A(t)$ and $B(t,s)$ are closed operators. It follows that $A(t)$ and $B(t,s)$ are in the set of bounded operators from Y to X, $B(Y, \mho)$, for $0 \le t < \infty$ and $0 \le s \le t < \infty$, respectively. Further, $A(t)$ and $B(t,s)$ are continuous on $0 \le t < \infty$ and $0 \le s \le t < \infty$, respectively, into $B(Y, \mho)$.

Definition 2.11. A resolvent operator for (2.1) is a bounded operator valued function $R(t,s) \in B(\mho)$, $0 \le s \le t < \infty$, the space of bounded linear operators on $\mho$, having the following properties.

(i) $R(t,s)$ is strongly continuous in s and t, $R(s,s) = I$, $0 \le s < \infty$, and $\|R(t,s)\| \le Me^{\beta(t-s)}$, for some constants M and β,
(ii) $R(t,s)Y \subset Y$, $R(t,s)$ is strongly continuous in s and t on Y,

(iii) For each $x \in D(A(t))$, $R(t,s)x$ is strongly continuously differentiable in t and s and

$$\frac{\partial R}{\partial t}(t,s)x = A(t)R(t,s)x + \int_s^t B(t,r)R(r,s)xdr,$$
$$\frac{\partial R}{\partial s}(t,s)x = -R(t,s)A(s)x - \int_s^t R(t,r)B(r,s)xdr,$$

with $\frac{\partial R}{\partial t}(t,s)x$ and $\frac{\partial R}{\partial s}(t,s)x$ strongly continuous on $0 \leq s \leq t < \infty$. Here, $R(t,s)$ can be extracted from the evolution operator of the generator $A(t)$. For nonautonomous differential equations in a Banach space, the resolvent operator is analogous to an evolution operator. The definition of the resolvent operator yields a number of results.

Theorem 2.12 ([12]). *Let $F \subset X$ be a bounded closed convex subset and $N : F \to F$ be a generalized β-condensing continuous mapping, where β is a nonsingular measure of noncompactress defined on the subsets of X. Then the set*

$$\mathrm{Fix}(N) = \{x \in F : x = N(x)\},$$

is nonempty.

3. Mild Solutions with Instantaneous Impulses

Firstly we define a generalized measure of noncompactness in the space $\mathcal{X}\times\mathcal{X}$. Let us fix a nonempty bounded subset S of the space $\mathcal{X}$. For $u \in S, T > 0,\ \epsilon > 0,\ \kappa_1, \kappa_2 \in [-T,T]$, such that $|\kappa_1 - \kappa_2| \leq \epsilon$. We denote $\omega^T(u,\epsilon)$ the modulus of continuity of the function u on the interval $[-T,T]$, namely,

$$\begin{array}{lcl} \omega^T(u,\epsilon) & = & \sup\{\|e^{-\kappa_1}u(\kappa_1) - e^{-\kappa_2}u(\kappa_2)\| \ ;\ \kappa_1,\kappa_2 \in [-T,T]\}, \\ \omega_0(S) & = & \lim_{T\to+\infty}\lim_{\epsilon\to 0}\sup_{u\in S}\{\omega^T(u,\epsilon)\}. \end{array}$$

If t is fixed from $\mathbb{R}$ let us denote for the set $S(t) = \{u(t) \in \mho \ ;\ u \in S\}$

$$d^{\Delta}(S(t)) = \sup\{\|u(t) - u(s)\| \ ;\ u \in S,\ t,s > T\}, \text{ and } D^{\Delta}(S) = \lim_{T\to\infty}\sup d^{\Delta}(S(t)).$$

Consider χ_{BPC^*} defined on the family of subset of $S = S_1 \times S_2 \subset \mathcal{X}\times\mathcal{X}$ by the formula

$$\chi_{BPC^*}(S_1\times S_2) = \begin{pmatrix}\chi_1(S_1)\\ \chi_1(S_2)\end{pmatrix} = \begin{pmatrix}\omega_0(S_1) + D^{\Delta}(S_2)\\ \omega_0(S_2) + D^{\Delta}(S_1)\end{pmatrix}.$$

As in [12, 21], the function χ_{BPC^*} is a generalized measure of noncompactness on the space $\mathcal{X}\times\mathcal{X}$.

Definition 3.1. A function $(\vartheta,\varpi) \in \mathcal{X}\times\mathcal{X}$ is called a mild solution of problem (1.1) if it verifies

$$\vartheta(t) = \begin{cases} R_1(t,0)(\Phi_1(0) - \Lambda_1(0,\Phi_1,\Phi_2)) + \Lambda_1(t,\vartheta_{\rho_1(t,\vartheta_t)},\varpi_{\rho_2(t,\varpi_t)}) \\ + \int_0^t R_1(t,s)f_1\left(\theta,\vartheta_{\rho_1(s,\vartheta_s)},\varpi_{\rho_2(s,\varpi_s)},H_1(\vartheta(s),\varpi(s))\right)ds, \\ + \sum_{0<t_k<t} R_1(t-t_k)\Delta_k(\vartheta(t_k{}^-),\varpi(t_k{}^-)), \text{ if } t \in \Omega, \\ \Phi_1(t), \text{ if } t \in \mathbb{R}_-. \end{cases}$$

And

$$\varpi(t)=\begin{cases} R_2(t,0)(\Phi_2(0)-\Lambda_2(0,\Phi_1,\Phi_2))+\Lambda_2(t,\vartheta_{\rho_1(t,\vartheta_t)},\varpi_{\rho_2(t,\varpi_t)}) \\ +\int_0^t R_2(t,s)f_2\left(\theta,\vartheta_{\rho_1(s,\vartheta_s)},\varpi_{\rho_2(s,\varpi_s)},H_2(\vartheta(s),\varpi(s))\right)ds, \\ +\sum_{0<t_k<t}R_2(t-t_k)\widetilde{\Delta}_k(\vartheta(t_k{}^-),\varpi(t_k{}^-)) \text{ if } t\in\Omega, \\ \Phi_2(t), \text{ if } t\in\mathbb{R}_-. \end{cases}$$

The hypotheses:

$(H1)$ The functions $f_i:\Omega\times\S\times\S\times\mho\to\mho$ are Carathéodory functions and there exist functions $\xi_i,\ \widehat{\xi_i}\in L^1(\Omega,\mathbb{R}^+)$, and a continuous nondecreasing functions $\psi_i,\widehat{\psi_i}\ :\Omega\to(0,+\infty)$ such that

$$\begin{aligned}\|f_i(t,\vartheta_1,\vartheta_2,\vartheta_3)-f_i(t,\widehat{\vartheta_1},\widehat{\vartheta_2},\widehat{\vartheta_3})\|\leq{}&\xi_i(t)\psi_i(\|\vartheta_1-\widehat{\vartheta_1}\|_\S+\|\vartheta_2-\widehat{\vartheta_2}\|_\S)\\&+\widehat{\xi_i}(t)\widehat{\psi_i}(\|\vartheta_3-\widehat{\vartheta_3}\|),\end{aligned}$$

for $\vartheta_1,\widehat{\vartheta_1},\ \vartheta_2\ ,\widehat{\vartheta_2}\in\S,\ \vartheta_3\ ,\widehat{\vartheta_3}\in\mho$. And, for every $\varrho_1,\varrho_2\geq 0$ we have

$$\lim_{t\to+\infty}\sup_{t\in\Omega}\int_0^t e^{-\mu_i(t-s)}\left[\varrho_i\xi_i(s)+\varrho_i\xi_i(s)\right]ds=0,$$

and

$$f_i^*=\sup_{t\in\Omega}\left\{\int_0^t\|f_i(s,0,0,H_i(0,0))\|ds\right\}<\infty.$$

$(H2)$ The functions $h_i:D_{h_i}\times\mho\times\mho\to\mho$ are continuous and there exists a continuous functions $h_{c_i}\ :\ D_{h_i}\to(0,+\infty)$ such that, for each $(t,s)\in D_{h_i}$ and $\vartheta_3,\ \widehat{\vartheta_3},\ \vartheta_4,\ \widehat{\vartheta_4}\in\mho$:

$$\|h_i(t,s,\vartheta_3,\vartheta_4)-h_i(t,s,\widehat{\vartheta}_3,\widehat{\vartheta}_4)\|\leq h_{c_i}(t,s)(\|\vartheta_3-\widehat{\vartheta_3}\|+\|\vartheta_4-\widehat{\vartheta_4}\|).$$

$$\max\left\{\sup_{D_h}\{h_{c_i}(t,s)\},\sup_{D_{h_i}}\{\|h_i(t,s,0)\|\}\right\}=\max\{h_{c_i}^*,h_i^*\}<\infty.$$

$(H3)$ The functions $\Lambda_i:\Omega\times\S\times\S\to\mho$ are continuous and for any bounded set $B\in\S$, the function $\{t\to\Lambda_i(t,x_t,y_t),\ (x,y)\in B\times B\}$ is equicontinuous on each compact interval of $[0;+\infty)$. Also we assume that there exists continuous functions $L_{\Lambda_i}\ :\ \Omega\to\mathbb{R}^+$, and a nondecreasing continuous functions $\widehat{\varphi_i}:\mathbb{R}^+\to\mathbb{R}^+$, such that

$$\|\Lambda_i(t,\vartheta_1,\vartheta_2)-\Lambda_i(t,\widehat{\vartheta_1},\widehat{\vartheta_2})\|\leq L_{\Lambda_i}(t)\widehat{\varphi_i}(\|\vartheta_1-\widehat{\vartheta_1}\|_\S+\|\vartheta_2-\widehat{\vartheta_2}\|_\S),$$

for all $\vartheta_1,\widehat{\vartheta_1},\ \vartheta_2\ ,\widehat{\vartheta_2}\in\S$, with

$$L_{\Lambda_i}^*=\sup_{t\in\Omega}\{L_{\Lambda_i}(t)\}<+\infty,\ \Lambda_i^*=\|\Lambda_i(t,w_0,\widehat{w}_0)\|.$$

$(H4)$ There exist $M_{R_i}\geq 1,\mu_i\geq 0$, such that

$$\|R_i(t,s)\|_{B(\mho)}\leq M_{R_i}e^{-\mu_i(t-s)}.$$

$(H5)$ The functions $Q_k^i,\ \mho\times\mho\to\mho$ are continuous and there exist positive constants $L_{Q_k^i},\ k\in\mathbb{N}$, such that

$$\|Q_k^i(\vartheta_3,\vartheta_4)-Q_k^i(\widehat{\vartheta_3},\widehat{\vartheta}_4)\|\leq L_{Q_k^i}(\|\vartheta_3-\widehat{\vartheta}_3\|+\|\vartheta_4-\widehat{\vartheta}_4\|),$$

$$\text{for all } \vartheta_3,\ \widehat{\vartheta}_3,\ \vartheta_4,\ \widehat{\vartheta}_4\in\mho,\ k\in\mathbb{N}.$$

And

$$\sum_{k\geq 0}\|Q_k^i(0,0)\| = Q_k^i < +\infty,$$

where

$$Q_k^i = \begin{cases} \Delta_k & \text{if } i=1, \\ \widetilde{\Delta}_k & \text{if } i=2. \end{cases}$$

(H_Φ) Set $\mathcal{R}(\rho^-) = \{\rho(s,\varphi) : (s,\varphi) \in \Omega \times \mathcal{B}, \rho(s,\varphi) \leq 0\}$. We assume that $\rho : \Omega \times \mathcal{B} \to \mathbb{R}$ is continuous. Moreover we assume the following assumption and hypothesis:

- The function $t \to \Phi_t$ is continuous from $\mathcal{R}(\rho^-)$into $\mathcal{B}$ and there exists a continuous and bounded function $L^\Phi : \mathcal{R}(\rho^-) \to (0,\infty)$ such that

$$\|\Phi_t\|_{\mathcal{B}} \leq L^\Phi(t)\|\Phi\|_{\mathcal{B}}, \text{ for every } t \in \mathcal{R}\left(\rho^-\right).$$

Remark 3.2. The condition (H_Φ), is verified by continuous and bounded functions. See [18], for more information.

Lemma 3.3 ([17]). *If $y : \mathbb{R} \to \mho$ is a function such that $y_0 = \Phi$, then*

$$\|y_s\|_{\mathcal{B}} \leq \left(M + \mathcal{L}^\Phi\right)\|\Phi\|_{\mathcal{B}} + L\sup\{|y(\theta)| \,;\, \theta \in [0,\max\{0,s\}]\}, \quad s \in \mathcal{R}\left(\rho^-\right) \cup \Omega,$$

where $\mathcal{L}^\Phi = \sup_{t\in\mathcal{R}(\rho^-)} \mathcal{L}^\Phi(t)$.

Now, we define the matrix

$$M_\Xi = \begin{pmatrix} \varsigma_1 & \varsigma_2 \\ \varsigma_5 & \varsigma_6 \end{pmatrix}.$$

Where $\varsigma_1, \varsigma_2, \varsigma_5, \varsigma_6$ are constants, they will be specific later.

Theorem 3.4. *If the conditions $(H1)-(H5)$ and (H_Φ) are satisfied, and the matrix M_Ξ converges to zero. Then, the system (1.1) has at least one mild solution.*

Proof. Consider the operator $\Xi : \mathcal{X} \times \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ given by:

$$\Xi(\vartheta(t),\varpi(t)) = (\Xi_1(\vartheta(t),\varpi(t)), \Xi_2(\vartheta(t),\varpi(t))),$$

where

$$\Xi_i(\vartheta(t),\varpi(t)) = \begin{cases} R_i(t,0)(\Phi_i(0) - \Lambda_i(0,\Phi_1,\Phi_2)) + \Lambda_i(t,\vartheta_{\rho_1(t,\vartheta_t)},\varpi_{\rho_2(t,\varpi_t)}) \\ + \int_0^t R_i(t,s) f_i\left(s,\vartheta_{\rho_1(s,\vartheta_s)},\varpi_{\rho_2(s,\varpi_s)}, H_i(\vartheta(s),\varpi(s))\right) ds \\ + \sum_{0<t_k<t} R_i(t-t_k) Q_k^i(\vartheta(t_k^-),\varpi(t_k^-)), \text{ if } t \in \Omega, \\ \Phi_i(t), \text{ if } t \in \mathbb{R}_-. \end{cases}$$

Clearly, the fixed points of the operator Ξ are mild solutions of the problem (1.1), so we shall check that operator Ξ satisfies all conditions of Theorem 2.12.

Let B_θ be defined by:

$$B_\theta = \{(\vartheta,\varpi) \in \mathcal{X} \times \mathcal{X} : (\|\vartheta\|_{\mathcal{X}}, \|\varpi\|_{\mathcal{X}}) \leq (\theta_1,\theta_2)\},$$

with $\theta_i > \|\Phi_i\|_{\S}$. Then, the set B_θ is bounded, closed and convex of $\mathcal{X} \times \mathcal{X}$.
We have divided the proof into 3 steps:

Step 1 : $\Xi(B_\theta) \subset B_\theta$.
For each $(\vartheta, \varpi) \in B_\theta$ and $t \in \mathcal{R}(\rho^-) \cup \Omega$, it follows that

$$\begin{aligned}\|\vartheta_t\|_{\S} &\leq \left(M_1(t) + \mathcal{L}^{\Phi_1}\right)\|\Phi_1\|_{\S} + L_1(t)\sup_{[0,s]}\|\vartheta(t)\| \\ &\leq \aleph_1\theta_1 + \left(\aleph_1 + \mathcal{L}^{\Phi_1}\right)\|\Phi_1\|_{\S} = \aleph_1^*,\end{aligned}$$

and

$$\|\varpi_t\|_{\S} \leq \aleph_2\theta_2 + \left(\aleph_2 + \mathcal{L}^{\Phi_2}\right)\|\Phi_2\|_{\S} = \aleph_2^*.$$

And, we get

$$\|H_i(\vartheta, \varpi)(s)\| \leq a(h_{c_i}^*\theta_1 + \bar{h}_{c_i}^*\theta_2 + h_i^*) = H_i^*.$$

Also, we have

$$\|\Lambda_i(t, \vartheta_{\rho_1(t,\vartheta_t)}, \varpi_{\rho_2(t,\varpi_t)}) - \Lambda_i(t,0,0)\| \leq L_{\Lambda_i}(t)\widehat{\varphi}_i(\|\vartheta_{\rho_1(t,\vartheta_t)}\|_{\S} + \|\varpi_{\rho_2(t,\varpi_t)}\|_{\S}).$$

And

$$\|Q_k^i(u,v)\| \leq m_k^i(\|u\|_{\mathcal{X}}) + \widetilde{m}_k^i(\|v\|_{\mathcal{X}}) + \|Q_k^i(0,0)\|.$$

If $t \in \Omega$, then we have

$$\begin{aligned}\|\Xi_i(\vartheta,\varpi)(t)\| &\leq M_{R_i}(\|\Phi_i(0)\|_{\S} + \left(L_{\Lambda_i}^*\widehat{\varphi}(\|\Phi_1\|_{\S} + \|\Phi_2\|_{\S}) + {\Lambda_i}^*\right)) \\ &\quad + M_{R_i}\int_0^t \xi_i(s)\psi_i(\aleph_i^*) + \widehat{\xi}_i(s)\widehat{\psi}_i(H_i^*)dsd + M_{R_i}f_i^* + \left(L_{\Lambda_i}^*\widehat{\varphi}(\aleph_1^* + \aleph_2^*) + \Lambda_i^*\right) \\ &\quad + M_{R_i}\sum_{k\geq 0}(m_k^i(\|\vartheta\|_{\mathcal{X}}) + \widetilde{m}_k^i(\|\varpi\|_{\mathcal{X}})) + M_{R_i}\|Q_k^i(0,0)\|. \\ &\leq M_{R_i}\left(\|\Phi_i(0)\|_{\S} + \left(L_{\Lambda_i}^*\widehat{\varphi}(\|\Phi_1\|_{\S} + \|\Phi_2\|_{\S}) + {\Lambda_i}^*\right) + \|\xi_i\|_{L^1}\psi_i(\aleph_i^*) + \|\widehat{\xi}_i\|_{L^1}\widehat{\psi}_i(H_i^*) + f_i^*\right) \\ &\quad + \left(L_{\Lambda_i}^*\widehat{\varphi}(\aleph_1^* + \aleph_2^*) + \Lambda_i^*\right) + M_{R_i}\sum_{k\geq 0}L_{Q_k^i}^* + M_{R_i}\sum_{k\geq 0}\|Q_k^i(0,0)\|.\end{aligned}$$

We define

$$\begin{aligned}\varsigma_1 &= M_{R_1}\left(\|\xi_1\|_{L^1}\aleph_1 + \|\widehat{\xi}_1\|_{L^1}ah_{c_1}^* + \sum_{k\geq 0}L_{Q_k^1}^*\right) + \aleph_1 L_{\Lambda_1}^*. \\ \varsigma_2 &= M_{R_1}\left(\|\widehat{\xi}_1\|_{L^1}a\overline{h}_{c_1}^* + \sum_{k\geq 0}L_{Q_k^1}^*\right) + \aleph_2 L_{\Lambda_1}^*. \\ \varsigma_3 &= \left(\aleph_1 + \mathcal{L}^{\Phi_1}\right)\|\Phi_1\|_{\S}\|\xi_1\|_{L^1} + (ah_1^* + f_1^*)\|\widehat{\xi}_1\|_{L^1}. \\ \varsigma_4 &= L_{\Lambda_1}^*\left(\left(\aleph_1 + \mathcal{L}^{\Phi_1}\right)\|\Phi_1\|_{\S} + \left(\aleph_2 + \mathcal{L}^{\Phi_2}\right)\|\Phi_2\|_{\S}\right)) + \Lambda_1^*. \\ \varsigma_5 &= M_{R_2}\left(\|\widehat{\xi}_2\|_{L^1}ah_{c_2}^* + \sum_{k\geq 0}L_{Q_k^2}^*\right) + \aleph_1 L_{\Lambda_2}^*. \\ \varsigma_6 &= M_{R_2}\left(\|\xi_2\|_{L^1}\aleph_2 + \|\widehat{\xi}_2\|_{L^1}a\overline{h}_{c_2}^* + \sum_{k\geq 0}L_{Q_k^2}^*\right) + \aleph_2 L_{\Lambda_2}^*. \\ \varsigma_7 &= \left(\aleph_2 + \mathcal{L}^{\Phi_2}\right)\|\Phi_2\|_{\S}\|\xi_2\|_{L^1} + (ah_2^* + f_2^*)\|\widehat{\xi}_2\|_{L^1}. \\ \varsigma_8 &= \left(\left(\aleph_1 + \mathcal{L}^{\Phi_1}\right)\|\Phi_1\|_{\S} + \left(\aleph_2 + \mathcal{L}^{\Phi_2}\right)\|\Phi_2\|_{\S}\right)L_{\Lambda_2}^* + \Lambda_2^*.\end{aligned}$$

Then, we obtain

$$\begin{pmatrix}\|\vartheta\|_{\mathcal{X}} \\ \|\varpi\|_{\mathcal{X}}\end{pmatrix} \leq M_\Xi\begin{pmatrix}\|\vartheta\|_{\mathcal{X}} \\ \|\varpi\|_{\mathcal{X}}\end{pmatrix} + \begin{pmatrix}P_1 \\ P_2\end{pmatrix},$$

where

$$P_1 = M_{R_1}\left(\|\Phi_1(0)\|_{\S} + \left(L^*_{\Lambda_1}\widehat{\varphi}(\|\Phi_1\|_{\S} + \|\Phi_2\|_{\S}) + {\Lambda_1}^*\right) + \varsigma_3\right) + \varsigma_4,$$
$$P_2 = M_{R_2}\left(\|\Phi_2(0)\|_{\S} + \left(L^*_{\Lambda_2}\widehat{\varphi}(\|\Phi_1\|_{\S} + \|\Phi_2\|_{\S}) + {\Lambda_2}^*\right) + \varsigma_7\right) + \varsigma_8.$$

Since M_Ξ converges to zero. From Lemma 2.6, $(I - M_\Xi)$ is invertible and its inverse $(I - M_\Xi)^{-1}$ has nonnegative elements. Hence,

$$(1 - M_\Xi)^{-1}\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} \leq \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}.$$

If $t \in \mathbb{R}^-$, then we get

$$\begin{aligned} \|\Xi_i(\vartheta, \varpi)(t)\| &= \|\Phi_i(t)\| \\ &\leq \|\Phi_i\|_{\S}. \end{aligned}$$

Therefore,

$$\|\Xi(\vartheta, \varpi)\|_{\mathcal{X}\times\mathcal{X}} \leq \theta.$$

Thus, $\Xi(B_\theta) \subset B_\theta$ and $\Xi(B_\theta)$ is bounded.

Step 2: Ξ is continuous.

Let $(\vartheta_m, \varpi_m)_{m\in\mathbb{N}}$ be a sequence such that $(\vartheta_m, \varpi_m) \to (\vartheta^*, \varpi^*)$ in B_θ. At the first, we study the convergence of the sequences $\left(\vartheta^m_{\rho(s,\vartheta^m_s)}\right)_{m\in\mathbb{N}}$, $s \in \Omega$. If $s \in \Omega$, such that $\rho(s, \vartheta_s) > 0$, then we have

$$\begin{aligned} \left\|\vartheta^m_{\rho(s,\vartheta^m_s)} - \vartheta^*_{\rho(s,\vartheta^*_s)}\right\|_{\S} &\leq \left\|\vartheta^m_{\rho(s,\vartheta^m_s)} - \vartheta^*_{\rho(s,\vartheta^m_s)}\right\|_{\S} + \left\|\vartheta^*_{\rho(s,\vartheta^m_s)} - \vartheta^*_{\rho(s,\vartheta^*_s)}\right\|_{\S} \\ &\leq L\|\vartheta_m - \vartheta^*\| + \left\|\vartheta^*_{\rho(s,\vartheta^n_s)} - \vartheta^*_{\rho(s,\vartheta^*_s)}\right\|_{\S}, \end{aligned}$$

which proves that $\vartheta^m_{\rho(s,\vartheta^m_s)} \to \vartheta^*_{\rho(s,\vartheta_s)}$ in $\S$, as $m \to \infty$, for every $s \in \Omega$ such that $\rho(s, \vartheta_s) > 0$. Similarly, if $\rho(s, \vartheta_s) < 0$, we get

$$\left\|\vartheta^n_{\rho(s,\vartheta^m_s)} - \vartheta^*_{\rho(s,\vartheta_s)}\right\|_{\S} = \left\|\Phi^m_{\rho(s,\vartheta^m_s)} - \Phi_{\rho(s,\vartheta^*_s)}\right\|_{\S} = 0,$$

which also shows that $\vartheta^m_{\rho(s,\vartheta^m_s)} \to \vartheta^*_{\rho(s,\vartheta_s)}$ in $\S$, as $m \to \infty$, for every $s \in \Omega$ such that $\rho(s, \vartheta_s) < 0$. Then, for $t \in I_0$, we have

$$\begin{aligned} &\|(\Xi_i(\vartheta^m, \varpi^m))(t) - (\Xi_i(\vartheta^*, \varpi^*))(t)\| \\ &\quad\leq \|\Lambda_i(t, \vartheta^m_{\rho_1(t,\vartheta^m_t)}, \varpi^m_{\rho_2(t,\varpi^m_t)}) - \Lambda_i(t, (\vartheta^*_{\rho_1(t,\vartheta^*_t)}), (\varpi^*_{\rho_2(t,\varpi^*_t)}))\| \\ &\qquad + M_{R_i}\int_0^t \|f_i(s, \vartheta^m_{\rho_1(s,\vartheta^m_s)}, \varpi^m_{\rho_2(s,\varpi^m_s)}, H_i(\vartheta^m, \varpi^m)(s)) \\ &\qquad - f_i(s, \vartheta^*_{\rho_1(s,\vartheta^*_s)}, \varpi^*_{\rho_2(s,\varpi^*_s)}, H_i(\vartheta^*, \varpi^*)(s))\|ds \\ &\qquad + M_{R_i}\sum_{k\geq 0}\|Q^i_k(\vartheta_n(t), \varpi_n(t)) - Q^i_k(\vartheta^*(t), \varpi^*(t))\|. \end{aligned}$$

From $(H1)$-$(H3)$, we obtain

$$h_i(t, s, (\vartheta^m, \varpi^m)(s)) \to h_i(t, s, (\vartheta^*, \varpi^*)(s)),\ as\ m \to +\infty.$$

And

$$\|h_i(t, s, (\vartheta^m, \varpi^m)(s)) - h_i(t, s, (\vartheta^*, \varpi^*)(s))\| \leq h^*_{c_i}(\|\vartheta^m - \vartheta^*\|_{\mathcal{X}} + \|\varpi^m - \varpi^*\|_{\mathcal{X}}).$$

By the Lebesgue dominated convergence theorem, we have

$$\int_0^t h_i(t, s, (\vartheta^m, \varpi^m)(s))ds \xrightarrow[m\to+\infty]{} \int_0^t h_i(t, s, (\vartheta^*, \varpi^*)(s))ds.$$

Then, we get

$$f_i(s, \vartheta^m_{\rho_1(s,\vartheta^m_s)}, \varpi^m_{\rho_2(s,\varpi^m_s)}, H_i(\vartheta^m, \varpi^m)(s)) \xrightarrow[m\to+\infty]{} f_i(s, \vartheta^*_{\rho_1(s,\vartheta^*_s)}, \varpi^*_{\rho_2(s,\varpi^*_s)}, H_i(\vartheta^*, \varpi^*)(s)).$$

Also, the continuity of Q_k^i gives

$$Q_k^i(\vartheta_m(\cdot),\varpi_m(\cdot)) \to Q_k^i(\vartheta^*(\cdot),\varpi^*(\cdot)) \; as \; m \to +\infty.$$

The same way about Λ_i, we obtain

$$\|(\Xi_i(\vartheta^m,\varpi^m))(t) - (\Xi_i(\vartheta^*,\varpi^*))(t)\| \to 0, \; as \; m \to +\infty.$$

If $t \in \mathbb{R}^-$, we get

$$\|(\Xi_i(\vartheta^m,\varpi^m))(t) - (\Xi_i(\vartheta^*,\varpi^*))(t)\| \quad = \quad 0.$$

Thus, Ξ is continuous.

Step 3: The set $\Xi(B_\theta)$ is equicontinuous.
For $\Pi \subset B_\theta$, $T > 0$, $(\vartheta,\varpi) \in \Pi$ and $\kappa_1,\kappa_2 \in [0,T]$, we have

$$\begin{aligned}
&\|\Xi_i(\vartheta,\varpi)(\kappa_1) - \Xi_i(\vartheta,\varpi)(\kappa_2)\| \\
&\quad \leq \|R_i(\kappa_1,0) - R_i(\kappa_2,0)\| \left(\|\Phi_1(0)\|_{\S} + \left(L^*_{\Lambda_i}\widehat{\varphi}(\|\Phi_1\|_{\S} + \|\Phi_2\|_{\S}) + {\Lambda_i}^*\right)\right) \\
&\qquad + \|\Lambda_i(\kappa_1,\vartheta_{\rho_1(\kappa_1,\vartheta_{\kappa_1})},\varpi_{\rho_2(\kappa_1,\varpi_{\kappa_1})}) - \Lambda_i(\kappa_2,\vartheta_{\rho_1(\kappa_2,\vartheta_{\kappa_2})},\varpi_{\rho_2(\kappa_2,\varpi_{\kappa_2})})\| \\
&\qquad + \int_{\kappa_1}^{\kappa_2} \|R_i(\kappa_1,s)\| \|f_i\left(s,\vartheta_{\rho_1(s,\vartheta_s)},\varpi_{\rho_2(s,\varpi_s)},H_i(\vartheta(s),\varpi(s))\right)\|ds \\
&\qquad + \int_0^{\kappa_2} \|R_i(\kappa_1,s) - R_i(\kappa_2,s)\| \|f_i\left(s,\vartheta_{\rho_1(s,\vartheta_s)},\varpi_{\rho_2(s,\varpi_s)},H_i(\vartheta(s),\varpi(s))\right)\|ds \\
&\qquad + \sum_{0<t_k<\kappa_1} \|R_i(\kappa_1 - t_k) - R_i(\kappa_2 - t_k)\| \|Q_k(\vartheta({t_k}^-),\varpi({t_k}^-))\| \\
&\qquad + M_{R_i} \sum_{\kappa_1<t_k<\kappa_2} \|Q_k^i(\vartheta({t_k}^-),\varpi({t_k}^-))\| \\
&\quad \leq \|R_i(\kappa_1,0) - R_i(\kappa_2,0)\| \left(\|\Phi_1(0)\|_{\S} + \left(L^*_{\Lambda_i}\widehat{\varphi}(\|\Phi_1\|_{\S} + \|\Phi_2\|_{\S}) + {\Lambda_i}^*\right)\right) \\
&\qquad + \|\Lambda_i(\kappa_1,\vartheta_{\rho_1(\kappa_1,\vartheta_{\kappa_1})},\varpi_{\rho_2(\kappa_1,\varpi_{\kappa_1})}) - \Lambda_i(\kappa_2,\vartheta_{\rho_1(\kappa_2,\vartheta_{\kappa_2})},\varpi_{\rho_2(\kappa_2,\varpi_{\kappa_2})})\| \\
&\qquad + \int_{\kappa_1}^{\kappa_2} \psi_i(\aleph_i^*)\xi_i(s) + \widehat{\psi}_i(H_i^*)\widehat{\xi}_i(s)ds + \int_{\kappa_1}^{\kappa_2} \|f_i(s,0,0,H_i(0,0))\|ds \\
&\qquad + \int_{\kappa_1}^{\kappa_2} \|R_i(\kappa_1,s) - R_i(\kappa_2,s)\|(\psi_i(\aleph_i^*)\xi_i(s) + \widehat{\psi}_i(H_i^*)\widehat{\xi}_i(s))ds \\
&\qquad + \int_{\kappa_1}^{\kappa_2} \|R_i(\kappa_1,s) - R_i(\kappa_2,s)\| \|f_i(s,0,0,H_i(0,0))\|ds \\
&\qquad + \sum_{0<t_k<\kappa_1} \|R_i(\kappa_1 - t_k) - R_i(\kappa_2 - t_k)\| \|Q_k(\vartheta({t_k}^-),\varpi({t_k}^-))\| \\
&\qquad + M_{R_i} \sum_{\kappa_1<t_k<\kappa_2} \|Q_k^i(\vartheta({t_k}^-),\varpi({t_k}^-))\|.
\end{aligned}$$

By the strong continuity of $R_i(\cdot)$ and $(H3)$, we get

$$\|\Xi_i(\vartheta,\varpi)(\kappa_1) - \Xi_i(\vartheta,\varpi)(\kappa_2)\| \to 0, \; as \; \kappa_1 \to \kappa_2.$$

And for $t \in \mathbb{R}^-$, we obtain

$$\|\Xi_i(\vartheta,\varpi)(\kappa_1) - \Xi_i(\vartheta,\varpi)(\kappa_2)\| = \|\Phi_i(\kappa_1) - \Phi_i(\kappa_2)\| \to 0, \; as \; \kappa_1 \to \kappa_2.$$

Hence, the set $\Xi(\Pi)$ is equicontinuous, then $\omega_0(\Xi(\Pi)) = 0$.
On the other hand, for $\vartheta, \widehat{\vartheta},\ \varpi, \widehat{\varpi} \in \Pi$ and $t \in \Omega$, we have

$$\begin{aligned}
&\|(\Xi_i(\vartheta,\varpi))(t) - (\Xi_i(\widehat{\vartheta},\widehat{\varpi}))(t)\| \\
&\quad \leq \|\Lambda_i(t,\vartheta_{\rho(t,\vartheta_t)},\varpi_{\rho(t,\varpi_t)}) - \Lambda_i(t,\widehat{\vartheta}_{\rho(t,\widehat{\vartheta}_t)},\widehat{\varpi}_{\rho(t,\widehat{\varpi}_t)})\| \\
&\qquad + \int_0^t \|R_i(t,s)\| \|f_i\left(s,\vartheta_{\rho(s,\vartheta_s)},\varpi_{\rho(s,\varpi_s)},H_i(\vartheta,\varpi)(s)\right) \\
&\qquad - f_i\left(s,\widehat{\vartheta}_{\rho(s,\widehat{\vartheta}_s)},\widehat{\varpi}_{\rho(t,\widehat{\varpi}_t)},H_i(\widehat{\vartheta},\widehat{\varpi})(s)\right)\| ds \\
&\qquad + M_{R_i}\sum_{k\geq 0} L_{Q_k^i}(\|\vartheta - \widehat{\vartheta}\|_{\mathcal{X}} + \|\varpi - \widehat{\varpi}\|_{\mathcal{X}}) \\
&\quad \leq L^*_{\Lambda_i}\left(\|\vartheta_{\rho(t,\vartheta_t)} - \widehat{\vartheta}_{\rho(t,\widehat{\vartheta}_t)}\| + \|\varpi_{\rho(t,\varpi_t)} - \widehat{\varpi}_{\rho(t,\widehat{\varpi}_t)}\|\right) \\
&\qquad + M_{R_i}\int_0^t e^{-\mu_i(t-s)}\left(\psi_i(\aleph_i^*)\xi_i(s) + \psi_i(H_i^*)\xi_i(s)\right) \\
&\qquad + M_{R_i}\sum_{k\geq 0} L_{Q_k^i}(\|\vartheta - \widehat{\vartheta}\|_{\mathcal{X}} + \|\varpi - \widehat{\varpi}\|_{\mathcal{X}}).
\end{aligned}$$

When t tends to $+\infty$, we get

$$D^{\Delta}(\Xi_i(\Pi))) \leq \left(L^*_{\Lambda_i} + M_{R_i}\sum_{k\geq 0} L_{Q_k^i}\right)\left(D^{\Delta}(\Pi_1) + D^{\Delta}(\Pi_2)\right).$$

If $t \in \mathbb{R}^-$, we have

$$\|\Xi_i(\vartheta,\varpi)(t) - \Xi_i(\widehat{\vartheta},\widehat{\varpi})(t)\| = 0.$$

Thus,

$$\chi_{BPC^*}(\Xi(\Pi)) \leq \widetilde{M}_{\Xi}\begin{pmatrix}\chi_1(\Pi_1)\\ \chi_1(\Pi_2)\end{pmatrix},$$

such that,

$$\widetilde{M}_{\Xi} = \begin{pmatrix} L^*_{\Lambda_1} + M_{R_1}\sum_{k\geq 0} L_{Q_k^1} & L^*_{\Lambda_1} + M_{R_1}\sum_{k\geq 0} L_{Q_k^1} \\ L^*_{\Lambda_2} + M_{R_2}\sum_{k\geq 0} L_{Q_k^2} & L^*_{\Lambda_2} + M_{R_2}\sum_{k\geq 0} L_{Q_k^2}\end{pmatrix}.$$

We have $|\widetilde{M}_{\Xi}| < M_{\Xi}$, then from Theorem 2.7, the spectral radius $\rho(\widetilde{M}_{\Xi}) \in (0,1)$. Therefore, $\widetilde{M}_{\Xi}$ converge to zero. Thus, Ξ is generalized χ_{BPC^*}-condensing operator. Applying now Theorem 2.12, we conclude that Ξ has at least one fixed point, which is a mild solution of problem (1.1). □

4. Mild Solutions with Noninstantaneous Impulses

Definition 4.1. A function $(\vartheta,\varpi) \in BPC(\mathbb{R},\mho) \times BPC(\mathbb{R},\mho) = \mathcal{Y} \times \mathcal{Y}$ is called a mild solution of problem (1.2) if it satisfies

$$\vartheta(t)=\begin{cases} R_1(t,0)(\Phi_1(0)-\Lambda_1(0,\Phi_1,\Phi_2))+\Lambda_1(t,\vartheta_{\rho_1(t,\vartheta_t)},\varpi_{\rho_2(t,\varpi_t)}) \\ +\int_0^t R_1(t,s)f_1\left(\theta,\vartheta_{\rho_1(s,\vartheta_s)},\varpi_{\rho_2(s,\varpi_s)},H_1(\vartheta(s),\varpi(s))\right)ds, \text{ if } t\in I_0, \\ R_1(t,s_k)\left(\Theta_k\left(s_k,\vartheta\left(t_k^-\right),\varpi\left(t_k^-\right)\right)-\Lambda_1(s_k,\vartheta_{\rho_1(s_k,\vartheta_{s_k})},\varpi_{\rho_2(s_k,\varpi_{s_k})})\right) \\ +\int_{s_k}^t R_1(t,s)f_1\left(\theta,\vartheta_{\rho_1(s,\vartheta_s)},\varpi_{\rho_2(s,\varpi_s)},H_1(\vartheta(s),\varpi(s))\right)ds \\ +\Lambda_1(t,\vartheta_{\rho_1(t,\vartheta_t)},\varpi_{\rho_2(t,\varpi_t)}), \text{ if } t\in I_k,\ k\in\mathbb{N}, \\ \Theta_k\left(t,\vartheta\left(t_k^-\right),\varpi\left(t_k^-\right)\right) \text{ if } t\in\Omega_k,\ k\in\mathbb{N}, \\ \Phi_1(t), \text{ if } t\in\mathbb{R}_-, \end{cases}$$

and

$$\varpi(t)=\begin{cases} R_2(t,0)(\Phi_2(0)-\Lambda_2(0,\Phi_1,\Phi_2))+\Lambda_2(t,\vartheta_{\rho_1(t,\vartheta_t)},\varpi_{\rho_2(t,\varpi_t)}) \\ +\int_0^t R_2(t,s)f_2\left(\theta,\vartheta_{\rho_1(s,\vartheta_s)},\varpi_{\rho_2(s,\varpi_s)},H_2(\vartheta(s),\varpi(s))\right)ds, \text{ if } t\in I_0, \\ R_2(t,s_k)\left(\Xi_k\left(s_k,\vartheta\left(t_k^-\right),\varpi\left(t_k^-\right)\right)-\Lambda_2(s_k,\vartheta_{\rho_1(s_k,\vartheta_{s_k})},\varpi_{\rho_2(s_k,\varpi_{s_k})})\right) \\ +\int_{s_k}^t R_2(t,s)f_2\left(\theta,\vartheta_{\rho_1(s,\vartheta_s)},\varpi_{\rho_2(s,\varpi_s)},H_2(\vartheta(s),\varpi(s))\right)ds \\ +\Lambda_2(t,\vartheta_{\rho_1(t,\vartheta_t)},\varpi_{\rho_2(t,\varpi_t)}), \text{ if } t\in I_k,\ k\in\mathbb{N}, \\ \Xi_k\left(t,\vartheta\left(t_k^-\right),\varpi\left(t_k^-\right)\right) \text{ if } t\in\Omega_k,\ k\in\mathbb{N}, \\ \Phi_2(t), \text{ if } t\in\mathbb{R}_-. \end{cases}$$

We introduce the following assumption to establish our results:

$(H6)$ The functions $Z_k^i,\ \Omega_k\times\mho\times\mho\to\mho$ are continuous and there exist functions $L_{Z_k^i}:\Omega\to(0,+\infty),\ k\in\mathbb{N}$, such that

$$\|Z_k^i(\theta,\vartheta_3,\vartheta_4)-Z_k^i(\theta,\widehat{\vartheta_3},\widehat{\vartheta}_4)\|\leq L_{Z_k^i}(\theta)(\|\vartheta_3-\widehat{\vartheta}_3\|+\|\vartheta_4-\widehat{\vartheta}_4\|),$$

for all $\vartheta_3,\ \widehat{\vartheta}_3,\ \vartheta_4,\ \widehat{\vartheta}_4\in\mho,\ k\in\mathbb{N}$. And, let

$$\left(\max_{k\in\mathbb{N}}\sup_{\theta\in\Omega}\left\{L_{Z_k^i}(\theta),\ k\in\mathbb{N}\right\},Z_k^i(0,0)\right)=\left(L^*_{Z_k^i},Z_k^i\right)<+\infty,$$

where

$$Z_k^i=\begin{cases}\Theta_k & \text{if } i=1\\ \Xi_k & \text{if } i=2.\end{cases}$$

Now, we define the matrix

$$M_{\alpha_1}=\begin{pmatrix}\sigma_1 & \sigma_2\\ \sigma_5 & \sigma_6\end{pmatrix},$$

where $\sigma_1,\sigma_2,\sigma_5,\sigma_6$ are constants to be specific later.

Theorem 4.2. *If the conditions $(H1)-(H4)$, $(H6)$ and (H_Φ) are satisfied, and the matrix M_{α_1} converges to zero with $\max_{i=1,2}\left\{M_{R_i}\left(L^*_{Z_k^i}+L^*_{\Lambda_i}\aleph_i\right)\right\}<1$, then the system (1.2) has at least one mild solution.*

Proof. Firstly consider the generalized measure of noncompactness χ_{BPC} defined on the family of subset of $S = S_1 \times S_2 \subset BPC(\Omega, \mho) \times BPC(\Omega, \mho)$ by the formula

$$\chi_{BPC}(S_1 \times S_2) = \begin{pmatrix} \chi_1(S_1) \\ \chi_1(S_2) \end{pmatrix} = \begin{pmatrix} \omega_0(S_1) + D^{\Delta}(S_2) \\ \omega_0(S_2) + D^{\Delta}(S_1) \end{pmatrix}.$$

Now, transform the problem (1.2) into a fixed point problem, consider the operator $\Upsilon : BPC(\Omega, \mho) \times BPC(\Omega, \mho) \to BPC(\Omega, \mho) \times BPC(\Omega, \mho)$ define by:

$$\Upsilon(\vartheta(t), \varpi(t)) = (\Upsilon_1(\vartheta(t), \varpi(t)), \Upsilon_2(\vartheta(t), \varpi(t))),$$

where

$$\Upsilon_i(\vartheta(t), \varpi(t)) = \begin{cases} R_i(t,0)(\Phi_i(0) - \Lambda_i(0, \Phi_1, \Phi_2)) + \Lambda_i(t, \vartheta_{\rho_1(t,\vartheta_t)}, \varpi_{\rho_2(t,\varpi_t)}) \\ + \int_0^t R_i(t,s) f_i\left(s, \vartheta_{\rho_1(s,\vartheta_s)}, \varpi_{\rho_2(s,\varpi_s)}, H_i(\vartheta(s), \varpi(s))\right) ds, \text{ if } t \in I_0, \\ R_i(t, s_k)\left(Z_k^i\left(s_k, \vartheta\left(t_k^-\right), \varpi\left(t_k^-\right)\right) - \Lambda_i(s_k, \vartheta_{\rho_1(s_k,\vartheta_{s_k})}, \varpi_{\rho_2(s_k,\varpi_{s_k})})\right) \\ + \int_{s_k}^t R_i(t,s) f_i\left(s, \vartheta_{\rho_1(s,\vartheta_s)}, \varpi_{\rho_2(s,\varpi_s)}, H_i(\vartheta(s), \varpi(s))\right) ds \\ + \Lambda_i(t, \vartheta_{\rho_1(t,\vartheta_t)}, \varpi_{\rho_2(t,\varpi_t)}), \text{ if } t \in I_k, \ k \in \mathbb{N}, \\ Z_k^i\left(t, \vartheta\left(t_k^-\right), \varpi\left(t_k^-\right)\right) \text{ if } t \in \Omega_k, \ k \in \mathbb{N}, \\ \Phi_i(t), \text{ if } t \in \mathbb{R}_-, \end{cases}$$

and

$$Z_k^i = \begin{cases} \Psi_k & \text{if } i = 1 \\ \widehat{\Psi}_k & \text{if } i = 2. \end{cases}$$

Clearly, the fixed points of the operator Υ are mild solutions of the problem (1.2), so we shall check that operator Υ satisfies all conditions of Theorem 2.12.
Let B_δ be defined by:

$$B_\delta = \{(\vartheta, \varpi) \in \mathcal{Y} \times \mathcal{Y} : (\|\vartheta\|_{\mathcal{Y}}, \|\varpi\|_{\mathcal{Y}}) \leq (\delta_1, \delta_2)\},$$

with $\delta_i > \|\Phi_i\|_{\S}$. The set B_δ is bounded, closed and convex of $\mathcal{Y} \times \mathcal{Y}$.
We have divided the proof into 4 steps:

Step 1 : $\Upsilon(B_\delta) \subset B_\delta$.
For each $(\vartheta, \varpi) \in B_\delta$ and $t \in \mathcal{R}(\rho^-) \cup \Omega$, it follows that

$$\|\vartheta_t\|_{\S} \leq \aleph_1 \delta_1 + \left(\aleph_1 + \mathcal{L}^{\Phi_1}\right) \|\Phi_1\|_{\S} = \aleph_1^*,$$
$$\|\varpi_t\|_{\S} \leq \aleph_2 \delta_2 + \left(\aleph_2 + \mathcal{L}^{\Phi_2}\right) \|\Phi_2\|_{\S} = \aleph_2^*,$$

and

$$\|H_i(\vartheta, \varpi)(s)\| \leq a(h_{c_i}^* \delta_1 + \bar{h}_{c_i}^* \delta_2 + h_i^*) = H_i^*.$$

Then, we have the following cases:

- $Case\ 1:\ t \in I_0$. We have

$$\begin{aligned}\|\Upsilon_i(\vartheta,\varpi)(t)\| &\leq M_{R_i}(\|\Phi_i(0)\|_{\S} + \left(L^*_{\Lambda_i}\widehat{\varphi}(\|\Phi_1\|_{\S} + \|\Phi_2\|_{\S}) + {\Lambda_i}^*\right))\\ &\quad + M_{R_i}\int_0^t \xi_i(s)\psi_i(\aleph_i^*) + \widehat{\xi}_i(s)\widehat{\psi}_i(H_i^*)ds\\ &\quad + M_{R_i}f_i^* + \left(L^*_{\Lambda_i}\widehat{\varphi}(\aleph_1^* + \aleph_2^*) + \Lambda_i^*\right)\\ &\leq M_{R_i}\Bigg(\|\Phi_i(0)\|_{\S} + \left(L^*_{\Lambda_i}\widehat{\varphi}(\|\Phi_1\|_{\S} + \|\Phi_2\|_{\S}) + {\Lambda_i}^*\right)\\ &\quad + \|\xi_i\|_{L^1}\psi_i(\aleph_i^*) + \|\widehat{\xi}_i\|_{L^1}\widehat{\psi}_i(H_i^*) + f_i^*\Bigg)\\ &\quad + \left(L^*_{\Lambda_i}\widehat{\varphi}(\aleph_1^* + \aleph_2^*) + \Lambda_i^*\right).\end{aligned}$$

We define

$$\begin{aligned}\sigma_1 &= M_{R_1}(\|\xi_1\|_{L^1}\aleph_1 + \|\widehat{\xi}_1\|_{L^1}ah^*_{c_1}) + \aleph_1 L^*_{\Lambda_1},\\ \sigma_2 &= M_{R_1}(\|\widehat{\xi}_1\|_{L^1}a\overline{h}^*_{c_1}) + \aleph_2 L^*_{\Lambda_1},\\ \sigma_3 &= \left(\aleph_1 + \mathcal{L}^{\Phi_1}\right)\|\Phi_1\|_{\S}\|\xi_1\|_{L^1} + (ah_1^* + f_1^*)\|\widehat{\xi}_1\|_{L^1},\\ \sigma_4 &= (\left(\aleph_1 + \mathcal{L}^{\Phi_1}\right)\|\Phi_1\|_{\S} + \left(\aleph_2 + \mathcal{L}^{\Phi_2}\right)\|\Phi_2\|_{\S})L^*_{\Lambda_1} + \Lambda_1^*.\\ \sigma_5 &= M_{R_2}(\|\widehat{\xi}_2\|_{L^1}ah^*_{c_2}) + \aleph_1 L^*_{\Lambda_2},\\ \sigma_6 &= M_{R_2}(\|\xi_2\|_{L^1}\aleph_2 + \|\widehat{\xi}_2\|_{L^1}a\overline{h}^*_{c_2}) + \aleph_2 L^*_{\Lambda_2},\\ \sigma_7 &= \left(\aleph_2 + \mathcal{L}^{\Phi_2}\right)\|\Phi_2\|_{\S}\|\xi_2\|_{L^1} + (ah_2^* + f_2^*)\|\widehat{\xi}_2\|_{L^1},\\ \sigma_8 &= (\left(\aleph_1 + \mathcal{L}^{\Phi_1}\right)\|\Phi_1\|_{\S} + \left(\aleph_2 + \mathcal{L}^{\Phi_2}\right)\|\Phi_2\|_{\S})L^*_{\Lambda_2} + \Lambda_2^*.\end{aligned}$$

Therefore,

$$\begin{pmatrix}\|\vartheta\|_{\mathcal{Y}}\\ \|\varpi\|_{\mathcal{Y}}\end{pmatrix} \leq M_{\alpha_1}\begin{pmatrix}\|\vartheta\|_{\mathcal{Y}}\\ \|\varpi\|_{\mathcal{Y}}\end{pmatrix} + \begin{pmatrix}W_1\\ W_2\end{pmatrix},$$

where

$$\begin{aligned}W_1 &= M_{R_1}\left(\|\Phi_1(0)\|_{\S} + \left(L^*_{\Lambda_1}\widehat{\varphi}(\|\Phi_1\|_{\S} + \|\Phi_2\|_{\S}) + {\Lambda_1}^*\right) + \sigma_3\right) + \sigma_4,\\ W_2 &= M_{R_2}\left(\|\Phi_2(0)\|_{\S} + \left(L^*_{\Lambda_2}\widehat{\varphi}(\|\Phi_1\|_{\S} + \|\Phi_2\|_{\S}) + {\Lambda_2}^*\right) + \sigma_7\right) + \sigma_8.\end{aligned}$$

- $Case\ 2\ :\ t \in I_k$. We have

$$\|Z_k^i(.,u(.))\| \leq L^*_{Z_k^i}\varphi(\delta_1 + \delta_2) + Z_k^{0,i}.$$

Then,

$$\begin{aligned}\|\Upsilon_i(\vartheta,\varpi)(t)\| \leq M_{R_i}\Bigg(&l^*_{Z_k^i}\varphi(\delta_1 + \delta_2) + Z_k^{0,i} + L^*_{\Lambda_i}\widehat{\varphi}(\aleph_1^* + \aleph_2^*) + \Lambda_i^* + \|\xi_i\|_{L^1}\psi_i(\aleph_i^*)\\ &+ \|\widehat{\xi}_i\|_{L^1}\widehat{\psi}_i(H_i^*) + f_i^*\Bigg) + L^*_{\Lambda_i}\left(\widehat{\varphi}(\aleph_1^* + \aleph_2^*)\right) + \Lambda_i^* = \Delta_2^{\delta_i}.\end{aligned}$$

Then, we obtain

$$\begin{pmatrix}\|\vartheta\|_{\mathcal{Y}}\\ \|\varpi\|_{\mathcal{Y}}\end{pmatrix} \leq M_{\alpha_2}\begin{pmatrix}\|\vartheta\|_{\mathcal{Y}}\\ \|\varpi\|_{\mathcal{Y}}\end{pmatrix} + \begin{pmatrix}\widetilde{W}_1\\ \widetilde{W}_2\end{pmatrix},$$

where

$$\begin{aligned}\widetilde{W}_1 &= M_{R_1}\left(Z_k^{0,1} + \Lambda_1^* + \sigma_3\right) + \sigma_4,\\ \widetilde{W}_2 &= M_{R_2}\left(Z_k^{0,2} + \Lambda_2^* + \sigma_7\right) + \sigma_8,\end{aligned}$$

and

$$M_{\alpha_2} = \begin{pmatrix} M_{R_1}(L^*_{Z_k^1} + L^*_{\Lambda_1}\aleph_1) + \sigma_1 & M_{R_1}(L^*_{Z_k^1} + L^*_{\Lambda_1}\aleph_2) + \sigma_2 \\ M_{R_2}(L^*_{Z_k^2} + L^*_{\Lambda_2}\aleph_1) + \sigma_5 & M_{R_2}(L^*_{Z_k^2} + L^*_{\Lambda_2}\aleph_2) + \sigma_6 \end{pmatrix}.$$

We have

$$M_{\alpha_2} = M_{\alpha_1} + \begin{pmatrix} M_{R_1}(L^*_{Z_k^1} + L^*_{\Lambda_1}\aleph_1) & M_{R_1}(L^*_{Z_k^1} + L^*_{\Lambda_1}\aleph_2) \\ M_{R_2}(L^*_{Z_k^2} + L^*_{\Lambda_2}\aleph_1) & M_{R_2}(L^*_{Z_k^2} + L^*_{\Lambda_2}\aleph_2) \end{pmatrix}.$$

Since

$$\max_{i=1,2}\left\{M_{R_i}\left(L^*_{Z_k^i} + L^*_{\Lambda_i}\aleph_i\right)\right\} < 1,$$

the from Lemma 2.8, we deduce that M_{α_2} converges to zero.

- $Case\ 3\ :\ t \in \Omega_k$. We get

$$\|\Upsilon_i(\vartheta,\varpi)(t)\| \quad \leq \quad L^*_{Z_k^i}\varphi(\delta_1+\delta_2) + Z_k^{0,i} = \Delta_3^{\delta_i}.$$

Then, we obtain

$$\begin{pmatrix} \|\vartheta\|_{\mathcal{Y}} \\ \|\varpi\|_{\mathcal{Y}} \end{pmatrix} \leq M_{\alpha_3}\begin{pmatrix} \|\vartheta\|_{\mathcal{Y}} \\ \|\varpi\|_{\mathcal{Y}} \end{pmatrix} + \begin{pmatrix} Z_k^{0,1} \\ Z_k^{0,2} \end{pmatrix},$$

where

$$M_{\alpha_3} = \begin{pmatrix} L^*_{Z_k^1} & L^*_{Z_k^1} \\ L^*_{Z_k^2} & L^*_{Z_k^2} \end{pmatrix}.$$

We have $|M_{\alpha_3}| < M_{\alpha_2}$, then $\rho(M_{\alpha_3}) \in (0,1)$. Therefore, M_{α_3} converge to zero.

- $Case\ 4\ :\ t \in \mathbb{R}^-$. We have

$$\|\Upsilon_i(\vartheta,\varpi)(t)\| \quad = \quad \|\Phi_i(t)\| \leq \|\Phi_i\|_{\S}.$$

Thus,

$$\begin{pmatrix} \|\vartheta\|_{\mathcal{Y}} \\ \|\varpi\|_{\mathcal{Y}} \end{pmatrix} \quad \leq \quad \widetilde{A_1}\begin{pmatrix} \|\vartheta\|_{\mathcal{Y}} \\ \|\varpi\|_{\mathcal{Y}} \end{pmatrix} + \widetilde{B_1},$$

such that

$$\widetilde{A_1} = \begin{cases} M_{\alpha_1};\ \text{if } t \in I_0, \\ \\ M_{\alpha_2};\ \text{if } t \in I_k,\ k \in N_1^m, \\ \\ M_{\alpha_3};\ \text{if } t \in \Omega_k,\ k \in N_1^m, \end{cases}$$

and

$$\widetilde{B_1} = \begin{cases} \binom{W_1}{W_2};\ \text{if } t \in I_0, \\ \\ \binom{\widetilde{W}_1}{\widetilde{W}_2};\ \text{if } t \in I_k,\ k \in N_1^m, \\ \\ \binom{Z_k^{0,1}}{Z_k^{0,2}};\ \text{if } t \in \Omega_k,\ k \in N_1^m. \end{cases}$$

Since for $j = 1,2,3$, M_{α_j} converges to zero, from Lemma 2.6 $\left(I - M_{\alpha_j}\right)$ are invertible and their inverses $\left(I - M_{\alpha_j}\right)^{-1}$ has nonnegative elements. Hence,

$$\left(1 - \widetilde{A_1}\right)^{-1}\widetilde{B_1} \leq \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix}.$$

Therefore,

$$\|\Upsilon(\vartheta,\varpi)\|_{\mathcal{Y}\times\mathcal{Y}} \quad \leq \quad \delta.$$

Thus, $\Upsilon(B_\delta) \subset B_\delta$ and $\Upsilon(B_\delta)$ is bounded.

Step 2: Υ is continuous.
Let $(\vartheta_m, \varpi_m)_{m\in\mathbb{N}}$ be a sequence such that $(\vartheta_m, \varpi_m) \to (\vartheta^*, \varpi^*)$ in B_δ.
Similarly to Step 2 in the proof of Theorem 3.4, the sequences $\left(\vartheta^m_{\rho(s,\vartheta^m_s)}\right)_{m\in\mathbb{N}}$, converge to $\vartheta^*_{\rho(s,\vartheta_s)}$ in §, as $m \to \infty$ for every $s \in \Omega$. Then, for $t \in I_0$, we have

$$\begin{aligned}
&\|(\Upsilon_i(\vartheta^m, \varpi^m))(t) - (\Upsilon_i(\vartheta^*, \varpi^*))(t)\| \\
&\quad \le \|\Lambda_i(t, \vartheta^m_{\rho_1(t,\vartheta^m_t)}, \varpi^m_{\rho_2(t,\varpi^m_t)}) - \Lambda_i(t, (\vartheta^*_{\rho_1(t,\vartheta^*_t)}), (\varpi^*_{\rho_2(t,\varpi^*_t)}))\| \\
&\qquad + M_{R_i}\int_0^t \|f_i(s, \vartheta^m_{\rho_1(s,\vartheta^m_s)}, \varpi^m_{\rho_2(s,\varpi^m_s)}, H_i(\vartheta^m, \varpi^m)(s)) \\
&\qquad - f_i(s, \vartheta^*_{\rho_1(s,\vartheta^*_s)}, \varpi^*_{\rho_2(s,\varpi^*_s)}, H_i(\vartheta^*, \varpi^*)(s))\|ds.
\end{aligned}$$

From $(H1)-(H3)$, we obtain

$$h_i(t, s, (\vartheta^m, \varpi^m)(s)) \to h_i(t, s, (\vartheta^*, \varpi^*)(s)),\ as\ m \to +\infty,$$

and

$$\|h_i(t, s, (\vartheta^m, \varpi^m)(s)) - h_i(t, s, (\vartheta^*, \varpi^*)(s))\| \le h^*_{c_i}(\|\vartheta^m - \vartheta^*\|_{\mathcal{Y}} + \|\varpi^m - \varpi^*\|_{\mathcal{Y}}).$$

By the Lebesgue dominated convergence theorem, we have

$$\int_0^t h_i(t, s, (\vartheta^m, \varpi^m)(s))ds \xrightarrow[m\to+\infty]{} \int_0^t h_i(t, s, (\vartheta^*, \varpi^*)(s))ds.$$

Then, we get

$$f_i(s, \vartheta^m_{\rho_1(s,\vartheta^m_s)}, \varpi^m_{\rho_2(s,\varpi^m_s)}, H_i(\vartheta^m, \varpi^m)(s)) \xrightarrow[m\to+\infty]{} f_i(s, \vartheta^*_{\rho_1(s,\vartheta^*_s)}, \varpi^*_{\rho_2(s,\varpi^*_s)}, H_i(\vartheta^*, \varpi^*)(s)).$$

The same way about Λ_i, we get

$$\|(\Upsilon_i(\vartheta^m, \varpi^m))(t) - (\Upsilon_i(\vartheta^*, \varpi^*))(t)\| \to 0,\ as\ m \to +\infty.$$

Now, for $t \in I_k$, we have

$$\begin{aligned}
&\|(\Upsilon_i(\vartheta^m, \varpi^m))(t) - (\Upsilon_i(\vartheta^*, \varpi^*))(t)\| \\
&\quad \le M_{R_i}\|Z^i_k(s_k, \vartheta^m(t^-_k), \varpi^m(t^-_k)) - Z^i_k(s_k, \vartheta^*(t^-_k), \varpi^*(t^-_k))\| \\
&\qquad + M_{R_i}\|\Lambda_i(s_k, \vartheta^m_{\rho_1\left(s_k,\vartheta^m_{s_k}\right)}, \varpi^m_{\rho_2\left(s_k,\varpi^m_{s_k}\right)}) \\
&\qquad - \Lambda_i(s_k, (\vartheta^*_{\rho_1\left(s_k,\vartheta^*_{s_k}\right)}), (\varpi^*_{\rho_2\left(s_k,\varpi^*_{s_k}\right)}))\| \\
&\qquad + \|\Lambda_i(t, \vartheta^m_{\rho_1(t,\vartheta^m_t)}, \varpi^m_{\rho_2(t,\varpi^m_t)}) - \Lambda_i(t, (\vartheta^*_{\rho_1(t,\vartheta^*_t)}), (\varpi^*_{\rho_2(t,\varpi^*_t)}))\| \\
&\qquad + M_{R_i}\int_{s_k}^t \|f_i(s, \vartheta^m_{\rho_1(s,\vartheta^m_s)}, \varpi^m_{\rho_2(s,\varpi^m_s)}, H_i(\vartheta^m, \varpi^m)(s)) \\
&\qquad - f_i(s, \vartheta^*_{\rho_1(s,\vartheta^*_s)}, \varpi^*_{\rho_2(s,\varpi^*_s)}, H_i(\vartheta^*, \varpi^*)(s))\|ds.
\end{aligned}$$

Similarly, by the continuity of $h_i, f_i, \Lambda_i\ and\ Z^i_k$, we get

$$\|(\Upsilon_i(\vartheta^m, \varpi^m))(t) - (\Upsilon_i(\vartheta^*, \varpi^*))(t)\| \to 0,\ as\ m \to +\infty.$$

If $t \in \Omega_k$, we get

$$\begin{aligned}&\|(\Upsilon_i(\vartheta^m, \varpi^m))(t) - (\Upsilon_i(\vartheta^*, \varpi^*))(t)\| \\ &\quad \leq \|Z_k^i(t_k, \vartheta^m(t_k^-), \varpi^m(t_k^-)) - Z_k^i(t_k, \vartheta^*(t_k^-), \varpi^*(t_k^-))\|.\end{aligned}$$

The continuity of Z_k^i gives

$$\|(\Upsilon_i(\vartheta^m, \varpi^m))(t) - (\Upsilon_i(\vartheta^*, \varpi^*))(t)\| \to 0, \ as \ m \to +\infty.$$

And for $t \in \mathbb{R}^-$, we have

$$\|(\Upsilon_i(\vartheta^m, \varpi^m))(t) - (\Upsilon_i(\vartheta^*, \varpi^*))(t)\| \quad = \quad 0.$$

Thus, Υ is continuous.

Step 3: The set $\Upsilon(B_\delta)$ is equicontinuous.
For $\Pi \subset B_\delta$, $T > 0$, $k_0 \in \mathbb{N}$ with $T \geq t_{k_0}$, then for $(\vartheta, \varpi) \in \Pi$ and $\kappa_1, \kappa_2 \in I_0$, we have

$$\begin{aligned}&\|\Upsilon_i(\vartheta, \varpi)(\kappa_1) - \Upsilon_i(\vartheta, \varpi)(\kappa_2)\| \\ &\quad \leq \|R_i(\kappa_1, 0) - R_i(\kappa_2, 0)\| \left(\|\Phi_1(0)\|_{\S} + \left(L_{\Lambda_i}^* \widehat{\varphi}(\|\Phi_1\|_{\S} + \|\Phi_2\|_{\S}) + {\Lambda_i}^*\right)\right) \\ &\qquad + \|\Lambda_i(\kappa_1, \vartheta_{\rho_1(\kappa_1, \vartheta_{\kappa_1})}, \varpi_{\rho_2(\kappa_1, \varpi_{\kappa_1})}) - \Lambda_i(\kappa_2, \vartheta_{\rho_1(\kappa_2, \vartheta_{\kappa_2})}, \varpi_{\rho_2(\kappa_2, \varpi_{\kappa_2})})\| \\ &\qquad + \| \int_{\kappa_1}^{\kappa_2} \|R_i(\kappa_1, s)\| \|f_i\left(s, \vartheta_{\rho_1(s, \vartheta_s)}, \varpi_{\rho_2(s, \varpi_s)}, H_i(\vartheta(s), \varpi(s))\right)\| ds \\ &\qquad + \int_0^{\kappa_2} \|R_i(\kappa_1, s) - R_i(\kappa_2, s)\| \|f_i\left(s, \vartheta_{\rho_1(s, \vartheta_s)}, \varpi_{\rho_2(s, \varpi_s)}, H_i(\vartheta(s), \varpi(s))\right)\| ds \\ &\quad \leq \|R_i(\kappa_1, 0) - R_i(\kappa_2, 0)\| \left(\|\Phi_1(0)\|_{\S} + \left(L_{\Lambda_i}^* \widehat{\varphi}(\|\Phi_1\|_{\S} + \|\Phi_2\|_{\S}) + {\Lambda_i}^*\right)\right) \\ &\qquad + \|\Lambda_i(\kappa_1, \vartheta_{\rho_1(\kappa_1, \vartheta_{\kappa_1})}, \varpi_{\rho_2(\kappa_1, \varpi_{\kappa_1})}) - \Lambda_i(\kappa_2, \vartheta_{\rho_1(\kappa_2, \vartheta_{\kappa_2})}, \varpi_{\rho_2(\kappa_2, \varpi_{\kappa_2})})\| \\ &\qquad + \int_{\kappa_1}^{\kappa_2} \psi_i(\aleph_i^*)\xi_i(s) + \widehat{\psi}_i(H_i^*)\widehat{\xi}_i(s) ds + \int_{\kappa_1}^{\kappa_2} \|f_i(s, 0, 0, H_i(0, 0))\| ds \\ &\qquad + \int_{\kappa_1}^{\kappa_2} \|R_i(\kappa_1, s) - R_i(\kappa_2, s)\| (\psi_i(\aleph_i^*)\xi_i(s) + \widehat{\psi}_i(H_i^*)\widehat{\xi}_i(s)) ds \\ &\qquad + \int_{\kappa_1}^{\kappa_2} \|R_i(\kappa_1, s) - R_i(\kappa_2, s)\| \|f_i(s, 0, 0, H_i(0, 0))\| ds.\end{aligned}$$

By the strong continuity of $R_i(\cdot)$ and assumption $(H3)$, we get

$$\|\Upsilon_i(\vartheta, \varpi)(\kappa_1) - \Upsilon_i(\vartheta, \varpi)(\kappa_2)\| \to 0, \ as \ \kappa_1 \to \kappa_2.$$

Now, if $\kappa_1, \kappa_2 \in I_k$, we have

$$\begin{aligned}
&\|\Upsilon_i(\vartheta,\varpi)(\kappa_1) - \Upsilon_i(\vartheta,\varpi)(\kappa_2)\| \\
&\quad \leq \|R_i(\kappa_1,0) - R_i(\kappa_2,0)\| \Big(l^*_{Z^i_k}\varphi(\delta_1+\delta_2) + Z^{0,i}_k + L^*_{\Lambda_i}\widehat{\varphi}(\aleph^*_1+\aleph^*_2) + \Lambda^*_i \\
&\qquad + \|\xi_i\|_{L^1}\psi_i(\aleph^*_i) + \|\widehat{\xi_i}\|_{L^1}\widehat{\psi_i}(H^*_i) + f^*_i \Big) \\
&\qquad + \|\Lambda_i(\kappa_1, \vartheta_{\rho_1(\kappa_1,\vartheta_{\kappa_1})}, \varpi_{\rho_2(\kappa_1,\varpi_{\kappa_1})}) - \Lambda_i(\kappa_2, \vartheta_{\rho_1(\kappa_2,\vartheta_{\kappa_2})}, \varpi_{\rho_2(\kappa_2,\varpi_{\kappa_2})})\| \\
&\qquad + \int_{\kappa_1}^{\kappa_2} \psi_i(\aleph^*_i)\xi_i(s) + \widehat{\psi_i}(H^*_i)\widehat{\xi_i}(s)ds \\
&\qquad + \int_{\kappa_1}^{\kappa_2} \|f_i(s,0,0,H_i(0,0))\|ds \\
&\qquad + \int_{\kappa_1}^{\kappa_2} \|R_i(\kappa_1,s) - R_i(\kappa_2,s)\|(\psi_i(\aleph^*_i)\xi_i(s) + \widehat{\psi_i}(H^*_i)\widehat{\xi_i}(s))ds \\
&\qquad + \int_{\kappa_1}^{\kappa_2} \|R_i(\kappa_1,s) - R_i(\kappa_2,s)\|\|f_i(s,0,0,H_i(0,0))\|ds.
\end{aligned}$$

Similarly, we get

$$\|\Upsilon_i(\vartheta,\varpi)(\kappa_1) - \Upsilon_i(\vartheta,\varpi)(\kappa_2)\| \to 0,\ as\ \kappa_1 \to \kappa_2.$$

If $\kappa_1, \kappa_2 \in \Omega_k$, from $(H3)$, the set $\left\{Z^i_k(t,\vartheta(t^-_k),\varpi(t^-_k))\right\}_{k=1}^{+\infty}$ is equicontinuous, then we get

$$\begin{aligned}
\|\Upsilon_i(\vartheta,\varpi)(\kappa_1) - \Upsilon_i(\vartheta,\varpi)(\kappa_2)\| &= \|Z^i_k(\kappa_1,\vartheta(t^-_k),\varpi(t^-_k)) - Z^i_k(\kappa_2,\vartheta(t^-_k),\varpi(t^-_k))\| \\
&\longrightarrow 0,\ as\ \kappa_1 \to \kappa_2.
\end{aligned}$$

Hence, the set $\Upsilon(\Pi)$ is equicontinuous, then $\omega_0(\Upsilon(\Pi)) = 0$.
On the other hand, for $\vartheta, \widehat{\vartheta},\ \varpi, \widehat{\varpi} \in \Pi$ and $t \in I_0$, we have

$$\begin{aligned}
\|(\Upsilon_i(\vartheta,\varpi))(t) - (\Upsilon_i(\widehat{\vartheta},\widehat{\varpi}))(t)\| &\leq \|\Lambda_i(t, \vartheta_{\rho(t,\vartheta_t)}, \varpi_{\rho(t,\varpi_t)}) - \Lambda_i(t, \widehat{\vartheta}_{\rho(t,\widehat{\vartheta}_t)}, \widehat{\varpi}_{\rho(t,\widehat{\varpi}_t)})\| \\
&\quad + \int_0^t \|R_i(t,s)\|\|f_i\left(s, \vartheta_{\rho(s,\vartheta_s)}, \varpi_{\rho(s,\varpi_s)}, H_i(\vartheta,\varpi)(s)\right) \\
&\quad - f_i\left(s, \widehat{\vartheta}_{\rho(s,\widehat{\vartheta}_s)}, \widehat{\varpi}_{\rho(t,\widehat{\varpi}_t)}, H_i(\widehat{\vartheta},\widehat{\varpi})(s)\right)\|ds \\
&\leq L^*_{\Lambda_i}\left(\|\vartheta_{\rho(t,\vartheta_t)} - \widehat{\vartheta}_{\rho(t,\widehat{\vartheta}_t)}\| + \|\varpi_{\rho(t,\varpi_t)} - \widehat{\varpi}_{\rho(t,\widehat{\varpi}_t)}\|\right) \\
&\quad + M_{R_i}\int_0^t e^{-\mu_i(t-s)}\left(\psi_i(\aleph^*_i)\xi_i(s) + \psi_i(H^*_i)\xi_i(s)\right).
\end{aligned}$$

When t tend to $+\infty$, we get

$$D^{\Delta}(\Upsilon_i(\Pi))(t)) \leq \left(L^*_{\Lambda_i}\right)\left(D^{\Delta}(\Pi_1(t)) + D^{\Delta}(\Pi_2(t))\right).$$

Now, for $t \in I_k$, we have

$$\begin{aligned}
&\|(\Upsilon_i(\vartheta,\varpi))(t) - (\Upsilon_i(\widehat{\vartheta},\widehat{\varpi}))(t)\| \\
&\quad \leq M_{R_i} e^{-\mu_i(t-s_k)} \Big(\|Z_k^i(s_k,\vartheta(t_k^-),\varpi(t_k^-)) - Z_k^i(s_k,\widehat{\vartheta}(t_k^-),\widehat{\varpi}(t_k^-))\| \\
&\qquad + L^*_{\Lambda_i}(\|\vartheta_{\rho(s_k,\vartheta_{s_k})} - \widehat{\vartheta}_{\rho(s_k,\widehat{\vartheta}_{s_k})}\| + \|\varpi_{\rho(s_k,\varpi_{s_k})} - \widehat{\varpi}_{\rho(s_k,\widehat{\varpi}_{s_k})}\|) \Big) \\
&\qquad + M_{R_i} \int_{s_k}^{t} e^{-\mu_i(t-s)} \left(\psi_i(\aleph_i^*)\xi_i(s) + \psi_i(H_i^*)\xi_i(s)\right).
\end{aligned}$$

When t tends to $+\infty$, we get

$$D^{\Delta}(\Upsilon_i(\Pi))) \leq \left(L^*_{\Lambda_i}\right) \Big(D^{\Delta}(\Pi_1) + D^{\Delta}(\Pi_2) \Big).$$

For $t \in \Omega_k$, we obtain

$$\begin{aligned}
\|\Upsilon_i(\vartheta,\varpi)(t) - \Upsilon_i(\widehat{\vartheta},\widehat{\varpi})(t)\| &\leq \|Z_k(t_k,\vartheta(t_k^-),\varpi(t_k^-)) - Z_k((t_k,\widehat{\vartheta}(t_k^-)),\widehat{\varpi}(t_k^-))\| \\
&\leq L^*_{Z_k^i}(\|\vartheta(t) - \widehat{\vartheta}(t)\| + \|\varpi(t) - \widehat{\varpi}(t)\|).
\end{aligned}$$

Then,

$$D^{\Delta}(\Upsilon_i(\Pi))(t)) \leq L^*_{Z_k^i} \Big(D^{\Delta}(\Pi_1(t)) + D^{\Delta}(\Pi_2(t)) \Big).$$

If $t \in \mathbb{R}^-$, we have

$$\|\Upsilon_i(\vartheta,\varpi)(t) - \Upsilon_i(\widehat{\vartheta},\widehat{\varpi})(t)\| = 0.$$

Thus,

$$\chi_{BPC}(\Upsilon\Pi) \leq \widetilde{A} \begin{pmatrix} \chi_1(\Pi_1) \\ \chi_1(\Pi_2) \end{pmatrix},$$

such that

$$\widetilde{A} = \begin{cases} \begin{pmatrix} L^*_{\Lambda_1} & L^*_{\Lambda_1} \\ L^*_{\Lambda_2} & L^*_{\Lambda_2} \end{pmatrix}; \text{ if } t \in I_k,\ k \in N_0^m, \\ \\ \begin{pmatrix} L^*_{Z_k^1} & L^*_{Z_k^1} \\ L^*_{Z_k^2} & L^*_{Z_k^2} \end{pmatrix}; \text{ if } t \in \Omega_k,\ k \in N_1^m. \end{cases}$$

We have $|\widetilde{A}| < M_{\alpha_1}$, then $\rho(\widetilde{A}) \in (0,1)$. Therefore, $\widetilde{A}$ converges to zero. Thus Υ is generalized χ_{BPC}-condensing operator. Applying now Theorem 2.12, we conclude that Υ has at least one fixed point, which is a mild solution of problem (1.2). □

Remark 4.3. We can use the technique of fixed points theorems in classical Banach space with Banach, Schauder or Darbo theorems. But in this case we need to assume some conditions witch are strong compared to what we assumed, for example in Proof of Theorem 3.4 if we work in classical Banach space and if we want to prove that Ξ is a contraction (Banach fixed point theorem) we need to assume that $\varsigma_1 + \varsigma_2 + \varsigma_5 + \varsigma_6 < 1$, or when we want to prove that $\Xi(B_\theta) \subset B_\theta$ (Schauder or Darbo fixed point theorems). However in our analysis we used the approach matrices that converge to zero and we just needed to assume that matrix M_Ξ converge to zero, even in the worst case, we only need to assume that $\max(\varsigma_1, \varsigma_2, \varsigma_5, \varsigma_6) < 1$, and that always stays more general.

5. An Example

Consider the following class of partial integro-differential system:

$$\begin{cases}
\frac{\partial}{\partial t}\left(\zeta_1(t,x)-\widetilde{D}(t,s,\zeta_1,\zeta_2,\varepsilon_3)\right)=\varepsilon_7(t)\frac{\partial^2}{\partial^2 x}\left(\zeta_1(t,x)-\widetilde{D}(t,s,\zeta_1,\zeta_2,\varepsilon_3)\right)\\
\quad-\int_t^0\Gamma_1(t,s)\varepsilon_7(s)\frac{\partial^2}{\partial^2 x}\left(\zeta_1(s,x)-\widetilde{D}(s,\iota,\zeta_1,\zeta_2,\varepsilon_3)\right)ds\\
\quad+\frac{e^{(-\pi(\mu_1+1)+1)t+1}\cos^2(t+1)}{(1+t^2)^2(2+\sin(1)-\cos(1))}\left(\int_0^1\frac{\cos(s)e^{-s-t}}{99}(\zeta_1(s,x)+\zeta_2(s,x))ds\right)\\
\quad+\frac{\cos^2(t+1)}{99e^{-(1-\pi(\mu_1+1))t}}\int_{-\infty}^{-t}\frac{e^{-D_1(t,\tau,\varepsilon_1,\varepsilon_2)-D_2(t,\tau,\varepsilon_1,\varepsilon_2)}}{((t+\tau)^2+1)^2}d\tau,\ \text{ if } t\in I\ \text{ and }\ x\in(0,1),\\
\\
\frac{\partial}{\partial t}\left(\zeta_2(t,x)-\widetilde{D}(t,s,\zeta_1,\zeta_2,\varepsilon_6)\right)=\varepsilon_7(t)\frac{\partial^2}{\partial^2 x}\left(\zeta_2(t,x)-\widetilde{D}(t,s,\zeta_1,\zeta_2,\varepsilon_6)\right)\\
\quad-\int_t^0\Gamma_2(t,s)\varepsilon_7(s)\frac{\partial^2}{\partial^2 x}\left(\zeta_2(s,x)-\widetilde{D}(s,\iota,\zeta_1,\zeta_2,\varepsilon_6)\right)ds+\varepsilon_7(t)\zeta_2(t,x)\\
\quad+\frac{e^{(t+1)^3-\pi(\mu_2+1)t}}{(1+\coth^2(t))(e^\pi-1)}\int_0^1\frac{e^{s-(t+1)^3}}{111}\big(|\zeta_1(t,x)|+|\zeta_2(t,x)|\big)ds\\
\quad+\int_{-\infty}^{-t}\frac{\sinh^2(t)\cos(t+\tau)(D_1(t,\tau,\varepsilon_1,\varepsilon_2)+D_2(t,\tau,\varepsilon_1,\varepsilon_2))}{111e^{\pi(\mu_2+1)t}(1+2\sinh^2(t))}d\tau, t\in I,\ x\in(0,1),\\
\\
\zeta_1(t,0)=\zeta_1(t,1)=\zeta_2(t,0)=\zeta_2(t,1)=0,\quad \text{for } t\in I,\\
\zeta_1(t,x)=\frac{1}{77+e^t}\sin\left(\zeta_1\left(2k^-,x\right)\right),\ \text{if } t\in\Omega_k,\ \ x\in(0,1),\\
\zeta_2(t,x)=\frac{1}{113}\cos\left(\zeta_2\left(2k^-\right)\right),\ \text{if } t\in\Omega_k,\ \ x\in(0,1),\\
\zeta_1(t,x)=\Phi_1(t,x),\ \text{if } t\in\mathbb{R}_-\ \text{ and }\ x\in(0,1),\\
\zeta_2(t,x)=\Phi_2(t,x),\ \text{if } t\in\mathbb{R}_-\ \text{ and }\ x\in(0,1),
\end{cases}\tag{5.1}$$

where $I=\mathbb{R}^+$, $I_k=(2k-1,2k]$, $\Omega_k=(2k,2k+1]$; $k\in\mathbb{N}$, $q:\mathbb{R}\to\mathbb{R}$ and $\Gamma_i:\mathbb{R}^+\times\mathbb{R}^+\mapsto\mathbb{R}$ are continuous, $\varepsilon_1,\ \varepsilon_2\ \varepsilon_4,\ \varepsilon_5,\ \varepsilon_7:(0,+\infty)\to(0,+\infty)$, $\varepsilon_3,\ \varepsilon_6\ :\Omega\times\Omega\ \to(0,+\infty)$, $\int_{-\infty}^t\varepsilon_3(t,s)ds=m_{\varepsilon_3}$, $\int_{-\infty}^t\varepsilon_6(t,s)ds=m_{\varepsilon_6}$, with $m_{\varepsilon_3},m_{\varepsilon_6}\in\left(0,\frac{\pi}{33}\right)$,
and

$$D_i(t,s,\varepsilon_1,\varepsilon_2)=\zeta_i\left(s-\varepsilon_1(t)\varepsilon_2\left(\int_0^\pi q(r)|\zeta_i(t,r)|^2dr\right),x\right).$$

$$\widetilde{D}(t,\zeta_1,\zeta_2,\varepsilon_3)=\int_{-\infty}^t\frac{\varepsilon_3(t,s)\big(D_1(t,s,\varepsilon_1,\varepsilon_2)+D_2(t,s,\varepsilon_4,\varepsilon_5)\big)}{((t+s)^2+1)}ds.$$

Let

$$\mathcal{H}:=L^2(0,1)=\left\{\wp_1:(0,1)\longrightarrow\mathbb{R}:\int_0^1|\wp_1(x)|^2dx<\infty\right\},$$

be the Hilbert space with the scalar product $\langle\wp_1,\wp_2\rangle=\int_0^1\wp_1(x)\wp_2(x)dx$, and the phase space $\S$ be $BUC\left(\mathbb{R}^-,\mathcal{H}\right)$, the space of bounded uniformly continuous functions endowed with the following norm: $\|\psi\|_\S=\sup_{-\infty<\tau\leq0}\|\psi(\tau)\|_{L^2},\psi\in\S$. It is well known that $\S$ satisfies the axioms (A_1) and (A_2) with $K=1$ and $L(t)=M(t)=1$, (see [18]), and put $Y=BPC(\mathbb{R},X)$.
We define the operator A_i as follows:

$$D(A_i)=\left\{\wp_1\in L^2(0,1)\ /\ \wp_1,\ \wp_1{}''\in L^2(0,1)\ ,\ \wp_1(0)=\wp_1(1)=0\right\},$$

and

$$(A_i(t)\wp_1)(x)=\varepsilon_7(t)\frac{\partial^2\wp_1(x)}{\partial x^2}.$$

It is well known that $A_i(t)$ generates $(S(t,s))_{0\leq s\leq t}$, which are compact with $\|S_i(t,s)\|\leq e^{-\varepsilon_i^2(t-s)}$, and for some $b_i>\frac{1}{\varepsilon_i^2}$, we assume that $\|\Gamma_i(t,s)\|\leq\frac{e^{-\varepsilon_i^2(t-s)}}{b_i}$, and $\|\Gamma_i'(t,s)\|\leq\frac{e^{-\varepsilon_i^2(t-s)}}{b_i^2}$. It follows

from [13], that $\|R_i(t,s)\| \leq e^{-\varkappa_i t}$, where $\varkappa_i = 1 - b_i^{-1}$.
Let $B_i(t,s) : \mathcal{H} \mapsto \mathcal{H}$ be such that:

$$B_i(t,s)z = \Gamma_i(t,s)A_i z, \ for\ 0 \leq s \leq t, \ z \in D(\mathrm{A_i}).$$

More appropriate conditions on operators B_i, $(H4)$ hold with $M_{R_i} = 1$ and $\mu_i = 1 - b_i^{-1}$.

We put $\zeta(t)(x) = \zeta(t,x)$, for $t \in \mathbb{R}^+$, and we define

$$\begin{aligned}
&\mathrm{f}_1(t, \overline{\phi}_1, \overline{\phi}_2, H_1(\phi_1,\phi_2)) \\
&\quad = \frac{e^{(-\pi(\mu_1+1)+1)t+1}\cos^2(t+1)}{99(1+t^2)^2(2+\sin(1)-\cos(1))}\left(\int_0^1 \frac{\cos(s)e^{-s-t}}{99}(\phi_1(s)+\phi_2(s))ds\right) \\
&\qquad + \frac{\cos^2(t+1)}{99e^{-(-\pi(\mu_1+1))+1)t}}\int_{-\infty}^{-t}\frac{e^{-D_1(t,\tau,\varepsilon_1,\varepsilon_2)-D_2(t,\tau,\varepsilon_1,\varepsilon_2)}}{\left((t+\tau)^2+1\right)^2}d\tau,
\end{aligned}$$

$$\begin{aligned}
&\mathrm{f}_2(t, \overline{\phi}_1, \overline{\phi}_2, H_2(\phi_1,\phi_2)) \\
&\quad = \frac{e^{(t+1)^3-\pi(\mu_2+1)t}}{(1+\coth^2(t))(e^{\pi}-1)}\int_0^1 \frac{e^{s-(t+1)^3}}{111}\big(|\phi_1(t)|+|\phi_2(t)|\big)ds \\
&\qquad + \int_{-\infty}^{-t}\frac{\sinh^2(t)\cos(t+\tau)(D_1(t,\tau,\varepsilon_1,\varepsilon_2)+D_2(t,\tau,\varepsilon_1,\varepsilon_2))}{111e^{\pi(\mu_2+1)t}(1+2\sinh^2(t))}d\tau,
\end{aligned}$$

$$\begin{aligned}
\Lambda_1(t,\phi_{\rho_1(t,\phi_t)},\widehat{\phi}_{\rho_2(t,\widehat{\phi}_t)}) &= \widetilde{D}(t,\zeta_1,\zeta_2,\varepsilon_3),\ \Lambda_2(t,\phi_{\rho_1(t,\phi_t)},\widehat{\phi}_{\rho_2(t,\widehat{\phi}_t)}) \\
&= \widetilde{D}(t,\zeta_1,\zeta_2,\varepsilon_6),
\end{aligned}$$

$$\begin{aligned}
H_1(\phi_1,\phi_2) &= \int_0^1 \frac{\cos(s)e^{-s-t}}{99}(\phi_1(s)+\phi_2(s))ds.\ H_2(\phi_1,\phi_2) \\
&= \int_0^1 \frac{e^{s-(t+1)^3}}{111}(|\phi_1(s)|+|\phi_2(s)|)ds,
\end{aligned}$$

and

$$\begin{aligned}
\Theta_k(t,\phi_1(t)) &= \frac{1}{77+e^t}\sin\left(\phi_1\left(2k^-\right)\right),\ \Xi_k(t,\phi_2(t)) \\
&= \frac{1}{113+\ln(t+e)}\cos\left(\phi_2\left(2k^-\right)\right).
\end{aligned}$$

Using these definitions we can represent the system (5.1) in the abstract form (1.2).
For $t \in I$, we have

$$\begin{aligned}
&|f_1(t,\varkappa_t,\nu_t,H_1(\varkappa(t),\nu(t))) - f_1(t,\widetilde{\varkappa}_t,\widetilde{\nu}_t,H_1(\widetilde{\varkappa}(t),\widetilde{\nu}(t)))| \\
&\quad \leq \frac{\cos^2(t+1)e^{-\pi(\mu_1+1)t}}{99(1+t^2)^2}\left(\|\varkappa-\widetilde{\varkappa}\|_{\S}+\|\nu-\widetilde{\nu}\|_{\S}+\|\varkappa(t)-\widetilde{\varkappa}(t)\|+\|\nu(t)-\widetilde{\nu}(t)\|\right),
\end{aligned}$$

and

$$\begin{aligned}
&|f_2(t,\varkappa_t,\nu_t,H_2(\varkappa(t),\nu(t))) - f_2(t,\widetilde{\varkappa}_t,\widetilde{\nu}_t,H_2(\widetilde{\varkappa}(t),\widetilde{\nu}(t)))| \\
&\quad \leq \frac{e^{-\pi(\mu_2+1)t}}{111(1+\coth^2(t))}\left(\|\varkappa-\widetilde{\varkappa}\|_{\S}+\|\nu-\widetilde{\nu}\|_{\S}+\|\varkappa(t)-\widetilde{\varkappa}(t)\|+\|\nu(t)-\widetilde{\nu}(t)\|\right).
\end{aligned}$$

So, for $i=1,2$, $\psi_i(t)=\phi_i(t)=t$, are continuous nondecreasing functions from $(0,+\infty)$ to $(0,+\infty)$. And, we have

$$\xi_1(t)=\widehat{\xi}_1(t)=\frac{\cos^2(t+1)e^{-\pi(\mu_1+1)t}}{99(1+t^2)^2}\in L^1(\Omega,\mathbb{R}^+),\ \lim_{t\to+\infty}\sup_{t\in\Omega}\int_0^t e^{-\mu_1(t-s)}\xi_1(s)ds=0,$$

and

$$\xi_2(t)=\widehat{\xi}_2(t)=\frac{e^{-\pi(\mu_2+1)t}}{111(1+\coth^2(t))}\in L^1(\Omega,\mathbb{R}^+),\ \lim_{t\to+\infty}\sup_{t\in\Omega}\int_0^t e^{-\mu_2(t-s)}\xi_2(s)ds=0.$$

Now, about h_i,Λ_i,Θ_k and Ξ_k, we have

$$\|h_1(t,s,(\varkappa_1,\varkappa_2)(t))-h_1(t,s,(\widehat{\varkappa}_1,\widehat{\varkappa}_2)(t))\|\leq\frac{|\cos(s)|}{99e^{s+t}}\left(\|\varkappa_1(t)-\varkappa_2(t)\|+\|\widehat{\varkappa}_1(t)-\widehat{\varkappa}_2(t)\|\right),$$

$$\|h_2(t,s,(\varkappa_1,\varkappa_2)(t))-h_2(t,s,(\widehat{\varkappa}_1,\widehat{\varkappa}_2)(t))\|\leq\frac{e^{s-(t+1)^3}}{111}\left(\|\varkappa_1(t)-\varkappa_2(t)\|+\|\widehat{\varkappa}_1(t)-\widehat{\varkappa}_2(t)\|\right),$$

$$\|\Xi_k(t,(\varkappa_1,\widehat{\varkappa}_1)(t))-\Xi_k(t,(\varkappa_2,\widehat{\varkappa}_2)(t))\|\leq\frac{(\|\varkappa_1(t)-\varkappa_2(t)\|+\|\widehat{\varkappa}_1(t)-\widehat{\varkappa}_2(t)\|)}{113+\ln(t+e)},$$

$$\|\Lambda_1(t,\varkappa_t,\nu_t)-\Lambda_1(t,\widetilde{\varkappa}_t,\widetilde{\nu}_t)\|\leq\frac{1}{t^2+1}\int_{-\infty}^t\varepsilon_3(t,s)ds(\|\varkappa-\widetilde{\varkappa}\|_{\S}+\|\nu-\widetilde{\nu}\|_{\S}),$$

$$\|\Lambda_2(t,\varkappa_t,\nu_t)-\Lambda_2(t,\widetilde{\varkappa}_t,\widetilde{\nu}_t)\|\leq\frac{1}{t^2+1}\int_{-\infty}^t\varepsilon_6(t,s)ds(\|\varkappa-\widetilde{\varkappa}\|_{\S}+\|\nu-\widetilde{\nu}\|_{\S}),$$

$$\|\Theta_k(t,(\varkappa_1,\widehat{\varkappa}_1)(t))-\Theta_k(t,(\varkappa_2,\widehat{\varkappa}_2)(t))\|\leq\frac{1}{77+e^t}(\|\varkappa_1(t)-\varkappa_2(t)\|+\|\widehat{\varkappa}_1(t)-\widehat{\varkappa}_2(t)\|),$$

$$h_{c_1}=\bar{h}_{c_1}=\frac{1}{99},\ h_{c_2}=\bar{h}_{c_2}=\frac{1}{111},\ (L^*_{\Theta_k},L^*_{\Xi_k})=\left(\frac{1}{78},\frac{1}{114}\right),$$

$$\|\xi_1\|_{L^1}=\|\widehat{\xi}_1\|_{L^1}\simeq 4,9.10^{-4},\ \|\xi_2\|_{L^1}=\|\widehat{\xi}_2\|_{L^1}\simeq 2,7.10^{-4}.$$

Also, we have

$$\sigma_1\simeq 4,9\times10^{-4}+m_{\varepsilon_3},\ \sigma_2\simeq 0,049.10^{-4}+m_{\varepsilon_3},\ \sigma_5\simeq 2,7.10^{-4}+m_{\varepsilon_6},\ \sigma_6\simeq 0,024.10^{-4}+m_{\varepsilon_6}.$$

Then,

$$M_{\alpha_1}\leq\begin{pmatrix}0.09569 & 0.0952\\ 0.0952 & 0.09547\end{pmatrix}.$$

We have for all $p\geq 449$

$$\begin{pmatrix}0.09569 & 0.0952\\ 0.0952 & 0.09547\end{pmatrix}^p=0,$$

and

$$\max_{i=1,2}\left\{M_{R_i}\left(L^*_{Z^i_k}+L^*_{\Lambda_i}\aleph_i\right)\right\}=\max\left(\frac{1}{78}+m_{\varepsilon_3},\frac{1}{114}+m_{\varepsilon_6}\right)\simeq 0.10802<1.$$

Thus, $M_{\alpha_j};\ j=1,2,3$, converges to zero. Therefore, all conditions of Theorems 4.2 are verified. Then, the problem (5.1) has at least one mild solution.

Now, concerning the applicability of the case when we have instantaneous impulsive, we can use the same previous example, but change the following equations:

$$\begin{cases}\zeta_1(t,x)=\frac{1}{77+e^t}\sin\left(\zeta_1\left(2k^-,x\right)\right),\ \text{if } t\in\Omega_k,\ \ x\in(0,1),\\ \zeta_2(t,x)=\frac{1}{113}\cos\left(\zeta_2\left(2k^-\right)\right),\ \text{if } t\in\Omega_k,\ \ x\in(0,1),\end{cases}\tag{5.2}$$

with

$$\begin{cases} \Delta\zeta_1(t_k,x) = \frac{\alpha_k\zeta_1(t_k^-,x)}{70\sqrt{1+|\zeta_1(t_k^-,x)|+|\zeta_2(t_k^-,x)|}}, \text{ for } t_k = 2^k, k\in\mathbb{N}, \text{ and } x\in(0,1). \\ \Delta\zeta_2(t_k,x) = \frac{\beta_k\zeta_2(t_k^-,x)}{110\sqrt{1+|\zeta_1(t_k^-,x)|+|\zeta_2(t_k^-,x)|}}, \text{ for } t_k = 2^k, k\in\mathbb{N}, \text{ and } x\in(0,1), \end{cases} \tag{5.3}$$

where $\alpha_k = 7^{-k},\ \beta_k = 11^{-k}$, $\widetilde{\Omega} = I\backslash\{2^k\}_{k\in\mathbb{N}}$ and the first two equations they will be defined in $\widetilde{\Omega}$. Now, we put

$$\begin{aligned} \Delta_k(\phi_1(t_k^-),\phi_2(t_k^-)) &= \frac{\alpha_k\phi_1(t_k^-)}{70\sqrt{1+|\phi_1(t_k^-)|+|\phi_2(t_k^-)|}}, \\ \widetilde{\Delta}_k(x(t_k^-),\varphi(t_k^-)) &= \frac{\beta_k\phi_2(t_k^-)}{110\sqrt{1+|\phi_1(t_k^-)|+|\phi_2(t_k^-)|}}. \end{aligned}$$

Using these definitions we can represent the system (5.1) in the abstract form (1.1).
On the other hand, we get

$$\begin{aligned} \|\Delta_k(\varkappa_1,\varkappa_2) - \aleph_k(\widehat{\varkappa}_1,\widehat{\varkappa}_2)\| &\leq \frac{\alpha_k}{70}\left(\|\varkappa_1-\varkappa_2\| + \|\widehat{\varkappa}_1-\widehat{\varkappa}_2\|\right), \\ \|\widetilde{\Delta}_k(\varkappa_1,\varkappa_2) - \widetilde{\aleph}_k(\widehat{\varkappa}_1,\widehat{\varkappa}_2)\| &\leq \frac{\beta_k}{110}\left(\|\varkappa_1-\varkappa_2\| + \|\widehat{\varkappa}_1-\widehat{\varkappa}_2\|\right), \end{aligned}$$

Also, we have

$$\begin{aligned} \varsigma_1 &\simeq 4,9\times10^{-4} + \sum_{k\geq0}\frac{\alpha_k}{70} + m_{\varepsilon_3},\ \varsigma_2 \simeq 0,049.10^{-4} + \sum_{k\geq0}\frac{\alpha_k}{70} + m_{\varepsilon_3}. \\ \varsigma_5 &\simeq 2,7.10^{-4} + \sum_{k\geq0}\frac{\beta_k}{110} + m_{\varepsilon_6},\ \varsigma_6 \simeq 0,024.10^{-4} + \sum_{k\geq0}\frac{\beta_k}{110} + m_{\varepsilon_6}. \end{aligned}$$

Then, we get

$$0 < M_\Xi \leq \begin{pmatrix} 0.11235 & 0.11186 \\ 0.10546 & 0.10519 \end{pmatrix}.$$

And, we have for all $p \geq 488$

$$\begin{pmatrix} 0.11235 & 0.11186 \\ 0.10546 & 0.10519 \end{pmatrix}^p = 0.$$

Thus, $\rho(M_\Xi) < 1$, then the matrix M_Ξ converges to zero. Therefore, all conditions of Theorems 3.4 are verified. Then, the problem (5.1) has at least one mild solution.

Statements and Declarations

This article does not contain any studies with human participants or animals performed by any of the authors. It is declared that authors has no competing interests. The study was carried out in collaboration of all authors. All authors read and approved the final manuscript. Funding was not available. Data sharing not applicable to this paper as no data sets were generated or analyzed during the current study.

References

[1] S. Abbas and M. Benchohra. *Advanced Functional Evolution Equations and Inclusions*, volume 39 of *Developments in Mathematics*. Springer, Cham, 2015.

[2] L. Bai and J. J. Nieto. Variational approach to differential equations with not instantaneous impulses. *Applied Mathematics Letters*, 73:44–48, 2017.

[3] M. Benchohra, F. Bouazzaoui, E. Karapinar, and A. Salim. Controllability of second order functional random differential equations with delay. *Mathematics*, 10(7):Article ID 1120, 2022.

[4] M. Benchohra, J. Henderson, and S. Ntouyas. *Impulsive Differential Equations and Inclusions*, volume 2 of *Contemporary Mathematics and Its Applications*. Hindawi Publishing Corporation, New York, 2006.

[5] N. Benkhettou, K. Aissani, A. Salim, M. Benchohra, and C. Tunc. Controllability of fractional integro-differential equations with infinite delay and non-instantaneous impulses. *Applied Analysis and Optimization*, 6(1):79–94, 2022.
[6] O. Bolojan and R. Precup. Implicit first order differential systems with nonlocal conditions. *Electronic Journal of Qualitative Theory of Differential Equations*, 2014(69):1–13, 2014.
[7] O. Bolojan-Nica, G. Infante, and R. Precup. Existence results for systems with coupled nonlocal initial conditions. *Nonlinear Analysis: Theory, Methods & Applications*, 94:231–242, 2014.
[8] S. Bouriah, A. Salim, and M. Benchohra. On nonlinear implicit neutral generalized Hilfer fractional differential equations with terminal conditions and delay. *Topological Algebra and its Applications*, 10(1):77–93, 2022.
[9] P. Chen, X. Zhang, and Y. Li. Existence of mild solutions to partial differential equations with non-instantaneous impulses. *Electronic Journal of Differential Equations*, 2016(241):1–11, 2016.
[10] C. Derbazi, Z. Baitiche, M. Benchohra, and J. R. Graef. Extremal solutions to a coupled system of nonlinear fractional differential equations with ψ-Caputo fractional derivatives. *Journal of Mathematics and Applications*, 44:19–34, 2021.
[11] J. P. C. dos Santos. On state-dependent delay partial neutral functional integro-differential equations. *Applied Mathematics and Computation*, 216(5):1637–1644, 2010.
[12] J. R. Graef, J. Henderson, and A. Ouahab. *Topological Methods for Differential Equations and Inclusions*. Monographs and Research Notes in Mathematics. CRC Press, Boca Raton, FL, 2019.
[13] R. C. Grimmer. Resolvent operators for integral equations in a banach space. *Transactions of the American Mathematical Society*, 273(1):333–349, 1982.
[14] J. K. Hale and J. Kato. Phase space for retarded equations with infinite delay. *Funkcialaj Ekvacioj*, 21(1):11–41, 1978.
[15] X. Hao and L. Liu. Mild solution of semilinear impulsive integro-differential evolution equation in Banach spaces. *Mathematical Methods in the Applied Sciences*, 40(13):4832–4841, 2017.
[16] A. Heris, A. Salim, M. Benchohra, and E. Karapınar. Fractional partial random differential equations with infinite delay. *Results in Physics*, 37:Article ID 105557, 2022.
[17] E. Hernández, R. Sakthivel, and S. Tanaka Aki. Existence results for impulsive evolution differential equations with state-dependent delay. *Electronic Journal of Differential Equations*, 2008(28):1–11, 2008.
[18] Y. Hino, S. Murakami, and T. Naito. *Functional-Differential Equations with Infinite Delay*, volume 1473 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1991.
[19] S. Krim, A. Salim, S. Abbas, and M. Benchohra. On implicit impulsive conformable fractional differential equations with infinite delay in b-metric spaces. *Rendiconti del Circolo Matematico di Palermo. Second Series (2)*, 72(4):2579–2592, 2023.
[20] O. Nica and R. Precup. On the nonlocal initial value problem for first order differential systems. *Studia Universitatis Babeş-Bolyai Mathematica*, 56(3):113–125, 2011.
[21] L. Olszowy. Existence of mild solutions for semilinear differential equations with nonlocal and impulsive conditions. *Central European Journal of Mathematics*, 12(4):623–635, 2014.
[22] R. Precup. The role of matrices that are convergent to zero in the study of semilinear operator systems. *Mathematical and Computer Modelling*, 49(3-4):703–708, 2009.
[23] R. Precup and A. Viorel. Existence results for systems of nonlinear evolution equations. *International Journal of Pure and Applied Mathematics*, 47(2):199–206, 2008.
[24] A. Salim, M. Benchohra, and J. E. Lazreg. Nonlocal k-generalized ψ-Hilfer impulsive initial value problem with retarded and advanced arguments. *Applied Analysis and Optimization*, 6(1):21–47, 2022.
[25] A. Salim, M. Benchohra, J. E. Lazreg, and E. Karapınar. On k-generalized ψ-hilfer impulsive boundary value problem with retarded and advanced argument. *Journal of Mathematical Extension*, 15:1–39, 2021.
[26] A. Salim, M. Boumaaza, and M. Benchohra. Random solutions for mixed fractional differential equations with retarded and advanced arguments. *Journal of Nonlinear and Convex Analysis*, 23(7):1361–1375, 2022.
[27] R. S. Varga. *Matrix Iterative Analysis*, volume 27 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, expanded edition, 2000.
[28] D. Yang and J. Wang. Integral boundary value problems for nonlinear non-instantaneous impulsive differential equations. *Journal of Applied Mathematics and Computing*, 55(1):59–78, 2017.