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ON BOYD-WONG TYPE CONTRACTION IN GENERALIZED METRIC SPACES AND ITS APPLICATION TO THE STABILITY OF SOME NONLINEAR DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we introduce a Boyd-Wong type contraction in generalized metric spaces and prove the existence of a unique fixed point for such mappings. We investigate the stability of a general class of nonlinear differential equations in the sense of Hyers–Ulam and Hyers–Ulam–Rassias using our obtained fixed point result. Our results extend and improve some existence results and stability results in literature. Some illustrative examples are given to show the usability of our main results.

Keywords. Boyd-Wong type contraction, fixed point, generalized metric space, stability, fractional inclusions.

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1. Introduction

The study of data dependence in the theory of differential equations is a fundamental area of research, focusing on the sensitivity of solutions to perturbations in initial data and parameters. This field encompasses concepts such as monotonicity, continuity, and differentiability of solutions. Among these, Ulam stability has gained significant attention over the last few decades as a specialized framework for studying how closely an approximate solution remains to an exact one. The origin of this subject traces back to 1940, when S. M. Ulam posed a celebrated question regarding the stability of functional equations in a talk at the University of Wisconsin. In 1941, D. H. Hyers provided the first partial solution for linear equations in Banach spaces using what is now known as the "direct method". Hyers demonstrated that for every approximately additive mapping, there exists a unique exact additive mapping nearby. In 1978, T. M. Rassias generalized this result by allowing the Cauchy difference to be unbounded, a concept now widely termed Hyers–Ulam–Rassias stability. While the direct method was pioneering, it often required restrictive assumptions and complex manual constructions of solutions, particularly for nonlinear systems. To address these limitations, researchers shifted toward the fixed point technique, which offers a more unified and flexible framework for proving stability. A pivotal development in this direction was the introduction of generalized complete metric spaces (GCMS) by Luxemburg (1958), where the distance between two points is permitted to be infinity. Building on this notion, Diaz and Margolis [3] proved a fundamental theorem of the alternative for strictly contractive operators in GCMS. This theorem provides a powerful tool for establishing the existence and uniqueness of fixed points in spaces that may consist of disjoint metric components. In 2010, Soon-Mo Jung [6] utilized this fixed point approach to investigate the stability of first-order differential equations of the form $y' = F(x, y)$. However, Jung's original results (Theorem 3.1 in [6]) were bound by the requirement that the product of certain constants must satisfy $KL < 1$ (or $Lr < 1$), where L is the Lipschitz constant. This condition meant that stability could only be guaranteed for systems with high

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contraction rates or within restricted intervals. Subsequently, in 2020, Başcı, Misir, and Öğrekçi ([2]) provided a significant improvement to Jung's work. By introducing a weighted metric that incorporated an exponential function $(e^{-M(x-x_0)})$, they succeeded in modifying the contraction properties of the operator. This creative manipulation of the metric space allowed them to completely remove the restrictive condition $KL < 1$, establishing stability results under much fewer and weaker assumptions than previously required. Despite the advancements made in ([2]), their methodology and most existing literature remain dependent on a fixed Lipschitz constant (L). Such linear contraction conditions are often too rigid for many complex nonlinear models. In the present paper, we aim to extend and generalize the stability results of both Jung and Başcı et al. by introducing a Boyd-Wong type contraction in the framework of generalized complete metric spaces. By replacing the fixed Lipschitz constant with an altering distance function, we establish stability criteria for a broader class of nonlinear differential equations where the "rate of contraction" may vary dynamically. Our approach not only unifies the previous results but also covers "edge cases" where classical Lipschitzian methods fail due to the contraction ratio approaching unity.

Let I be an interval of the real numbers and $F : I \times \mathbb{R} \to \mathbb{R}$ be a given mapping. Let for any differentiable function $f : I \to \mathbb{R}$ satisfying the inequality

$$|f'(x) - F(x, f(x))| \leq \varepsilon$$

for all $x \in I$ and some $\varepsilon > 0$, there exists a solution $y_0 : I \to \mathbb{R}$ of the differential equation

$$y'(x) = F(x, y(x)) \tag{1.1}$$

such that

$$|f(x) - y_0(x)| \leq K(\varepsilon),$$

for any $x \in I$, where $K : (0, \infty) \to (0, \infty)$ is a function. Then we say that the differential equation (1.1) has the Hyers-Ulam stability. The next results are obtained independently in order to provide sufficient conditions which under them the Hyers-Ulam stability is true.

Theorem 1.1. [2] *Assume that $F : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function which satisfies a Lipschitz condition*

$$|F(x, y_1) - F(x, y_2)| \leq L|y_1 - y_2|$$

for all $x \in I$ and all $y_1, y_2 \in \mathbb{R}$, where $L > 0$ is a Lipschitz constant. If a continuously differentiable function $y : I \to \mathbb{R}$ satisfies

$$|y'(x) - F(x, y(x))| \leq \varepsilon$$

for all $x \in I$ and some $\varepsilon \geq 0$, then there exists a unique solution y_0 of (1.1) satisfying

$$|y(x) - y_0(x)| \leq (1 + L)r\varepsilon,$$

for all $x \in I$.

Theorem 1.2. [6] *Given $c \in \mathbb{R}$ and $r > 0$, let I denote a closed ball of radius r and centered at c, that is, $I = \{x \in \mathbb{R} | c - r \leq x \leq c + r\}$ and let $F : I \times \mathbb{R} \to \mathbb{R}$ be a continuous function which satisfies the Lipschitz condition 1.4 for all $x \in I$ and $y, z \in \mathbb{R}$, where L is a constant with $0 < Lr < 1$. If a continuously differentiable function $y : I \to \mathbb{R}$ satisfies the differential inequality*

$$|y'(x) - F(x, y(x)| \leq \varepsilon \tag{1.2}$$

for all $x \in I$ and for some $\varepsilon > 0$, then there exists a unique continuous function $y_0 : I \to \mathbb{R}$ satisfying the integral equation (1.7) (y_0 is a solution to (1.1)) and

$$|y(x) - y_0(x)| \leq \frac{r}{1 - Lr}\varepsilon \tag{1.3}$$

for any $x \in I$.

If in the definition of Hyers-Ulam stability, we replace ε and $K(\varepsilon)$ by $\varphi(x)$ and $\Phi(x)$, respectively, where $\varphi, \Phi : I \to [0, \infty)$ are functions not depending on f and y_0 explicitly, then we say that the corresponding differential equation has the Hyers-Ulam-Rassias stability (or the generalized Hyers-Ulam stability). We may apply these terminologies for other equations. We refer to [7, 6, 10, 11], for more details of the Hyers-Ulam stability and the Hyers-Ulam-Rassias stability. In [6], Jung studied the Hyers-Ulam stability and Hyers-Ulam-Rassias stability result for the differential equation (1.1) by using the Lipschitz condition on the function F. The following result due to Jung [6].

Theorem 1.3. [6] *For given real numbers a and b with $a < b$, let $I = [a, b]$ be a closed interval and $c \in I$. Let K and L be positive constants with $0 < KL < 1$. Assume that $F : I \times \mathbb{R} \to \mathbb{R}$ is a continuous mapping which satisfies a Lipschitz condition*

$$|F(x, y) - F(x, z)| \leq L|y - z| \tag{1.4}$$

for all $x \in I$ and $y, z \in \mathbb{R}$. If a continuously differentiable function $y : I \to \mathbb{R}$ satisfies

$$|y'(x) - F(x, y(x))| \leq \phi(x) \tag{1.5}$$

for all $x \in I$, where $\phi : I \to (0, \infty)$ is a continuous function with

$$|\int_c^x \phi(t)dt| \leq K\phi(x) \tag{1.6}$$

for each $x \in I$, then there exists a unique continuous function $y_0 : I \to \mathbb{R}$ such that

$$y_0(x) = y(c) + \int_c^x F(\tau, y_0(\tau))d\tau \tag{1.7}$$

(consequently, y_0 is a solution to 1.1)) and

$$|y(x) - y_0(x)| \leq \frac{K}{1 - KL}\phi(x) \tag{1.8}$$

for all $x \in I$.

In this paper, we first introduce a new contraction which is called Boyd-Wong type contraction, for the function F and, by using it, the Hyers-Ulam stability and Hyers-Ulam-Rassias stability for the differential equation (1.1) under suitable conditions are stated. By providing examples, it is shown that they are real generalization of Jung's results [6].

2. Preliminaries and Auxiliary Notions

The following definition plays a crucial role in this paper.

Definition 2.1. [9]

For a nonempty set X, the function $d : X \times X \to [0, +\infty]$ is called a generalized metric on X if and only if d satisfies:

(M1) $d(x, y) = 0$ if and only if $x = y$;
(M2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(M3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

We remark that the only difference between generalized metric and usual metric is that the range of the mapping d is permitted to include the infinity. This concept differs from the usual concept of a metric space by the fact that not every two points in X have necessarily a finite distance.

The following examples illustrate Definition 2.1

Example 2.2. Let $X = \mathbb{R} = Q \cup Q^c$, the real numbers, and $d : X \times X \to [0, \infty]$ be defined by

$$d(x, y) = \begin{cases} 0, & x = y; \\ 1, & x \neq y, (x, y) \in Q \times Q\ ; \\ \infty, & Otherwise\ . \end{cases}$$

It is obvious that d is a generalized metric and if $\{x_n\}$ is a Cauchy sequence in X then there exists positive integer n_0 such that $x_n = x_{n_0}$, for each $n \geq n_0$ which implies the sequence $\{x_n\}$ is a convergent sequence. Hence the generalized metric space (X, d) is complete.

We need the following example in the next.

Example 2.3. Let X be the set of all continuous functions on $I = [a, b]$. If we define $d : X \times X \to [0, \infty]$ by

$$d(f, g) := \inf\{C \in [0, \infty] : |f(x) - g(x)|e^{-M(x-x_0)} \leq C\Phi(x)\ , x \in I\},$$

where M is a positive real number, $x_0 \in I$, and $\Phi : I \to (0, \infty)$ a continuous function. It is easy to check that d satisfies the axioms $M2, M3$ of a generalized metric but not necessarily in $M1$. If we take the function Φ such $\inf \Phi$ equals to zero and f, g two elements of X with $\inf |f - g| > 0$, then $d(f, g) = \infty$. Hence, in this case, $d(f, g) = 0$ for $f = g$ and $d(f, g) = \infty$ when with $\inf |f - g| > 0$, which is similar to the discrete metric on a set.

We now introduce one of the fundamental results of fixed point theory. For the proof, we refer the readers to [3] and the references therein. This theorem plays an important role in proving the stability for the differential equation (1.1) in [6]. Before doing it, we need the following definition which is an extension of the definition of Banach contraction from usual metric spaces to the generalized metric spaces.

Definition 2.4. [1, 3, 4]
Let (X, d) be a generalized metric space and $T : X \to X$ a map. The map T is called a contraction, if there is a constant (Lipschitz constant) $r \in (0, 1)$ such that $d(Tu, Tv) \leq rd(u, v)$ for all $(u, v) \in X \times X$ with $d(u, v) < \infty$.

It is obvious that if T is a contraction mapping then it is uniformly continuous.
The next result is a new version of the first Theorem given in [3]. Moreover, The Banach's contraction mapping theorem [1] and Luxemburg's contraction mapping theorem [9] for generalized metric spaces are special cases of it.

Theorem 2.5. *Let (Δ, δ) be a generalized complete metric space. Assume that $\chi : \Delta \to \Delta$ is a contraction mapping with the (Lipschitz) constant $L < 1$. If there exists a nonnegative integer k such that $\delta(\chi^k\xi, \chi^{k+1}\xi) < \infty$ for some $\xi \in \Delta$, then the followings are true:*

(i) *The sequence $\{\chi^n\xi\}$ converges to a fixed point ξ^* of χ;*
(ii) *ξ^* is the unique fixed point of χ in*

$$\Delta^* = \{\nu \in \Delta | \delta(\chi^k\xi, \nu) < \infty\};$$

(iii) *If $\nu \in \Delta^*$, then*

$$\delta(\nu, \xi^*) \leq \frac{1}{1 - L}\delta(\chi\nu, \nu).$$

In [6], Jung presented the following Hyers-Ulam-Rassias stability of the differential equation (1.1) in an unbounded interval.

Theorem 2.6. [6] *For given real numbers a and b, let I denote either $(-\infty, b]$ or $\mathbb{R}$ or $[a, \infty)$. Set either $c = a$ for $I = [a, \infty)$ or $c = b$ for $I = (-\infty, b]$ or c is a fixed real number if $I = \mathbb{R}$. Let K and L be positive constants with $0 < KL < 1$. Assume that $F : I \times R \to R$ is a continuous function which is Lipschitz with respect to the second variable (the Lipschitz condition 1.4). If a continuously differentiable function $y : I \to \mathbb{R}$ satisfies the differential inequality (1.5) for all $x \in I$, where $\phi : I \to (0, \infty)$ is a continuous function satisfying the condition (1.6) for any $x \in I$, then there exists a unique continuous function $y_0 : I \to \mathbb{R}$ which satisfies (1.7) and (1.8) for all $x \in I$.*

3. Boyd-Wong Fixed Point Theorem

Theorem 2.5 plays a crucial role in the references [2, 6]. Hence it is valuable if one can extend and improve it. The main goal of this section is to extend it by using the Boyd-Wong^s condition. In fact, we eliminate the condition $0 < KL < 1$ given in [6] and we also replace the Lipschitz condition by the Boyd-Wong condition. Moreover, we introduce a new version of the contraction such that Definition2.4 is a special case of it. The following notion is needed in the sequel.

Denote by Φ the set of all functions $\varphi : [0, \infty) \to [0, \infty)$ so that

(δ_1) φ is nondecreasing and upper semi-continuous,
(δ_2) $\varphi(t) < t$ for all $t > 0$.

The functions having the above mentioned properties are called altering distance functions. The simple example $\varphi(t) = [t]$, for all $t \geq 0$, is upper semi-continuous but $\varphi(t) < t$ for all $t > 0$ is not true. This means that the second item is independent of the first item in the definition of the alerting distance function. Moreover, if we take $\varphi(t) = |sin(t)|$ then it is continuous and $\varphi(t) < t$ for all $t > 0$ while it is not nondecreasing on $[0, \infty)$. Finally, it is easy to verify that any altering distance function ϕ has the properties $\phi(0) = 0$ and $\lim_{n\to\infty} \phi^n(t) = 0$ for all $t \geq 0$ and the class of the altering distance functions, that is Φ, is a convex subset of the class of all functions $\varphi : [0, \infty) \to [0, \infty)$.

Some examples of elements in Φ are the following functions defined on $[0, \infty)$:

(i) $\varphi_1(t) = \frac{t}{t+1}$,

(ii) $\varphi_2(t) = \ln(t + 1)$,

(iii) $\varphi_3(t) = kt\,(0 \leq k < 1)$,

(iv) $\varphi_4(t) = \frac{t}{a+bt}, a > 1, b > 0$.

The next definition is an extension of Definition2.4

Definition 3.1. Let (Δ, δ) be a generalized metric space and $\chi : \Delta \to \Delta$ be a self-mapping. We say that χ is a generalized Boyd-Wong type contraction if there exists $\varphi \in \Phi$ such that

$$\delta(\chi\xi, \chi\nu) \leq \varphi(\delta(\xi, \nu)), \tag{3.1}$$

for all $\xi, \nu \in \Delta$ with $\delta(\xi, \nu) < \infty$.

Remark that Definition2.4 is a special case of Definition3.1 because it is enough we take $\varphi(t) = rt$ for all $t \in [0, \infty)$.

The next result is an extension of Theorem 2.5 for alternating distance functions. Moreover, it is obvious that the Banach contraction theorem is a special case of it.

Theorem 3.2. *Let (Δ, δ) be a generalized complete metric space. Assume that $\chi : \Delta \to \Delta$ is a generalized Boyd-Wong type contraction with the altering distance function φ. If there exists a nonnegative integer k such that $\delta(\chi^k\xi, \chi^{k+1}\xi) < \infty$ for some $\xi \in \Delta$, then the followings are true:*

(i) *The sequence $\{\chi^n\xi\}$ converges to a fixed point ξ^* of χ;*

(ii) *If* $M = \Sigma_{i=0}^{\infty}\varphi^i(\delta(\chi^k\xi, \chi^{k+1}\xi)) < \infty$, *then* ξ^* *is the unique fixed point of* χ *in*

$$\Delta^* = \{\nu \in \Delta | \delta(\chi^k\xi, \nu) < \infty\};$$

(iii) *If* $\nu \in \Delta^*$, *then*

$$\delta(\nu, \xi^*) - \varphi(\delta(\nu, \xi^*)) \leq \delta(\chi\nu, \nu).$$

Moreover, if the function $\beta(t) = t - \varphi(t)$ *is increasing and onto on* $[0, \infty)$, *then*

$$\delta(\nu, \xi^*) \leq \beta^{-1}(\delta(\chi\nu, \nu)).$$

Proof. We provide a sequence $\{\xi_n\}$ in Δ with $\xi_0 = \xi, \xi_n = \chi^n\xi, n = 1, 2, \cdots$. Obviously, $\xi_{n+1} = \chi\xi_n$ for all $n \in \mathbb{N} \cup \{0\}$. Now

$$\delta(\xi_{k+1}, \xi_{k+2}) = \delta(\chi\xi_k, \chi\xi_{k+1}) \leq \varphi(\delta(\xi_k, \xi_{k+1})) \leq \delta(\xi_k, \xi_{k+1}) = \delta(\chi^k\xi, \chi^{k+1}\xi) < \infty$$

and

$$\delta(\xi_{k+2}, \xi_{k+3}) = \delta(\chi\xi_{k+1}, \chi\xi_{k+2}) \leq \varphi(\delta(\xi_{k+1}, \xi_{k+2})) \leq \delta(\xi_{k+1}, \xi_{k+2}) < \infty.$$

Continuing this process, we see that $\delta(\xi_n, \xi_{n+1}) < \infty$ for all $n \in \mathbb{N}$ with $n \geq k$ and $\{\delta(\xi_n, \xi_{n+1})\}_{n\geq k}$ is a decreasing sequence of nonnegative numbers with

$$\delta(\xi_{n+1}, \xi_{n+2}) \leq \varphi(\delta(\xi_n, \xi_{n+1})) \quad \text{for all } n \geq k. \tag{3.2}$$

Thus there exists $r \geq 0$ such that $\lim_{n\to\infty} \delta(\xi_n, \xi_{n+1}) = r$. We are going to show that $r = 0$. Suppose to the contrary, $r > 0$. Taking $\limsup_{n\to\infty}$ both sides of (3.2) and using the upper semicontinuity of φ, we get $r \leq \varphi(r)$ which is contradicted by (δ_2). Thus $r = 0$, that is $\lim_{n\to\infty} \delta(\xi_n, \xi_{n+1}) = 0$. We claim that $\{\xi_n\}$ is a Cauchy sequence. If $\{\xi_n\}$ is not Cauchy, then there are $\varepsilon > 0$ and subsequences $\{\xi_{m_i}\}$ and $\{\xi_{n_i}\}$ of $\{\xi_n\}$ so that

$$n_i > m_i > i \geq k,\ \delta(\xi_{m_i}, \xi_{n_i}) \geq \varepsilon \tag{3.3}$$

and

$$\delta(\xi_{m_i}, \xi_{n_i-1}) < \varepsilon.$$

Using (3.3), we get

$$\varepsilon \leq \delta(\xi_{m_i}, \xi_{n_i}) \leq \delta(\xi_{m_i}, \xi_{n_i-1}) + \delta(\xi_{n_i-1}, \xi_{n_i}) < \varepsilon + \delta(\xi_{n_i-1}, \xi_{n_i}).$$

If $i \to \infty$, then

$$\lim_{i\to\infty} \delta(\xi_{m_i}, \xi_{n_i}) = \varepsilon.$$

Also, we have

$$\begin{aligned}
&\delta(\xi_{m_i}, \xi_{n_i}) - \delta(\xi_{m_i}, \xi_{m_i+1}) - \delta(\xi_{n_i}, \xi_{n_i+1}) \\
&\leq \delta(\xi_{m_i+1}, \xi_{n_i+1}) \\
&\leq \delta(\xi_{m_i}, \xi_{m_i+1}) + \delta(\xi_{m_i}, \xi_{n_i}) + \delta(\xi_{n_i}, \xi_{n_i+1}).
\end{aligned}$$

As $i \to \infty$, we find

$$\lim_{i\to\infty} \delta(\xi_{m_i+1}, \xi_{n_i+1}) = \varepsilon.$$

From (3.1), we get

$$\delta(\xi_{m_i+1}, \xi_{n_i+1}) \leq \varphi(\delta(\xi_{m_i}, \xi_{n_i})). \tag{3.4}$$

Taking limsup in both sides of the last inequality and using the properties of φ we get

$$\varepsilon \leq \varphi(\varepsilon) < \varepsilon,$$

which is a contradiction.

Thus, $\{\xi_n\}$ is a Cauchy sequence in the complete metric space (Δ, δ). Hence, there is $\xi^* \in \Delta$ so that

$$\lim_{n\longrightarrow\infty} \delta(\xi_n, \xi^*) = 0. \tag{3.5}$$

Finally, we claim that $\chi\xi^* = \xi^*$. From 3.5, there exists $N \in \mathbb{N}$ so that $\delta(\xi_n, \xi^*) < \infty$ for all $n \geq N$. Now

$$\begin{aligned} \delta(\xi^*, \chi\xi^*) &\leq \delta(\xi^*, \xi_{n+1}) + \delta(\xi_{n+1}, \chi\xi^*) \\ &= \delta(\xi^*, \xi_{n+1}) + \delta(\chi\xi_n, \chi\xi^*) \\ &\leq \delta(\xi^*, \xi_{n+1}) + \varphi(\delta(\xi_n, \xi^*)). \end{aligned} \tag{3.6}$$

Taking $\limsup_{n\to\infty}$ on both sides of (3.6), we get $\delta(\xi^*, \chi\xi^*) = 0$, that is $\chi\xi^* = \xi^*$. So ξ^* is a fixed point of χ. Now we shall show that $\xi^* \in \Delta^*$. For any $m \in \mathbb{N}$ we have

$$\begin{aligned} \delta(\xi_k, \xi_{k+m}) &\leq \Sigma_{i=0}^{m-1}\delta(\xi_{k+i}, \xi_{k+i+1}) \\ &\leq \Sigma_{i=0}^{m-1}\varphi^i(\delta(\xi_k, \xi_{k+1})) \\ &= \Sigma_{i=0}^{m-1}\varphi^i(\delta(\chi^k\xi, \chi^{k+1}\xi)) \\ &\leq \Sigma_{i=0}^{\infty}\varphi^i(\delta(\chi^k\xi, \chi^{k+1}\xi)) = M < \infty. \end{aligned}$$

Taking $m \to \infty$ we get $\delta(\xi_k, \xi^*) \leq M < \infty$. Thus $\xi^* \in \Delta^*$. Now we shall show that ξ^* is the unique fixed point of χ in Δ^*. Let ζ^* be another fixed point of χ in Δ^*. Then we have $\delta(\xi^*, \zeta^*) \leq \delta(\xi^*, \chi^k\xi) + \delta(\chi^k\xi, \zeta^*) < \infty$. If $\xi^* \neq \zeta^*$, from (3.1), we get

$$\delta(\xi^*, \zeta^*) = \delta(\chi\xi^*, \chi\zeta^*) \leq \varphi(\delta(\xi^*, \zeta^*)) < \delta(\xi^*, \zeta^*),$$

which is a contradiction. Thus $\xi^* = \zeta^*$. To prove (iii), let ν is an element of Δ^*. Then $\delta(\nu, \chi^k\xi) < \infty$ and so $\delta(\nu, \xi^*) \leq \delta(\nu, \chi^k\xi) + \delta(\chi^k\xi, \xi^*) < \infty$. Therefore

$$\begin{aligned} \delta(\nu, \xi^*) &\leq \delta(\nu, \chi\nu) + \delta(\chi\nu, \xi^*) \\ &= \delta(\nu, \chi\nu) + \delta(\chi\nu, \chi\xi^*) \\ &\leq \delta(\nu, \chi\nu) + \varphi(\delta(\nu, \xi^*)). \end{aligned}$$

Thus, $\delta(\nu, \xi^*) - \varphi(\delta(\nu, \xi^*)) \leq \delta(\chi\nu, \nu)$. Moreover, if the function $\beta(t) = t - \varphi(t)$ be increasing and onto on $[0, \infty)$, then

$$\delta(\nu, \xi^*) \leq \beta^{-1}(\delta(\chi\nu, \nu)).$$

□

Remark. Since in a metric space (Δ, d) we have always $d(x, y) < \infty$ for all $x, y \in X$ and particularly $d(x, \chi x) < \infty$ for any $x \in X$, where $\chi : \Delta \to \Delta$ is a self-mapping, so it follows from the proof of Theorem3.2 that we can rewrite Theorem3.2 without the condition $\Sigma_{i=0}^{\infty}\varphi^i(\delta(\chi^k x, \chi^{k+1}x)) < \infty$ as follows.

Corollary 3.3. *Let (Δ, d) be a complete metric space. Assume that $\chi : \Delta \to \Delta$ is a generalized Boyd-Wong type contraction with the altering distance function φ. Then the followings are true:*

(i) *For any $\xi \in \Delta$, the sequence $\{\chi^n\xi\}$ converges to a fixed point ξ^* of χ;*
(ii) *ξ^* is the unique fixed point of χ in Δ.*
(iii) *For any $\nu \in \Delta$,*

$$\delta(\nu, \xi^*) - \varphi(\delta(\nu, \xi^*)) \leq \delta(\chi\nu, \nu).$$

Moreover, if the function $\beta(t) = t - \varphi(t)$ be increasing and onto on $[0, \infty)$, then

$$\delta(\nu, \xi^*) \leq \beta^{-1}(\delta(\chi\nu, \nu)).$$

4. Hyers–Ulam Stability of Boyd-Wong Type Contractions

The following definition is needed in the sequel.

Definition 4.1. Let (Δ, δ) be a generalized metric space. The mapping $\chi : \Delta \to \Delta$ is called a generalized Hyers–Ulam stable mapping on Δ if there exists $\psi : [0, \infty) \to [0, \infty)$ such that for any $\xi \in \Delta$ and $\varepsilon > 0$ with $\delta(\xi, \chi\xi) < \varepsilon$, there exists a fixed point ξ^* of χ such that

$$\delta(\xi, \xi^*) \leq \psi(\varepsilon).$$

The following theorem provides sufficient conditions which under them a self-mapping on a generalized complete metric space has Hyers–Ulam stability property.

Theorem 4.2. *Let (Δ, δ) be a generalized complete metric space. Assume that $\chi : \Delta \to \Delta$ is a generalized Boyd-Wong type contraction with the altering distance function φ. If $\delta(\xi, \chi\xi) < \varepsilon$ for some $\xi \in \Delta$ and $\varepsilon > 0$, then the followings are true:*

$\acute{b}$ $(\acute{i})$ The sequence $\{\chi^n\xi\}$ converges to a point ξ^* which $\delta(\xi^*, \chi\xi^*) = 0$ (that is ξ^* is a fixed point of χ);

$(\acute{i})$ If $M = \Sigma_{i=0}^{\infty}\varphi^i(\varepsilon) < \infty$, then ξ^* is the unique fixed point of χ in

$$\Delta^* = \{\nu \in \Delta | \delta(\xi, \nu) < \infty\};$$

$$\delta(\xi, \xi^*) - \varphi(\delta(\xi, \xi^*)) \leq \varepsilon.$$

Moreover, if the function $\beta(t) = t - \varphi(t)$ is (strictly) increasing and onto on $[0, \infty)$, then

$$\delta(\xi, \xi^*) \leq \beta^{-1}(\varepsilon),$$

that is, χ is a generalized Hyers–Ulam stable mapping on Δ with $\psi = \beta^{-1}$.

Proof. It follows from $\delta(\chi^k\xi, \chi^{k+1}\xi) = \delta(\xi, \chi\xi) < \epsilon$ that we can assume $k = 0$ in Theorem3.2. Thus $\delta(\chi^k\xi, \chi^{k+1}\xi) < \infty$. By our hypotheses, it is easy to see that the conditions of Theorem3.2 are true for $k = 0$. Hence, the assertion $(\acute{i})$ of the theorem is obtained by the part (i) of Theorem3.2. To show $(i\acute{i})$, let $M = \Sigma_{i=0}^{\infty}\varphi^i(\varepsilon) < \infty$. Then

$$\Sigma_{i=0}^{\infty}\varphi^i(\delta(\chi^k\xi, \chi^{k+1}\xi)) = \Sigma_{i=0}^{\infty}\varphi^i(\delta(\xi, \chi\xi)) < \Sigma_{i=0}^{\infty}\varphi^i(\epsilon) < \infty.$$

Now,$(i\acute{i})$ follows from the condition of (ii) in Theorem3.2. So ξ^* is the unique fixed point in

$$\Delta^* = \{\nu \in \Delta | \delta(\chi^k\xi, \nu) < \infty\} = \{\nu \in \Delta | \delta(\xi, \nu) < \infty\}.$$

Also $(ii\acute{i})$ is a direct consequence of Theorem3.2 by considering $\delta(\chi^k\xi, \xi) = \delta(\xi, \xi) = 0 < \infty$. In fact, $\xi \in \Delta^*$ and (iii) in Theorem 3.2 imply

$$\delta(\xi, \xi^*) - \varphi(\delta(\xi, \xi^*) \leq \delta(\chi\xi, \xi) < \epsilon$$

and so

$$\delta(\xi, \xi^*) - \varphi(\delta(\xi, \xi^*) \leq \epsilon. \tag{4.1}$$

Therefore $(ii\acute{i})$ holds. Finally, it follows the (strictly)increasing of $\beta(t) = t - \varphi(t)$ the inverse of it exists and then by taking β^{-1} on both sides of the last inequality, we deduce that

$$\delta(\xi, \xi^*) \leq \beta^{-1}(\epsilon).$$

This completes the proof. □

5. Application to Hyers–Ulam Stability of Differential Equations

The following result is an application of Theorem 4.2 to Hyers–Ulam stability of differential equation (1.1).

Theorem 5.1. *Given $c \in \mathbb{R}$ and $r > 0$, let I denotes a closed interval of radius r and centered at c, that is, $I = \{x \in \mathbb{R} | c - r \leq x \leq c + r\}$ and let $F : I \times \mathbb{R} \to \mathbb{R}$ be a continuous function which satisfies the Boyd-Wong type contractive condition*

$$|F(x, y) - F(x, z)| \leq \frac{1}{r}\varphi(|y - z|) \tag{5.1}$$

for all $x \in I$ and $y, z \in \mathbb{R}$. If a continuously differentiable function $y : I \to \mathbb{R}$ satisfies the differential inequality

$$|y'(x) - F(x, y(x)| \leq \varepsilon \tag{5.2}$$

for all $x \in I$ and for some $\varepsilon > 0$, then there exists a unique continuous function $y_0 : I \to \mathbb{R}$ satisfying the equation

$$y_0(x) = y(c) + \int_c^x F(\tau, y_0(\tau))d\tau$$

for all $x \in I$ (y_0 is a solution to (1.1)). Moreover, if the function $\beta(t) = t - \varphi(t)$ be increasing and onto on $[0, \infty)$, then

$$|y(x) - y_0(x)| \leq \beta^{-1}(r\varepsilon) \tag{5.3}$$

for any $x \in I$.

Proof. Assume that $\Delta = C(I, \mathbb{R})$ be the space of all continuous real valued functions $f : I \to \mathbb{R}$. Define $\delta : X \times X \to [0, \infty)$ by

$$\delta(f, g) = \inf\{C \in [0, \infty] : |f(x) - g(x)| \leq C \ \forall x \in I\}. \tag{5.4}$$

From [6], we know that (Δ, δ) is a complete generalized metric space. Also since the function $|f - g|$ is continuous on the closed interval $[c - r, c + r]$, thus it is bounded and so $\delta(f, g) < \infty$. So, $(\Delta = C(I, \mathbb{R}), \delta)$ is a complete metric space. Let us define an operator $\Lambda : \Delta \to \Delta$ by

$$(\Lambda f)(x) = y(c) + \int_c^x F(\tau, f(\tau))d\tau \ (x \in I)$$

for any $f \in \Delta$. We know claim that Λ is a Boyd-wong type contraction on Δ. For any $f, g \in \Delta$, let $C_{fg} \in [0, \infty]$ be an arbitrary constant with $\delta(f, g) \leq C_{fg}$, that is, by (5.4), we have $|f(x) - g(x)| \leq C_{fg} \ \forall x \in I$. Then, we have

$$\begin{aligned} |\Lambda f(x) - \Lambda g(x)| &= |\int_c^x [F(\tau, f(\tau)) - F(\tau, g(\tau))]d\tau| \\ &\leq \int_c^x |F(\tau, f(\tau)) - F(\tau, g(\tau))|d\tau \\ &\leq \int_c^x \tfrac{1}{r}\varphi(|f(\tau) - g(\tau)|)d\tau \\ &\leq \varphi(C_{fg}). \end{aligned}$$

Thus, $|\Lambda f(x) - \Lambda g(x)| \leq \varphi(\delta(f, g))$ for all $x \in I$. Therefore $\delta(\Lambda f, \Lambda g) \leq \varphi(\delta(f, g))$. Thus Λ is a Boyd-wong type contraction on Δ. Since $|y'(x) - F(x, y(x)| \leq \varepsilon$ for all $x \in I$, therefore

$$-\varepsilon \leq y'(x) - F(x, y(x) \leq \varepsilon \quad \text{for all } x \in I.$$

Integrating from all sides of the above inequality in $[c, x]$, we obtain:

$$-\varepsilon[x - c] \leq y(x) - y(c) - \int_c^x F(\tau, y(\tau)) \leq \varepsilon[x - c] \quad \text{for all } x \in I.$$

Therefore

$$|y(x) - y(c) - \int_c^x F(\tau, y(\tau))| \leq \varepsilon[x - c] \leq r\varepsilon \quad \text{for all } x \in I.$$

Thus, $\delta(y, \Lambda y) \leq r\varepsilon$. So, by Corollary 3.3, there exists a unique fixed point y_0 of Λ in Δ and $\delta(y, y_0) - \varphi(\delta(y, y_0)) \leq \delta(y, \Lambda y) \leq r\varepsilon$. Moreover, if the function $\beta(t) = t - \varphi(t)$ be increasing and onto on $[0, \infty)$, then $\delta(y, y_0) \leq \beta^{-1}(\delta(y, \Lambda y)) \leq \beta^{-1}(r\varepsilon)$, that is,

$$y_0(x) = y(c) + \int_c^x F(\tau, y_0(\tau))d\tau$$

for all $x \in I$ (y_0 is a solution to (1.1)) and

$$|y(x) - y_0(x)| \leq \beta^{-1}(r\varepsilon) \tag{5.5}$$

for any $x \in I$. □

The following example shows that Theorem 5.1 improves Theorem 1.2 in the case that the function $F : I \times \mathbb{R} \to \mathbb{R}$ given in the differential equation (1.1) does not provide necessarily a Lipschitz condition but provide a weaker condition of Boyd-Wong type.

Example 5.2. Suppose that in the differential equation (1.1) the function $F(x, y)$ equals to (assuming $K = 1$ for simplicity)

$$F(x, y) = \frac{1}{K}\frac{y}{1+y} + p(x), \forall x, y \in I,$$

where I contains zero. In order to examine the Lipschitz condition of F with respect to the second variable with Lipschitz constant $L < 1$ we need to verify the following inequality, for all $y, z \in I$,

$$F(x, y) - F(x, z)| = \frac{|y - z|}{|(1 + y)(1 + z)|} \leq l|y - z|.$$

If in the last inequality y and z tend to zero then $|F(x, y) - F(x, z)|$ will go to zero which means that there is no a fixed number like L, strictly less than 1, such that $|F(x, y) - F(x, z)| = \frac{|y-z|}{|(1+y)(1+z)|} \leq l|y - z|$, for all $y, z \in I$. This means F fails to the Lipschitz condition with Lipschitz constant $L < 1$. Thus for each $x \in I$, the function $y \to F(x, y)$ does not fulfil in the Banach contraction principle. It is easy to see that F satisfies in Boyd-Wong condition by taking $\varphi(t) = \frac{t}{1+t}$, for all $t \geq 0$, as an altering distance function. In fact

$$|F(x, y) - F(x, z)| \leq \frac{|y - z|}{1 + |y - z|} = \varphi(|y - z|), \forall x, y, z \in I.$$

Also, it is straightforward to check that F satisfies all the hypotheses of Theorem5.1. Consequently, Theorem5.1 implies that the differential equation (1.1) has Hyers–Ulam stability with respect to the (mentioned above) F.

6. Application to Hyers–Ulam-Rassias Stability of Differential Equations

The following theorem is an application of Theorem 4.2 to having the Hyers–Ulam-Rassias stability property of differential equation (1.1). It is an improvement of Theorem1.3 replacing the Lipschitz condition by the Boyd-Wong type contraction condition.

Theorem 6.1. *Let $a < b$, $I = [a, b]$, $c \in I$, $K > 0$ and $\varphi \in \Phi$. Suppose that $F : I \times \mathbb{R} \to \mathbb{R}$ a continuous function which satisfies a Boyd-Wong type contraction condition*

$$|F(x, y) - F(x, z)| \leq \frac{1}{K}\varphi(|y - z|) \tag{6.1}$$

for all $x \in I$ and $y, z \in \mathbb{R}$, where $\varphi(\lambda t) \leq \lambda\varphi(t)$ for all $t \geq 0$ and $\lambda \in \phi([a, b])$.

If a continuously differentiable function $y : I \to \mathbb{R}$ satisfies the differential inequality

$$|y'(x) - F(x, y(x)| \leq \phi(x) \tag{6.2}$$

for all $x \in I$, where $\phi : I \to \mathbb{R}$ is a continuous function with

$$|\int_c^x \phi(t)dt| \leq K\phi(x)$$

and $M = \Sigma_{i=0}^{\infty}\varphi^i(K) < \infty$, then there exists a unique continuous function $y_0 : I \to \mathbb{R}$ satisfying equation

$$y_0(x) = y(c) + \int_c^x F(\tau, y_0(\tau))d\tau$$

for all $x \in I$ (y_0 is a solution to (1.1)) and if the function $\beta(t) = t - \varphi(t)$ is (strictly) increasing onto $[0, \infty)$, then

$$|y(x) - y_0(x)| \leq \beta^{-1}(K)\phi(x) \tag{6.3}$$

for any $x \in I$.

Proof. Assume that $\Delta = C(I, \mathbb{R})$ denotes the space of all continuous real valued functions $f : I \to \mathbb{R}$. Define $\delta : X \times X \to [0, \infty]$ by

$$\delta(f, g) = inf\{C \in [0, \infty] : |f(x) - g(x)| \leq C\phi(x)\ \forall x \in I\}. \tag{6.4}$$

From [6], we know that (Δ, δ) is a complete generalized metric space. Let us define $\Lambda : \Delta \to \Delta$ by

$$(\Lambda f)(x) = y(c) + \int_c^x F(\tau, f(\tau))d\tau\ (x \in I)$$

for any $f \in \Delta$. We claim that Λ is a Boyd-wong type contraction on Δ. For any $f, g \in \Delta$, let $C_{fg} \in [0, \infty]$ be an arbitrary constant with $\delta(f, g) \leq C_{fg}$, that is, by (6.4), we have $|f(x) - g(x)| \leq C_{fg}\phi(x)\ \forall x \in I$. Then, for any $x \in I$, we have

$$\begin{aligned}
|\Lambda f(x) - \Lambda g(x)| &= |\int_c^x [F(\tau, f(\tau)) - F(\tau, g(\tau))]d\tau \\
&\leq \int_c^x |F(\tau, f(\tau)) - F(\tau, g(\tau))|d\tau \\
&\leq \int_c^x \tfrac{1}{K}\varphi(|f(\tau) - g(\tau)|)d\tau \\
&\leq \tfrac{1}{K}\int_c^x \varphi(C_{fg}\phi(\tau))d\tau \\
&\leq \tfrac{1}{K}\int_c^x \phi(\tau)\varphi(C_{fg})d\tau \\
&\leq \tfrac{1}{K}\varphi(C_{fg})|\int_c^x \phi(\tau)d\tau| \\
&\leq \tfrac{1}{K}\varphi(C_{fg})K\phi(x) \\
&= \varphi(C_{fg})\phi(x)
\end{aligned}$$

Thus, $\delta(\Lambda f, \Lambda g) \leq \varphi(\delta(f, g))$ for all $x \in I$.

Therefore, Λ is a Boyd-wong type contraction on Δ. Since $|y'(x) - F(x, y(x)| \leq \phi(x)$ for all $x \in I$, therefore

$$-\phi(x) \leq y'(x) - F(x, y(x) \leq \phi(x) \quad \text{for all } x \in I.$$

Integrating from all sides of the above inequality in $[c, x]$, we obtain:

$$|y(x) - y(c) - \int_c^x F(\tau, y(\tau))d\tau| \leq |\int_c^x \phi(\tau)d\tau| \leq K\phi(x) \quad \text{for all } x \in I.$$

Thus $\delta(y, \Lambda y) \leq K < \infty$. Therefore $\Sigma_{i=0}^{\infty}\varphi^i(\delta(y, \Lambda y)) \leq \Sigma_{i=0}^{\infty}\varphi^i(K) < \infty$. Note that here $k = 0$ and so $\Lambda^k y = y$ and $\delta(\Lambda^k y, \Lambda^{k+1} y) = \delta(y, \Lambda y)$. So by Theorem 4.2, there exists a unique fixed point y_0 of Λ in

$$\Delta^* = \{f \in \Delta | \delta(y, f) < \infty\} = \Delta$$

and if the function $\beta(t) = t - \varphi(t)$ is (strictly) increasing and onto on $[0, \infty)$, then $\delta(y, y_0) \leq \beta^{-1}(\delta(y, \Lambda y)) \leq \beta^{-1}(K)$, that is,

$$y_0(x) = y(c) + \int_c^x F(\tau, y_0(\tau))d\tau$$

for all $x \in I$ (y_0 is a solution to (1.1)) and

$$|y(x) - y_0(x)| \leq \beta^{-1}(K)\phi(x) \tag{6.5}$$

for any $x \in I$. □

Remark 6.2. If, in Theorem 6.1, we add the condition $\phi(x) \geq m$ for a positive constant $m > 0$, then we can remove the condition $M = \Sigma_{i=0}^{\infty}\varphi^i(K) < \infty$. In the other words, the result of Theorem 6.1 is true when we replace $M = \Sigma_{i=0}^{\infty}\varphi^i(K) < \infty$ by $\phi(x) \geq m$ for some $m > 0$.

The following theorem is an application of Theorem 4.2 in order to establish Hyers–Ulam-Rassias stability of differential equation (1.1) when the interval I is unbounded. Moreover, it is a new version of Theorem 2.6.

Theorem 6.3. *Let either $I = (-\infty, b]$ or $I = [a, \infty)$ or $I = \mathbb{R}$ where $a, b \in \mathbb{R}$. Set either $c = a$ for $I = [a, \infty)$ or $c = b$ for $I = (\infty, b]$ or c is a fixed real number if $I = \mathbb{R}$. Suppose there exist $K > 0, \varphi \in \Phi$ and let $F : I \times \mathbb{R} \to \mathbb{R}$ be a continuous function which satisfies the Boyd-Wong type contraction condition*

$$|F(x, y) - F(x, z)| \leq \frac{1}{K}\varphi(|y - z|) \tag{6.6}$$

for all $x \in I$ and $y, z \in \mathbb{R}$, where $\varphi(\lambda t) \leq \lambda\varphi(t)$ for all $t \geq 0$ and $\lambda \in \phi(I)$. If a continuously differentiable function $y : I \to \mathbb{R}$ satisfies the differential inequality

$$|y'(x) - F(x, y(x)| \leq \phi(x) \tag{6.7}$$

for all $x \in I$, where $\phi : I \to \mathbb{R}$ is a continuous function with

$$|\int_c^x \phi(t)dt| \leq K\phi(x)$$

and $M = \sum_{i=0}^{\infty}\varphi^i(K) < \infty$, then there exists a unique continuous function $y_0 : I \to \mathbb{R}$ satisfying equation

$$y_0(x) = y(c) + \int_c^x F(\tau, y_0(\tau))d\tau \tag{6.8}$$

for all $x \in I$ (y_0 is a solution to (1.1)) and if the function $\beta(t) = t - \varphi(t)$ be increasing and onto on $[0, \infty)$, then

$$|y(x) - y_0(x)| \leq \beta^{-1}(K)\phi(x) \tag{6.9}$$

for any $x \in I$.

Proof. We give the proof only in the case that $I = \mathbb{R}$. The proof of the other cases are similar. For any $n \in \mathbb{N}$, set $I_n = [c - n, c + n]$. (We set $I_n = [b - n, b]$ for $I = (-\infty, b]$ and $I_n = [a, a + n]$ for $I = [a, \infty)$.) According to Theorem 6.1, there exists a unique continuous function $y_n : I_n \to \mathbb{R}$ such that

$$y_n(x) = y(c) + \int_c^x F(\tau, y_n(\tau))d\tau$$

and

$$|y(x) - y_n(x)| \leq \beta^{-1}(K)\phi(x) \tag{6.10}$$

for any $x \in I_n$. The uniqueness of y_n implies that if $x \in I_n$, then

$$y_n(x) = y_{n+1}(x) = y_{n+2}(x) = \cdots .$$

For any $x \in \mathbb{R}$, let us define $n(x) \in \mathbb{N}$ as

$$n(x) = \min\{n \in \mathbb{N} | x \in I_n\}.$$

Moreover, we define a function $y_0 : \mathbb{R} \to \mathbb{R}$ by

$$y_0(x) = y_{n(x)}(x). \tag{6.11}$$

We know from theorem 3.1 of [6] that y_0 is a continuous function and y_0 satisfies (6.8). To show (6.9), since $x \in I_{n(x)}(x)$ for every $x \in \mathbb{R}$, by (6.10) and (6.11), we have

$$|y(x) - y_0(x)| = |y(x) - y_{n(x)}(x)| \leq \beta^{-1}(K)\phi(x) \tag{6.12}$$

for any $x \in \mathbb{R}$. Finally, we show that y_0 is unique. Let $z_0 : \mathbb{R} \to \mathbb{R}$ be another continuous function which satisfies (6.8) and (6.9), with z_0 in place of y_0, for all $x \in \mathbb{R}$. Suppose x is an arbitrary real number. Since the restrictions $y_0|_{I_{n(x)}} (= y_{n(x)})$ and $z_0|_{I_{n(x)}}$ both satisfy (6.8) and (6.9) for all $x \in I_{n(x)}$, the uniqueness of $y_{n(x)} = y_0|_{I_{n(x)}}$ implies that

$$y_0(x) = y_0|_{I_{n(x)}}(x) = z_0|_{I_{n(x)}}(x) = z_0(x).$$

The proof is completed. □

Remark 6.4. The result of the previous Theorem is still valid if we replace the condition $M = \Sigma_{i=0}^{\infty}\varphi^i(K) < \infty$ by $\phi(x) \geq m$ for a positive constant $m > 0$.

7. Examples

In this section, we show some clear examples for Theorems 5.1, 6.1 and 6.3. this examples is in the jung 's article as the function that uses $Ly(x)$ is a linear function , but instead of a linear function , we use a fractional function $F(x, y) = \frac{1}{K}\frac{|y|}{1+|y|}$this is not the case with Lipschitz condition but Boyd-Wong condition is. these examples illustrate that the equation is not the result of Jung's theorem , but agrees with our theorem .

Example 7.1. Consider the positive constants K and ε with $1 \leq \varepsilon < 2K$. Let $I = [\varepsilon, 2K]$ be a closed interval. Given a polynomial $p(x)$, we assume that a continuously differentiable function $y : I \to \mathbb{R}$ satisfies

$$|y'(x) - \frac{1}{K}\frac{|y(x)|}{1 + |y(x)|} - p(x)| \leq x$$

for all $x \in I$. Let $c \in I$ be a fixed number. If we set $F(x, y) = \frac{1}{K}\frac{|y|}{1+|y|} + p(x)$ and $\phi(x) = x$, then the above inequality has the same form with (6.7). Moreover, we obtain

$$|\int_c^x \phi(t)dt| = |\int_c^x tdt| = \frac{x^2}{2} - \frac{c^2}{2} \leq \frac{x^2}{2} \leq Kx = K\phi(x).$$

(Since $x \in [c, 2K]$, we have $x \leq 2K$ and so $\frac{x}{2} \leq K$. Multiplying with x, we get $\frac{x^2}{2} \leq Kx$)
To show (6.7), we have

$$\begin{aligned} |F(x, y) - F(x, z)| &= \frac{1}{K}|\frac{|y|}{|y| + 1} - \frac{|z|}{|z| + 1}| \\ &= \frac{1}{K}|\frac{|y| - |z|}{(|y| + 1)(|z| + 1)}| \\ &\leq \frac{1}{K}\frac{|y - z|}{|y - z| + 1} \\ &= \frac{1}{K}\varphi(|y - z|), \end{aligned}$$

where $\varphi(t) = \frac{t}{t+1}$.

Moreover, since $\phi(x) = x \geq c$ on $[c, 2K]$, thus we need not to have $M = \Sigma_{i=0}^{\infty}\varphi^i(K) < \infty$. Also we have $\varphi(\lambda t) = \frac{\lambda t}{\lambda t+1} \leq \frac{\lambda t}{t+1} = \lambda\varphi(t)$ for any $\lambda \in [1, \infty)$ and so for $\lambda \in [c, 2K)$ $(t \geq 0)$.

According to Theorem 6.1, there exists a unique continuous function $y_0 : I \to \mathbb{R}$ such that

$$y_0(x) = y(c) + \int_c^x (\frac{1}{K}\frac{|y_0(\tau)|}{1+|y_0(\tau)|} + p(\tau))d\tau$$

for all $x \in I$, that is, y_0 is a solution to

$$y_0'(x) = \frac{1}{K}\frac{|y_0(x)|}{1+|y_0(x)|} + p(x) \tag{7.1}$$

for any $x \in I$ and

$$|y(x) - y_0(x)| \leq \beta^{-1}(K)x = \frac{K + \sqrt{K^2+4K}}{2}x \tag{7.2}$$

for any $x \in I$.

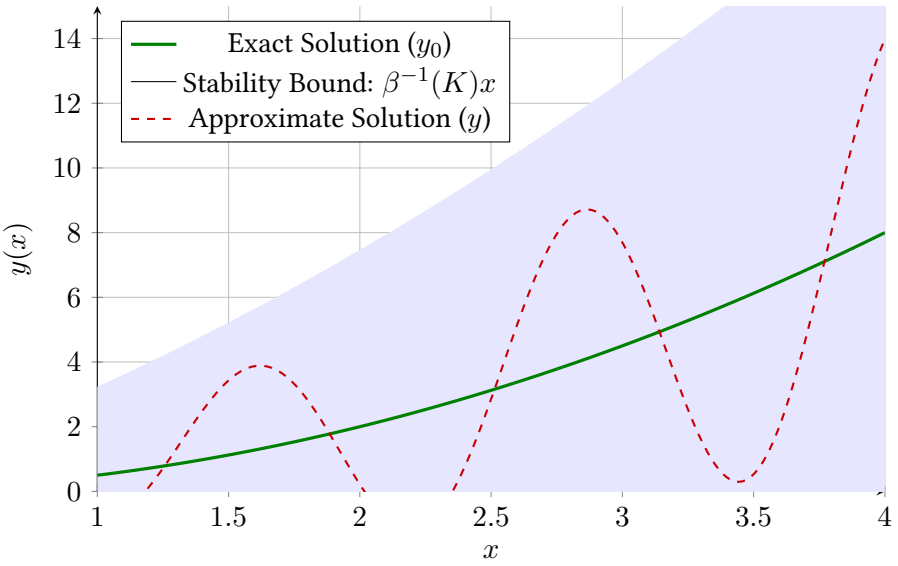


FIGURE 1. HUR Stability Visualization for Example 7.1

Example 7.2. Consider a real constant $a > 1$. Let $I = [c, \infty)$ be a closed interval, where $c \geq 0$. Given a polynomial $p(x)$, we assume that a continuously differentiable function $y : I \to \mathbb{R}$ satisfies

$$|y'(x) - (\ln a)\frac{|y(x)|}{1+|y(x)|} - p(x)| \leq a^x$$

for all $x \in I$. If we set $F(x, y) = (\ln a)\frac{|y|}{1+|y|} + p(x)$ and $\phi(x) = a^x$, then the above inequality has the same form with (6.7). Moreover, we obtain

$$|\int_c^x a^t dt| \leq \frac{1}{\ln a}a^x = K\phi(x),$$

where $K = \frac{1}{\ln a}$. To show (6.6), we have

$$\begin{aligned}|F(x,y)-F(x,z)| &= (\ln a)|\frac{|y|}{|y|+1}-\frac{|z|}{|z|+1}|\\ &= \frac{1}{K}|\frac{(|y|-|z|)}{(|y|+1)(|z|+1)}|\\ &\leq \frac{1}{K}|\frac{|y-z|}{|yz|+|y|+|z|+1}|\\ &\leq \frac{1}{K}\frac{|y-z|}{|y-z|+1}\\ &= \frac{1}{K}\varphi(|y-z|),\end{aligned}$$

where $\varphi(t)=\frac{t}{t+1}$.

Moreover, since $\phi(x)=a^x\geq 1$ on $[c,\infty)$, thus we need not to have $M=\Sigma_{i=0}^{\infty}\varphi^i(K)<\infty$. Also we have $\varphi(\lambda t)=\frac{\lambda t}{\lambda t+1}\leq\frac{\lambda t}{t+1}=\lambda\varphi(t)$ for any $\lambda\in[1,\infty)$ and so for $\lambda\in[a^c,\infty)=\phi([c,\infty))$ $(t\geq 0)$.

According to Theorem 6.3, there exists a unique continuous function $y_0: I\to\mathbb{R}$ such that

$$y_0(x)=y(c)+\int_c^x(\ln a.\frac{y_0(\tau)}{1+y_0(\tau)}+p(\tau))d\tau$$

for all $x\in I$, that is, y_0 is a solution to

$$y_0'(x)=(\ln a)\frac{y_0(x)}{1+y_0(x)}+p(x) \tag{7.3}$$

for any $x\in I$ and

$$|y(x)-y_0(x)|\leq\beta^{-1}(K)a^x=\beta^{-1}(\frac{1}{\ln a})a^x=\frac{1+\sqrt{1+4\ln a}}{2\ln a}a^x \tag{7.4}$$

for any $x\in I$.

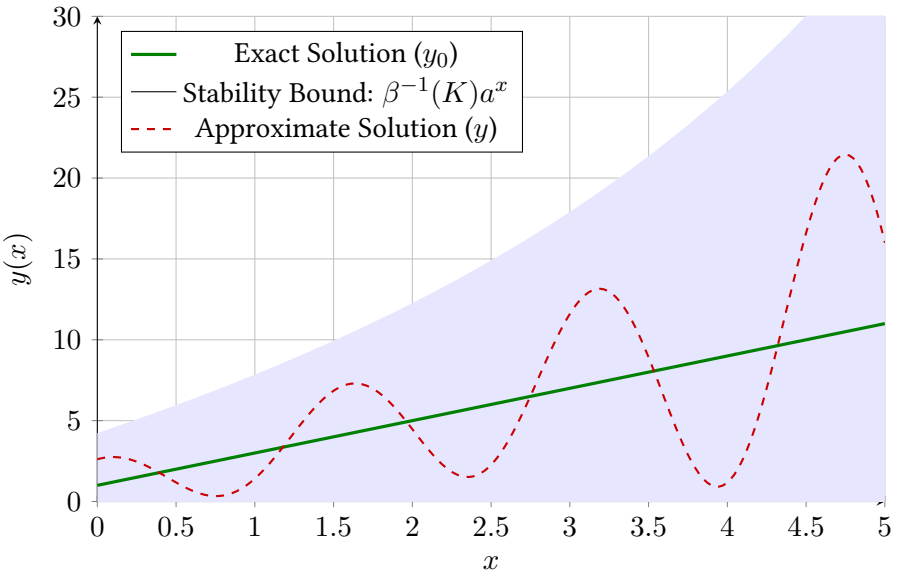


FIGURE 2. HUR Stability on Unbounded Interval (Example 7.2)

Example 7.3. Consider the interval $I = [c - r, c + r]$ be a closed interval, where $c, r \in \mathbb{R}$ and $r > 0$. Given a polynomial $p(x)$, we assume that a continuously differentiable function $y : I \to \mathbb{R}$ satisfies

$$|y'(x) - \frac{1}{r}\frac{|y(x)|}{1+|y(x)|} - p(x)| \leq \varepsilon$$

for all $x \in I$. If we set $F(x, y) = \frac{1}{r}\frac{|y|}{1+|y|} + p(x)$, then the above inequality has the same form with (5.2). To show (5.1), we have

$$\begin{aligned} |F(x,y) - F(x,z)| &= \frac{1}{r}|\frac{|y|}{|y|+1} - \frac{|z|}{|z|+1}| \\ &= \frac{1}{r}|\frac{(|y| - |z|)}{(|y|+1)(|z|+1)}| \\ &\leq \frac{1}{r}\frac{|y-z|}{|y-z|+1} \\ &= \frac{1}{r}\varphi(|y-z|), \end{aligned}$$

where $\varphi(t) = \frac{t}{t+1}$.

Acording to Theorem 5.1, there exists a unique continuous function $y_0 : I \to \mathbb{R}$ such that

$$y_0(x) = y(c) + \int_c^x (\frac{1}{r}\frac{y_0(\tau)}{1+y_0(\tau)} + p(\tau))d\tau$$

for all $x \in I$, that is, y_0 is a solution to

$$y_0'(x) = \frac{1}{r}\frac{y_0(x)}{1+y_0(x)} + p(x) \tag{7.5}$$

for any $x \in I$ and

$$|y(x) - y_0(x)| \leq \beta^{-1}(r\varepsilon) = \frac{r\varepsilon + \sqrt{(r\varepsilon)^2 + 4r\varepsilon}}{2} \tag{7.6}$$

for any $x \in I$.

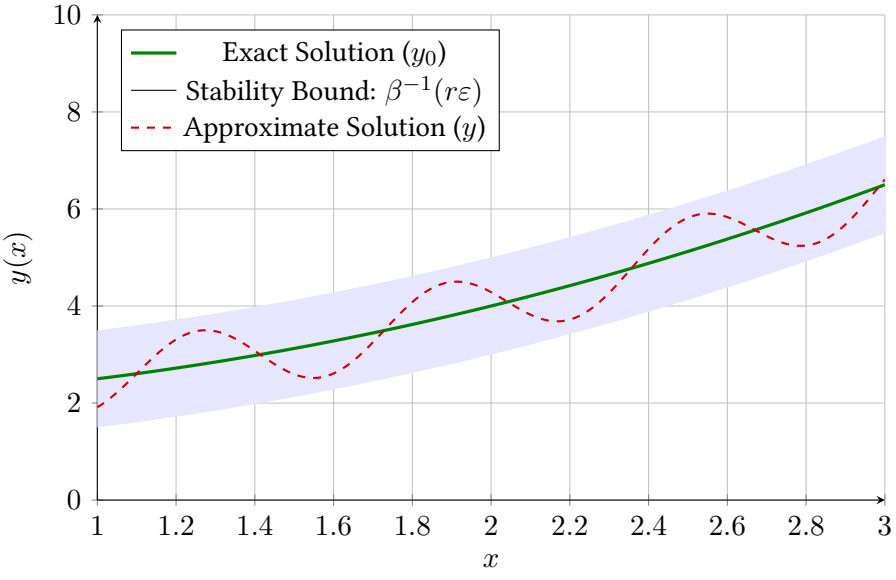


FIGURE 3. Hyers-Ulam Stability Visualization (Example 7.3)

8. conclusion

In this paper, we introduced a Boyd-Wong-type contraction in generalized metric spaces and proved the existence of a unique fixed point for such mappings. We investigated the stability of a general class of nonlinear differential equations in the sense of Hyers–Ulam and Hyers–Ulam–Rassias using our fixed point results. Our results extend and improve some existing results and stability results in the literature. Some illustrative examples have been given to show the usability of our main results.

We proposed in the future to establish a coupled fixed point theorem for two-variable mappings on generalized metric spaces satisfying a Boyd-Wong type contraction. Also, we proposed to investigate the solvability of systems of differential equations when the right-hand functions apprise a Boyd-Wong type contraction rather than it apprise a Lipschitz contraction condition.

Statements and Declarations

The authors declare that they have no conflict of interest, and the manuscript has no associated data.

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