APPLICABLE NONLINEAR ANALYSIS
Volume 4 (2027), No. 1, 67–88
DOI: https://doi.org/10.69829/apna-027-0401-ta05
Tulipa Opera Scholarum

QUASILINEAR PARABOLIC EQUATIONS IN DIVERGENCE FORM WITH SINGULAR COEFFICIENTS: THE p-GRADIENT-FORM-BOUNDEDNESS APPROACH

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Abstract. We study the initial-boundary value problem for a class of quasilinear parabolic equations in divergence form with natural growth of the lower-order term and singular coefficients. The structural assumptions are those of the Ladyzhenskaya–Solonnikov–Uraltseva pk-class, except that the classical anisotropic integrability condition on the gradient coefficient is replaced by a *p-gradient-form-boundedness* condition tailored to the energy structure. This condition contains the classical L^r case ($r > n+2$) as a special instance, while also allowing coefficients with weak-L^{n+2} singularities.

Under these assumptions we give a self-contained proof of local and global a priori estimates: a Caccioppoli-type inequality, local boundedness via De Giorgi–Ladyzhenskaya iteration, and a global energy bound under a smallness condition. Existence of weak solutions is obtained by the Galerkin method combined with the Minty–Browder technique for strong convergence of gradients. Maximum and comparison principles are established, and the Hölder continuity of bounded weak solutions is proved both in the interior and up to the boundary for Lipschitz cylinders. The results hold on arbitrary bounded space-time cylinders without any regularity hypothesis on the boundary.

The paper systematically develops the p-gradient-form-boundedness framework and shows that the essential tools of the De Giorgi–Ladyzhenskaya theory remain valid under this weaker condition.

Keywords. Quasilinear parabolic equation, divergence form, natural growth, gradient-form- boundedness, De Giorgi iteration, Hölder continuity, singular coefficients, maximum principle, existence of weak solutions.

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1. Introduction

The study of second-order quasilinear parabolic equations in divergence form,

$$\partial_t u - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} a_i(x,t,u,\nabla u) + b(x,t,u,\nabla u) = f, \tag{1.1}$$

stands at the intersection of nonlinear analysis, geometric flows, and the modelling of diffusion processes with singular or degenerate coefficients. The classical treatise by Ladyzhenskaya, Solonnikov and Uraltseva [1] (hereafter LSU) provides a complete existence, uniqueness and regularity theory for such equations under the structural conditions of the pk-class: the principal part exhibits polynomial growth of order $p-1$ in the gradient, while the lower-order term b satisfies natural growth of order p. The gradient-dependent part of b is controlled by a measurable function μ_1 belonging to the anisotropic Lebesgue space $L^{q,r}(Q_T)$ with $\frac{n}{q} + \frac{2}{r} < 1$. This sharp integrability condition, combined with the parabolic Sobolev embedding, permits the singular term $\iint \mu_1 |\nabla u|^{p-1}|u|$ to be absorbed into

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2020 Mathematics Subject Classification: 35K59, 35B45, 35B65, 35K20, 35D30, 35K67.
Accepted: July 26, 2026.

the coercivity of the principal part via Young's inequality, thus enabling the De Giorgi–Ladyzhenskaya iterative machinery.

However, in a wide range of physically motivated problems the coefficient μ_1 exhibits singularities that are stronger than those permitted by the LSU condition. Examples arise in fluid dynamics (Navier–Stokes equations with rough forcing), image processing (total variation flow with irregular texture), and biological aggregation models, where μ_1 may belong to a weak Lebesgue space $L^{n+2,\infty}$ or even be a measure not absolutely continuous with respect to the Lebesgue measure. Such singularities are excluded by the classical LSU framework, yet they are precisely the ones that appear naturally when one regularises or approximates discontinuous coefficients.

A major breakthrough for linear elliptic and parabolic equations was achieved by Mazya and Verbitsky [5], who introduced the concept of *form-boundedness*. A vector field b is called form-bounded with respect to the Dirichlet form $\int |\nabla h|^2$ if for every $\beta > 0$ there exists $c(\beta)$ such that

$$\int |b|^2 h^2 \le \beta \int |\nabla h|^2 + c(\beta) \int h^2, \qquad \forall\, h \in H^1.$$

This condition, strictly weaker than $b \in L^n$ (or L^{n+2} in the parabolic case), suffices to absorb the singular drift $\int b \cdot \nabla u\, u$ via the Cauchy–Schwarz inequality. Subsequently, Kovalenko, Semenov and Liskevich [6, 7, 9] used form-boundedness to construct strongly continuous semigroups with Gaussian upper bounds for the corresponding heat kernels, while Sturm [8] proved Harnack inequalities for parabolic operators with singular lower-order terms. These works demonstrated that form-boundedness is the natural substitute for Lebesgue integrability when dealing with rough coefficients.

For quasilinear equations with p-growth ($p \ge 2$), the problem of extending the form-boundedness philosophy is both timely and nontrivial. Recent years have witnessed an intense activity on singular parabolic equations of p-Laplacian type: Kinzebulatov [15] analysed the Laplacian with singular drift in a critical borderline case via form-boundedness techniques; Zhang [16] established fundamental regularity results for parabolic equations with a singular lower-order term; Chen and DiBenedetto [17] obtained Hölder estimates for solutions of singular parabolic equations with measurable coefficients; Liao [18] studied parabolic De Giorgi classes and their regularity properties; and Park and Shin [19] derived sharp regularity estimates for singular parabolic measure data problems. All these contributions indicate a growing need for a unified theory that captures singularities beyond the classical LSU regime while preserving the essential tools of the De Giorgi–Ladyzhenskaya iteration.

In this paper we introduce the *p-gradient-form-boundedness* condition (Definition 2.1):

$$\iint_{Q_T} \mu_1^p\, |u|^p \,\mathrm{d}x\,\mathrm{d}t \le \varepsilon \iint_{Q_T} |\nabla u|^p \,\mathrm{d}x\,\mathrm{d}t + C_\varepsilon \iint_{Q_T} |u|^p \,\mathrm{d}x\,\mathrm{d}t \qquad \forall\, u \in L^p(0,T;W_0^{1,p}(\Omega)).$$

This condition is the precise p-energy analogue of the quadratic form-boundedness of Mazya and Verbitsky. It is exactly tailored to absorb the singular term $\iint \mu_1 |\nabla u|^{p-1}|u|$ via Hölder's inequality and the basic energy structure of the equation. We prove (Lemma 2.3) that the classical LSU integrability $\mu_1 \in L^r(Q_T)$, $r > n+2$, implies this condition, so that our framework strictly generalises the classical one. Moreover, functions with μ_1^p in the weak Lebesgue space $L^{(n+2)/p,\infty}(Q_T)$ also satisfy it, thereby covering borderline singularities that are inaccessible by the LSU theory.

The main contribution of the present work is to demonstrate that the entire De Giorgi–Ladyzhenskaya programme for quasilinear parabolic equations — global energy estimates, Caccioppoli inequalities, local boundedness via De Giorgi iteration, existence via the Galerkin method, maximum and comparison principles, and interior Hölder continuity — can be carried out under the sole assumption of p-gradient-form-boundedness. Our proofs are self-contained and rely only on elementary inequalities and the parabolic Sobolev embedding; no sophisticated harmonic analysis or capacity theory is required. The results hold for arbitrary bounded space-time cylinders, without any regularity hypothesis

on the boundary, so that existence and interior regularity are established on arbitrary bounded open sets. Boundary Hölder continuity is proved for Lipschitz domains.

The importance of this advance is threefold. First, it demonstrates that the powerful machinery developed by Ladyzhenskaya, Solonnikov and Uraltseva can be substantially weakened in the assumptions while retaining its full strength — a fact that was known for linear equations but remained open for the quasilinear p-case. Second, the new condition is natural from the point of view of energy estimates and is robust under perturbations, making it suitable for the analysis of more complex systems (e.g. coupled chemotaxis models, non-Newtonian flows with rough data). Third, by bridging the classical LSU theory and the modern form-boundedness approach, we provide a unifying framework that simplifies and extends recent works that required ad hoc integrability hypotheses.

The paper is organised as follows. Section 2 introduces notation, the function spaces, Steklov averages, the p-gradient-form-boundedness condition, and the proof that it is implied by $\mu_1 \in L^r$ with $r > n+2$. Section 3 states the initial-boundary value problem and the structural assumptions. Section 4 contains the a priori estimates: global energy, Caccioppoli inequality, and local boundedness. The proofs of the Caccioppoli inequality and the De Giorgi iteration are given in full detail. In Section 5 we prove existence of weak solutions by the Galerkin method, together with the compactness and the Minty–Browder argument. Section 6 establishes a weak maximum principle and a comparison principle. Section 7 contains the regularity theory: the logarithmic lemma, the oscillation decay proposition, and the Hölder continuity. Section 8 discusses boundary regularity and the problem on rough domains. An appendix collects the standard auxiliary lemmas.

2. Notation and Functional Spaces

2.1. **Basic notation.** Let $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) be a bounded open set and $T > 0$. We denote by Q_T the space-time cylinder $\Omega \times (0,T)$. For a measurable function $u : Q_T \to \mathbb{R}$, we write $u(t)$ for the mapping $x \mapsto u(x,t)$. The parabolic boundary of Q_T is $\partial_p Q_T = (\overline{\Omega} \times \{0\}) \cup (\partial\Omega \times [0,T])$.

For $p \geq 2$, $L^p(Q_T)$ is the usual Lebesgue space with norm $\|u\|_{L^p(Q_T)} = (\iint_{Q_T} |u|^p \,\mathrm{d}x\,\mathrm{d}t)^{1/p}$. The Sobolev space $W^{1,p}(\Omega)$ is defined in the standard way, and $W^{1,p}_0(\Omega)$ is the closure of $C^\infty_c(\Omega)$ in the $W^{1,p}$ norm. Its dual is denoted $W^{-1,p'}(\Omega)$ with $p' = p/(p-1)$.

The energy space for the Cauchy–Dirichlet problem is

$$V_p(Q_T) := C([0,T]; L^2(\Omega)) \cap L^p(0,T; W^{1,p}_0(\Omega)),$$

equipped with the norm

$$\|u\|_{V_p(Q_T)} := \sup_{0\leq t\leq T} \|u(\cdot,t)\|_{L^2(\Omega)} + \|\nabla u\|_{L^p(Q_T)}.$$

2.2. **Steklov averaging.** To handle the time derivative rigorously, we use Steklov averages. For $v \in L^1(Q_T)$ and $h > 0$ define

$$v_h(x,t) := \frac{1}{h}\int_t^{t+h} v(x,s)\,\mathrm{d}s, \qquad (x,t) \in \Omega \times (0, T-h).$$

The following properties are classical (see [1, Chapter II, §4]):

(1) If $v \in L^p(Q_T)$, then $v_h \to v$ in $L^p(Q_{T-\delta})$ for every $\delta > 0$ as $h \to 0$.
(2) If $v \in C([0,T]; L^2(\Omega))$, then $v_h(\cdot,t) \to v(\cdot,t)$ in $L^2(\Omega)$ uniformly in t.
(3) For $u \in V_p(Q_T)$, the Steklov average u_h belongs to $L^\infty(0,T-h; W^{1,p}_0(\Omega))$, and its time derivative exists as $\partial_t u_h(x,t) = \frac{1}{h}(u(x,t+h) - u(x,t)) \in L^\infty(0,T-h; L^2(\Omega))$.

In the weak formulation, we will first test with Steklov averages and then pass to the limit $h \to 0$.

Smooth test functions are taken from

$$\mathcal{D}(Q_T) := \big\{\varphi \in C^\infty(\overline{Q_T}) : \varphi = 0 \text{ on } \partial\Omega \times [0,T],\ \varphi(\cdot,T) = 0\big\}.$$

2.3. **Intrinsic cylinders.** The equation has a natural scaling: if u solves $\partial_t u - \Delta_p u = 0$, then $u_\lambda(x,t) = u(\lambda x, \lambda^p t)$ is also a solution. Therefore we work with intrinsic cylinders

$$Q_\rho(x_0,t_0) := B_\rho(x_0) \times (t_0 - \rho^p, t_0),$$

where $B_\rho(x_0)$ is the Euclidean ball of radius ρ. When the centre is clear we simply write Q_ρ. All cylinders are assumed to satisfy $Q_{2R} \Subset Q_T$, i.e. the closure is contained in Q_T.

2.4. **The p-gradient-form-boundedness condition.**

Definition 2.1. A measurable function $\mu_1 \geq 0$ on Q_T is called *p-gradient-form-bounded* if for every $\varepsilon > 0$ there exists a constant $C_\varepsilon \geq 0$ such that

$$\iint_{Q_T} \mu_1^p(x,t)\,|u|^p \,\mathrm{d}x\,\mathrm{d}t \leq \varepsilon \iint_{Q_T} |\nabla u|^p \,\mathrm{d}x\,\mathrm{d}t + C_\varepsilon \iint_{Q_T} |u|^p \,\mathrm{d}x\,\mathrm{d}t \qquad \forall\, u \in L^p(0,T;W_0^{1,p}(\Omega)). \tag{2.1}$$

The class of such functions is denoted by $\mathcal{F}_p(Q_T)$.

Remark 2.2. Inequality (2.1) is homogeneous of degree p. By Hölder's inequality,

$$\iint \mu_1 |\nabla u|^{p-1}|u| \leq \Big(\iint |\nabla u|^p\Big)^{\frac{p-1}{p}} \Big(\iint \mu_1^p |u|^p\Big)^{\frac{1}{p}}.$$

Thus (2.1) directly implies the bound on the singular energy term

$$\iint \mu_1 |\nabla u|^{p-1}|u| \leq \varepsilon \iint |\nabla u|^p + C_\varepsilon \iint |u|^p,$$

which is the exact form needed in the energy estimates.

2.5. **From L^r to p-gradient-form-boundedness.** The following lemma shows that the classical LSU condition implies our condition.

Lemma 2.3 (Splitting Lemma)**.** *Let $2 \leq p < n+2$ and assume $\mu_1 \in L^r(Q_T)$ with $r > n+2$. Then $\mu_1 \in \mathcal{F}_p(Q_T)$.*

Proof. Since Q_T has finite measure, $L^r(Q_T) \subset L^{n+2}(Q_T)$ for $r > n+2$. Thus $\mu_1^p \in L^{(n+2)/p}(Q_T)$. Given $\varepsilon > 0$, we decompose μ_1^p into a bounded part and a part with small $L^{(n+2)/p}$ norm. Concretely, for any $\eta > 0$ set

$$\nu_1 = \min\{\mu_1^p, M\}, \qquad \nu_2 = \mu_1^p - \nu_1,$$

where M is chosen large enough so that $\|\nu_2\|_{L^{(n+2)/p}(Q_T)} \leq \eta$. Such M exists because $\mu_1^p \in L^{(n+2)/p}(Q_T)$ and one can use the absolute continuity of the integral. Then $\nu_1 \in L^\infty(Q_T)$ with $\|\nu_1\|_{L^\infty} \leq M$.

Take any $u \in L^p(0,T;W_0^{1,p}(\Omega))$. We have

$$\iint \mu_1^p |u|^p = \iint \nu_1 |u|^p + \iint \nu_2 |u|^p \leq M \iint |u|^p + \iint \nu_2 |u|^p.$$

For the second term, apply Hölder's inequality with exponents $\frac{n+2}{p}$ and $\frac{p^*}{p}$ (where $p^* = \frac{p(n+2)}{n+2-p}$ is the parabolic Sobolev exponent):

$$\iint \nu_2 |u|^p \leq \|\nu_2\|_{L^{(n+2)/p}(Q_T)} \Big(\iint |u|^{p^*}\Big)^{p/p^*}.$$

By the parabolic Sobolev embedding (Theorem 2.5) and the Poincaré inequality, $\big(\iint |u|^{p^*}\big)^{p/p^*} \leq C_S^p \iint |\nabla u|^p$. Thus

$$\iint \mu_1^p |u|^p \leq M \iint |u|^p + \eta C_S^p \iint |\nabla u|^p.$$

Now choose $\eta = \varepsilon / C_S^p$ and set $C_\varepsilon = M$. This yields (2.1). □

Remark 2.4. The same proof works for μ_1^p in the weak Lebesgue space $L^{(n+2)/p,\infty}(Q_T)$, which is strictly larger than $L^{(n+2)/p}$. Thus our condition is strictly weaker than the classical L^r assumption.

2.6. **Parabolic Sobolev embedding.** We recall the following well-known result (see [1, Chapter II, §3], [2]).

Theorem 2.5 (Parabolic Sobolev embedding). *Let $2 \le p < n+2$ and define $p^* = \frac{p(n+2)}{n+2-p}$. There exists a constant C_S depending on n, p, Ω, T such that for every $u \in L^p(0,T; W_0^{1,p}(\Omega))$,*

$$\|u\|_{L^{p^*}(Q_T)} \le C_S\big(\|\nabla u\|_{L^p(Q_T)} + \|u\|_{L^p(Q_T)}\big).$$

If u additionally vanishes on $\partial\Omega \times (0,T)$ in the sense of traces, the Poincaré inequality yields the pure gradient form $\|u\|_{L^{p^*}(Q_T)} \le C_S' \|\nabla u\|_{L^p(Q_T)}$.

3. The Quasilinear Parabolic Equation and Structural Assumptions

3.1. **Statement of the problem.** We study the initial-boundary value problem

$$\partial_t u - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} a_i(x,t,u,\nabla u) + b(x,t,u,\nabla u) = f \quad \text{in } Q_T = \Omega \times (0,T), \tag{3.1}$$

with homogeneous Dirichlet condition

$$u = 0 \quad \text{on } \partial\Omega \times (0,T), \tag{3.2}$$

and initial condition

$$u(x,0) = u_0(x) \quad \text{a.e. in } \Omega. \tag{3.3}$$

The data are $f \in L^{p'}(Q_T)$, $u_0 \in L^2(\Omega)$, and the coefficients $a_i, b : Q_T \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ are Carathéodory functions (measurable in (x,t) for all (u,ξ) and continuous in (u,ξ) for a.e. (x,t)).

3.2. **Structural assumptions.**

Assumption 3.1 (Principal part). There exist constants $\nu > 0$, $\mu \ge 0$, $c_1 \ge 0$ and non-negative functions $\varphi_1, \varphi_2 \in L^1(Q_T)$ such that for a.e. $(x,t) \in Q_T$ and all $u \in \mathbb{R}$, $\xi \in \mathbb{R}^n$:

(A1) (Coercivity) $\displaystyle\sum_{i=1}^{n} a_i(x,t,u,\xi)\,\xi_i \ge \nu|\xi|^p - \varphi_1(x,t) - c_1|u|^p$.

(A2) (Growth) $|a_i(x,t,u,\xi)| \le \mu|\xi|^{p-1} + \varphi_2(x,t) + c_1|u|^{p-1}$, $i = 1,\dots,n$.

(A3) (Monotonicity) There exists $\nu_0 > 0$ such that for all $\xi, \eta \in \mathbb{R}^n$,

$$\sum_{i=1}^{n} \big(a_i(x,t,u,\xi) - a_i(x,t,u,\eta)\big)(\xi_i - \eta_i) \ge \nu_0|\xi - \eta|^p.$$

Assumption 3.2 (Lower-order term). There exist constants $c_2 \ge 0$, a non-negative function $\varphi_3 \in L^1(Q_T)$ and a p-gradient-form-bounded function $\mu_1 \ge 0$ on Q_T (i.e. $\mu_1 \in \mathcal{F}_p(Q_T)$) such that for a.e. $(x,t) \in Q_T$ and all $u \in \mathbb{R}$, $\xi \in \mathbb{R}^n$:

(B1) (Sign condition) $b(x,t,u,\xi)\,u \ge -\mu_1(x,t)\,|\xi|^{p-1}|u| - \varphi_3(x,t)$.

(B2) (Growth) $|b(x,t,u,\xi)| \le \mu_1(x,t)\,|\xi|^{p-1} + \varphi_3(x,t) + c_2|u|^{p-1}$.

Remark 3.3. Condition (A3) is used only for the existence proof (strong convergence of gradients); it is satisfied by the p-Laplacian $a_i(\xi) = |\xi|^{p-2}\xi_i$ and many other models. The p-gradient-form-boundedness of μ_1 is our main new condition. By Lemma 2.3, it is strictly weaker than the classical LSU condition, while still guaranteeing that all integrals in the weak formulation are finite (see the justification after Definition 3.4).

3.3. **Weak formulation.** Due to the natural growth of b, the integral $\iint b\varphi$ may diverge for general $\varphi \in L^p(0,T;W_0^{1,p}(\Omega))$ unless φ is bounded. We therefore use bounded test functions from $\mathcal{D}(Q_T)$.

Definition 3.4 (Weak solution)**.** A function $u \in V_p(Q_T)$ is called a *weak solution* of problem (3.1)–(3.3) if $u(x,0) = u_0(x)$ a.e. in Ω and for every test function $\varphi \in \mathcal{D}(Q_T)$,

$$\begin{aligned}\iint_{Q_T} \Big(-u\,\partial_t\varphi + \sum_{i=1}^{n} a_i(x,t,u,\nabla u)\,\partial_i\varphi + b(x,t,u,\nabla u)\,\varphi\Big)\,\mathrm{d}x\,\mathrm{d}t \\ = \iint_{Q_T} f\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega u_0(x)\,\varphi(x,0)\,\mathrm{d}x.\end{aligned} \tag{3.4}$$

The absolute convergence of the integrals is justified as follows. The term $\iint u\,\partial_t\varphi$ is finite because $u \in L^2(Q_T)$. For the principal part, (A2) and the fact that $\nabla u \in L^p$, $\varphi \in C^\infty$ give integrability. The lower-order term is the delicate one. By (B2),

$$|b(x,t,u,\nabla u)\,\varphi| \le \mu_1|\nabla u|^{p-1}|\varphi| + \varphi_3|\varphi| + c_2|u|^{p-1}|\varphi|.$$

The last two terms are integrable. For the singular part we use Hölder's inequality:

$$\iint \mu_1|\nabla u|^{p-1}|\varphi| \le \Big(\iint |\nabla u|^p\Big)^{\frac{p-1}{p}} \Big(\iint \mu_1^p|\varphi|^p\Big)^{\frac{1}{p}}.$$

Since $\varphi \in \mathcal{D}(Q_T) \subset L^p(0,T;W_0^{1,p}(\Omega))$ (the smooth function vanishes on the lateral boundary and at $t=T$, so it belongs to the energy space), we can apply Definition 2.1 with $u=\varphi$ to obtain

$$\iint \mu_1^p|\varphi|^p \le \varepsilon \iint |\nabla\varphi|^p + C_\varepsilon \iint |\varphi|^p < \infty.$$

Thus $\iint \mu_1|\nabla u|^{p-1}|\varphi|$ is finite. This argument uses only the p-gradient-form-boundedness of μ_1 and the smoothness of φ; no additional integrability condition on μ_1 is required.

4. A Priori Estimates

Throughout this section, u denotes a weak solution. The constants C may change from line to line but depend only on the structural data.

4.1. Global energy estimate.

Lemma 4.1 (Energy estimate)**.** *For a.e. $\tau \in (0,T)$,*

$$\frac{1}{2}\int_\Omega u^2(\tau)\,\mathrm{d}x + \frac{\nu}{2}\iint_{Q_\tau} |\nabla u|^p \le \frac{1}{2}\|u_0\|_{L^2}^2 + C_*\iint_{Q_\tau} |u|^p + \Phi + \frac{1}{2}\iint_{Q_\tau} |f|^{p'}, \tag{4.1}$$

where $\Phi = \|\varphi_1+\varphi_3\|_{L^1(Q_T)}$ and $C_* = C_{\nu/2} + c_1 + \frac{1}{2}$, with $C_{\nu/2}$ from Definition 2.1 applied to the singular term via Hölder's inequality.

Proof. Using Steklov averages and passing to the limit (as justified in LSU), we obtain for a.e. $\tau \in (0,T)$

$$\frac{1}{2}\int_\Omega u^2(\tau)\,\mathrm{d}x + \iint_{Q_\tau} \sum a_i(x,t,u,\nabla u)\partial_i u + \iint_{Q_\tau} b(x,t,u,\nabla u)\,u = \frac{1}{2}\int_\Omega u_0^2\,\mathrm{d}x + \iint_{Q_\tau} fu.$$

By (A1) and (B1),

$$\sum a_i\partial_i u \ge \nu|\nabla u|^p - \varphi_1 - c_1|u|^p, \qquad b\,u \ge -\mu_1|\nabla u|^{p-1}|u| - \varphi_3.$$

Thus

$$\frac{1}{2}\int_\Omega u^2(\tau)\,\mathrm{d}x + \nu\iint_{Q_\tau}|\nabla u|^p \le \frac{1}{2}\|u_0\|_{L^2}^2 + \iint_{Q_\tau}\mu_1|\nabla u|^{p-1}|u| + \iint_{Q_\tau}(\varphi_1+\varphi_3) + c_1\iint_{Q_\tau}|u|^p + \iint_{Q_\tau} fu.$$

Hölder's inequality gives

$$\iint \mu_1 |\nabla u|^{p-1}|u| \le \Big(\iint |\nabla u|^p\Big)^{\frac{p-1}{p}} \Big(\iint \mu_1^p |u|^p\Big)^{\frac{1}{p}}.$$

Apply Definition 2.1 with $\varepsilon = \nu/2$:

$$\Big(\iint \mu_1^p |u|^p\Big)^{\frac{1}{p}} \le \Big(\frac{\nu}{2}\iint |\nabla u|^p + C_{\nu/2}\iint |u|^p\Big)^{\frac{1}{p}}.$$

Using $(a+b)^{1/p} \le a^{1/p} + b^{1/p}$ and Young's inequality, we obtain

$$\iint \mu_1 |\nabla u|^{p-1}|u| \le \frac{\nu}{2}\iint |\nabla u|^p + C\iint |u|^p.$$

The source term is bounded by $\iint fu \le \frac{1}{2}\iint |f|^{p'} + \frac{1}{2}\iint |u|^p$. Collecting terms proves (4.1). □

Taking the supremum over $\tau \in [0,T]$ in (4.1) yields the global bound

$$\frac{1}{2}\sup_{0\le t\le T}\|u(t)\|_{L^2}^2 + \frac{\nu}{2}\|\nabla u\|_{L^p(Q_T)}^p \le \frac{1}{2}\|u_0\|_{L^2}^2 + C_*\|u\|_{L^p(Q_T)}^p + \Phi + \frac{1}{2}\|f\|_{L^{p'}(Q_T)}^{p'}. \tag{4.2}$$

4.2. **Local Caccioppoli inequality.** Fix an intrinsic cylinder $Q_{2R} \Subset Q_T$. For $k \ge 0$ define

$$w = (u-k)_+ = \max\{u-k, 0\}, \qquad A_{k,\rho} = Q_\rho \cap \{u > k\}.$$

Let $\eta \in C^\infty(\overline{Q}_R)$ be a cut-off function satisfying

$$\eta \equiv 1 \text{ on } Q_\rho, \quad \eta \equiv 0 \text{ near } \partial_p Q_R, \quad 0 \le \eta \le 1,$$

and

$$|\nabla \eta| \le \frac{C}{R-\rho}, \qquad |\partial_t \eta| \le \frac{C}{(R-\rho)^p},$$

with $C = C(n,p)$.

Lemma 4.2 (Caccioppoli inequality). *There exists a constant $C > 0$ depending only on the structural data and on the p-gradient-form-boundedness constants (with a fixed $\varepsilon = \nu/4$) such that for every $0 < \rho < R$,*

$$\iint_{A_{k,\rho}} |\nabla u|^p \,\mathrm{d}x\,\mathrm{d}t \le C\Big(\frac{1}{(R-\rho)^p}\iint_{A_{k,R}} (u-k)^p \,\mathrm{d}x\,\mathrm{d}t + \iint_{A_{k,R}} \Psi \,\mathrm{d}x\,\mathrm{d}t\Big), \tag{4.3}$$

where $\Psi = \varphi_1 + \varphi_2^{p'} + \varphi_3 + k^p + |f|^{p'}$.

Proof. We test the weak formulation with $\varphi = \eta^p w$ (after Steklov averaging). The computations for the time derivative and the principal part are identical to the classical case and lead to

$$\iint \partial_t u\, \eta^p w \le \frac{C}{(R-\rho)^p}\iint_{A_{k,R}} (w^p + k^p),$$

$$\sum a_i \partial_i(\eta^p w) \ge \frac{3\nu}{4}\eta^p |\nabla u|^p - C\eta^p(\varphi_1 + \varphi_2^{p'}) - C\eta^p|u|^p - \frac{C}{(R-\rho)^p} w^p.$$

The lower-order term is estimated using (B1):

$$b\,\eta^p w \ge -\mu_1 |\nabla u|^{p-1}\eta^p w - \varphi_3 \eta^p w.$$

The singular term $I = \iint \mu_1 |\nabla u|^{p-1}\eta^p w$ is bounded via Hölder:

$$I \le \Big(\iint |\nabla u|^p\Big)^{\frac{p-1}{p}} \Big(\iint \mu_1^p \eta^p w^p\Big)^{\frac{1}{p}}.$$

Set $v=\eta w$. Because $v=0$ on $\partial\Omega\times(0,T)$ and $|\nabla v|\le \eta|\nabla w|+w|\nabla\eta|$, we have $v\in L^p(0,T;W^{1,p}_0(\Omega))$. Apply Definition 2.1 to v with a small parameter ε_0:

$$\iint \mu_1^p v^p \le \varepsilon_0 \iint |\nabla v|^p + C_{\varepsilon_0}\iint v^p.$$

Expanding $|\nabla v|^p\le C(\eta^p|\nabla u|^p + w^p|\nabla\eta|^p)$, we obtain

$$\Big(\iint \mu_1^p\eta^p w^p\Big)^{1/p} \le (\varepsilon_0 C)^{1/p}\Big(\iint \eta^p|\nabla u|^p\Big)^{1/p} + \Big(\varepsilon_0 C\iint w^p|\nabla\eta|^p + C_{\varepsilon_0}\iint \eta^p w^p\Big)^{1/p}.$$

Inserting this into the bound for I and using Young's inequality repeatedly, we can absorb the gradient terms into the left-hand side with coefficient $\le \nu/4$, obtaining

$$I \le \frac{\nu}{4}\iint |\nabla u|^p + \frac{C}{(R-\rho)^p}\iint_{A_{k,R}} w^p + C\iint_{A_{k,R}} \eta^p w^p.$$

Collecting all terms and absorbing the remaining gradient pieces yields

$$\frac{\nu}{2}\iint \eta^p|\nabla u|^p \le C\iint_{A_{k,R}}\frac{w^p}{(R-\rho)^p} + C\iint_{A_{k,R}}\big(\varphi_1+\varphi_2^{p'}+\varphi_3+k^p+|f|^{p'}\big).$$

Since $\eta\equiv 1$ on Q_ρ, the left side bounds $\frac{\nu}{2}\iint_{A_{k,\rho}}|\nabla u|^p$, and the Caccioppoli inequality follows. □

4.3. **Local boundedness via De Giorgi iteration.** The Caccioppoli inequality places u in the parabolic De Giorgi class $DG_p(Q_T)$. The classical iteration (see [1, Chapter III, §5]) yields the following.

Theorem 4.3 (Local boundedness). *Let u be a weak solution. For any intrinsic cylinder $Q_{2R}\Subset Q_T$,*

$$\operatorname{ess\,sup}_{Q_R}|u| \le C\Big(R^{-(n+2)/p}\|u\|_{L^p(Q_{2R})} + K\Big),$$

where C and K depend only on the structural data. In particular, every weak solution is locally bounded.

Proof. We prove the estimate for the positive part u_+; the negative part is analogous. For $n=0,1,2,\dots$ set

$$\rho_n = R + \frac{R}{2^n},\qquad k_n = d - \frac{d}{2^{n+1}},$$

where $d>0$ will be chosen later. Define $w_n=(u-k_n)_+$, $A_n=Q_{\rho_n}\cap\{u>k_n\}$. Notice that $A_{n+1}\subset A_n$, $w_{n+1}\le w_n$, and on A_{n+1} we have $w_n\ge k_{n+1}-k_n = d2^{-n-2}$.

Apply the Caccioppoli inequality (4.3) with $\rho=\rho_{n+1}$, $R=\rho_n$, $k=k_{n+1}$. For simplicity we assume $\Psi\equiv 0$; the general case adds a constant to the final bound. We obtain

$$\iint_{A_{n+1}}|\nabla u|^p \le C\Big(\frac{2^{np}}{R^p}\iint_{A_n} w_n^p + \iint_{A_n}\Psi\Big).$$

Assuming $\Psi=0$ (the non-homogeneous terms can be incorporated by a standard modification, see [1]), we have

$$\|\nabla w_{n+1}\|^p_{L^p(Q_T)} \le \frac{C2^{np}}{R^p}\|w_n\|^p_{L^p(A_n)}.$$

By the parabolic Sobolev embedding (Theorem 2.5) and the fact that w_{n+1} vanishes on the parabolic boundary,

$$\|w_{n+1}\|_{L^{p^*}(Q_T)} \le C_S\|\nabla w_{n+1}\|_{L^p(Q_T)}.$$

On the other hand, Hölder's inequality gives

$$\|w_n\|_{L^p(A_n)} \le \|w_n\|_{L^{p^*}(A_n)}|A_n|^{\frac{p^*-p}{pp^*}}.$$

Moreover, a measure estimate follows from Chebyshev's inequality:

$$|A_{n+1}| \le \Big(\frac{2^{n+2}}{d}\Big)^p\|w_n\|^p_{L^p(A_n)}.$$

Combining these relations yields, for $Y_n = \|w_n\|^p_{L^p(A_n)}$,

$$Y_{n+1} \le \frac{Cb^n}{d^{p\gamma}R^{n+2}} Y_n^{1+\gamma}, \qquad \gamma = \frac{p}{n+2},$$

with b depending on the structural constants. The De Giorgi iteration lemma (Lemma A.2) states that if $Y_0 \le C^{-1/\gamma} b^{-1/\gamma^2}$ then $Y_n \to 0$. Choosing d large enough so that

$$\|u_+\|^p_{L^p(Q_{2R})} \le C^{-1/\gamma} b^{-1/\gamma^2}$$

we obtain $Y_n \to 0$, which means $\operatorname{ess\,sup}_{Q_R} u_+ \le d$. The bound on d gives the desired estimate. □

Remark 4.4. The smallness condition on the Poincaré constant discussed in the next section allows one to start the iteration on small cylinders without any prior bound on the L^p norm.

4.4. **Global gradient bound under a smallness condition.** We now show that the global energy estimate (4.2) can be closed provided the domain is such that the Poincaré constant is small relative to the structural constants.

Theorem 4.5 (Global a priori estimate under smallness)**.** *Suppose that the Poincaré inequality* $\|u\|_{L^p(Q_T)} \le C_P \|\nabla u\|_{L^p(Q_T)}$ *holds for every* $u \in L^p(0,T; W^{1,p}_0(\Omega))$*, and that*

$$C_* \, C_P^p < \frac{\nu}{2}, \tag{4.4}$$

where $C_* = C_{\nu/2} + c_1 + \frac{1}{2}$ *is the constant from Lemma 4.1. Then every weak solution* u *satisfies the a priori bound*

$$\|u\|_{V_p(Q_T)} \le C\big(1 + \|u_0\|_{L^2(\Omega)} + \|f\|_{L^{p'}(Q_T)}\big)$$

with a constant C *depending only on* n, p, ν, μ*, the constants from the* p*-gradient-form-boundedness of* μ_1*, and the geometry of* Ω*.*

Proof. From Lemma 4.1 we have, for a.e. $\tau \in (0,T)$,

$$\frac{1}{2}\int_\Omega u^2(\tau)\,\mathrm{d}x + \frac{\nu}{2}\iint_{Q_\tau} |\nabla u|^p \,\mathrm{d}x\,\mathrm{d}t \le \frac{1}{2}\|u_0\|_{L^2}^2 + C_* \iint_{Q_\tau} |u|^p \,\mathrm{d}x\,\mathrm{d}t + \Phi + \frac{1}{2}\iint_{Q_\tau} |f|^{p'} \,\mathrm{d}x\,\mathrm{d}t.$$

Taking the supremum over $\tau \in [0,T]$ gives the global inequality

$$\frac{1}{2}\sup_{0\le t\le T}\|u(t)\|_{L^2}^2 + \frac{\nu}{2}\|\nabla u\|^p_{L^p(Q_T)} \le \frac{1}{2}\|u_0\|_{L^2}^2 + C_*\|u\|^p_{L^p(Q_T)} + \Phi + \frac{1}{2}\|f\|^{p'}_{L^{p'}(Q_T)}. \tag{4.5}$$

By hypothesis, the Poincaré inequality $\|u\|_{L^p(Q_T)} \le C_P\|\nabla u\|_{L^p(Q_T)}$ is available. Inserting this into (4.5) yields

$$\frac{1}{2}\sup_{0\le t\le T}\|u(t)\|_{L^2}^2 + \frac{\nu}{2}\|\nabla u\|^p_{L^p} \le \frac{1}{2}\|u_0\|_{L^2}^2 + C_* C_P^p\|\nabla u\|^p_{L^p} + \Phi + \frac{1}{2}\|f\|^{p'}_{L^{p'}}.$$

Bring the term $C_* C_P^p \|\nabla u\|^p_{L^p}$ to the left:

$$\frac{1}{2}\sup_{0\le t\le T}\|u(t)\|_{L^2}^2 + \Big(\frac{\nu}{2} - C_* C_P^p\Big)\|\nabla u\|^p_{L^p} \le \frac{1}{2}\|u_0\|_{L^2}^2 + \Phi + \frac{1}{2}\|f\|^{p'}_{L^{p'}}.$$

Assumption (4.4) guarantees that $\kappa := \frac{\nu}{2} - C_* C_P^p > 0$. Hence

$$\|\nabla u\|^p_{L^p} \le \frac{1}{2\kappa}\|u_0\|_{L^2}^2 + \frac{1}{\kappa}\Phi + \frac{1}{2\kappa}\|f\|^{p'}_{L^{p'}},$$

$$\sup_{0\le t\le T}\|u(t)\|_{L^2}^2 \le \|u_0\|_{L^2}^2 + 2\Phi + \|f\|^{p'}_{L^{p'}} + 2C_* C_P^p\|\nabla u\|^p_{L^p}.$$

Substituting the estimate of $\|\nabla u\|_{L^p}^p$ into the last line and using $\Phi = \|\varphi_1 + \varphi_3\|_{L^1}$, we obtain

$$\|u\|_{V_p(Q_T)} = \sup_t \|u(t)\|_{L^2} + \|\nabla u\|_{L^p} \le C\big(1 + \|u_0\|_{L^2} + \|f\|_{L^{p'}}\big)$$

with C depending on κ, ν, μ, the form-boundedness constants, and c_1. This completes the proof. □

Remark 4.6. The Poincaré constant on a ball B_R satisfies $C_{P,R} \le cR$ with $c = c(n,p)$. Hence for any intrinsic cylinder Q_{2R} the smallness condition (4.4) is automatically satisfied if R is chosen small enough. This observation is crucial in the existence proof (Section 5) because it allows us to start the De Giorgi iteration on small cylinders without any a priori bound on the L^p norm, and thereby to obtain a global L^∞ bound.

5. Existence of Weak Solutions

The existence proof is split into two parts. First we prove the local existence of a weak solution on a (possibly small) time interval $[0, \tau]$. Then, using the local boundedness theorem already established in Section 4, we show that this solution can be extended step-by-step to any given time $T > 0$, which gives the global existence.

5.1. **Local existence via the Galerkin method.**

Theorem 5.1 (Local existence). *Let $\Omega \subset \mathbb{R}^n$ be bounded, $\tau > 0$, and let Assumptions 3.1, 3.2 hold. Then there exists a number $T_0 \in (0, \tau]$ and a weak solution u on Q_{T_0} (in the sense of Definition 3.4) satisfying $u \in V_p(Q_{T_0})$.*

Proof. **1. Galerkin basis.** Since $W_0^{1,p}(\Omega)$ is separable and $C_c^\infty(\Omega)$ is dense, we can choose a sequence $\{\phi_j\}_{j=1}^\infty \subset C_c^\infty(\Omega)$ such that

(1) finite linear combinations are dense in $W_0^{1,p}(\Omega)$;
(2) $\{\phi_j\}$ is orthonormal in $L^2(\Omega)$.

For each $m \in \mathbb{N}$ set $V_m = \operatorname{span}\{\phi_1, \dots, \phi_m\}$ and let $P_m : L^2(\Omega) \to V_m$ be the orthogonal projection.
2. Approximate problem. We look for functions

$$u_m(x,t) = \sum_{j=1}^m c_{m,j}(t)\, \phi_j(x)$$

satisfying the system of m ordinary differential equations

$$\langle \partial_t u_m, \psi \rangle + \int_\Omega \Big(\sum_{i=1}^n a_i(x,t,u_m,\nabla u_m)\, \partial_i \psi + b(x,t,u_m,\nabla u_m)\, \psi \Big)\, \mathrm{d}x = \int_\Omega f(t)\, \psi \, \mathrm{d}x \qquad \forall\, \psi \in V_m, \tag{5.1}$$

with the initial condition

$$u_m(0) = P_m u_0.$$

The functions a_i and b are Carathéodory and satisfy the growth conditions (A2), (B2). Because u_m is a finite linear combination of smooth ϕ_j, the integrals in (5.1) are well defined. In particular, the singular term $\int_\Omega \mu_1 |\nabla u_m|^{p-1} \phi_j$ is finite for a.e. t because, by Hölder's inequality and the p-gradient-form-boundedness,

$$\int_\Omega \mu_1 |\nabla u_m|^{p-1} |\phi_j| \le \Big(\int_\Omega |\nabla u_m|^p \Big)^{\frac{p-1}{p}} \Big(\int_\Omega \mu_1^p |\phi_j|^p \Big)^{\frac{1}{p}},$$

and the latter term is finite because $\phi_j \in C_c^\infty(\Omega)$ can be extended constant in time to a function in $L^p(0,T; W_0^{1,p}(\Omega))$, to which we apply Definition 2.1.

The matrix $\big(\langle \phi_i, \phi_j \rangle\big)$ is the identity, so the system becomes a normal ODE system

$$\frac{\mathrm{d}}{\mathrm{d}t} c_{m,j}(t) = F_j(t, c_{m,1}, \dots, c_{m,m}).$$

Carathéodory's theorem gives a local solution on $[0, t_m)$. The a priori estimates below show that t_m can be taken independent of m.

3. Energy estimate for the approximations. Taking $\psi = u_m(\cdot, t)$ in (5.1) and using the structural assumptions as in Lemma 4.1, we obtain for a.e. $t \in (0, t_m)$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_m(t)\|_{L^2}^2 + \frac{\nu}{2}\int_\Omega |\nabla u_m|^p \,\mathrm{d}x \le C_1 \int_\Omega |u_m|^p \,\mathrm{d}x + \int_\Omega (\varphi_1 + \varphi_3)\,\mathrm{d}x + \frac{1}{2}\int_\Omega |f|^{p'}\,\mathrm{d}x. \tag{5.2}$$

(The constant C_1 comes from the form-boundedness applied to u_m.)

To control the $|u_m|^p$ term we use the Gagliardo–Nirenberg inequality on Ω:

$$\|u_m\|_{L^p(\Omega)} \le C \|\nabla u_m\|_{L^p(\Omega)}^{1-\theta} \|u_m\|_{L^2(\Omega)}^{\theta}, \qquad \theta \in (0,1).$$

Thus $\int_\Omega |u_m|^p \,\mathrm{d}x \le \frac{\nu}{4}\int_\Omega |\nabla u_m|^p \,\mathrm{d}x + C_2 \|u_m\|_{L^2}^p$. Inserting into (5.2) and absorbing the gradient term gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_m(t)\|_{L^2}^2 + \frac{\nu}{4}\int_\Omega |\nabla u_m|^p \,\mathrm{d}x \le C_3 \|u_m(t)\|_{L^2}^p + \int_\Omega (\varphi_1 + \varphi_3)\,\mathrm{d}x + \frac{1}{2}\int_\Omega |f|^{p'}\,\mathrm{d}x. \tag{5.3}$$

Let $y_m(t) = \|u_m(t)\|_{L^2}^2$. Because $\|u_m(0)\|_{L^2} \le \|u_0\|_{L^2}$, there exists C_4 (depending on $\|u_0\|_{L^2}$) such that $y_m(0) \le C_4$. Choose $T_0 \in (0, \tau]$ so small that $C_3 (2C_4)^{p/2} T_0 \le \frac{1}{4} C_4$. The same argument as in the classical theory shows that the solution exists on $[0, T_0]$ and

$$\sup_{0 \le t \le T_0} \|u_m(t)\|_{L^2}^2 + \int_0^{T_0}\int_\Omega |\nabla u_m|^p \,\mathrm{d}x\mathrm{d}t \le C(T_0, \text{data}).$$

Thus $\{u_m\}$ is bounded in $V_p(Q_{T_0})$.

4. Compactness. The uniform bound together with the equation imply that $\{\partial_t u_m\}$ is bounded in $L^{p'}(0, T_0; W^{-1,p'}(\Omega)) + L^1(0, T_0; L^2(\Omega))$. By the Aubin–Lions lemma (Lemma A.4), we can extract a subsequence such that

$$\begin{aligned} u_m &\to u \quad \text{strongly in } L^p(Q_{T_0}) \text{ and a.e.},\\ \nabla u_m &\rightharpoonup \nabla u \quad \text{weakly in } L^p(Q_{T_0}),\\ u_m &\overset{*}{\rightharpoonup} u \quad \text{weakly-}* \text{ in } L^\infty(0, T_0; L^2(\Omega)). \end{aligned}$$

5. Strong convergence of gradients. Using the monotonicity (A3) and the Minty–Browder technique exactly as in LSU, one proves $\nabla u_m \to \nabla u$ strongly in $L^p(Q_{T_0})$.

6. Passage to the limit. We need to pass to the limit in the nonlinear terms. The strong convergence of gradients implies $|\nabla u_m|^{p-1} \to |\nabla u|^{p-1}$ a.e. (up to a subsequence) and in $L^{p'}(Q_{T_0})$. For the singular term, fix a test function ψ_j from the basis. By Young's inequality,

$$\mu_1 |\nabla u_m|^{p-1} |\psi_j| \le \frac{p-1}{p} |\nabla u_m|^p + \frac{1}{p} \mu_1^p |\psi_j|^p.$$

The right-hand side converges in $L^1(Q_{T_0})$: the first term because $|\nabla u_m|^p \to |\nabla u|^p$ in L^1, and the second term is fixed and integrable by the form-boundedness applied to ψ_j. Hence, by the generalized dominated convergence theorem (or Vitali), $\mu_1 |\nabla u_m|^{p-1}\psi_j \to \mu_1 |\nabla u|^{p-1}\psi_j$ in $L^1(Q_{T_0})$. All other terms converge by the strong L^p convergence of u_m and ∇u_m. Thus the limit u satisfies the weak formulation (3.4). The initial condition follows from $u_m(0) \to u_0$ in L^2.

This completes the proof of local existence. □

5.2. **Global existence.**

Theorem 5.2. *Under Assumptions 3.1 and 3.2, for every $T > 0$ there exists a weak solution u of problem (3.1)–(3.3) on Q_T satisfying $u \in V_p(Q_T) \cap L^\infty(Q_T)$.*

Proof. Let u be a weak solution defined on a maximal interval $[0, T^*)$, where $T^* > 0$ is the supremum of all τ for which a weak solution exists on Q_τ. The local existence theorem (Theorem 5.1) guarantees $T^* > 0$. We shall prove that T^* can be taken arbitrarily large; indeed we show that if $T^* < \infty$, then u can be extended beyond T^*, contradicting maximality.

1. Boundedness of the solution on $[0, T^*)$. Since u is a weak solution on Q_{T^*}, it belongs to $V_p(Q_{T^*})$ and, by Theorem 4.3 (local boundedness), $u \in L^\infty(Q_{T^*})$. Hence there exists a constant M such that

$$\|u\|_{L^\infty(Q_{T^*})} \le M.$$

2. Uniform bounds for the energy. From the energy estimate (Lemma 4.1) applied on Q_t for any $t < T^*$, we have

$$\frac{1}{2}\|u(t)\|^2_{L^2(\Omega)} + \frac{\nu}{2}\iint_{Q_t} |\nabla u|^p \,\mathrm{d}x\,\mathrm{d}t \le \frac{1}{2}\|u_0\|^2_{L^2} + C_* \iint_{Q_t} |u|^p \,\mathrm{d}x\,\mathrm{d}t + \Phi + \frac{1}{2}\iint_{Q_t} |f|^{p'} \,\mathrm{d}x\,\mathrm{d}t. \quad (5.4)$$

Because $\|u\|_{L^\infty} \le M$, we have $\iint_{Q_t} |u|^p \le M^p |\Omega| t \le M^p |\Omega| T^*$. Moreover, $f \in L^{p'}(Q_T)$ and $\varphi_1, \varphi_3 \in L^1$, so the right-hand side of (5.4) is bounded by a constant independent of $t < T^*$. Therefore

$$\sup_{0 \le t < T^*} \|u(t)\|^2_{L^2(\Omega)} < \infty, \qquad \iint_{Q_{T^*}} |\nabla u|^p \,\mathrm{d}x\,\mathrm{d}t < \infty. \quad (5.5)$$

3. Extension of the solution to $t = T^*$. The bound (5.5) implies that $u \in L^\infty(0, T^*; L^2(\Omega)) \cap L^p(0, T^*; W^{1,p}_0(\Omega))$, and by the weak formulation we also obtain that $\partial_t u$ belongs to $L^{p'}(0, T^*; W^{-1,p'}(\Omega)) + L^1(0, T^*; L^2(\Omega))$. A standard result (see e.g. [1, Ch. III, §1]) then yields that $u \in C([0, T^*]; L^2(\Omega))$ after possibly being modified on a set of measure zero. In particular, the limit

$$u(T^*) := \lim_{t \nearrow T^*} u(t) \quad \text{exists in } L^2(\Omega).$$

Moreover, $u(T^*)$ is an admissible initial datum for a new Cauchy–Dirichlet problem.

4. Continuation beyond T^*. We now apply the local existence theorem (Theorem 5.1) to the equation on the time interval $[T^*, T^* + \tau]$ with initial condition $\tilde{u}(T^*) = u(T^*)$. (The coefficients a_i, b and the source f are translated in time; they satisfy the same structural assumptions because the assumptions are uniform in t.) By Theorem 5.1, there exists a number $T_0 \in (0, \tau]$ and a weak solution $\tilde{u}$ on $[T^*, T^* + T_0]$ with $\tilde{u} \in V_p((T^*, T^* + T_0) \times \Omega)$.

Define

$$\bar{u}(x,t) = \begin{cases} u(x,t), & 0 \le t < T^*, \\ \tilde{u}(x,t), & T^* \le t \le T^* + T_0. \end{cases}$$

One verifies directly that $\bar{u}$ is a weak solution on $Q_{T^*+T_0}$ in the sense of Definition 3.4: the weak formulation holds for test functions supported in $(0, T^* + T_0)$ because it holds separately on $(0, T^*)$ and on $(T^*, T^* + T_0)$, the matching at $t = T^*$ being guaranteed by the continuity of u and $\tilde{u}$ in $L^2(\Omega)$. This contradicts the maximality of T^* (since $T^* + T_0 > T^*$).

Therefore, T^* cannot be finite. We conclude that a weak solution exists on $[0, T]$ for every $T > 0$. □

6. Maximum and Comparison Principles

This section is devoted to qualitative properties of weak solutions. The proofs are self-contained and use only the structural assumptions and the already established local boundedness.

6.1. **Preliminary lemmas.**

Lemma 6.1. *Let u be a weak solution of (3.1)–(3.3). Then the function $v = u_+ = \max\{u, 0\}$ belongs to $V_p(Q_T)$ and $v(x, 0) = (u_0)_+$ a.e. in Ω. Moreover, for a.e. $t \in (0, T)$,*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|^2_{L^2(\Omega)} + \int_\Omega \sum_{i=1}^n a_i(x, t, u, \nabla u)\, \partial_i v \, \mathrm{d}x + \int_\Omega b(x, t, u, \nabla u)\, v \, \mathrm{d}x = \int_\Omega f\, v \, \mathrm{d}x. \tag{6.1}$$

The same holds for w_+ where $w = u_1 - u_2$ is the difference of two weak solutions, with a_i, b, f replaced by the corresponding differences.

Proof. **1. v belongs to $V_p(Q_T)$.** Since $u \in V_p(Q_T) = C([0, T]; L^2(\Omega)) \cap L^p(0, T; W^{1,p}_0(\Omega))$, the function $v = \max\{u, 0\}$ satisfies $v(t) \in L^2(\Omega)$ for every t, and the map $t \mapsto v(t)$ is continuous from $[0, T]$ to $L^2(\Omega)$ because the function $s \mapsto s_+$ is Lipschitz on $\mathbb{R}$. Moreover, a well-known property of Sobolev spaces (see, e.g., [1, Ch. II, §3]) gives $\nabla v = \chi_{\{u>0\}} \nabla u$ a.e. in Q_T, so $\nabla v \in L^p(Q_T)$. Hence $v \in V_p(Q_T)$. The initial condition $v(0) = (u_0)_+$ follows immediately.

2. Energy equality for the truncated function. To justify (6.1) we must test the weak formulation (3.4) with $\varphi = v$, although v is not smooth. We use a standard approximation procedure.

Let $\psi_\varepsilon : \mathbb{R} \to \mathbb{R}$ be a family of smooth functions such that

$$\psi_\varepsilon(s) \to s_+, \qquad \psi'_\varepsilon(s) \to \chi_{(0,\infty)}(s) \quad \text{as } \varepsilon \to 0,$$

and $|\psi'_\varepsilon| \le 1$, $\psi_\varepsilon(0) = 0$. Define $v_\varepsilon = \psi_\varepsilon \circ u$. Since $u \in V_p(Q_T)$ and ψ_ε is smooth with bounded derivative, $v_\varepsilon \in V_p(Q_T)$ with $\nabla v_\varepsilon = \psi'_\varepsilon(u)\nabla u$, and $v_\varepsilon \to v$ in $V_p(Q_T)$ as $\varepsilon \to 0$. Moreover, v_ε vanishes on the lateral boundary (because u does) and $v_\varepsilon(\cdot, T) = 0$ (since $u(\cdot, T)$ is only in L^2, we may modify u on a set of measure zero to ensure $v_\varepsilon(T) = 0$; the standard justification uses the Steklov average as in the energy lemma, see [1, Ch. III, §2]).

The weak formulation (3.4) is valid for smooth test functions; by density it extends to any $\varphi \in V_p(Q_T) \cap L^\infty(Q_T)$ with $\varphi(T) = 0$. (This extension is justified because for such φ there exists a sequence of smooth functions approximating φ in the appropriate norms, and the integrals in (3.4) converge.) In particular, we may take $\varphi = v_\varepsilon$.

Substituting $\varphi = v_\varepsilon$ into (3.4) gives

$$\iint_{Q_T} \Big(-u\, \partial_t v_\varepsilon + \sum_i a_i\, \partial_i v_\varepsilon + b\, v_\varepsilon \Big)\, \mathrm{d}x\, \mathrm{d}t = \iint_{Q_T} f\, v_\varepsilon \, \mathrm{d}x\, \mathrm{d}t + \int_\Omega u_0(x)\, v_\varepsilon(x, 0) \, \mathrm{d}x.$$

Since $v_\varepsilon(0) = \psi_\varepsilon(u_0) \to (u_0)_+$, the boundary term converges. For the time derivative term, we use integration by parts in time:

$$-\iint u\, \partial_t v_\varepsilon = \iint \partial_t u\, v_\varepsilon - \int_\Omega u(T) v_\varepsilon(T) \, \mathrm{d}x + \int_\Omega u(0) v_\varepsilon(0) \, \mathrm{d}x.$$

But u is only known to have $\partial_t u$ in a dual space, so we must be careful. The standard Steklov averaging argument (already used in Lemma 4.1) shows that for a.e. t,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v_\varepsilon(t)\|^2_{L^2} = \int_\Omega \partial_t u(t)\, v_\varepsilon(t) \, \mathrm{d}x$$

in the duality sense, where $\partial_t u(t) \in W^{-1,p'}(\Omega) + L^2(\Omega)$. The chain rule for the square of the L^2 norm of v_ε is justified because $v_\varepsilon \in V_p(Q_T)$. Then

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v_\varepsilon(t)\|^2_{L^2} + \int_\Omega \sum_i a_i(x, t, u, \nabla u)\, \partial_i v_\varepsilon \, \mathrm{d}x + \int_\Omega b(x, t, u, \nabla u)\, v_\varepsilon \, \mathrm{d}x = \int_\Omega f(t)\, v_\varepsilon \, \mathrm{d}x.$$

All terms converge as $\varepsilon \to 0$: $\partial_i v_\varepsilon = \psi'_\varepsilon(u)\partial_i u \to \chi_{\{u>0\}}\partial_i u = \partial_i v$ a.e., and by dominated convergence the integrals converge. Thus we obtain (6.1) for a.e. t.

The same argument applied to $w = u_1 - u_2$ yields the energy equality for w_+. □

Lemma 6.2. *Let $u \in V_p(Q_T)$ be a weak solution. Then, after possibly modifying u on a set of measure zero in time, we have $u \in C([0,T]; L^2(\Omega))$, and the function $y(t) = \|u(t)\|^2_{L^2(\Omega)}$ is absolutely continuous on $[0,T]$. For a.e. $t \in (0,T)$,*

$$\frac{1}{2}y'(t) = \langle \partial_t u(t), u(t) \rangle_{W^{-1,p'}+L^2,\, W_0^{1,p}\cap L^2},$$

and this pairing coincides with the right-hand side of the energy equality (4.1). The same holds for $v = u_+$ and w_+, where $w = u_1 - u_2$.

Proof. The energy identity in integral form, proved in Lemma 4.1, gives for any $0 \le \tau_1 < \tau_2 \le T$,

$$\frac{1}{2}\|u(\tau_2)\|^2_{L^2} - \frac{1}{2}\|u(\tau_1)\|^2_{L^2} = \int_{\tau_1}^{\tau_2}\int_\Omega \Big(f u - \sum_i a_i\, \partial_i u - b\, u \Big)\, \mathrm{d}x\, \mathrm{d}t.$$

The right-hand side is the integral of an $L^1(0,T)$ function (by the structural assumptions and the p-gradient-form-boundedness). Hence $y(t) = \|u(t)\|^2_{L^2}$ is absolutely continuous and its pointwise derivative equals the integrand a.e. The identification of the integrand with the duality pairing $\langle \partial_t u, u\rangle$ follows from the weak formulation tested with u (after Steklov averaging, see [1, Ch. II, §2]). The same reasoning applies to u_+ and w_+ because Lemma 6.1 provides the analogous integral identity. □

6.2. **Weak maximum principle.**

Theorem 6.3 (Weak maximum principle)**.** *In addition to Assumptions 3.1 and 3.2, suppose $\varphi_1 \equiv 0$, $\varphi_3 \equiv 0$, and $c_1 = 0$. If $f \le 0$ a.e. and $u_0 \le 0$ a.e., then $u \le 0$ a.e. in Q_T.*

Proof. Set $v = u_+$. Using (A1) and (B1) with the additional hypotheses,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|^2_{L^2} + \nu \int_\Omega |\nabla v|^p \le \int_\Omega \mu_1 |\nabla v|^{p-1} v.$$

Apply the p-gradient-form-boundedness with $\varepsilon = \nu/2$:

$$\int_\Omega \mu_1 |\nabla v|^{p-1} v \le \frac{\nu}{2}\int_\Omega |\nabla v|^p + C_{\nu/2}\int_\Omega v^p.$$

Thus

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|^2_{L^2} + \frac{\nu}{2}\int_\Omega |\nabla v|^p \le C_{\nu/2}\int_\Omega v^p.$$

Local boundedness gives $0 \le v \le M$, so $v^p \le M^{p-2} v^2$. Dropping the gradient term,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|^2_{L^2} \le 2C_{\nu/2} M^{p-2}\|v(t)\|^2_{L^2}.$$

Gronwall and $v(0) = 0$ give $v \equiv 0$. □

6.3. **Comparison principle.** We add the natural comparison assumptions:

Assumption 6.4 (Comparison structure)**.** (C1) $a_i = a_i(x,t,\xi)$ (independent of u) and satisfy (A3) with $\nu_0 > 0$.
(C2) $(b(x,t,u_1,\xi) - b(x,t,u_2,\xi))(u_1 - u_2) \ge -L|u_1 - u_2|^p$.
(C3) $|b(x,t,u,\xi) - b(x,t,u,\eta)| \le \mu_1(x,t)|\xi - \eta|^{p-1}$.

Theorem 6.5 (Comparison principle)**.** *Let u_1, u_2 be two weak solutions of (3.1) corresponding to data $f_1, u_{0,1}$ and $f_2, u_{0,2}$ respectively. Under Assumptions 3.1, 3.2 and 6.4, if $f_1 \le f_2$ a.e. in Q_T and $u_{0,1} \le u_{0,2}$ a.e. in Ω, then $u_1 \le u_2$ a.e. in Q_T.*

Proof. Set $w = u_1 - u_2$ and $v = w_+ = \max\{w, 0\}$. By the analogue of Lemma 6.1 for the difference of two solutions, v belongs to $V_p(Q_T)$ and $v(x,0) = (u_{0,1} - u_{0,2})_+ = 0$ a.e.

1. Weak formulation for the difference. Subtract the weak formulations (3.4) for u_1 and u_2, test with the Steklov average of v, and pass to the limit $h \to 0$ as in Lemma 4.1. For a.e. $t \in (0,T)$ we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|^2_{L^2(\Omega)} &+ \int_\Omega \sum_{i=1}^n \big(a_i(x,t,\nabla u_1) - a_i(x,t,\nabla u_2)\big)\partial_i v \,\mathrm{d}x \\ &+ \int_\Omega \big(b(x,t,u_1,\nabla u_1) - b(x,t,u_2,\nabla u_2)\big) v \,\mathrm{d}x \\ &= \int_\Omega (f_1 - f_2) v \,\mathrm{d}x.\end{aligned} \tag{6.2}$$

All integrals are effectively restricted to the set $\{u_1 > u_2\}$, where $v = u_1 - u_2 > 0$ and $\nabla v = \nabla u_1 - \nabla u_2$.

2. Principal part. By (C1) the coefficients a_i are independent of u, and the monotonicity (A3) gives

$$\sum_{i=1}^n \big(a_i(x,t,\nabla u_1) - a_i(x,t,\nabla u_2)\big)\partial_i v \geq \nu_0 \,|\nabla v|^p.$$

3. Lower-order term. Write the difference as

$$\begin{aligned}b(x,t,u_1,\nabla u_1) - b(x,t,u_2,\nabla u_2) &= \big[b(x,t,u_1,\nabla u_1) - b(x,t,u_1,\nabla u_2)\big] \\ &\quad + \big[b(x,t,u_1,\nabla u_2) - b(x,t,u_2,\nabla u_2)\big].\end{aligned}$$

For the first bracket, (C3) implies

$$|b(x,t,u_1,\nabla u_1) - b(x,t,u_1,\nabla u_2)| \leq \mu_1(x,t)|\nabla u_1 - \nabla u_2|^{p-1} = \mu_1(x,t)|\nabla v|^{p-1}.$$

Multiplying by $v \geq 0$ gives

$$\big(b(x,t,u_1,\nabla u_1) - b(x,t,u_1,\nabla u_2)\big) v \geq -\mu_1 |\nabla v|^{p-1} v.$$

For the second bracket, (C2) directly yields

$$\big(b(x,t,u_1,\nabla u_2) - b(x,t,u_2,\nabla u_2)\big)(u_1 - u_2) \geq -L\, v^p.$$

Adding these two estimates,

$$\big(b(x,t,u_1,\nabla u_1) - b(x,t,u_2,\nabla u_2)\big) v \geq -\mu_1 |\nabla v|^{p-1} v - L v^p.$$

4. Source term. Since $f_1 \leq f_2$ a.e., we have $(f_1 - f_2)v \leq 0$.

5. Energy inequality for the difference. Substituting the estimates of steps 2–4 into (6.2) yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|^2_{L^2} + \nu_0 \int_\Omega |\nabla v|^p \,\mathrm{d}x \leq \int_\Omega \mu_1 |\nabla v|^{p-1} v \,\mathrm{d}x + L \int_\Omega v^p \,\mathrm{d}x. \tag{6.3}$$

6. Absorption of the singular term. For a.e. fixed t, $v(\cdot,t) \in W^{1,p}_0(\Omega)$. Apply the p-gradient-form-boundedness (Definition 2.1) to $v(\cdot,t)$ with $\varepsilon = \nu_0/2$:

$$\int_\Omega \mu_1 |\nabla v|^{p-1} v \,\mathrm{d}x \leq \frac{\nu_0}{2} \int_\Omega |\nabla v|^p \,\mathrm{d}x + C_{\nu_0/2} \int_\Omega v^p \,\mathrm{d}x.$$

Inserting this into (6.3) gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|^2_{L^2} + \frac{\nu_0}{2} \int_\Omega |\nabla v|^p \,\mathrm{d}x \leq \big(C_{\nu_0/2} + L\big) \int_\Omega v^p \,\mathrm{d}x. \tag{6.4}$$

7. Boundedness and Gronwall. By Theorem 4.3 (local boundedness), both u_1 and u_2 are locally bounded; hence their difference is bounded. There exists a constant M such that $0 \leq v \leq M$ a.e. in

Q_T. Consequently, $v^p \le M^{p-2} v^2$ a.e. Dropping the non-negative gradient term in (6.4) we obtain the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|_{L^2}^2 \le 2\big(C_{\nu_0/2} + L\big) M^{p-2}\, \|v(t)\|_{L^2}^2 =: K\, \|v(t)\|_{L^2}^2.$$

The function $y(t) = \|v(t)\|_{L^2}^2$ is absolutely continuous (by the analogue of Lemma 6.2 for the difference of two solutions), $y(0) = 0$, and $y'(t) \le K\, y(t)$ for a.e. $t \in (0,T)$. Integrating from 0 to t yields

$$y(t) \le K \int_0^t y(s)\, \mathrm{d}s.$$

Gronwall's inequality (integral form) implies $y(t) = 0$ for all $t \in [0,T]$. Hence $v \equiv 0$, i.e. $u_1 \le u_2$ a.e. in Q_T. □

7. Hölder Continuity of Solutions

We now prove that bounded weak solutions are locally Hölder continuous. For simplicity we assume the inhomogeneous terms vanish: $\varphi_1 = \varphi_3 = 0$, $c_1 = c_2 = 0$, $f = 0$. The general case follows by a perturbation argument (see [1, Ch. III, §10]). Let u be a bounded weak solution in $Q_{2R} \Subset Q_T$, $M = \operatorname{ess\,sup}_{Q_{2R}} |u|$.

7.1. Logarithmic estimate.

For $k \ge 0$ and $\varepsilon > 0$ define $w_\varepsilon = (u-k)_+ + \varepsilon > 0$.

Lemma 7.1 (Logarithmic Lemma). *Let η be a cut-off as in the Caccioppoli inequality. There exists $C > 0$ depending only on the structural data such that*

$$\iint_{Q_\rho} |\nabla \log w_\varepsilon|^p \,\mathrm{d}x\,\mathrm{d}t \le C\Big(\frac{R^{n+2}}{(R-\rho)^p} + 1\Big).$$

Proof. Test the weak formulation with

$$\varphi = \eta^p \big(w_\varepsilon^{1-p} - (M+\varepsilon)^{1-p} \big).$$

The time derivative and principal part estimates are identical to the classical case; they yield

$$\iint \partial_t u\, \varphi \ge -\frac{C R^{n+2}}{(R-\rho)^p},$$
$$\sum a_i \partial_i \varphi \le -\frac{p-1}{2} \nu\, \eta^p |\nabla \log w_\varepsilon|^p + C |\nabla \eta|^p.$$

The lower-order term is split as

$$\iint b\, \varphi \le \iint \mu_1 |\nabla u|^{p-1} \eta^p w_\varepsilon^{1-p} + (M+\varepsilon)^{1-p} \iint \mu_1 |\nabla u|^{p-1} \eta^p.$$

For the first singular term we use the p-gradient-form-boundedness applied to $\eta \log w_\varepsilon$ (after a suitable truncation), giving the bound $\frac{\nu}{8} \iint \eta^p |\nabla \log w_\varepsilon|^p + C \iint \eta^p (\log w_\varepsilon)^p$. The second integral is bounded by $C|Q_R|$ since $|\log w_\varepsilon| \le C(M,\varepsilon)$. For the constant part we apply Young's inequality:

$$(M+\varepsilon)^{1-p} \iint \mu_1 |\nabla u|^{p-1} \eta^p \le \frac{\nu}{8} \iint |\nabla u|^p + C \iint \mu_1^p \eta^p.$$

To bound $\iint \mu_1^p \eta^p$ we invoke Definition 2.1 with the function $u = \eta$ (which belongs to $L^p(0,T; W_0^{1,p}(\Omega))$ because η vanishes on the lateral boundary):

$$\iint \mu_1^p \eta^p \le \varepsilon \iint |\nabla \eta|^p + C_\varepsilon \iint \eta^p \le C \frac{R^{n+2}}{(R-\rho)^p} + C R^{n+2}.$$

Collecting all terms and absorbing the $\frac{\nu}{4}$ gradient part into the left-hand side yields the desired estimate. □

7.2. **Oscillation decay.** Define the oscillation of u on an intrinsic cylinder Q_ρ as

$$\omega(\rho) = \operatorname{ess\,sup}_{Q_\rho} u - \operatorname{ess\,inf}_{Q_\rho} u.$$

Proposition 7.2 (Oscillation decay). *There exist constants $\sigma \in (0,1)$ and $\alpha > 0$, depending only on the structural data, such that for any intrinsic cylinder $Q_{2r} \Subset Q_T$,*

$$\omega(r) \le \sigma\, \omega(2r) + C r^{\alpha}. \tag{7.1}$$

Proof. As explained at the beginning of Section 7, we may assume $\varphi_1 = \varphi_3 = 0$, $c_1 = c_2 = 0$, $f = 0$; the general case adds only harmless constants (see [1, Ch. III, §10]). Let u be a bounded weak solution in $Q_{2r} \Subset Q_T$ and set

$$M_{2r} = \operatorname{ess\,sup}_{Q_{2r}} u, \qquad m_{2r} = \operatorname{ess\,inf}_{Q_{2r}} u, \qquad \omega_{2r} = M_{2r} - m_{2r}.$$

If $\omega_{2r} = 0$ the statement is trivial. Otherwise replace u by $\dfrac{2(u - m_{2r})}{\omega_{2r}} - 1$; then

$$\operatorname{ess\,inf}_{Q_{2r}} u = -1, \qquad \operatorname{ess\,sup}_{Q_{2r}} u = 1, \qquad \omega_{2r} = 2.$$

Both the Caccioppoli inequality (Lemma 4.2) and the logarithmic estimate (Lemma 7.1) are invariant under such affine transformations, so we work with this normalized solution.

Throughout the proof we denote by γ the universal exponent

$$\gamma = \frac{p}{n+2} > 0.$$

Constants C may change from line to line but depend only on the structural data and on the constants appearing in Lemma 4.2 and Lemma 7.1.

1. First alternative.

Lemma 7.3. *There exists a constant $\delta_0 \in (0,1)$ such that if for some $k \ge 0$ and $\rho \in (0,1]$ we have*

$$|A_{k,\rho}| \le \delta_0 |Q_\rho|, \qquad A_{k,\rho} = Q_\rho \cap \{u > k\},$$

then

$$\operatorname{ess\,sup}_{Q_{\rho/2}} u \le k + \frac{1}{2}\big(\operatorname{ess\,sup}_{Q_\rho} u - k\big).$$

Proof. Let $M_\rho = \operatorname{ess\,sup}_{Q_\rho} u$. If $k \ge M_\rho$ there is nothing to prove; we assume $k < M_\rho$. Set

$$\rho_n = \frac{\rho}{2} + \frac{\rho}{2^{n+1}}, \qquad k_n = k + (M_\rho - k)\Big(1 - \frac{1}{2^{n+1}}\Big), \qquad n = 0, 1, 2, \dots$$

Then $\rho_0 = \rho$, $\rho_\infty = \rho/2$, $k_0 = k + \frac{1}{2}(M_\rho - k)$, $k_\infty = M_\rho$. Define $v_n = (u - k_n)_+$ and $A_n = Q_{\rho_n} \cap \{u > k_n\}$.

Apply the Caccioppoli inequality (4.3) with $\Psi = 0$ (the data have been removed) on the pair $Q_{\rho_{n+1}} \subset Q_{\rho_n}$. Since $\rho_n - \rho_{n+1} = \rho/2^{n+2}$,

$$\iint_{A_{n+1}} |\nabla u|^p \le \frac{C}{(\rho_n - \rho_{n+1})^p} \iint_{A_n} (u - k_n)^p = \frac{C\, 2^{np}}{\rho^p} \iint_{A_n} v_n^p.$$

Set $Y_n = \iint_{A_n} v_n^p$. Using the parabolic Sobolev embedding (Theorem 2.5) and Hölder's inequality as in the proof of the local boundedness (Theorem 4.3) we obtain

$$Y_{n+1} \le \frac{C\, b^n}{\rho^{n+2}} |A_n|^{\gamma} Y_n^{1+\gamma}, \qquad \gamma = \frac{p}{n+2},$$

where $b > 1$ depends only on the structural constants.

The measure of A_n is estimated by Chebyshev's inequality:

$$|A_{n+1}| \le \Big(\frac{2^{n+2}}{M_\rho - k}\Big)^p Y_n.$$

Since $|A_n| \le |A_0| = |A_{k,\rho}|$, we may replace $|A_n|^\gamma$ by $|A_{k,\rho}|^\gamma$ and then invoke the De Giorgi iteration lemma (Lemma A.2). The condition $Y_n \to 0$ (which implies $\operatorname{ess\,sup}_{Q_{\rho/2}} u \le k$) is satisfied if

$$Y_0 \le C^{-\frac{1}{\gamma}} b^{-\frac{1}{\gamma^2}} \rho^{\frac{n+2}{\gamma}} |A_{k,\rho}|^{-1}.$$

Because $Y_0 = \iint_{A_{k,\rho}} (u-k)^p \le (M_\rho - k)^p |A_{k,\rho}|$, this inequality holds when

$$|A_{k,\rho}| \le \delta_0 |Q_\rho|,$$

where δ_0 can be chosen as

$$\delta_0 = \big(C^{-\frac{1}{\gamma}} b^{-\frac{1}{\gamma^2}} \rho^{\frac{n+2}{\gamma}} |Q_\rho|^{-1} (M_\rho - k)^{-p}\big)^{\frac{1}{1+\gamma}}.$$

Since $\rho^{\frac{n+2}{\gamma}} = \rho^{n+2}$ and $|Q_\rho| = c_n \rho^{n+2}$, the powers of ρ cancel. Moreover, $M_\rho \le 1$ after normalization, so $(M_\rho - k)^{-p} \ge 1$. Hence δ_0 can be taken independent of ρ and k, depending only on the structural constants. This proves the first alternative. □

2. Second alternative.

Lemma 7.4. *There exist constants $\delta_1 \in (0,1)$ and $\lambda > 0$ such that if for some $\rho \in (0,1]$*

$$|\{u > 0\} \cap Q_\rho| \ge \delta_1 |Q_\rho|,$$

then

$$\operatorname{ess\,inf}_{Q_{\rho/2}} u \ge \lambda.$$

Proof. Let $v = 1 - u$, which is non-negative and satisfies an equation of the same type. The condition $|\{u > 0\} \cap Q_\rho| \ge \delta_1 |Q_\rho|$ becomes $|\{v < 1\} \cap Q_\rho| \ge \delta_1 |Q_\rho|$. Apply the logarithmic estimate (Lemma 7.1) to v with $k = 0$ and $\varepsilon > 0$ (to be chosen later). For any $\theta \in (0,1)$,

$$|\{v \le \theta\} \cap Q_{\rho/2}| \le \frac{C}{|\log \theta|^p} \big(1 + \frac{\rho^{n+2}}{(\rho/2)^p}\big) \le \frac{C}{|\log \theta|^p} |Q_{\rho/2}|.$$

(The logarithmic estimate gives a bound on the L^p norm of $\nabla \log(v + \varepsilon)$, which via Poincaré and measure arguments translates into the above measure estimate; see [1, Ch. III, Lemma 7.2] for the detailed derivation.)

Choose θ so small that $\frac{C}{|\log \theta|^p} \le \frac{1}{2}$. Then $|\{v \le \theta\} \cap Q_{\rho/2}| \le \frac{1}{2} |Q_{\rho/2}|$, which means $v > \theta$ on at least half of $Q_{\rho/2}$. A standard covering argument (or a second application of the logarithmic estimate on a covering of $Q_{\rho/2}$ by smaller cylinders) lifts this pointwise lower bound to the whole $Q_{\rho/2}$: there exists $\lambda > 0$, depending only on θ and the structural constants, such that $\operatorname{ess\,inf}_{Q_{\rho/2}} v \ge \lambda$. Hence $\operatorname{ess\,inf}_{Q_{\rho/2}} u \le 1 - \lambda$. Repeating the argument with v replaced by $u + 1$ (or applying the same reasoning to $-u$) gives a symmetric bound for the infimum. Thus $\operatorname{ess\,inf}_{Q_{\rho/2}} u \ge \lambda'$ for some $\lambda' > 0$. Taking $\lambda = \min\{\lambda', 1 - \lambda\}$ yields the claim. The constant δ_1 is exactly the value that makes the measure of $\{u > 0\}$ large enough to trigger the above estimate; it can be chosen as the same δ_0 from the first alternative or any smaller number. □

3. Proof of oscillation decay. Let $\delta = \min\{\delta_0, \delta_1\}$. Consider the cylinder Q_1 (i.e., radius 1). We distinguish two cases.

Case 1. $|\{u > 0\} \cap Q_1| \le \delta |Q_1|$. Then Lemma 7.3 with $k = 0$ and $\rho = 1$ gives $\operatorname{ess\,sup}_{Q_{1/2}} u \le \frac{1}{2}$. Since $\operatorname{ess\,inf}_{Q_{1/2}} u \ge -1$, $\omega(1/2) \le \frac{3}{2} = \frac{3}{4}\,\omega(2)$.

Case 2. $|\{u > 0\} \cap Q_1| > \delta |Q_1|$. Then Lemma 7.4 gives $\operatorname{ess\,inf}_{Q_{1/2}} u \geq \lambda$ for some $\lambda > 0$. Hence $\omega(1/2) \leq 1 - \lambda \leq (1-\lambda)\,\omega(2)$.

In both cases we have shown that

$$\omega(1/2) \leq \sigma\,\omega(2), \qquad \sigma = \max\{\tfrac{3}{4},\, 1-\lambda\} < 1.$$

By scaling, the same argument applies to any cylinder Q_{2r}: after the affine normalization the problem reduces exactly to the situation above. Therefore

$$\omega(r/2) \leq \sigma\,\omega(r) \qquad \forall\, r > 0 \text{ with } Q_{2r} \Subset Q_T.$$

Iterating this inequality m times on a chain of nested cylinders $Q_{r/2^k}$ yields

$$\omega(r/2^m) \leq \sigma^m \omega(r).$$

Standard dyadic arguments then give the full decay estimate (7.1) with $\alpha = 0$ in the homogeneous case. The general case with non-zero data adds a term Cr^α; see [1, Ch. III, §10] for the perturbation argument. □

7.3. **Hölder continuity.** Iterating the oscillation decay proposition gives the main regularity theorem.

Theorem 7.5 (Interior Hölder continuity)**.** *Let u be a bounded weak solution of (3.1) under Assumptions 3.1 and 3.2. Then u is locally Hölder continuous in Q_T: there exist $\alpha \in (0,1)$ and $C > 0$ such that for any $Q' \Subset Q_T$,*

$$|u(x_1,t_1) - u(x_2,t_2)| \leq C\big(|x_1 - x_2| + |t_1 - t_2|^{1/p}\big)^{\alpha}.$$

Proof. By a perturbation argument (see [1, Ch. III, §10]) it suffices to prove the statement under the additional assumptions $\varphi_1 = \varphi_3 = 0$, $c_1 = c_2 = 0$, $f = 0$. We work with these throughout the proof.

Let u be a bounded weak solution and fix a cylinder $Q_{2R} \Subset Q_T$. Proposition 7.2 together with the normalisation argument in its proof actually yields the sharper inequality

$$\omega(r/2) \leq \sigma\,\omega(r) \qquad \forall\, r \in (0, R], \tag{7.2}$$

where $\sigma \in (0,1)$ depends only on the structural data. (Indeed, when the inhomogeneous terms vanish, the constant Cr^α in the proposition disappears because the affine normalisation $u \mapsto \frac{2(u - m_{2r})}{\omega(2r)} - 1$ makes the equation homogeneous, and the iteration in the proof of the proposition yields exactly the geometric decay.)

Let $\beta = -\log_2 \sigma > 0$. For any $\rho \in (0, R]$ choose the integer $m \geq 0$ such that $2^{-m-1}R < \rho \leq 2^{-m}R$. Then by (7.2),

$$\omega(\rho) \leq \omega(2^{-m}R) \leq \sigma^m \omega(R) \leq 2^{\beta}\Big(\frac{\rho}{R}\Big)^{\beta} \omega(R).$$

Hence there is a constant $C_0 = 2^\beta \omega(R) R^{-\beta}$ such that for every $\rho \in (0, R]$,

$$\omega(\rho) \leq C_0 \rho^\beta. \tag{7.3}$$

Now let $Q' \Subset Q_T$ be any compact subset. There exists $R_0 > 0$ such that the intrinsic cylinder $Q_{2R_0}(x,t)$ is contained in Q_T for every $(x,t) \in Q'$, and Q' itself can be covered by finitely many cylinders $Q_{R_0}(x_i,t_i)$. On each such cylinder the estimate (7.3) holds with a uniform constant C_0 (because the structural constants are universal and u is bounded on Q_T).

Define the parabolic distance in $\mathbb{R}^n \times \mathbb{R}$ by

$$d_p\big((x_1,t_1),(x_2,t_2)\big) = \max\big\{|x_1 - x_2|\,,\ |t_1 - t_2|^{1/p}\big\}.$$

Take two points $(x_1,t_1), (x_2,t_2) \in Q'$ and set $\rho = d_p((x_1,t_1),(x_2,t_2))$. Without loss of generality we may assume $\rho \leq R_0$ (otherwise the Hölder estimate follows from the uniform boundedness of u on Q' by taking C large enough).

Consider the intrinsic cylinder $Q_\rho(x_1,t_1)$. By the definition of d_p, we have $(x_2,t_2)\in Q_\rho(x_1,t_1)$. Moreover, $Q_\rho(x_1,t_1)\subset Q_{2R_0}(x_1,t_1)\Subset Q_T$ because $\rho\le R_0$. Hence the oscillation of u on $Q_\rho(x_1,t_1)$ satisfies (7.3):

$$|u(x_1,t_1)-u(x_2,t_2)|\le \operatorname{ess\,sup}_{Q_\rho(x_1,t_1)} u-\operatorname{ess\,inf}_{Q_\rho(x_1,t_1)} u=\omega(\rho)\le C_0\rho^\beta.$$

By the definition of ρ, this is exactly

$$|u(x_1,t_1)-u(x_2,t_2)|\le C_0\max\bigl\{|x_1-x_2|\,,\ |t_1-t_2|^{1/p}\bigr\}^\beta\le C_0\bigl(|x_1-x_2|+|t_1-t_2|^{1/p}\bigr)^\beta.$$

Setting $\alpha=\beta$ and $C=C_0$ (depending on the data and Q') completes the proof. □

8. Boundary Regularity and The Problem on Rough Domains

8.1. **Boundary Hölder continuity for Lipschitz domains.** When Ω is a bounded Lipschitz domain, the local estimates proved in the interior can be carried over to the lateral boundary $\partial\Omega\times(0,T)$. The main idea is to straighten the boundary locally by a bi-Lipschitz map and then to apply the interior estimates in a half-cylinder.

Flattening of the boundary. Let $x_0\in\partial\Omega$. Since Ω is Lipschitz, there exist a neighbourhood U of x_0 and a bi-Lipschitz map $\Phi: U\to\mathbb{R}^n$ such that $\Phi(U\cap\Omega)=B_1^+:=\{y\in B_1(0): y_n>0\}$ and $\Phi(U\cap\partial\Omega)=B_1\cap\{y_n=0\}$. The map Φ and its inverse preserve the Sobolev spaces $W^{1,p}$ and the parabolic energy space V_p (up to an equivalence of norms).

Define the transformed functions

$$\tilde u(y,s)=u(\Phi^{-1}(y),s),\qquad \tilde a_i(y,s,\tilde u,\nabla\tilde u)=\text{(transformed coefficients)},\dots$$

A standard computation shows that $\tilde u$ satisfies an equation of the same type as (3.1) in the half-cylinder $B_1^+\times(0,T)$, with a zero Dirichlet condition on the flat part $\{y_n=0\}$. The transformed coefficients still satisfy Assumptions 3.1 and 3.2 because Φ is bi-Lipschitz; in particular the p-gradient-form-boundedness of μ_1 is preserved (the change of variables only introduces bounded factors into the inequality (2.1)).

Caccioppoli and logarithmic estimates up to the boundary. In the half-cylinder $Q_R^+=B_R^+\times(-R^p,0)$ (where $B_R^+=B_R\cap\{y_n>0\}$), the weak solution $\tilde u$ vanishes on the flat part $\{y_n=0\}$. For a level $k\ge0$ the truncated function $(\tilde u-k)_+$ also vanishes on $\{y_n=0\}$, hence it belongs to $L^p((-R^p,0);W_0^{1,p}(B_R^+))$. The test functions used in the Caccioppoli inequality (Lemma 4.2) and in the logarithmic estimate (Lemma 7.1) can be multiplied by a cut-off function that equals 1 on a smaller half-cylinder and vanishes on the curved part of the boundary, and they remain admissible because they vanish on $\{y_n=0\}$. The computations go through exactly as in the interior case, yielding the same inequalities with constants that depend on the Lipschitz character of $\partial\Omega$ (through the bi-Lipschitz constants) but are otherwise universal.

Oscillation decay and Hölder continuity. With the Caccioppoli and logarithmic estimates available in half-cylinders, the two- alternative proof of the oscillation decay (Proposition 7.2) can be repeated word for word for the transformed solution $\tilde u$ in Q_R^+. One obtains

$$\omega(r)\le\sigma\,\omega(2r)+Cr^\alpha$$

for cylinders centered at boundary points, where the oscillation is now measured in Q_ρ^+. Iteration then gives Hölder continuity of $\tilde u$ up to the flat boundary $\{y_n=0\}$, and mapping back with Φ^{-1} gives Hölder continuity of u up to $\partial\Omega\times[0,T]$.

Because the initial condition u_0 is only in $L^2(\Omega)$, the solution may not be Hölder continuous down to $t=0$; however the standard theory (see [1]) shows that for every $\delta>0$ the solution is Hölder continuous in $\overline{\Omega}\times[\delta,T]$. If u_0 is bounded and Hölder continuous itself, then the Hölder continuity holds up to $t=0$ as well. For the purposes of this paper we simply state the result for positive times.

Theorem 8.1 (Boundary Hölder continuity). *Let Ω be a bounded Lipschitz domain and let u be a bounded weak solution of (3.1)–(3.3) under Assumptions 3.1 and 3.2. Then for every $\delta > 0$, u is Hölder continuous in $\overline{\Omega} \times [\delta, T]$, i.e. $u \in C^{0,\alpha}(\overline{\Omega} \times [\delta, T])$ for some $\alpha \in (0,1)$ depending only on the structural data and the Lipschitz constant of $\partial\Omega$.*

8.2. **Existence on arbitrary bounded open sets.** The variational framework employed in the existence proof (Section 5) never uses any smoothness of $\partial\Omega$. Indeed, the Galerkin method requires only a dense sequence in $W_0^{1,p}(\Omega)$, which exists for any open set Ω, and the energy space $V_p(Q_T)$ is well defined. The a priori estimates of Section 4 are local and rely solely on the structural assumptions; the Caccioppoli inequality, De Giorgi iteration, and logarithmic lemma are proved on intrinsic cylinders strictly contained in Q_T, without touching the boundary.

Therefore, the existence theorem (Theorem 5.2) and the interior Hölder continuity (Theorem 7.5) remain valid for any bounded open set $\Omega \subset \mathbb{R}^n$, regardless of its boundary regularity. The solution u satisfies $u = 0$ on $\partial\Omega \times (0,T)$ in the sense of traces and is locally Hölder continuous in $\Omega \times (0,T]$.

Theorem 8.2 (Existence on rough domains). *Let $\Omega \subset \mathbb{R}^n$ be an arbitrary bounded open set, $T > 0$. Under Assumptions 3.1 and 3.2 there exists a weak solution $u \in V_p(Q_T) \cap L^\infty(Q_T)$ of (3.1)–(3.3). Moreover, u is locally Hölder continuous in $\Omega \times (0,T]$.*

Proof. The existence part is exactly the content of Theorem 5.2, whose proof uses only the separability of $W_0^{1,p}(\Omega)$ and the compact embedding $V_p(Q_T) \hookrightarrow L^p(Q_T)$; both hold for any bounded open set. The interior Hölder continuity is given by Theorem 7.5, which is a local statement and makes no reference to $\partial\Omega$. □

9. Conclusion

We have developed a comprehensive theory for quasilinear parabolic equations of p-Laplacian type under the sole assumption of p-gradient-form-boundedness for the singular coefficient μ_1. This condition, strictly weaker than the classical LSU integrability condition, allowed us to carry out the entire De Giorgi–Ladyzhenskaya programme: global and local a priori estimates, existence via Galerkin method, maximum and comparison principles, and Hölder continuity of weak solutions. The results hold on arbitrary bounded domains, with boundary regularity proved for Lipschitz cylinders. Future work will extend this framework to systems and to time-dependent coefficients with stronger singularities, and to the study of Harnack inequalities under form-boundedness.

Statements and Declarations

The author declares that he has no conflict of interest, and the manuscript has no associated data.

Appendix A. Technical Lemmas

Lemma A.1 (Young's inequality). *For $a, b \geq 0$, $p > 1$, $p' = p/(p-1)$, and any $\varepsilon > 0$, $ab \leq \varepsilon a^p + \varepsilon^{-p'/p}(p')^{-1}b^{p'}$.*

Lemma A.2 (De Giorgi iteration). *Let $\{y_n\}$ satisfy $y_{n+1} \leq Cb^n y_n^{1+\delta}$ with $C, b > 1$, $\delta > 0$. If $y_0 \leq C^{-1/\delta}b^{-1/\delta^2}$, then $y_n \to 0$.*

Lemma A.3 (Parabolic Poincaré inequality). *For $u \in L^p(0,T; W_0^{1,p}(\Omega))$, $\|u\|_{L^p(Q_T)} \leq C_P \|\nabla u\|_{L^p(Q_T)}$. If $\Omega = B_R$, then $C_P \leq cR$.*

Lemma A.4 (Aubin–Lions compactness). *Let X, Y, Z be Banach spaces such that $X \hookrightarrow Y$ is compact and $Y \hookrightarrow Z$ is continuous. Let $\{u_m\}$ be a bounded sequence in $L^p(0,T;X)$ ($1 \leq p < \infty$) and let $\{\partial_t u_m\}$ be bounded in $L^1(0,T;Z)$. Then $\{u_m\}$ is relatively compact in $L^p(0,T;Y)$.*

Proof. This is a classical result of J. Simon (Annali di Matematica Pura ed Applicata (4) **146** (1987), 65–96, Corollary 4). A direct proof can be given using the compactness of X in Y, which yields for every $\varepsilon > 0$ a constant C_ε with $\|v\|_Y \le \varepsilon \|v\|_X + C_\varepsilon \|v\|_Z$ for all $v \in X$. From the bound on $\partial_t u_m$ one obtains, for $h > 0$ and a.e. t, $\|u_m(t+h) - u_m(t)\|_Z \le \int_t^{t+h} \|\partial_t u_m(s)\|_Z \, \mathrm{d}s$. Combining these estimates and integrating in t shows that the $L^p(0,T;Y)$-norm of the translations $u_m(\cdot + h) - u_m(\cdot)$ tends to zero uniformly in m as $h \to 0$. The Fréchet–Kolmogorov–Riesz theorem (or the Simon compactness criterion) then yields the desired relative compactness. $\square$

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