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TOPOLOGICAL DYNAMICS FOR OPERATORS GENERATED BY GROUP ACTIONS

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Abstract. In this note, we characterize topologically multiple recurrence for weighted translation operators and a sequence of operators, generated by group actions. Some sufficient conditions for sequences of operators to be disjoint topologically transitive and disjoint topologically mixing are provided as well.

Keywords. Topologically multiple recurrence, disjoint topological transitivity, disjoint topological mixing, group action.

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1. Introduction

In [11, 12, 13], we investigated linear dynamics of weighted translations and a sequence of operators by group actions. These dynamical properties included topological transitivity and chaos for both translations and a sequence of operators, and disjoint topological transitivity only for translations. In this note, we will continue this study by tackling the notions of topologically multiple recurrence, and disjoint topological transitivity of a sequence of operators.

Linear dynamics has been studied intensely during the past five decades. These two books [2, 16] are the classic and important references in this direction. An operator T on a separable Banach space X is called *topologically multiply recurrent* if for every positive integer N and every nonempty open subset U of X, there is some $n \in \mathbb{N}$ such that $U \cap T^{-n}U \cap T^{-2n}U \cap \cdots \cap T^{-Nn}U \neq \emptyset$. In particular, T is said to be *recurrent* if $U \cap T^{-n}U \neq \emptyset$ holds for some $n \in \mathbb{N}$. In addition, we call T *topologically transitive* on X if given two nonempty open sets $U, V \subset X$, there exists $n \in \mathbb{N}$ such that $T^nU \cap V \neq \emptyset$. By definitions, topological transitivity implies recurrence. However, this is not the case for topologically multiple recurrence. Indeed, in [15], there exists a transitive weighted backward shift on $\ell^2(\mathbb{Z})$, which is not topologically multiply recurrent. That is why our aim in this note is to characterize topologically multiple recurrence.

It should be noted that the notions mentioned above can be extended from a single operator T to the case of a sequence of operators $(T_n)_{n\in\mathbb{N}}$. A sequence of operators $(T_n)_{n\in\mathbb{N}}$ is *topologically transitive* in [16] if given two nonempty open sets $U, V \subset X$, there exists $n \in \mathbb{N}$ such that $T_nU \cap V \neq \emptyset$. Likewise, in [9], we initiated the study of recurrence of $(T_n)_{n\in\mathbb{N}}$ below.

Definition 1.1. A sequence of operators $(T_n)_{n\in\mathbb{N}}$ on a Banach space X is called *topologically multiply recurrent* if for every positive integer N and every nonempty open set U in X, there is some $n \in \mathbb{N}$ such that

$$U \cap T_n^{-1}U \cap T_{2n}^{-1}U \cap \cdots \cap T_{Nn}^{-1}U \neq \emptyset.$$

If the condition $U \cap T_n^{-1}U \neq \emptyset$ is satisfied, then $(T_n)_{n\in\mathbb{N}}$ is *recurrent*.

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In 2007, Bernal-González, Bès and Peris in [3] and [7] respectively introduced the study of disjoint topological transitivity, which is one of the generalization of topological transitivity. Since then, disjoint linear dynamics on various spaces was studied by many authors (see [5, 6, 18, 19, 20]). Here we recall its definition for sequences of operators $(T_{1,n})_{n=1}^{\infty}, (T_{2,n})_{n=1}^{\infty}, \cdots, (T_{N,n})_{n=1}^{\infty}$.

Definition 1.2. ([7, Definition 2.1]) Given $N \geq 2$, the sequences of operators $(T_{1,n})_{n=1}^{\infty}, (T_{2,n})_{n=1}^{\infty}, \cdots, (T_{N,n})_{n=1}^{\infty}$ acting on a Banach space X are *disjoint topologically transitive* (in short, *d-topologically transitive*) if given nonempty open sets $U, V_1, \cdots, V_N \subset X$, there is some $n \in \mathbb{N}$ such that

$$\emptyset \neq U \cap T_{1,n}^{-1}(V_1) \cap T_{2,n}^{-1}(V_2) \cap \cdots \cap T_{N,n}^{-1}(V_N).$$

If the above condition is satisfied from some n onwards, then $(T_{1,n})_{n=1}^{\infty}, (T_{2,n})_{n=1}^{\infty}, \cdots, (T_{N,n})_{n=1}^{\infty}$ are said to be *disjoint topologically mixing*.

Another goal of this note is to give the sufficient condition for sequences of operators to be disjoint topologically transitive. We first recall our setting of actions of locally compact groups on measure spaces.

Definition 1.3. Let G be a locally compact group with the identity element e. Let X be a locally compact Hausdorff space, and let μ be a Borel nonnegative measure on X. Then a continuous function

$$G \times X \longrightarrow X, \quad (s, x) \mapsto sx, \quad (s \in G, x \in X)$$

is called an *action* of G on the measure space (X, μ) if

(i) for each $x \in X$, $ex = x$;
(ii) for each $s, t \in G$ and $x \in X$, $s(tx) = (st)x$;
(iii) for each $s \in G$, the self-mapping $x \mapsto sx$ on X is Borel measurable and measure-preserving, that is, for each $s \in G$ and Borel subset $E \subseteq X$, $sE := \{sx : x \in E\}$ is also a Borel subset of X and $\mu(sE) = \mu(E)$.

If this is this case, we say G acts on (X, μ).

We note that if $X = G$ is a locally compact group with the the Haar measure μ, then G acts on (G, μ) itself naturally. Besides, let $X = G/H$ be the quotient space where G is a locally compact group G and H is a closed subgroup of G satisfying the modular function $\Delta_H = \Delta_G|_H$. Then there exists a Borel measure μ on G/H such that $\mu(gE) = \mu(E)$ for each $g \in G$ and each Borel set $E \subseteq G/H$ [14], and G acts on $(G/H, \mu)$. More examples of group actions can be found in [11, 12, 13].

In what follows, we assume that G is a locally compact group and (X, μ) is a Borel measure space. For each $1 \leq p < \infty$, the space $L^p(X, \mu)$ is the usual Lebesgue space on (X, μ), where X is second countable locally compact Hausdorff so that $L^p(X, \mu)$ is separable [4, Proposition 2.3]. A bounded continuous function $w : X \longrightarrow (0, \infty)$ is called a *weight* on X. Now let G act on (X, μ), and $a \in G$. Then the weighted translation operator $T_{a,w} : L^p(X, \mu) \longrightarrow L^p(X, \mu)$ is defined by

$$T_{a,w} f(x) := w(x) f(a^{-1}x)$$

where $f \in L^p(X, \mu)$ and $x \in X$. If w^{-1} is bounded, then the inverse $S_{a,w}$ of $T_{a,w}$ can be given by

$$S_{a,w} f(x) = \frac{1}{w(ax)} f(ax).$$

Moreover, one can define a sequence of bounded linear operators $C_n : L^p(X, \mu) \longrightarrow L^p(X, \mu)$ by

$$C_n = \frac{1}{2}(T_{a,w}^n + S_{a,w}^n)$$

for all $n \in \mathbb{Z}$. Here only the dynamics of $(C_n)_{n \in \mathbb{N}}$ is needed to be considered due to $C_n = C_{-n}$ for all $n \in \mathbb{Z}$. Morover, this sequence $(C_n)_{n \in \mathbb{Z}}$ can be regarded as a cosine operator function by letting $C_n := \mathcal{C}(n)$. Indeed, a mapping $\mathcal{C}$ from the real line into the space of continuous operators on a Banach

space X is called a *cosine operator function* if it satisfies $\mathcal{C}(0) = I$, and $2\mathcal{C}(t)\mathcal{C}(s) = \mathcal{C}(t+s) + \mathcal{C}(t-s)$ for all $s, t \in \mathbb{R}$. Linear dynamical properties for cosine operator functions were investigated in [1, 8, 17].

Our aim in this paper is to characterize topologically multiple recurrence for a single operator $T_{a,w}$ and a sequence of operators $(C_n)_{n\in\mathbb{N}}$ in Section 2, and give the sufficient condition for given $N \geq 2$ sequences of operators $(C_{1,n})_{n\in\mathbb{N}}, (C_{2,n})_{n\in\mathbb{N}}, \cdots, (C_{N,n})_{n\in\mathbb{N}}$ to be disjoint topologically transitive in Section 3.

2. Recurrence

In this section, we will first provide the results of recurrence. In [11, 12, 13], we applied the aperiodicity of the group element $a \in G$ to obtain the arguments. Here we abide by the same assumption on $a \in G$. An element $a \in G$ is called *aperiodic* with respect to the given action if for any compact subset $K \subseteq X$ with $\mu(K) > 0$, there exists $N \in \mathbb{N}$ such that $K \cap a^{\pm n}K = \emptyset$ for all $n \geq N$. This characterization of aperiodicity is originally inspired by the group case in [10]. Indeed, if $X = G$, then the definition of aperiodicity in the action is identified with that of the group case. See [10, 11, 12, 13] for more discussion on both cases of actions and groups.

Now, we are ready to state and prove the main results. To simplify the notations, we let

$$\varphi_n(x) := \prod_{j=1}^{n} w(a^j x) \quad \text{and} \quad \widetilde{\varphi}_n(x) := \left(\prod_{j=0}^{n-1} w(a^{-j}x)\right)^{-1}.$$

Theorem 2.1. *Let G act on (X, μ) and let $a \in G$ be aperiodic. Let w be a weight on X such that w^{-1} is bounded. Let $1 \leq p < \infty$. Then the following conditions are equivalent.*

(i) $T_{a,w}$ is topologically multiply recurrent on $L^p(X, \mu)$.

(ii) For each $N \in \mathbb{N}$ and each compact subset K of X with $\mu(K) > 0$, there exist a sequence of Borel sets (E_k) in K and a strictly increasing sequence $(n_k) \subset \mathbb{N}$ such that $\lim_{k\longrightarrow\infty} \mu(E_k) = \mu(K)$ and for $1 \leq l \leq N$,

$$\lim_{k\longrightarrow\infty} \int_{E_k} \varphi_{ln_k}^p(x) d\mu(x) = \lim_{k\longrightarrow\infty} \int_{E_k} \widetilde{\varphi}_{ln_k}^p(x) d\mu(x) = 0$$

Proof. (ii) $\Rightarrow$ (i). Let U be a non-empty open subset of $L^p(X, \mu)$. It is known that the space $C_c(X)$ of all continuous functions on X with compact support is dense in $L^p(X, \mu)$. Hence one can find $f \in C_c(X) \cap U$. Let K be the compact support of f, and let χ_K be the characteristic function of K. Then there exist (E_k) in K and (n_k) satisfy condition (ii). Therefore, we have the following estimates. First, for each $l = 1, 2, \cdots, N$,

$$\begin{aligned}
\|T_{a,w}^{ln_k}(f\chi_{E_k})\|_p^p &= \int_X |T_{a,w}^{ln_k}(f\chi_{E_k})(x)|^p d\mu(x) \\
&= \int_X |w(x)w(a^{-1}x)\cdots w(a^{-(ln_k-1)}x) f(a^{-ln_k}x)\chi_{E_k}(a^{-ln_k}x)|^p d\mu(x) \\
&= \int_X |w(a^{ln_k}x)w(a^{ln_k-1}x)\cdots w(ax) f(x)\chi_{E_k}(x)|^p d\mu(x) \\
&\leq \|f\|_\infty^p \int_{E_k} \varphi_{ln_k}^p(x) d\mu(x) \to 0
\end{aligned}$$

as $k \to \infty$. Likewise,

$$\begin{aligned}
\|S_{a,w}^{ln_k}(f\chi_{E_k})\|_p^p &= \int_X |S_{a,w}^{ln_k}(f\chi_{E_k})(x)|^p d\mu(x) \\
&= \int_X \frac{1}{|w(ax)w(a^2x)\cdots w(a^{ln_k}x)|} |f(a^{ln_k}x)\chi_{E_k}(a^{ln_k}x)|^p d\mu(x) \\
&= \int_X \frac{1}{|w(a^{-(ln_k-1)}x)w(a^{-(ln_k-2)}x)\cdots w(x)|} |f(x)\chi_{E_k}(x)|^p d\mu(x) \\
&\le \|f\|_\infty^p \int_{E_k} \widetilde{\varphi}_{ln_k}^p(x) d\mu(x) \to 0
\end{aligned}$$

as $k \to \infty$, for $1 \le l \le N$. In addition,

$$\begin{aligned}
\|f - f\chi_{E_k}\|_p^p &= \int_X |f(x) - f(x)\chi_{E_k}(x)|^p d\mu(x) \\
&= \int_X |f(x)\chi_{K\setminus E_k}(x)|^p d\mu(x) \\
&= \int_{K\setminus E_k} |f(x)|^p d\mu(x) \\
&\le \|f\|_\infty^p \mu(K \setminus E_k) \to 0
\end{aligned}$$

as $k \to \infty$. Now let

$$v_k = f\chi_{E_k} + S_{a,w}^{n_k}(f\chi_{E_k}) + S_{a,w}^{2n_k}(f\chi_{E_k}) + \cdots + S_{a,w}^{Nn_k}(f\chi_{E_k}).$$

Then

$$\|v_k - f\|_p \le \|f\chi_{E_k} - f\|_p + \|S_{a,w}^{n_k}(f\chi_{E_k})\|_p + \|S_{a,w}^{2n_k}(f\chi_{E_k})\|_p + \cdots + \|S_{a,w}^{Nn_k}(f\chi_{E_k})\|_p$$

and

$$\begin{aligned}
\|T_{a,w}^{ln_k}v_k - f\|_p \le\ & \|T_{a,w}^{ln_k}(f\chi_{E_k})\|_p + \|T_{a,w}^{(l-1)n_k}(f\chi_{E_k})\|_p + \cdots + \|T_{a,w}^{n_k}(f\chi_{E_k})\|_p \\
& + \|f\chi_{E_k} - f\|_p + \|S_{a,w}^{n_k}(f\chi_{E_k})\|_p + \cdots + \|S_{a,w}^{(N-l)n_k}(f\chi_{E_k})\|_p.
\end{aligned}$$

Therefore

$$\lim_{k\to\infty} \|v_k - f\|_p = \lim_{k\to\infty} \|T_{a,w}^{ln_k}v_k - f\|_p = 0$$

for $1 \le l \le N$, which says

$$U \cap T_{a,w}^{-n_k}U \cap T_{a,w}^{-2n_k}U \cap \cdots \cap T_{a,w}^{-Nn_k}U \ne \emptyset.$$

(i) $\Rightarrow$ (ii). Let $\chi_K \in L^p(X,\mu)$ be the characteristic function of a compact set $K \subset X$ with $\mu(K) > 0$. Since a is aperiodic, there is some M such that $K \cap a^{\pm n}K = \emptyset$ for $n > M$. Let $\varepsilon \in (0,1)$, and let $U = \{g \in L^p(X,\mu) : \|g - \chi_K\|_p < \varepsilon^2\}$. Given some $N \in \mathbb{N}$, there exists $m > M$ such that

$$U \cap T_{a,w}^{-m}U \cap T_{a,w}^{-2m}U \cap \cdots \cap T_{a,w}^{-Nm}U \ne \emptyset$$

due to the fact $T_{a,w}$ is topologically multiply recurrent. Hence there exists $f \in L^p(X,\mu)$ such that

$$\|f - \chi_K\|_p < \varepsilon^2 \quad \text{and} \quad \|T_{a,w}^{lm}f - \chi_K\|_p < \varepsilon^2$$

for $l = 1, 2, \ldots, N$. Let $A = \{x \in K : |f(x) - 1| \ge \varepsilon\}$. Then

$$|f(x)| > 1 - \varepsilon \qquad (x \in K \setminus A) \qquad \text{and} \qquad \mu(A) < \varepsilon^p$$

by

$$\begin{aligned}\varepsilon^{2p} &> \|f-\chi_K\|_p^p \\ &= \int_X |f(x)-\chi_K(x)|^p d\mu(x) \\ &\geq \int_A |f(x)-1|^p d\mu(x) \\ &\geq \varepsilon^p \mu(A).\end{aligned}$$

Let $B_{l,m}=\{x\in K : |T_{a,w}^{lm}f(x)-1|\geq\varepsilon\}$ for $1\leq l\leq N$. Then

$$|T_{a,w}^{lm}f(x)|>1-\varepsilon \qquad (x\in K\setminus B_{l,m}) \qquad \text{and} \qquad \mu(B_{l,m})<\varepsilon^p$$

by

$$\begin{aligned}\varepsilon^{2p} &> \|T_{a,w}^{lm}f-\chi_K\|_p^p \\ &= \int_X |T_{a,w}^{lm}f(x)-\chi_K(x)|^p d\mu(x) \\ &\geq \int_{B_{l,m}} |T_{a,w}^{lm}f(x)-1|^p d\mu(x) \\ &\geq \varepsilon^p \mu(B_{l,m}).\end{aligned}$$

Now set

$$E_m = K\setminus(A\cup B_{1,m}\cup B_{2,m}\cup\cdots\cup B_{N,m}).$$

Then

$$\begin{aligned}\mu(K\setminus E_m) &= \mu(A\cup B_{1,m}\cup B_{2,m}\cup\cdots\cup B_{N,m}) \\ &\leq \mu(A)+\mu(B_{1,m})+\mu(B_{2,m})\cdots+\mu(B_{N,m})<(N+1)\varepsilon^p.\end{aligned}$$

On the other hand, by applying $K\cap a^{\pm m}K=\emptyset$,

$$\begin{aligned}\varepsilon^{2p} &> \|T_{a,w}^{lm}f-\chi_K\|_p \\ &= \int_X |T_{a,w}^{lm}f(x)-\chi_K(x)|^p d\mu(x) \\ &\geq \int_{a^{lm}E_m} |w(x)w(a^{-1}x)\cdots w(a^{-(lm-1)}x)f(a^{-lm}x)-\chi_K(x)|^p d\mu(x) \\ &= \int_{E_m} |w(a^{lm}x)w(a^{lm-1}x)\cdots w(ax)f(x)-\chi_K(a^{lm}x)|^p d\mu(x) \\ &= \int_{E_m} |w(a^{lm}x)w(a^{lm-1}x)\cdots w(ax)f(x)|^p d\mu(x) \\ &= \int_{E_m} |\varphi_{lm}(x)f(x)|^p d\mu(x) \\ &> (1-\varepsilon)^p \int_{E_m} \varphi_{lm}^p(x) d\mu(x)\end{aligned}$$

which implies

$$\int_{E_m} \varphi_{lm}^p(x) d\lambda(x) < \frac{\varepsilon^{2p}}{(1-\varepsilon)^p}.$$

Similarly, one can obtain

$$\int_{E_m} \widetilde{\varphi}_{lm}^p(x) d\mu(x) < \frac{\varepsilon^{2p}}{(1-\varepsilon)^p}.$$

Indeed,

$$\begin{aligned}
\varepsilon^{2p} &> \|f-\chi_K\|_p^p \\
&= \int_X |S_{a,w}^{lm}(T_{a,w}^{lm}f)(x)-\chi_K(x)|^p|d\mu(x) \\
&\geq \int_{a^{-lm}E_m} \frac{1}{|w(ax)w(a^2x)\cdots w(a^{lm}x)|^p}|(T_{a,w}^{lm}f)(a^{lm}x)|^p d\mu(x) \\
&= \int_{E_m} \frac{1}{|w(a^{-(lm-1)}x)w(a^{-(lm-2)x})\cdots w(x)|^p}|(T_{a,w}^{lm}f)(x)|^p d\mu(x) \\
&= \int_{E_m} \widetilde{\varphi}_{lm}^p(x)|(T_{a,w}^{lm}f)(x)|^p d\mu(x) \\
&> (1-\varepsilon)^p \int_{E_m} \widetilde{\varphi}_{lm}^p(x)d\mu(x).
\end{aligned}$$

Combining all these, condition (ii) follows. □

If $N=1$ in Theorem 2.1, then one can obtain the characterization for $T_{a,w}$ to be recurrent immediately.

Corollary 2.2. *Let G act on (X,μ) and let $a\in G$ be aperiodic. Let w be a weight on X such that w^{-1} is bounded. Let $1\leq p<\infty$. Then the following conditions are equivalent.*

(i) $T_{a,w}$ is recurrent on $L^p(X,\mu)$.

(ii) For each compact subset K of X with $\mu(K)>0$, there exist a sequence of Borel sets (E_k) in K and a strictly increasing sequence $(n_k)\subset\mathbb{N}$ such that $\lim_{k\longrightarrow\infty}\mu(E_k)=\mu(K)$ and

$$\lim_{k\longrightarrow\infty}\int_{E_k}\varphi_{n_k}^p(x)d\mu(x)=\lim_{k\longrightarrow\infty}\int_{E_k}\widetilde{\varphi}_{n_k}^p(x)d\mu(x)=0$$

Example 2.3. Let H be a closed subgroup of a locally compact group G so that G acts on the quotient space G/H with a Borel measure μ. Let $a\in G$ be aperiodic, and let w be a weight on G/H. Define the weighted translation $T_{a,w}: L^p(G/H,\mu)\to L^p(G/H,\mu)$ by

$$T_{a,w}f(xH):=w(xH)f(a^{-1}xH)\qquad (f\in L^p(G/H,\mu), x\in G).$$

For $x\in G$, let

$$\varphi_n(xH):=\prod_{j=1}^{n}w(a^jxH)\quad\text{and}\quad\widetilde{\varphi}_n(xH):=\left(\prod_{j=0}^{n-1}w(a^{-j}xH)\right)^{-1}.$$

Then $T_{a,w}$ is recurrent if, and only if, for each compact subset K of G/H with $\mu(K)>0$, there exist a sequence of Borel sets (E_k) in K and a strictly increasing sequence $(n_k)\subset\mathbb{N}$ such that $\lim_{k\longrightarrow\infty}\mu(E_k)=\mu(K)$ and

$$\lim_{k\longrightarrow\infty}\int_{E_k}\varphi_{n_k}^p(xH)d\mu(xH)=\lim_{k\longrightarrow\infty}\int_{E_k}\widetilde{\varphi}_{n_k}^p(x)d\mu(xH)=0$$

Next, we give the sufficient condition for a sequence of operators $(C_n)_{n\in\mathbb{N}}$, generated by $T_{a,w}$ and $S_{a,w}$, to be topologically multiply recurrent without the assumption of aperiodicity of $a\in G$.

Theorem 2.4. *Let G act on (X,μ) and let $a\in G$. Let w be a weight on X such that w^{-1} is bounded. Let $1\leq p<\infty$, and let $T_{a,w}$ be a weighted translation operator on $L^p(X,\mu)$ with its inverse $S_{a,w}$. Let $C_n:=\frac{1}{2}(T_{a,w}^n+S_{a,w}^n)$. Then* (ii) ⇒ (i).

(i) $(C_n)_{n\in\mathbb{N}_0}$ is topologically multiply recurrent on $L^p(X,\mu)$.

(ii) For each $N \in \mathbb{N}$ and each compact subset K of X with $\mu(K) > 0$, there exist a sequence of Borel sets (E_k) in K and a strictly increasing sequence $(n_k) \subset \mathbb{N}$ such that $\lim_{k\longrightarrow\infty} \mu(E_k) = \mu(K)$ and for $1 \leq s \leq 2N$,

$$\lim_{k\longrightarrow\infty} \int_{E_k} \varphi^p_{sn_k}(x)d\mu(x) = \lim_{k\longrightarrow\infty} \int_{E_k} \widetilde{\varphi}^p_{sn_k}(x)d\mu(x) = 0.$$

Proof. Let U be a non-empty open subset of $L^p(X,\mu)$. By the fact that the space $C_c(X)$ of all continuous functions on X with compact support is dense in $L^p(X,\mu)$, we can find $f \in C_c(X) \cap U$. Let K be the compact support of f, and let χ_K be the characteristic function of K. Then there exist (E_k) in K and (n_k) satisfy condition (ii). As in the proof of Theorem 2.1, for $1 \leq s \leq 2N$,

$$\lim_{k\to\infty} \|T^{sn_k}_{a,w}(f\chi_{E_k})\|_p = \lim_{k\to\infty} \|S^{sn_k}_{a,w}(f\chi_{E_k})\|_p = \lim_{k\to\infty} \|f - f\chi_{E_k}\|_p = 0.$$

Now let

$$\begin{aligned} v_k &= f\chi_{E_k} + T^{n_k}_{a,w}(f\chi_{E_k}) + T^{2n_k}_{a,w}(f\chi_{E_k}) + \cdots + T^{Nn_k}_{a,w}(f\chi_{E_k}) \\ &\quad + S^{n_k}_{a,w}(f\chi_{E_k}) + S^{2n_k}_{a,w}(f\chi_{E_k}) + \cdots + S^{Nn_k}_{a,w}(f\chi_{E_k}). \end{aligned}$$

Then

$$\|v_k - f\|_p \leq \|f\chi_{E_k} - f\|_p + \sum_{j=1}^{N} \|T^{jn_k}_{a,w}(f\chi_{E_k})\|_p + \sum_{j=1}^{N} \|S^{jn_k}_{a,w}(f\chi_{E_k})\|_p$$

and for $l = 1, 2, \cdots, N$,

$$\begin{aligned} \|2C_{ln_k}v_k - 2f\|_p \leq\ & \|2f\chi_{E_k} - 2f\|_p \\ & + \sum_{j=1}^{N+l} \|T^{jn_k}_{a,w}(f\chi_{E_k})\|_p + \sum_{j=1}^{N-l} \|S^{jn_k}_{a,w}(f\chi_{E_k})\|_p \\ & + \sum_{j=1}^{N+l} \|S^{jn_k}_{a,w}(f\chi_{E_k})\|_p + \sum_{j=1}^{N-l} \|T^{jn_k}_{a,w}(f\chi_{E_k})\|_p. \end{aligned}$$

Hence $\lim_{k\to\infty} v_k = f$ and $\lim_{k\to\infty} C_{ln_k}v_k = f$ for each $l = 1, 2, \cdots, N$. Therefore

$$U \cap C^{-1}_{n_k}(U) \cap C^{-1}_{2n_k}(U) \cap \cdots \cap C^{-1}_{Nn_k}(U) \neq \emptyset.$$

□

3. Disjoint Topological Transitivity

Given some $N \geq 2$, in this section, we will give the sufficient condition for sequences of operators $(C_{1,n})_{n\in\mathbb{N}}, (C_{2,n})_{n\in\mathbb{N}}, \cdots, (C_{N,n})_{n\in\mathbb{N}}$, generated by $T_{a_1,w_1}, T_{a_2,w_2}, \cdots, T_{a_N,w_N}$ respectively, to be disjoint topologically transitive. To simplify the notations, for each $l, n \in \mathbb{N}$, let

$$\varphi_{l,n}(x) := \prod_{j=1}^{n} w_l(a^j_l x) \quad \text{and} \quad \widetilde{\varphi}_{l,n}(x) := \left(\prod_{j=0}^{n-1} w_l(a^{-j}_l x)\right)^{-1}.$$

Theorem 3.1. *Let G act on (X,μ). Given some $N \geq 2$, let $a_l \in G$ and let w_l be a weight on X for $1 \leq l \leq N$. Let $1 \leq p < \infty$, and let $T_l := T_{a_l,w_l}$ be a weighted translation operator on $L^p(X,\mu)$ with its inverse $S_l := S_{a_l,w_l}$. Let $C_{l,n} = \frac{1}{2}(T^n_l + S^n_l)$. Then* (ii) $\Rightarrow$ (i).

(i) *$(C_{1,n})_{n\in\mathbb{N}}, (C_{2,n})_{n\in\mathbb{N}}, \cdots, (C_{N,n})_{n\in\mathbb{N}}$ are disjoint topologically transitive on $L^p(X,\mu)$.*

(ii) For each compact subset K of X with $\mu(K) > 0$, there exist a sequence of Borel sets (E_k) in K and a strictly increasing sequence $(n_k) \subset \mathbb{N}$ such that $\lim_{k\longrightarrow\infty} \mu(E_k) = \mu(K)$ and (for $1 \leq l \leq N$)

$$\lim_{k\to\infty} \int_{E_k} \varphi^p_{l,n_k}(x)d\mu(x) = 0,$$

$$\lim_{k\to\infty} \int_{E_k} \widetilde{\varphi}^p_{l,n_k}(x)d\mu(x) = 0$$

and (for $1 \leq l, s \leq N$,)

$$\lim_{k\to\infty} \int_{E_k} \varphi^p_{l,n_k}(a_s^{n_k}x) \cdot \varphi^p_{s,n_k}(x)d\mu(x) = 0,$$

$$\lim_{k\to\infty} \int_{E_k} \widetilde{\varphi}^p_{l,n_k}(a_s^{-n_k}x) \cdot \widetilde{\varphi}^p_{s,n_k}(x)d\mu(x) = 0,$$

and (for $1 \leq l, s \leq N$ with $l \neq s$,)

$$\lim_{k\to\infty} \int_{E_k} \varphi^p_{l,n_k}(a_s^{-n_k}x) \cdot \widetilde{\varphi}^p_{s,n_k}(x)d\mu(x) = 0,$$

$$\lim_{k\to\infty} \int_{E_k} \widetilde{\varphi}^p_{l,n_k}(a_s^{n_k}x) \cdot \varphi^p_{s,n_k}(x)d\mu(x) = 0.$$

Proof. For $1 \leq l \leq N$, let U and V_l be non-empty open subsets of $L^p(X, \mu)$. By the fact that the space $C_c(X)$ of continuous functions on X with compact support is dense in $L^p(X, \mu)$, we can pick $f \in C_c(X) \cap U$ and $g_l \in C_c(X) \cap V_l$ for each l. Let K be the union of the compact supports of f and all g_l. Then there exist $E_k \subset K$ and a sequence (n_k) satisfy condition (ii).

For $1 \leq l \leq N$, by using the condition $\lim_{k\longrightarrow\infty} \mu(E_k) = \mu(K)$, and applying the same arguments as in the proof of Theorem 2.1, we have

$$\lim_{k\to\infty} \|f\chi_{E_k} - f\|_p = \lim_{k\to\infty} \|g_l\chi_{E_k} - g_l\|_p = 0.$$

Besides, for $1 \leq l \leq N$,

$$\begin{aligned}
& \|T_l^{n_k}(g_l\chi_{E_k})\|_p^p \\
= & \int_X |T_l^{n_k}(g_l\chi_{E_k})(x)|^p d\mu(x) \\
= & \int_X |w_l(x)w_l(a_l^{-1}x) \cdots w_l(a_l^{-n_k+1}x)g_l(a_l^{-n_k}x)\chi_{E_k}(a_l^{-n_k}x)|d\mu(x) \\
= & \int_X |w_l(a_l^{n_k}x)w_l(a_l^{n_k-1}x) \cdots w_l(a_lx)g_l(x)\chi_{E_k}(x)|d\mu(x) \\
\leq & \|g_l\|_\infty^p \cdot \int_{E_k} \varphi^p_{l,n_k}(x)d\mu(x) \to 0
\end{aligned}$$

as $k \longrightarrow \infty$. Replacing g_l by f in the above estimate, $\lim_{k\longrightarrow\infty} \|T_l^{n_k}(f\chi_{E_k})\|_p^p = 0$.

Moreover, using the similar arguments to the iterates $S_l^{n_k}$ with the sequence $(\widetilde{\varphi}_{l,n_k})$, for $1 \leq l \leq N$,

$$\begin{aligned}
& \|S_l^{n_k}(g_l\chi_{E_k})\|_p^p \\
= & \int_X |S_l^{n_k}(g_l\chi_{E_k})(x)|^p d\mu(x) \\
= & \int_X \frac{1}{|w_l(a_l x)w_l(a_l^2 x)\cdots w_l(a_l^{n_k}x)|^p}|g_l(a_l^{n_k}x)\chi_{E_k}(a_l^{n_k}x)|^p d\mu(x) \\
= & \int_X \frac{1}{|w_l(a_l^{-n_k+1}x)w_l(a_l^{-n_k+2}x)\cdots w_l(x)|^p}|g_l(x)\chi_{E_k}(x)|^p d\mu(x) \\
\leq & \|g_l\|_\infty^p \cdot \int_{E_k} \widetilde{\varphi}_{l,n_k}^p(x)d\mu(x) \to 0
\end{aligned}$$

as $k \longrightarrow \infty$. Similarly, one has $\lim_{k\longrightarrow\infty}\|S_l^{n_k}(f\chi_{E_k})\|_p^p = 0$

In addition, for $1 \leq s, l \leq N$,

$$\begin{aligned}
& \|T_l^{n_k}(T_s^{n_k}g_s\chi_{E_k})\|_p^p \\
= & \int_X |w_l(x)w_l(a_l^{-1}x)\cdots w_l(a_l^{-(n_k-1)}x)|^p |T_s^{n_k}g_s\chi_{E_k}(a_l^{-n_k}x)|^p d\mu(x) \\
= & \int_X |w_l(x)w_l(a_l^{-1}x)\cdots w_l(a_l^{-(n_k-1)}x)|^p \cdot \\
& |w_s(a_l^{-n_k}x)w_s(a_s^{-1}a_l^{-n_k}x)\cdots w_s(a_s^{-(n_k-1)}a_l^{-n_k}x)|^p \cdot |g_s\chi_{E_k}(a_s^{-n_k}a_l^{-n_k}x)|^p d\mu(x) \\
= & \int_X |w_l(a_l^{n_k}a_s^{n_k}x)w_l(a_l^{n_k-1}a_s^{n_k}x)\cdots w_l(a_l a_s^{n_k}x)|^p \cdot \\
& |w_s(a_s^{n_k}x)w_s(a_s^{n_k-1}x)\cdots w_s(a_s x)| \cdot |g_s\chi_{E_k}(x)|d\mu(x) \\
= & \int_{E_k} \varphi_{l,n_k}^p(a_s^{n_k}x)\cdot\varphi_{s,n_k}^p(x)|g_s(x)|^p d\mu(x) \\
\leq & \|g_s\|_\infty^p \cdot \int_{E_k} \varphi_{l,n_k}^p(a_s^{n_k}x)\cdot\varphi_{s,n_k}^p(x)d\mu(x) \longrightarrow 0
\end{aligned}$$

as $k \longrightarrow \infty$, and

$$\begin{aligned}
& \|S_l^{n_k}(S_s^{n_k}g_s\chi_{E_k})\|_p^p \\
= & \int_X \frac{1}{|w_l(a_l x)w_l(a_l^2 x)\cdots w_l(a_l^{n_k}x)|^p}|S_s^{n_k}g_s\chi_{E_k}(a_l^{n_k}x)|^p d\mu(x) \\
= & \int_X \frac{1}{|w_l(a_l x)w_l(a_l^2 x)\cdots w_l(a_l^{n_k}x)|^p}\cdot \\
& \frac{1}{|w_s(a_s a_l^{n_k}x)w_s(a_s^2 a_l^{n_k}x)\cdots w_s(a_s^{n_k}a_l^{n_k}x)|}\cdot|g_s\chi_{E_k}(a_s^{n_k}a_l^{n_k}x)|^p d\mu(x) \\
= & \int_{E_k} \frac{1}{|w_l(a_l^{-(n_k-1)}a_s^{-n_k}x)w_l(a_l^{-(n_k-2)}a_s^{-n_k}x)\cdots w_l(a_s^{-n_k}x)|^p}\cdot \\
& \frac{1}{|w_s(xa_s^{-(n_k-1)})w_s(xa_s^{-(n_k-2)})\cdots w_s(x)|^p}\cdot|g_s(x)|^p d\mu(x) \\
= & \int_X \widetilde{\varphi}_{l,n_k}^p(a_s^{-n_k}x)\cdot\widetilde{\varphi}_{s,n_k}^p(x)|g_s\chi_{E_k}(x)|^p d\mu(x) \\
\leq & \|g_s\|_\infty^p \cdot \int_{E_k} \widetilde{\varphi}_{l,n_k}^p(a_s^{-n_k}x)\cdot\widetilde{\varphi}_{s,n_k}^p(x)d\mu(x) \longrightarrow 0
\end{aligned}$$

as $k \longrightarrow \infty$.

On the other hand, for $1 \leq s, l \leq N$ with $l \neq s$,

$$\begin{aligned}
& \|T_l^{n_k}(S_s^{n_k} g_s \chi_{E_k})\|_p^p \\
= & \int_X |w_l(x) w_l(a_l^{-1}x) \cdots w_l(a_l^{-(n_k-1)}x)|^p |S_s^{n_k} g_s \chi_{E_k}(a_l^{-n_k}x)|^p d\mu(x) \\
= & \int_X \frac{|w_l(x) w_l(a_l^{-1}x) \cdots w_l(a_l^{-(n_k-1)}x)|^p}{|w_s(a_s a_l^{-n_k}x) w_s(a_s^2 a_l^{-n_k}x) \cdots w_s(a_s^{n_k} a_l^{-n_k}x)|^p} \cdot |g_s \chi_{E_k}(a_s^{n_k} a_l^{-n_k}x)|^p d\mu(x) \\
= & \int_{E_k} \frac{|w_l(a_l^{n_k} a_s^{-n_k}x) w_l(a_l^{n_k-1} a_s^{-n_k}x) \cdots w_l(a_l a_s^{-n_k}x)|^p}{|w_s(a_s^{-n_k+1}x) w_s(a_s^{-n_k+2}x) \cdots w_s(x)|^p} \cdot |g_s(x)|^p d\mu(x) \\
= & \int_X \varphi_{l,n_k}^p(a_s^{-n_k}x) \cdot \widetilde{\varphi}_{s,n_k}^p(x) |g_s \chi_{E_k}(x)|^p d\mu(x) \\
\leq & \|g_s\|_\infty^p \cdot \int_{E_k} \varphi_{l,n_k}^p(a_s^{-n_k}x) \cdot \widetilde{\varphi}_{s,n_k}^p(x) d\mu(x) \longrightarrow 0
\end{aligned}$$

as $k \longrightarrow \infty$, and

$$\begin{aligned}
& \|S_l^{n_k}(T_s^{n_k} g_s \chi_{E_k})\|_p^p \\
= & \int_X \frac{1}{|w_l(a_l x) w_l(a_l^2 x) \cdots w_l(a_l^{n_k}x)|^p} |T_s^{n_k} g_s \chi_{E_k}(a_l^{n_k}x)|^p d\mu(x) \\
= & \int_X \frac{|w_s(a_l^{n_k}x) w_s(a_s^{-1} a_l^{n_k}x) \cdots w_s(a_s^{-(n_k-1)} a_l^{n_k}x)|^p}{|w_l(a_l x) w_l(a_l^2 x) \cdots w_l(a_l^{n_k}x)|^p} \cdot |g_s \chi_{E_k}(a_s^{-n_k} a_l^{n_k}x)|^p d\mu(x) \\
= & \int_X \frac{|w_s(a_s^{n_k}x) w_s(a_s^{n_k-1}x) \cdots w_s(a_s x)|^p}{|w_l(a_l^{-n_k+1} a_s^{n_k}x) w_l(a_l^{-n_k+2} a_s^{n_k}x) \cdots w_l(a_s^{n_k}x)|^p} \cdot |g_s \chi_{E_k}(x)|^p d\mu(x) \\
= & \int_{E_k} \widetilde{\varphi}_{l,n_k}^p(a_s^{n_k}x) \cdot \varphi_{s,n_k}^p(x) |g_s(x)|^p d\mu(x) \\
\leq & \|g_s\|_\infty^p \cdot \int_{E_k} \widetilde{\varphi}_{l,n_k}^p(a_s^{n_k}x) \cdot \varphi_{s,n_k}^p(x) d\mu(x) \longrightarrow 0
\end{aligned}$$

as $k \longrightarrow \infty$.

Combining all the estimates above, we are ready to achieve our goal. Indeed, for each $k \in \mathbb{N}$, let

$$\begin{aligned}
v_k = f\chi_{E_k} \quad & + \quad T_1^{n_k}(g_1 \chi_{E_k}) + T_2^{n_k}(g_2 \chi_{E_k}) + \cdots + T_N^{n_k}(g_N \chi_{E_k}) \\
& + \quad S_1^{n_k}(g_1 \chi_{E_k}) + S_2^{n_k}(g_2 \chi_{E_k}) + \cdots + S_N^{n_k}(g_N \chi_{E_k}).
\end{aligned}$$

Then

$$\|v_k - f\|_p \leq \|f\chi_{E_k} - f\|_p + \sum_{l=1}^N \|T_l^{n_k}(g_l \chi_{E_k})\|_p + \sum_{l=1}^N \|S_l^{n_k}(g_l \chi_{E_k})\|_p$$

and

$$\begin{aligned}
& \|2C_{l,n_k} v_k - 2g_l\|_p \\
\leq & \|T_l^{n_k}(f\chi_{E_k})\|_p + \sum_{s=1}^N \|T_l^{n_k} T_s^{n_k}(g_s \chi_{E_k})\|_p + \|g_l \chi_{E_k} - g_l\|_p + \sum_{s=1, s\neq l}^N \|T_l^{n_k} S_s^{n_k}(g_s \chi_{E_k})\|_p \\
& + \|S_l^{n_k}(f\chi_{E_k})\|_p + \sum_{s=1, s\neq l}^N \|S_l^{n_k} T_s^{n_k}(g_s \chi_{E_k})\|_p + \|g_l \chi_{E_k} - g_l\|_p + \sum_{s=1}^N \|S_l^{n_k} S_s^{n_k}(g_s \chi_{E_k})\|_p.
\end{aligned}$$

Hence $\lim_{k\to\infty} v_k = f$ and $\lim_{k\to\infty} C_{l,n_k} v_k = g_l$ for $l = 1, 2, \cdots, N$. Therefore we obtain

$$\emptyset \neq U \cap C_{1,n_k}^{-1}(V_1) \cap C_{2,n_k}^{-1}(V_2) \cap \cdots \cap C_{N,n_k}^{-1}(V_N).$$

□

By strengthening the condition (ii) of Theorem 3.1, the characterization of disjoint topological mixing for $(C_{1,n})_{n\in\mathbb{N}}, (C_{2,n})_{n\in\mathbb{N}}, \cdots, (C_{N,n})_{n\in\mathbb{N}}$ is given below.

Corollary 3.2. *Let G act on (X, μ). Given some $N \geq 2$, let $a_l \in G$ and let w_l be a weight on X for $1 \leq l \leq N$. Let $1 \leq p < \infty$, and let $T_l := T_{a_l,w_l}$ be a weighted translation operator on $L^p(X, \mu)$ with its inverse $S_l := S_{a_l,w_l}$. Let $C_{l,n} = \frac{1}{2}(T_l^n + S_l^n)$. Then* (ii) $\Rightarrow$ (i).

(i) *$(C_{1,n})_{n\in\mathbb{N}}, (C_{2,n})_{n\in\mathbb{N}}, \cdots, (C_{N,n})_{n\in\mathbb{N}}$ are disjoint topologically mixing on $L^p(X, \mu)$.*

(ii) For each compact subset K of X with $\mu(K) > 0$, there exists a sequence of Borel sets (E_k) in K such that $\lim_{n\longrightarrow\infty} \mu(E_n) = \mu(K)$
and (for $1 \leq l \leq N$)

$$\lim_{n\to\infty} \int_{E_n} \varphi_{l,n}^p(x) d\mu(x) = 0,$$

$$\lim_{n\to\infty} \int_{E_n} \widetilde{\varphi}_{l,n}^p(x) d\mu(x) = 0$$

and (for $1 \leq l, s \leq N$,)

$$\lim_{n\to\infty} \int_{E_n} \varphi_{l,n}^p(a_s^n x) \cdot \varphi_{s,n}^p(x) d\mu(x) = 0,$$

$$\lim_{n\to\infty} \int_{E_n} \widetilde{\varphi}_{l,n}^p(a_s^{-n} x) \cdot \widetilde{\varphi}_{s,n}^p(x) d\mu(x) = 0,$$

and (for $1 \leq l, s \leq N$ with $l \neq s$,)

$$\lim_{n\to\infty} \int_{E_n} \varphi_{l,n}^p(a_s^{-n} x) \cdot \widetilde{\varphi}_{s,n}^p(x) d\mu(x) = 0,$$

$$\lim_{n\to\infty} \int_{E_n} \widetilde{\varphi}_{l,n}^p(a_s^n x) \cdot \varphi_{s,n}^p(x) d\mu(x) = 0.$$

Example 3.3. Let $X = G = \mathbb{R}$ and let $a_1 = 2$ and $a_2 = 5$. Let $w_1 = w_2 := w$ be a weight on $\mathbb{R}$. Then $T_{2,w}$ and $T_{5,w}$ are given by

$$T_{2,w} f(x) := w(x) f(-2 + x) \quad \text{and} \quad T_{2,w} f(x) := w(x) f(-5 + x)$$

where $f \in L^p(\mathbb{R})$ and $x \in \mathbb{R}$. Let

$$C_{1,n} = \frac{1}{2}\left(T_{2.w}^n + S_{2,w}^n\right) \quad \text{and} \quad C_{2,n} = \frac{1}{2}\left(T_{5.w}^n + S_{5,w}^n\right)$$

where $S_{2,w}$ and $S_{5,w}$ are the inverses of $T_{2,w}$ and $T_{5,w}$ respectively. Then $(C_{1,n})_{n\in\mathbb{N}}, (C_{2,n})_{n\in\mathbb{N}}$ are disjoint topologically mixing if given each compact subset K of $\mathbb{R}$, there exists a sequence of Borel sets (E_k) in K such that $\lim_{n\longrightarrow\infty} \mu(E_n) = \mu(K)$
and (for $1 \leq l \leq 2$)

$$\lim_{n\to\infty} \int_{E_n} \varphi_{l,n}^p(x) dx = 0,$$

$$\lim_{n\to\infty} \int_{E_n} \widetilde{\varphi}_{l,n}^p(x) dx = 0$$

and (for $1 \leq l, s \leq 2$,)

$$\lim_{n\to\infty} \int_{E_n} \varphi_{l,n}^p(a_s n + x) \cdot \varphi_{s,n}^p(x) dx = 0,$$

$$\lim_{n\to\infty}\int_{E_n}\widetilde{\varphi}^p_{l,n}(-a_sn+x)\cdot\widetilde{\varphi}^p_{s,n}(x)dx=0,$$

and (for $1\le l,s\le 2$ with $l\neq s$,)

$$\lim_{n\to\infty}\int_{E_n}\varphi^p_{l,n}(-a_sn+x)\cdot\widetilde{\varphi}^p_{s,n}(x)dx=0,$$

$$\lim_{n\to\infty}\int_{E_n}\widetilde{\varphi}^p_{l,n}(a_sn+x)\cdot\varphi^p_{s,n}(x)dx=0.$$

4. Conclusion

In this paper, we mainly tackle the notions of topologically multiple recurrence, disjoint topological transitivity, and disjoint topological mixing for weighted translation operators and a sequence of operators, generated by group actions. All these results provide a class of examples including single operators and sequences of operators, which possess these dynamical properties under some conditions. This note generalizes and subsumes some previous work on this direction.

Statements and Declarations

The authors declare that they have no conflict of interest, and the manuscript has no associated data.

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References

[1] I. Akbarbaglu, M. R. Azimi, and V. Kumar. Topologically transitive sequence of cosine operators on Orlicz spaces. *Annals of Functional Analysis*, 12:1-14, 2021.
[2] F. Bayart and É. Matheron. *Dynamics of Linear Operators*, Cambridge Tracts in Mathematics 179. Cambridge University Press, Cambridge, 2009.
[3] L. Bernal-González. Disjoint hypercyclic operators. *Studia Mathematica*, 182:113-131, 2007.
[4] L. Bernal-González. Algebraic genericity of strict-order integrability. *Studia Mathematica*, 199:279-293, 2010.
[5] J. Bès, Ö. Martin, A. Peris, and S. Shkarin. Disjoint mixing operators. *Journal of Functional Analysis*, 263:1283-1322, 2012.
[6] J. Bès, Ö. Martin, and R. Sanders. Weighted shifts and disjoint hypercyclicity. *Journal of Operator Theory*, 72:15-40, 2014.
[7] J. Bès and A. Peris. Disjointness in hypercyclicity. *Journal of Mathematical Analysis and Applications*, 336:297-315, 2007.
[8] A. Bonilla and P. J. Miana. Hypercyclic and topologically mixing cosine functions on Banach spaces. *Proceedings of the American Mathematical Society*, 136:519-528, 2008.
[9] C.-C. Chen. Recurrence of cosine operator functions on groups. *Canadian Mathematical Bulletin*, 59:693-704, 2016.
[10] C.-C. Chen and C-H. Chu. Hypercyclic weighted translations on groups. *Proceedings of the American Mathematical Society*, 139:2839-2846, 2011.
[11] C.-C. Chen, S. M. Tabatabaie, and A. Mohammadi. Dynamics of weighted translations generated by group actions. *Filomat*, 34:2131-2139, 2020.
[12] C.-C. Chen, S. M. Tabatabaie, and A. Mohammadi. Chaotic cosine operator functions generated by group actions. *Iranian Journal of Science and Technology, Transaction A: Science*, 44:465-472, 2020.
[13] C.-C. Chen, S. M. Tabatabaie, and A. Mohammadi. Disjoint topological transitivity for weighted translations generated by group actions. *Mathematica Slovaca*, 71:1229-1240, 2021.
[14] G. B. Folland. *A Course in Abstract Harmonic Analysis*. CRC Press, New York, 1995.
[15] G. Costakis and I. Parissis. Szemerèdi's theorem, frequent hypercyclicity and multiple recurrence. *Mathematica Scandinavica*, 110:251-272, 2012.
[16] K.-G. Grosse-Erdmann and A. Peris. *Linear Chaos*, Universitext, Springer, 2011.
[17] T. Kalmes. Hypercyclicity and mixing for cosine operator functions generated by second order partial differential operators. *Journal of Mathematical Analysis and Applications*, 365:363-375, 2010.
[18] H. Salas. Dual disjoint hypercyclic operators. *Journal of Mathematical Analysis and Applications*, 374:106-117, 2011.

[19] H. Salas. The strong disjoint blow-up/collapse property. *Journal of Function Spaces and Applications*, 2013:Article ID 146517, 2013.
[20] V. Vlachou. Disjoint universality for families of Taylor-type operators. *Journal of Mathematical Analysis and Applications*, 448:1318-1330, 2017.