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APPROXIMATION SCHEMES FOR INVERSE HARMONIC-LIKE QUASI VARIATIONAL INEQUALITIES

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Abstract. Some new classes of inverse harmonic-like quasi variational inequalities are introduced and investigated. Using various techniques such as projection methods, auxiliary principle, dynamical systems coupled with finite difference approach, we suggest and analyze a number of new and known numerical techniques for solving the inverse harmonic-like quasi variational inequalities. Convergence analysis of these methods is investigated under suitable conditions. One can obtain a number of new classes of inverse harmonic-like quasi variational inequalities by interchanging the role of operators. Various special cases are discussed as applications of the main results. Several open problems are suggested for future research.

Keywords. Harmonic-like convex functions, Inverse quasi variational inequalities, iterative methods, Auxiliary principle technique, Dynamical system, convergence.

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1. Introduction

Variational inequality theory was introduced and considered in early sixties by Stampacchia [60] and Lions et al.[24], can be viewed as a novel extension and generalization of the variational principles. By variational principles,we mean maximum and minimum problems arising in game theory, mechanics,geometrical optics, general relativity theory, economics, transportation, differential geometry and related areas. It is amazing that a wide class of unrelated problems can be studied in the general and unified framework of variational inequalities. Variational inequalities have been generalized and extended in several directions using novel and innovative ideas to handle complicated and complex problems. Noor [31, 32] considered two new classes of variational inequalities involving two arbitrary operators in 1988, which are known as general variational inequalities and have applications in oceanography, non-positive and non symmetric differential equations theory.

It has been established that the variational inequalities are equivalent to the fixed point problems. Noor [36, 38] has proposed and suggested three step usng the updating the solution, known as Noor iterations, for finding the approximate solution. Suantai et. al. [62] have also considered some novel forward-backward algorithms for optimization and their applications to compressive sensing and image inpainting. Ashish et. al. [2, 3, 4], Cho et al. [9], Kwuni et al. [23] and Roy et al. [59] explored the Julia set and Mandelbrot set in Noor orbit using the Noor (three step) iterations. It is worth mentioning that Noor iterations have influenced the research in the fixed point theory and will continue to inspire further research in fractal geometry, chaos theory, coding, number theory, spectral geometry, dynamical systems, complex analysis, nonlinear programming, graphics and computer aided design. Noor (three-step) iterations contain Mann (one-step) iteration and Ishikawa (two-step) iterations as special

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cases. Inspired and motivated by the usefulness and applications of the splitting three-step methods, several classes of three-step approximation schemes for solving variational inequalities, fixed points and related problems are being investigated. It has been established [36, 58] that Noor (Three step) iterations, perform better than the (Ishikawa (two step) iterations and one step method Mann (one step) iteration.

In various cases, the underlying set may depends upon the solution explicitly or implicity, then the variational inequality is called the quasi-variational inequality, introduced and studied by Bensoussan and Lions [5] in in the field of impulse control.

To be more precise, for given nonlinear operators $\mathcal{T}, g : \mathcal{H} \leftarrow \mathcal{H}$ and convex valued closed convex set $\Omega(\mu)$, find $\mu \in \Omega(\mu) \subseteq \mathcal{H}$, such that

$$\langle \mathcal{T}\mu, \nu - g(\mu) \rangle \geq 0, \quad \forall \nu \in \Omega(\mu), \tag{1.1}$$

which is called the general quasi variational inequality considered by Noor [32] in 1988.

For $\mathcal{T} = I$, identity operator, the problem (1.1) reduces to finding $\mu \in \Omega(\mu)$ such that

$$\langle \mu, \nu - g(\mu) \rangle \geq 0, \quad \forall \nu \in \Omega(\mu), \tag{1.2}$$

which is known as inverse quasi variational inequality. It is worth mentioning that the inverse quasi variational inequalities have been used to consider a road pricing problem in which the environment impact problem due to traffic flow is taken into account to fix the road taxes and a bipartite market equilibrium problem. Also, a number of problems such as flow control problems, transportation, telecommunication networks, dynamic power price problem and least distance problem can be studied in the unified framework of inverse variational inequalities. For the applications, generalizations, numerical methods and other aspects of inverse variational inequalities, see [5, 7, 13, 16, 17, 18, 19, 27, 28, 48, 49, 50, 54, 59, 62, 64] and the references therein.

Several new generalizations and extensions of the convex functions and convex sets have been introduced to tackle unrelated complicated and complex problems in a unified manner. Harmonic functions and harmonic convex sets are important generalizations of the convex functions andconvex sets. Anderson et al. [1] have investigated several aspects of the harmonic convex functions. Harmonic analysis has become a vast subject with applications in areas as diverse as number theory, representation theory, signal processing, quantum mechanics, tidal analysis, spectral analysis, and neuroscience. The harmonic mean is generally used when there is a necessity to give greater weight to the smaller items. The harmonic mean is often used to calculate the average of the ratios or rates of the given values. It is the most appropriate measure for ratios and rates because it equalizes the weights of each data point. The harmonic means have novel applications in electrical circuits theory. It is known that the total resistance of a set of parallel resistors is obtained by adding up the reciprocals of the individual resistance values and then taking the reciprocal of their total. More precisely, if η_1 and η_2 are the resistances of two parallel resistors, then the total resistance is computed by the formula $(\frac{1}{\eta_1} + \frac{1}{\eta_2})^{-1} = \frac{\eta_1\eta_2}{\eta_1+\eta_2}$, which is half the harmonic means. The harmonic mean is employed in finance to determine the average multiples like the price-income ratio and data like price multiples. It is also practised by market professionals in order to determine the patterns like Fibonacci sequences. The harmonic mean are being used to suggest some iterative methods for solving nonlinear equations. Noor et al. [30, 51, 52, 53] introduced the new concepts of harmonic-like convex sets and harmonic-like convex functions, which can be viewed as an important generalization of harmonic convex sets and functions. They have discussed the applications of the harmonic-like convex functions by establishing the equivalence between the general harmonic-like nonlinear equations and the fixed point problem.

The projected dynamical systems associated with variational inequalities were considered by Dupuis et al.[14] and Nagurney et al.[25]. The novel feature of the projected dynamical system is that the its set of stationary points corresponds to the set of the corresponding set of the solutions of the variational

inequality problem. This dynamical system is associated with boundary value and initial value problem. Consequently, equilibrium and nonlinear problems arising in various branches in pure and applied sciences can now be studied in the setting of dynamical systems. It has been shown [13, 14, 26, 30, 44, 45, 46, 49, 50, 52, 54, 61, 62, 63, 64] that the dynamical systems are useful in developing some efficient numerical techniques for solving variational inequalities and related optimization problems.

Motivated and inspired by ongoing recent research in variational inequalities, we revisit the inverse harmonic-like quasi variational inequalities, which is a special case of quasi variational inequalities involving two arbitrary operators, introduced and studied by Noor [33] in 1988. Noor [33] established the equivalence between the quasi variational inequalities and fixed point problem, which has been used to consider an iterative method for solving harmonic-like quasi variational inequalities. We prove that the nonlinear programming problems and boundary value problems can be studied via the inverse harmonic-like quasi variational inequalities. Several special cases and applications of the inverse harmonic-like quasi variational inequalities are discussed in Section 2. In section 3, we discuss the unique existence of the solution as well as to suggest several inertial iterative method along with the convergence analysis. We also apply the auxiliary principle technique involving an arbitrary operator is used to discuss some iterative schemes for solving the inverse harmonic-like quasi variational inequalities in Section 5. Dynamical system approach is applied to study the stability of the solution and to suggest some iterative methods for solving the inverse quasi variational inequalities exploring the finite difference idea. Our results in this section can be viewed as significant refinement of the results in[25, 62, 63, 64] and the references therein.

One of the main purposes of this paper is to demonstrate the close connection among various classes of algorithms for the solution of the inverse harmonic-like quasi variational inequalities and to point out that researchers in different field of variational inequalities and optimization have been considering parallel paths. We would like to emphasize that the results obtained and discussed in this paper may motivate and bring a large number of novel, innovate potential applications, extensions and interesting topics in these areas. We have given only a brief introduction of this fast growing field. The interested reader is advised to explore this field further and discover novel and fascinating applications of inverse quasi variational inequalities in other areas of sciences such as machine learning, artificial intelligence, data analysis, fuzzy systems, random stochastic, financial analysis and related other optimization problems. It is expected the techniques and ideas of this paper may be starting point for further research.

2. Formulations and Basic Facts

Let Ω be a nonempty closed set in a real Hilbert space $\mathcal{H}$. We denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ be the inner product and norm, respectively. Let $g : \mathcal{H} \longrightarrow \mathcal{H}$ be a nonlinear operator.
First of all, we recall some concepts from convex analysis [1, 11, 29, 30], which are needed in the derivation of the main results.

Definition 2.1. A set Ω in $\mathcal{H}$ is said to be a harmonic-like general convex set, if

$$\frac{2w\mu}{w+\mu} + \lambda(g(\nu) - g(\mu)) \in \Omega, \qquad \forall w, \mu, \nu \in \Omega, \lambda \in [0,1].$$

Definition 2.2. A function Φ is said to be a harmonic-like general convex function, if

$$\Phi(\frac{2w\mu}{w+\mu} + \lambda(g(\nu) - g(\mu)) \leq (1-\lambda)\Phi(\mu) + \lambda\Phi(\nu), \quad \forall w, \mu, \nu \in \Omega, \quad \lambda \in [0,1].$$

Convex functions are closely related to the integral inequalities and variational inequalities. These type of inequalities have played crucial role in developing fields such as: numerical analysis, operations research, transportation, financial mathematics, structural analysis, dynamical systems, sensitivity analysis and machine learning.

If the harmonic-like general convex function Φ is differentiable, then $\mu \in \Omega$ is the minimum of the function Φ, if and only if, $\mu \in \Omega$ satisfies the inequality

$$\langle \Phi'(\frac{2w\mu}{w+\mu}), g(\nu) - g(\mu) \rangle \geq 0, \quad \forall w, \nu \in \Omega, \tag{2.1}$$

which is called the harmonic-like variational inequality.

In many applications, the convex set Ω depends upon the solution explicitly or implicitly. In such cases, variational inequality is called the quasi variational inequality. Let $\Omega : \mathcal{H} \longrightarrow \mathcal{H}$ be a set-valued mapping, which, for any element $\mu \in \mathcal{H}$, associates a convex-valued and closed set $\Omega(\mu) \subseteq \mathcal{H}$.
We now consider some new classes of inverse harmonic-like quasi variational inequalities, which include several new and known classes of inverse harmonic-like variational inequalities as special cases.

For a given nonlinear operator g, we consider the problem of finding $\mu \in \Omega(\mu)$, such that

$$\langle (\frac{2w\mu}{w+\mu}), g(\nu) - g(\mu) \rangle \geq 0, \quad \forall w, \nu \in \Omega(\mu), \tag{2.2}$$

which is called the general inverse harmonic-like quasi variational inequality.

For $w = \mu$, the problem (2.2) reduces to finding $\mu \in \Omega(\mu)$, such that

$$\langle \mu, g(\nu) - g(\mu) \rangle \geq 0, \quad \forall \nu \in \Omega(\mu), \tag{2.3}$$

which is called the inverse quasi variational inequality and is a special case of quasi variational inequality [32].

For $g = I$, the problem (2.2) reduces to finding $\nu \in \Omega(\mu)$, such that

$$\langle (\frac{2w\mu}{w+\mu}), \nu - \mu \rangle \geq 0, \quad \forall w, \nu \in \Omega(\mu), \tag{2.4}$$

which is also called the inverse harmonic-like quasi variational inequality. Consequently, it is evident that all the known results for inverse harmonic-like quasi variational inequalities are also valid for both types of inverse harmonic-like quasi variational inequalities. This is a surprising fascinating fact.

Special cases. We now point out some very important and interesting problems, which can be obtained as special cases of the problem (2.2).

(I). This problem (2.2) can be viewed as a problem of finding the minimum of general convex function [40]. Such type of problems have applications in optimization theory and imaging process in medical sciences and earthquake.

(II). For $w = \nu$, the problem (2.2) reduces to finding $\nu \in \Omega(\mu)$, such that

$$\langle (\frac{2\nu\mu}{\nu+\mu}), g(\nu) - g(\mu) \rangle \geq 0, \quad \forall w, \nu \in \Omega(\mu), \tag{2.5}$$

which is also called the inverse harmonic mean general quasi variational inequality.

(III). If $\Omega^*(\mu) = \{\mu \in \mathcal{H} : \langle \mu, \nu \rangle \geq 0, \quad \forall \nu \in \Omega(\mu)\}$ is a polar (dual) cone of a convex-valued cone $\Omega(\mu)$ in $\mathcal{H}$, then problem (2.2) is equivalent to finding $\mu \in \mathcal{H}$, such that

$$g(\mu) \in \Omega(\mu), \quad (\frac{2w\mu}{w+\mu}) \in \Omega^*(\mu) \quad \text{and} \quad \langle (\frac{2w\mu}{w+\mu}), g(\mu) \rangle = 0, \tag{2.6}$$

which is known as the inverse harmonic-like quasi complementarity problems and for, $w = \mu$, the problem (2.6) reduces to finding $\mu \in \mathcal{H}$, such that

$$g(\mu) \in \Omega(\mu), \quad \mu \in \Omega^*(\mu) \quad \text{and} \quad \langle \mu, g(\mu) \rangle = 0,$$

is called the inverse quasi complementarity problem [31].

For $\Omega(\mu) = \Omega$, the convex set, the problem (2.6) is equivalent to finding $\mu \in \mathcal{H}$ such that

$$g(\mu) \in \Omega, \quad (\frac{2w\mu}{w+\mu}) \in \Omega^* \quad \text{and} \quad \langle (\frac{2w\mu}{w+\mu}), g(\mu) \rangle = 0, \tag{2.7}$$

is called the inverse harmonic-like complementarity problem. Obviously inverse quasi complementarity problems include the inverse complementarity problems and linear complementarity problems. The complementarity problems were studied by Cottle et al.[10], Noor [32, 33, 34, 38] and Noor et al.[42, 55, 56, 57] in game theory, management sciences and quadratic programming as special cases. This inter relationship among these problems have played a major role in developing numerical results for these problems and their applications.

(III). If $\Omega(\mu) = \Omega$, where Ω is a convex set in $\mathcal{H}$, then problem (2.2) reduces to finding $\mu \in \Omega$ such that

$$\langle (\frac{2w\mu}{w+\mu}), g(\nu) - g(\mu) \rangle \geq 0, \quad \forall w, \nu \in \Omega, \tag{2.8}$$

is known as the inverse harmonic-like variational inequality.

Remark 2.3. It is worth mentioning that for appropriate and suitable choices of the operator g, set-valued convex set $\Omega(\mu)$ and the spaces, one can obtain several classes of inverse harmonic-like variational inequalities, inverse harmonic-like complementarity problems and optimization problems as special cases of the inverse harmonic-like quasi variational inequalities (2.2). This shows that the problem (2.2) is quite general and unifying one. For the applications, motivations, generalizations, extensions, numerical methods and other aspects of variational inequalities, inverse quasi variational inequalities and mathematical programming problems, see [1, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 27, 28, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 61, 62, 63, 64, 65] and the references therein. It is interesting problem to develop efficient implementable numerical methods for solving the general inverse harmonic-like quasi-variational inequalities and their variants forms.

We also need the following result, known as the projection Lemma(best approximation), which plays a crucial part in establishing the equivalence between the inverse harmonic-like quasi variational inequalities and the fixed point problems. This result is used in the analyzing the convergence analysis of the implicit and explicit methods for solving the variational inequalities and related optimization problems.

Lemma 2.1. [32] *Let $\Omega(\mu)$ be a closed and convex-valued set in $\mathcal{H}$. Then, for a given $z \in \mathcal{H}$, $\mu \in \Omega(\mu)$ satisfies the inequality*

$$\langle \mu - z, \nu - \mu \rangle \geq 0, \quad \forall \nu \in \Omega(\mu), \tag{2.9}$$

if and only if,

$$\mu = \Pi_{\Omega(\mu)}(z),$$

where $\Pi_{\Omega(\mu)}$ is implicit projection of $\mathcal{H}$ onto the closed convex-valued set $\Omega(\mu)$.

It is well known that the implicit projection operator $\Pi_{\Omega(\mu)}$ is not nonexpansive, but it is required to satisfy the following assumption, which plays an important part in the derivation of the results..

Assumption 2.1.

$$\|\Pi_{\Omega(\mu)}\omega - \Pi_{\Omega(\nu)}\omega\| \leq \eta\|\mu - \nu\|, \forall \mu, \nu, \omega \in \mathcal{H}, \tag{2.10}$$

where $\eta > 0$ is a constant.

Assumption 2.1 is used to prove the existence of a solution of inverse harmonic-like quasi variational inequalities as well as in analyzing the convergence of the iterative methods.

In many important applications, the convex-valued set $\Omega(\mu)$ can be written as

$$\Omega(\mu) = m(\mu) + \Omega,$$

is known as the moving convex set,where $m(\mu)$ is a point-point mapping and Ω is a convex set. In this case, we have

$$\Pi_{\Omega(\mu)}\omega = \Pi_{m(\mu)+\Omega} = m(\mu) + \Pi_{\Omega}[w - m(\mu)], \quad \forall \mu, w \in \Omega.$$

We note that, if $m(\mu)$ is a Lipschitz continuous mapping with constant $\gamma > 0$, then

$$\begin{aligned} \|\Pi_{\Omega(\mu)}w - \Pi_{\Omega(\nu)}w\| &= \|m(\mu) - m(\nu) + \Pi_{\Omega}[w - m(\mu)] - \Pi_{\Omega}[w - m(\nu)\| \\ &\leq 2\|m(\mu) - m(\nu)\| \leq 2\gamma\|\mu - \nu\|, \quad \forall \mu, \nu, w \in \Omega. \end{aligned}$$

which shows that Assumption 2.1 holds with $\eta = 2\gamma$.

Definition 2.4. An operator $\mathcal{T} : \mathcal{H} \to \mathcal{H}$ is said to be:

(1) strongly harmonic-like monotone, if there exists a constant $\alpha > 0$, such that

$$\langle \mathcal{T}(\frac{2w\mu}{w+\mu}) - \mathcal{T}(\frac{2w\nu}{w+\nu}), \mu - \nu\rangle \geq \alpha\|\mu - \nu\|^2, \quad \forall w, \mu, \nu \in \mathcal{H}.$$

(2) Lipschitz harmonic-like continuous, if there exists a constant $\beta > 0$, such that

$$\|\mathcal{T}(\frac{2w\mu}{w+\mu}) - \mathcal{T}(\frac{2w\nu}{w+\nu})\| \leq \beta\|\mu - \nu\|, \quad \forall w, \mu, \nu \in \mathcal{H}.$$

(3) harmonic-like monotone, if

$$\langle \mathcal{T}(\frac{2w\mu}{w+\mu}) - \mathcal{T}(\frac{2w\nu}{w+\nu}), \mu - \nu\rangle \geq 0, \quad \forall w, \mu, \nu \in \mathcal{H}.$$

(4) harmonic-like pseudo monotone, if

$$\langle \mathcal{T}(\frac{2w\mu}{w+\mu}), \nu - \mu\rangle \geq 0 \quad \Rightarrow \quad \langle \mathcal{T}(\frac{2w\nu}{w+\nu}), \nu - \mu\rangle \geq 0, \quad \forall w, \mu, \nu \in \mathcal{H}.$$

Remark 2.5. Every strongly harmonic-like monotone operator is a harmonic-like monotone operator and harmonic-like harmonic-like monotone operator is a pseudo monotone operator, but the converse is not true.

3. Iterative Methods and Convergence Analysis

In this section, we use the fixed point formulation to suggest and analyze some new implicit methods for solving the inverse harmonic-like quasi variational inequalities.

Using Lemma 2.1, one can show that the inverse harmonic-likequasi variational inequalities are equivalent to the fixed point problems.

Lemma 3.1. *The function $\mu \in \Omega(\mu)$ is a solution of the inverse harmonic-likequasi variational inequality (2.2), if and only if, $\mu \in \Omega(\mu)$ satisfies the relation*

$$g(\mu) = \Pi_{\Omega(\mu)}[g(\mu) - \rho(\frac{2w\mu}{w\mu})], \tag{3.1}$$

where $\Pi_{\Omega(\mu)}$ is the implicit projection operator and $\rho > 0$ is a constant.

Proof. Let $u \in \Omega(\mu)$ be a solution of the problem (2.2). Then

$$\langle \rho(\frac{2w\mu}{w+\mu}) + g(\mu) - g(\mu), g(\nu) - g(\mu)\rangle \geq 0, \quad \forall w, \nu \in \Omega(\mu).$$

Using Lemma 2.1, we have

$$g(\mu) = \Pi_{\Omega(\mu)}[g(\mu) - \rho(\frac{2w\mu}{w+\mu})],$$

the required result. □

Lemma 3.1 implies that the inverse harmonic-like quasi variational inequality (2.2) is equivalent to the fixed point problem (3.1). This equivalent fixed point formulation (3.1) will play an important role in deriving the main results.

From the equation (3.1), we have

$$\mu = \mu - g(\mu) + \Pi_{\Omega(\mu)}[g(\mu) - \rho(\frac{2w\mu}{w+\mu})].$$

We define the operator F associated with (3.1) as

$$F(\mu) = \mu - g(\mu) + \Pi_{\Omega(\mu)}[g(\mu) - \rho(\frac{2w\mu}{w+\mu})]. \tag{3.2}$$

To prove the unique existence of the solution of the problem (2.2), it is enough to show that the operator F defined by (3.2) has a fixed point.

Theorem 3.1. *Let the operator g be strongly harmonic-like monotone with constant $\sigma > 0$ and harmonic-like Lipschitz continuous with constant $\zeta > 0$, respectively. If the Assumption 2.1 holds and there exists a parameter $\rho > 0$, such that*

$$\rho < 1 - k, \quad k < 1, \tag{3.3}$$

where

$$\theta = \rho + k \tag{3.4}$$

$$k = \sqrt{1 - 2\sigma + \zeta^2} + \eta + \zeta. \tag{3.5}$$

then there exists a unique solution of the problem (2.2).

Proof. From Lemma 3.1, it follows that problems (3.1) and (2.2) are equivalent. Thus it is enough to show that the operator $F(u)$, defined by (3.2) has a fixed point.
For all $\nu \neq \mu \in \Omega(\mu)$, we have

$$\begin{aligned}
\|F(\mu) - F(\nu)\| &= \|\mu - \nu - (g(\mu) - g(\nu))\| + \Pi_{\Omega(\mu)}\|[g(\mu) - \rho(\frac{2w\mu}{w+\mu})] - \Pi_{\Omega(\nu)}[g(\nu) - \rho(\frac{2w\nu}{w+\nu})]\| \\
&= \|\nu - \mu - (g(\nu) - g(\mu))\| + \|\Pi_{\Omega(\mu)}[g(\nu) - \rho(\frac{2w\nu}{w+\nu})] - \Pi_{\Omega(\nu)}[g(\nu) - \rho(\frac{2w\nu}{w+\nu})]\| \\
&\quad + \|\Pi_{\Omega(\nu)}[g(\nu) - \rho(\frac{2w\nu}{w+\nu})] - \Pi_{\Omega(\nu)}[g(\mu) - \rho(\frac{2w\mu}{w+\mu})]\| \\
&\leq \|\mu - \nu - (g(\mu) - g(\nu))\| + \eta\|\nu - \mu\| + \|g(\nu) - g(\mu) - \rho((\frac{2w\nu}{w+\nu}) - (\frac{2w\mu}{w+\mu}))\| \\
&\leq \|\mu - \nu - (g(\mu) - g(\nu))\| + \eta\|\nu - \mu\| + \zeta\|\nu - \mu\| + \rho\|\nu - \mu\|.
\end{aligned} \tag{3.6}$$

Since the operator g is strongly monotone with constants $\sigma > 0$ and Lipschitz continuous with constant $\zeta > 0$, it follows that

$$\begin{aligned}
\|\mu - \nu - (g(\mu) - g(\nu))\|^2 &\leq \|\mu - \nu\|^2 - 2\langle g(\mu) - g(\nu), \mu - \nu\rangle + \zeta^2\|g(\mu) - g(\nu)\|^2 \\
&\leq (1 - 2\sigma + \zeta^2)\|\mu - \nu\|^2.
\end{aligned} \tag{3.7}$$

From (3.6) and (3.7), we have

$$\begin{aligned}\|F(\mu)-F(\nu)\| &\leq \left\{\sqrt{(1-2\sigma+\zeta^2)}+\eta+\zeta+\rho\right\}\|\mu-\nu\| \\ &= \theta\|\mu-\nu\|,\end{aligned}$$

where

$$\begin{aligned}\theta &= \rho+k \\ k &= \sqrt{1-2\sigma+\zeta^2}+\eta+\zeta.\end{aligned}$$

From (3.3), it follows that $\theta<1$, which implies that the operator $F(u)$ defined by (3.2) has a fixed point and it is the unique solution of (2.2). □

We now suggested and analyzed three step iterative methods for solving the inverse harmonic-like quasi variational inequality (2.2) applying the fixed point formulation (3.1). which are known as modi-

For a given μ_0, compute the approximate solution $\{\mu_{n+1}\}$ by the iterative schemes

$$y_n = (1-\gamma_n)\mu_n+\gamma_n\{\mu_n-g(\mu_n)+\Pi_{\Omega(\mu_n)}[g(\mu_n)-\rho(\frac{2w\mu_n}{w+\mu_n})]\} \tag{3.8}$$

$$z_n = (1-\beta_n)\mu_n+\beta_n\{y_n-g(y_n)+\Pi_{\Omega(y_n)}[g(y_n)-\rho(\frac{2wy_n}{w+y_n})]\} \tag{3.9}$$

$$\mu_{n+1} = (1-\alpha_n)\mu_n+\alpha_n\{z_n-g(z_n)+\Pi_{\Omega(z_n)}[z_n-\rho(\frac{2wz_n}{w+z_n})]\}. \tag{3.10}$$

fied Noor iterations.

For $\gamma_n=0$, Algorithm 3 reduces to: which is known as the Ishikawa iterative scheme for the problem

For a given μ_0, compute $\{\mu_{n+1}\}$ by the iterative schemes

$$\begin{aligned}z_n &= (1-\beta_n)\mu_n+\beta_n\{\mu_n-g(\mu_n)+\Pi_{\Omega(\mu_n)}[g(\mu_n)-\rho(\frac{2w\mu_n}{w+\mu_n})]\} \\ \mu_{n+1} &= (1-\alpha_n)\mu_n+\alpha_n\{z_n-g(z_n)+\Pi_{\Omega(\mu_n)}[g(z_n)-\rho(\frac{2wz_n}{w+z_n})\},\end{aligned}$$

(2.2).

Note that for $\gamma_n=0 \quad and \quad \beta_n=0$, Algorithm 3 is called the Mann iterative method, that is. We now

For a given μ_0, compute $\{\mu_{n+1}\}$ by the iterative schemes

$$\mu_{n+1}=(1-\beta_n)\mu_n+\beta_n\{\mu_n-g(\mu_n)+\Pi_{\Omega(\mu_n)}[g(\mu_n)-\rho(\frac{2w\mu_n}{w+\mu_n})]\}.$$

study the convergence analysis of Algorithm 3, which is the main motivation of our next result.

Theorem 3.2. *Let the operator g satisfy all the assumptions of Theorem 3.1. If the condition (3.3) holds, then the approximate solution $\{u_n\}$ obtained from Algorithm 3 converges to the exact solution $\mu\in\Omega(\mu)$ of the inverse harmonic-like quasi variational inequality (2.2) strongly in $\mathcal{H}$.*

Proof. From Theorem 3.1, we see that there exists a unique solution $\mu\in\Omega(\mu)$ of the inverse harmonic-like quasi variational inequalities (2.2). Let $\mu\in\Omega(\mu)$ be the unique solution of (2.2). Then, using Lemma

3.1, we have

$$\mu = (1-\alpha_n)\mu + \alpha_n\{\mu - g(\mu) + \Pi_{\Omega(\mu)}[g(\mu) - \rho(\frac{2w\mu}{w+\mu})]\} \tag{3.11}$$

$$= (1-\beta_n)\mu + \beta_n\{\mu - g(\mu) + \Pi_{\Omega(\mu)}[g(\mu) - \rho(\frac{2w\mu}{w+\mu})]\} \tag{3.12}$$

$$= (1-\gamma_n)\mu + \gamma_n\{\mu - g(\mu) + \Pi_{\Omega(\mu)}[g(\mu) - \rho(\frac{2w\mu}{w+\mu})]\}. \tag{3.13}$$

From (3.10),(3.11) and Assumption (2.1), we have

$$\begin{aligned}
\|\mu_{n+1} - \mu\| &= \|(1-\alpha_n)(\mu_n - \mu) + \alpha_n(z_n - \mu - (g(z_n) - g(\mu))) \\
&\quad +\alpha_n\Pi_{\Omega((z_n)}[g(z_n) - \rho(\frac{2wz_n}{w+z_n})] - \Pi_{(\mu)}[g(\mu) - \rho(\frac{2w\mu}{w+\mu})\}\| \\
&\leq (1-\alpha_n)\|\mu_n - \mu\| + \alpha_n\|z_n - \mu - (g(z_n) - g(\mu))\| \\
&\quad +\alpha_n\Pi_{\Omega((z_n)}[g(z_n) - \rho(\frac{2wz_n}{w+z_n})] - \Pi_{\Omega(z_n)}[g(\mu_n) - \rho(\frac{2w\mu}{w+\mu})\}\| \\
&\quad +\alpha_n\{\Pi_{(z_n)}[g(\mu_n) - \rho\mu] - \Pi_{\Omega(\mu)}[g(\mu) - \rho\mu\}\| \\
&\leq (1-\alpha_n)\|\mu_n - \mu\| + \alpha_n\|z_n - \mu - (g(z_n) - g(\mu))\| \\
&\quad +\alpha_n\|g(z_n) - g(\mu) - \rho(z_n - \mu)|| + \alpha_n\eta\|z_n - \mu\| \\
&\leq (1-\alpha_n)\|\mu_n - \mu\| + \alpha_n(k+\rho)\|z_n - \mu\| \\
&= (1-\alpha_n)||u_n - \mu\| + \alpha_n\theta\|z_n - \mu\|,
\end{aligned} \tag{3.14}$$

where θ is defined by (3.4).
In a similar way, from (3.8) and (3.12), we have

$$\begin{aligned}
\|z_n - \mu\| &\leq (1-\beta_n)\|\mu_n - \mu\| + 2\beta_n\theta\|y_n - \mu - (g(y_n) - g(\mu))\| \\
&\quad +\beta_n\|g(y_n) - g(\mu) - \rho(y_n - \mu)\| + \beta_n\eta\|y_n - \mu\| \\
&\leq (1-\beta_n)\|\mu_n - \mu\| + \beta_n(k+\rho)\|y_n - \mu\|, \\
&\leq (1-\beta_n)\|\mu_n - \mu|| + \beta_n\theta\|y_n - \mu\|,
\end{aligned} \tag{3.15}$$

where θ is defined by (3.3).
From (3.8) and (3.13), we obtain

$$\begin{aligned}
\|y_n - \mu\| &\leq (1-\gamma_n)\|\mu_n - \mu\| + \gamma_n\theta\|\mu_n - \mu\| \\
&\leq (1-(1-\theta)\gamma_n)\|\mu_n - \mu\| \\
&\leq ||\mu_n - \mu||.
\end{aligned} \tag{3.16}$$

From (3.15) and (3.16), we obtain

$$\begin{aligned}
\|w_n - \mu\| &\leq (1-\beta_n)\|\mu_n - \mu\| + \beta_n\theta\|\mu_n - \mu\| \\
&= (1-(1-\theta)\beta_n)\|\mu_n - \mu\| \\
&\leq ||\mu_n - \mu||.
\end{aligned} \tag{3.17}$$

Form the above equations, we have

$$\begin{aligned}
\|\mu_{n+1} - \mu\| &\leq (1-\alpha_n)\|\mu_n - \mu\| + \alpha_n\theta\|\mu_n - \mu\| \\
&= [1-(1-\theta)\alpha_n]\|\mu_n - \mu\| \\
&\leq \prod_{i=0}^{n}[1-(1-\theta)\alpha_i]\|\mu_0 - \mu\|.
\end{aligned}$$

Since $\sum_{n=0}^{\infty}\alpha_n$ diverges and $1-\theta>0$, we have $\prod_{i=0}^{n}[1-(1-\theta)\alpha_i]=0$. Consequently the sequence $\{u_n\}$ convergence strongly to μ. From (3.16), and (3.17), it follows that the sequences $\{y_n\}$ and $\{w_n\}$ also converge to μ strongly in $\mathcal{H}$. This completes the proof. □

We suggest new perturbed iterative schemes for solving the inverse harmonic-like quasi variational inequality (2.2). where $\{e_n\}$, $\{f_n\}$, and $\{h_n\}$ are the sequences of the elements of $\mathcal{H}$ introduced to

For a given μ_0, compute the approximate solution $\{\mu_n\}$ by the iterative schemes

$$\begin{aligned} y_n &= (1-\gamma_n)\mu_n+\gamma_n\{\mu_n-g(\mu_n)+\Pi_{\Omega(\mu_n)}[g(\mu_n)-\rho(\frac{2w\mu_n}{w+\mu_n})]\}+\gamma_n h_n\\ z_n &= (1-\beta_n)\mu_n+\beta_n\{y_n-g(y_n)+\Pi_{\Omega(y_n)}[g(y_n)-\rho(\frac{2wy_n}{w+y_n})]\}+\beta_n f_n\\ \mu_{n+1} &= (1-\alpha_n)\mu_n+\alpha_n\{z_n-g(z_n)+\Pi_{\Omega(z_n)}[g(z_n)-\rho(\frac{2wz_n}{w+z_n})\}+\alpha_n e_n, \end{aligned}$$

take into account possible inexact computations and $\Pi_{\Omega(\mu_n)}$ is the corresponding perturbed projection operator and the sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ satisfy

$$0\le\alpha_n,\beta_n,\gamma_n\le 1;\quad \forall n\ge 0,\quad \sum_{n=0}^{\infty}\alpha_n=\infty.$$

For $\gamma_n=0$, we obtain the perturbed Ishikawa iterative method and for $\gamma_n=0,\quad \beta_n=0$, we obtain the perturbed Mann iterative schemes for solving inverse harmonic-like quasi variational inequality (2.2).

Also, we can suggest the following iterative methods for solving the inverse harmonic-like quasi variational inequalities. which is known as the projection method.

For a given μ_0, compute μ_{n+1} by the iterative scheme

$$\mu_{n+1}=\mu_n-g(\mu_n)+\Pi_{\Omega(\mu_n)}[g(\mu_n)-\rho(\frac{2w\mu_n}{w+\mu_n})],\quad n=0,1,2,... \tag{3.18}$$

which is known as the implicit projection method and is equivalent to the following two-step method.

For a given μ_0, compute μ_{n+1} by the iterative scheme

$$\mu_{n+1}=\mu_n-g(\mu_n)+\Pi_{\Omega(\mu_{n+1})}[g(\mu_n)-\rho(\frac{2w\mu_{n+1}}{w+\mu_{n+1}})],\quad n=0,1,2,... \tag{3.19}$$

For a given μ_0, compute μ_{n+1} by the iterative scheme

$$\begin{aligned} z_n &= \mu_n-g(\mu_n)+\Pi_{\Omega(\mu_n)}[g(\mu_n)-\rho(\frac{2w\mu_n}{w+\mu_n})]\\ \mu_{n+1} &= \mu_n-g(\mu_n)+\Pi_{z_n)}[g(\mu_n)-\rho(\frac{2wz_n}{w+z_n})],\quad n=0,1,2,... \end{aligned}$$

which is known as the modified projection method and is equivalent to the iterative method. which

For a given μ_0, compute μ_{n+1} by the iterative scheme

$$\mu_{n+1} = \mu_n - g(\mu_n) + \Pi_{\Omega(\mu_{n+1})}[g(\mu_{n+1}) - \rho(\frac{2w\mu_{n+1}}{w+\mu_{n+1}})], \quad n = 0, 1, 2, ... \tag{3.20}$$

For a given μ_0, compute μ_{n+1} by the iterative scheme

$$\begin{aligned} z_n &= \mu_n - g(\mu_n) + \Pi_{\Omega(\mu_n)}[g(\mu_n) - \rho u_n] \\ \mu_{n+1} &= \mu_n - g(\mu_n) + \Pi_{\Omega(z_n)}[g(z_n) - \rho(\frac{2wz_n}{w+z_n})], \quad n = 0, 1, 2, ... \end{aligned}$$

is two-step predictor-corrector method for solving the problem (2.2).
We can rewrite the equation (3.1) as:

$$\mu = \mu - g(\mu) + \Pi_{\Omega(\mu)}[g(\frac{\mu+\mu}{2}) - \rho(\frac{2w\mu}{w+\mu})]. \tag{3.21}$$

This fixed point formulation is used to suggest the following implicit method. Applying the predictor-

[31]. For a given μ_0, compute μ_{n+1} by the iterative scheme

$$\mu_{n+1} = \mu_n - g(\mu_n) + \Pi_{\Omega(\mu_{n+1})}[g(\frac{\mu_n+\mu_{n+1}}{2}) - \rho(\frac{2w\mu_{n+1}}{w+\mu_{n+1}})]. \tag{3.22}$$

corrector technique, we suggest the following inertial iterative method for solving the problem (2.2) .

For a given μ_0, compute μ_{n+1} by the iterative scheme

$$\begin{aligned} z_n &= \mu_n - g(\mu_n) + \Pi_{\Omega(\mu_n)}[g(\mu_n) - \rho(\frac{2w\mu_n}{w+\mu_n})] \\ \mu_{n+1} &= \mu_n - g(\mu_n) + \Pi_{\Omega(z_n)}[g(\frac{z_n+\mu_n}{2}) - \rho(\frac{2wz_n}{w+z_n})]. \end{aligned}$$

For a parameter ξ, one can rewrite the (3.1) as

$$\mu = \mu - g(\mu) + \Pi_{\Omega(\mu)}[g((1-\xi)\mu + \xi\mu) - \rho(\frac{2w(1-\xi)\mu + \xi\mu)}{w + (1-\xi)\mu + \xi\mu})].$$

This equivalent fixed point formulation enables to suggest the following inertial method for solving the problem (2.3). which is equivalent to the following two-step method. Algorithm 3 is known as the inertial projection method.

We now suggest multi-step inertial methods for solving the inverse harmonic-like quasi variational inequalities (2.2). Algorithm 3 is a three-step modified inertial method for solving inverse harmonic-like quasi variational inequalities(2.2).

Similarly a four-step inertial method for solving the inverse harmonic-like quasi variational inequalities (2.2) is suggested. These multistep iterative methods can be viewed as novle geenralizations of the Noor (three- step) iterations. Using the technique of Noor et al.[55] and Jabeen et al [20], one can investigate the convergence analysis of these inertial projection methods. We would like to mention that Algorithm 3 and Algorithm 3 can be viewed as the generalizations of Noor (three-step) iterations [38, 39] for solving the inverse quasi variational inequalities. Similar multi-step hybrid iterative methods can be proposed and analyzed for solving system of inverse quasi variational inequalities, which is an interesting problem.

For a given $\mu_0, \mu_1 \in \Omega(\mu)$, compute μ_{n+1} by the iterative scheme

$$\mu_{n+1} = \mu_n - g(\mu_n) + \Pi_{\Omega(\mu_n)}[g((1-\xi)\mu_n + \xi\mu_{n-1}) - \rho(\frac{2w(1-\xi)\mu_n + \xi\mu_{n-1})}{w + (1-\xi)\mu_n + \xi\mu_{n-1}})], \quad n = 0, 1, 2, \ldots,$$

For given μ_0, μ_1, compute μ_{n+1} by the iterative scheme

$$\begin{aligned} z_n &= (1-\xi)u_n + \xi u_{n-1} \\ \mu_{n+1} &= \mu_n - g(\mu_n) + \Pi_{\Omega(\mu_n)}[g(z_n) - \rho(\frac{2wz_n}{w + z_n}]. \end{aligned}$$

For given μ_0, μ_1, compute μ_{n+1} by the recurrence relation

$$\begin{aligned} z_n &= \mu_n - \theta_n(\mu_n - \mu_{n-1}) \\ y_n &= (1-\beta_n)z_n + \beta_n\left\{z_n - g(z_n) + \Pi_{(z_n)}\left[g(\frac{z_n + \mu_n}{2}) - \rho(\frac{2w(z_n + \mu_n)}{2w + z_n + \mu_n}))\right]\right\}, \\ \mu_{n+1} &= (1-\alpha_n)y_n + \alpha_n\left\{y_n - g(y_n) + \Pi_{\Omega(y_n)}\left[g(\frac{z_n + y_n}{2}) - \rho(\frac{2w(y_n + z_n)}{2w + y_n + z_n})\right]\right\}, \end{aligned}$$

where $\theta_n \in [0,1], \forall n \geq 1$.

For given μ_0, μ_1, compute μ_{n+1} by the recurrence relation

$$\begin{aligned} z_n &= \mu_n - \theta_n(\mu_n - \mu_{n-1}), \\ y_n &= (1-\gamma_n)z_n + \gamma_n\left\{z_n - g(z_n) + \Pi_{(z_n)}\left[g(\frac{z_n + \mu_n}{2}) - \rho(\frac{2w(z_n + \mu_n}{2w + z_n + \mu_n})\right]\right\}, \\ t_n &= (1-\beta_n)y_n + \beta_n\left\{y_n - g(y_n) + \Pi_{\Omega(y_n)}\left[g(\frac{y_n + z_n + \mu_n}{3}) - \rho(\frac{2w(y_n + z_n + \mu_n)}{3w + y_n + z_n + \mu_n})\right]\right\}, \\ \mu_{n+1} &= (1-\alpha_n)z_n + \alpha_n\left\{z_n - g(z_n) + \Pi_{\Omega(z_n)}\left[g(\frac{z_n + y_n + t_n + \mu_n}{3}) - \rho(\frac{2w(z_n + y_n + t_n + \mu_n)}{4w + z_n + y_n + t_n + \mu_n})\right]\right\}, \end{aligned}$$

where $\alpha_n, \beta_n, \gamma_n, \theta_n \in [0,1], \quad \forall n \geq 1$.

4. Auxiliary Principle Technique

There are several techniques such as projection, resolvent, descent methods for solving the variational inequalities and their variant forms. None of these techniques can be applied for suggesting the iterative methods for solving the several nonlinear variational inequalities and equilibrium problems. To overcome these drawbacks, one usually applies the auxiliary principle technique, which is mainly due to Glowinski et al [15] as developed in [35, 50, 54, 55, 58], to suggest and analyze some proximal point methods for solving the inverse harmonic-like quasi variational inequalities (2.2).

We apply the auxiliary principle technique involving an arbitrary operator, which is mainly due to Noor [35], for finding the approximate solution of the inverse harmonic-like quasi variational inequalities (2.2).

For a given $\mu \in \Omega(\mu)$ satisfying (2.2), find $z \in \Omega(\mu)$ such that

$$\langle \rho(\frac{2w(z + \eta(\mu - z))}{w + z + \eta(\mu - z)}), \nu - g(z)\rangle + \langle M(z) - M(\mu), \nu - z\rangle \geq 0, \quad \forall w, \nu \in \Omega(\mu), \tag{4.1}$$

where $\rho > 0, \eta \in [0,1]$ are constants and M is an arbitrary operator. The inequality (7.1) is called the auxiliary inverse harmonic-like quasi variational inequality.

If $z = \mu$, then w is a solution of (2.2). This simple observation enables us to suggest the following iterative method for solving (2.2). Algorithm 4 is called the hybrid proximal point algorithm for solving the inverse harmonic-like quasi variational inequalities (2.2).

For a given $\mu_0 \in \Omega(\mu)$, compute the approximate solution μ_{n+1} by the iterative scheme

$$\langle \rho(\frac{2w(\mu_{n+1}+\eta(\mu_n-\mu_{n+1})}{w+\mu_{n+1}+\eta(\mu_n-\mu_{n+1}})), \nu - g(\mu_{n+1})\rangle + \langle M(\mu_{n+1}) - M(\mu_n), \nu - \mu_{n+1}\rangle \geq 0, \quad \forall w, \nu \in \Omega(\mu). \tag{4.2}$$

Special cases: We now discuss some special cases.
(I). For $\eta = 0$, Algorithm 4 reduces to is called the implicit iterative methods for solving the problem (2.2).

For a given μ_0, compute the approximate solution μ_{n+1} by the iterative scheme

$$\langle \rho(\frac{2w\mu_{n+1}}{w+\mu_{n+1}}), \nu - g(\mu_{n+1})\rangle + \langle M(\mu_{n+1}) - M(\mu_n), \nu - \mu_{n+1}\rangle \geq 0, \quad \forall w, \nu \in \Omega(\mu), \tag{4.3}$$

(II). If $\eta = 1$, then Algorithm 4 collapses to is called the explicit iterative method.

For a given μ_0, compute the approximate solution μ_{n+1} by the iterative scheme

$$\langle \rho(\frac{2w\mu_n}{w+\mu_n}), \nu - g(\mu_{n+1})\rangle + \langle M(\mu_{n+1}) - M(\mu_n), \nu - \mu_{n+1}\rangle \geq 0, \quad \forall \nu \in \Omega(\mu),$$

(III). For $\eta = \frac{1}{2}$, Algorithm 4 becomes: is known as the mid-point proximal method for solving the problem

For a given μ_0, compute the approximate solution μ_{n+1} by the iterative scheme

$$\langle \rho(\frac{2w(\mu_{n+1}+\mu_n}{2w+\mu_{n+1}+\mu_n}), \nu - g(\mu_{n+1})\rangle + \langle M(\mu_{n+1}) - M(\mu_n), \nu - \mu_{n+1}\rangle \geq 0, \quad \forall w, \nu \in \Omega(\mu),$$

(2.2).
For the convergence analysis of Algorithm 4, we need the following concepts.

Definition 4.1. An operator g is said to be pseudomontone, if

$$\langle (\frac{2w\mu}{w+\mu}), \nu - g(\mu)\rangle \geq 0, \quad \forall w, \nu \in \Omega(\mu),$$

implies that

$$-\langle \frac{2w\nu}{w+\nu}), g(\mu) - \nu\rangle \geq 0, \quad \forall w, \nu \in \Omega(\mu).$$

Theorem 4.1. *Let the operator g be a pseudo-monotone. Let the approximate solution μ_{n+1} obtained in Algorithm 4 converges to the exact solution $\mu \in \Omega(\mu)$ of the problem (2.2). If the operator M is strongly monotone with constant $\xi \geq 0$ and Lipschitz continuous with constant $\zeta \geq 0$, then*

$$\xi\|\mu_{n+1} - \mu_n\| \leq \zeta\|\mu - \mu_n\|. \tag{4.4}$$

Proof. Let $\mu \in \Omega(\mu)$ be a solution of the problem (2.2). Then

$$-\langle \rho(\frac{2w\nu}{w+\nu}), g(\mu) - \nu\rangle \geq 0, \quad \forall w, \nu \in \Omega(\mu), \tag{4.5}$$

since the operator g is a pseudo-monotone.
Takin $\nu = \mu_{n+1}$ in (4.5), we obtain

$$-\langle \rho(\frac{2w\mu_{n+1}}{w+\mu_{n+1}}), g(\mu) - \mu_{n+1}\rangle \geq 0. \tag{4.6}$$

Setting $\nu = \mu$ in (7.3), we have

$$\langle \rho(\frac{2w\mu_{n+1}}{w+\mu_{n+1}}), g(\mu) - \mu_{n+1}\rangle + \langle M(\mu_{n+1}) - M(\mu_n), \mu - \mu_{n+1}\rangle \geq 0. \tag{4.7}$$

Combining (4.7), (4.6) and (4.5), we have

$$\langle M(\mu_{n+1}) - M(\mu_n), \mu - \mu_{n+1}\rangle \quad \geq \quad -\langle \rho\mu_{n+1}, g(\mu) - u_{n+1}\rangle \geq 0. \tag{4.8}$$

From the equation (4.8), we have

$$\begin{aligned} 0 \quad &\leq \quad \langle M(\mu_{n+1}) - M(\mu_n), \mu - \mu_{n+1}\rangle \\ &= \quad \langle M(\mu_{n+1}) - M(\mu_n), \mu - \mu_n + \mu_n - u_{n+1}\rangle \\ &= \quad \langle M(\mu_{n+1}) - M(\mu_n), \mu - \mu_n\rangle + \langle M(\mu_{n+1} - M(\mu_n), \mu_n - \mu_{n+1}\rangle, \end{aligned}$$

which implies that

$$\langle M(\mu_{n+1} - M(\mu_n), \mu_{n+1} - \mu_n\rangle \leq \langle M(\mu_{n+1}) - M(\mu_n), \mu - \mu_n\rangle.$$

Now using the strongly monotonicity with constant $\xi > 0$ and Lipschitz continuity with constant ζ of the operator M, we obtain

$$\xi\|\mu_{n+1} - \mu_n\|^2 \leq \zeta\|\mu_{n+1} - \mu_n\|\|\mu_n - \mu\|.$$

Thus

$$\xi\|\mu_n - \mu_{n+1}\| \leq \zeta\|\mu_n - \mu\|,$$

the required result (7.4). □

Theorem 4.2. *Let H be a finite dimensional space and all the assumptions of Theorem 4.1 hold. Then the sequence $\{\mu_n\}_0^\infty$ given by Algorithm 4 converges to the exact solution $\mu \in \Omega(\mu)$ of (2.2).*

Proof. Let $\mu \in \Omega(\mu)$ be a solution of (2.2). From (7.4), it follows that the sequence $\{\|\mu - \mu_n\|\}$ is nonincreasing and consequently $\{u_n\}$ is bounded. Furthermore, we have

$$\xi\sum_{n=0}^{\infty}\|\mu_{n+1} - \mu_n\| \leq \zeta\|\mu_n - \mu\|,$$

which implies that

$$\lim_{n\to\infty}\|\mu_{n+1} - \mu_n\| = 0. \tag{4.9}$$

Let $\hat{\mu}$ be the limit point of $\{\mu_n\}_0^\infty$; whose subsequence $\{\mu_{n_j}\}_1^\infty$ of $\{\mu_n\}_0^\infty$ converges to $\hat{\mu} \in \Omega(\mu)$. Replacing w_n by μ_{n_j} in (4.3), taking the limit $n_j \longrightarrow \infty$ and using (4.9), we have

$$\langle \rho(\frac{2w\hat{\mu}}{w + \hat{\mu}}), \nu - g(\hat{\mu})\rangle \geq 0, \qquad \forall w, \nu \in \Omega(\mu),$$

which implies that $\hat{u}$ solves the problem (2.2) and

$$\|\mu_{n+1} - \mu\| \leq \|\mu_n - \mu\|.$$

Thus, it follows from the above inequality that $\{\mu_n\}_1^\infty$ has exactly one limit point $\hat{u}$ and

$$\lim_{n\to\infty}(\mu_n) = \hat{\mu}.$$

the required result. □

In recent years inertial type iterative methods have been applied to find the approximate solutions of variational inequalities and related optimizations. We again apply the modified auxiliary principle approach involving an arbitrary nonlinear operator to suggest some hybrid inertial proximal point schemes for solving the inverse harmonic-like quasi variational inequalities.

For a given $\mu \in \Omega(\mu)$ satisfying (2.2), find $w \in \Omega(\mu)$ such that

$$\begin{aligned} &\langle \rho(\frac{2w(z + \eta(\mu - z))}{w + z + \eta(\mu - z}\,)), \nu - g(z)\rangle \\ &\quad + \langle M(z) - M(\mu) + \alpha(\mu - \mu), \nu - z\rangle \geq 0, \quad \forall w, \nu \in \Omega(\mu), \end{aligned} \tag{4.10}$$

where $\rho > 0, \eta, \alpha \in [0, 1]$ are constants and M is a nonlinear operator.

Clearly $z = \mu$, implies that z is a solution of (2.2). This simple observation enables us to suggest the following

For given μ_0, μ_1, compute the approximate solution μ_{n+1} by the iterative scheme

$$\langle \rho(\frac{2w(\mu_{n+1}+\eta(\mu_n-\mu_{n+1}))}{w+\mu_{n+1}+\eta(\mu_n-\mu_{n+1}})), \nu - g(\mu_{n+1})\rangle$$
$$+\langle M(\mu_{n+1}) - M(\mu_n) + \alpha(\mu_n - \mu_{n-1}), \nu - \mu_{n+1}\rangle \geq 0, \quad \forall w, \nu \in \Omega(\mu).$$

iterative method for solving (2.2). Algorithm 4 is called the hybrid proximal point algorithm for solving the inverse harmonic-like quasi variational inequalities (2.2). For $\alpha = 0$, Algorithm 4 is exactly Algorithm 4.
For $M = I$, Algorithm 4 reduces to the following method:

For given μ_0, μ_1, compute the approximate solution μ_{n+1} by the iterative scheme

$$\langle \rho(\frac{2w(\mu_{n+1}+\eta(\mu_n-\mu_{n+1}))}{w+\mu_{n+1}+\eta(\mu_n-\mu_{n+1}})), \nu - g(\mu_{n+1})\rangle$$
$$+\langle \mu_{n+1} - \mu_n + \alpha(\mu_n - \mu_{n-1}), \nu - \mu_{n+1}\rangle \geq 0, \quad \forall w, \nu \in \Omega(\mu).$$

Remark 4.2. For different and suitable choice of the parameters ρ, η, α, operators g, M and convex-valued sets, one can recover new and known iterative methods for solving inverse harmonic-like quasi variational inequalities, inverse complementarity problems and related optimization problems. Using the technique and ideas of Theorem 4.1 and Theorem 4.2, one can analyze the convergence of Algorithm 4 and its special cases.

.

5. Dynamical Systems Technique

In this section, we consider the dynamical systems technique for solving the inverse harmonic-like quasi variational inequalities. The projected dynamical systems associated with variational inequalities were considered by Dupuis et al. [14] and Nagurney et al. [25]. It is worth mentioning that the dynamical system is a first order initial value problem. Consequently, variational inequalities and nonlinear problems arising in various branches in pure and applied sciences can now be studied via the differential equations. It has been shown that the dynamical systems are useful in developing some efficient numerical techniques for solving variational inequalities and related optimization problems. For applications, generalizations and numerical aspects, see [13, 14, 26, 30, 44, 45, 46, 49, 50, 52, 54, 61, 62, 63, 64]. We consider some new iterative methods for solving the inverse harmonic-likequasi variational inequalities. We investigate the convergence analysis of these new methods involving only the monotonicity of the operators.

We now define the residue vector $R(\mu)$ by the relation

$$R(\mu) = \Pi_{\Omega(\mu)}[g(\mu) - \rho(\frac{2w\mu}{w+\mu})] - g(\mu). \tag{5.1}$$

Invoking Lemma 3.1, one can easily conclude that $\mu \in \mathcal{H}$ is a solution of the problem(2.2), if and only if, $\mu \in \mathcal{H}$ is a zero of the equation

$$R(\mu) = 0. \tag{5.2}$$

We now consider a dynamical system associated with the inverse harmonic-likequasi variational inequalities. Using the equivalent formulation (3.1), we suggest a class of projection dynamical systems as

$$\frac{d\mu}{dt} = \lambda\{\Pi_{\Omega(\mu)}[g(\mu) - \rho(\frac{2w\mu}{w+\mu})] - g(\mu)\}, \quad \mu(t_0) = \alpha, \tag{5.3}$$

where λ is a parameter. The system of type (5.3) is called the projection dynamical system associated with the problem (2.2). Here the right hand is related to the projection and is discontinuous on the boundary. From the definition, it is clear that the solution of the dynamical system always stays in $\mathcal{H}$. This implies that the qualitative results such as the existence, uniqueness and continuous dependence of the solution of (2.2) can be studied.
The equilibrium point of the dynamical system (5.12) is defined as follows.

Definition 5.1. An element $\mu \in \mathcal{H}$, is an equilibrium point of the dynamical system (5.12), if,

$$\frac{d\mu}{dx} = 0.$$

Thus it is clear that $\mu \in \mathcal{H}$ is a solution of the inverse harmonic-like quasi variational inequality (2.2), if and only if, $\mu \in \mathcal{H}$ is an equilibrium point.
This implies that $\mu \in \Omega(\mu)$ is a solution of the inverse harmonic-like quasi variational inequality (2.2), if and only if, $\mu \in \Omega(\mu)$ is an equilibrium point.

Definition 5.2. [13, 24] The dynamical system is said to converge to the solution set S^* of (5.3), if , irrespective of the initial point, the trajectory of the dynamical system satisfies

$$\lim_{t\to\infty} \text{dist}(\mu(t), S^*) = 0, \tag{5.4}$$

where

$$\text{dist}(\mu, S^*) = \inf_{\nu\in S^*} \|\mu - \nu\|.$$

It is easy to see, if the set S^* has a unique point μ^*, then (5.4) implies that

$$\lim_{t\to\infty} \mu(t) = \mu^*.$$

If the dynamical system is still stable at μ^* in the Lyapunov sense, then the dynamical system is globally asymptotically stable at μ^*.

Definition 5.3. The dynamical system is said to be globally exponentially stable with degree η at μ^*, if, irrespective of the initial point, the trajectory of the system satisfies

$$\|\mu(t) - \mu^*\| \leq u_1\|\mu(t_0) - \mu^*\| exp(-\eta(t - t_0)), \quad \forall t \geq t_0,$$

where u_1 and η are positive constants independent of the initial point.

It is clear that the globally exponentially stability is necessarily globally asymptotically stable and the dynamical system converges arbitrarily fast.

Lemma 5.1. *(Gronwall Lemma)*[13] *Let $\hat{\mu}$ and $\hat{\nu}$ be real-valued nonnegative continuous functions with domain $\{t : t \leq t_0\}$ and let $\alpha(t) = \alpha_0(|t - t_0|)$, where α_0 is a monotone increasing function. If, for $t \geq t_0$,*

$$\hat{\mu} \leq \alpha(t) + \int_{t_0}^{t} \hat{\mu}(s)\hat{\nu}(s)ds,$$

then

$$\hat{\mu}(s) \leq \alpha(t)exp\{\int_{t_0}^{t} \hat{\nu}(s)ds\}.$$

We now establish that the trajectory of the solution of the projection dynamical system (5.3) converges to the unique solution of the inverse harmonic-like quasi variational inequality (2.2). The analysis is in the spirit of Noor [38, 39] and Xia and Wang [64].

Theorem 5.1. *Let the operator $g : H \longrightarrow H$ be Lipschitz continuous with constant $\zeta > 0$. If $\lambda(\eta + 2\zeta + \rho) < 1$ and Assumption 2.1 then, for each $\mu_0 \in \Omega\mu$, there exists a unique continuous solution $\mu(t)$ of the dynamical system (5.3) with $\mu(t_0) = \mu_0$ over $[t_0, \infty)$.*

Proof. Let

$$G(\mu) = \Pi_{\Omega(\mu)}[g(\mu) - \rho(\frac{2w\mu}{w + \mu})] - g(\mu)\}, \quad \forall w, \mu \in H.$$

where $\lambda > 0$ is a constant and $G(\mu) = \frac{d\mu}{dt}$.
$\forall \mu, \nu \in H$, we have

$$\begin{aligned}
\|G(\mu) - G(\nu)\| &\leq \lambda\{\|\Pi_{\Omega(\mu)}[g(\mu) - \rho(\frac{2w\mu}{w+\mu})] - \Pi_{\Omega(\nu)}[g(\nu) - \rho(\frac{2w\nu}{w+\nu})]\| + \|g(\mu) - g(\nu)\|\} \\
&= \lambda\{\|g(\mu) - g(\nu)\| + \|\Pi_{\Omega(\mu)}[g(\mu) - \rho(\frac{2w\mu}{w+\mu})] - \Pi_{\Omega(\mu)}[g(\nu) - \rho(\frac{2w\nu}{w+\nu})]\| \\
&\quad + \|\Pi_{\Omega(\mu)}[g(\nu) - \rho(\frac{2w\nu}{w+\nu})] - \Pi_{\Omega(\nu)}[g(\nu) - \rho(\frac{2w\nu}{w+\nu})]\|\} \\
&\leq \lambda\{\|g(\mu) - g(\nu)\| + \eta\|\mu - \nu\| + \|g(\mu) - g(\nu) - \rho((\frac{2w\mu}{w+\mu}) - (\frac{2w\nu}{w+\nu}))\} \\
&\leq \lambda\{\|g(\mu) - g(\nu)\| + \eta\|\mu - \nu\| + \{\|g(\mu) - g(\nu)\| + \rho\|\mu - \nu\|\} \\
&\leq \lambda\{(\eta + 2\zeta + \rho)\}\|\mu - \nu\|.
\end{aligned}$$

This implies that the operator $G(\mu)$ is a Lipschitz continuous with constant $\lambda\{(\eta + 2\zeta + \rho)\} < 1$ and for each $\mu \in \Omega(\mu)$, there exists a unique and continuous solution $\mu(t)$ of the dynamical system (5.3), defined on an interval $t_0 \leq t < T_1$ with the initial condition $\mu(t_0) = \mu_0$. Let $[t_0, T_1)$ be its maximal interval of existence. Then we have to show that $T_1 = \infty$. Consider , for any $\mu \in \Omega(\mu)$,

$$\begin{aligned}
\|G(\mu)\| = \|\frac{d\mu}{dt}\| &= \lambda\|[g(\mu) - \rho(\frac{2w\mu}{w+\mu})] - g(\mu)\| \\
&\leq \lambda\{\|\Pi_{\Omega(\mu)}[g(\mu) - \rho(\frac{2w\mu}{w+\mu})] - \Pi_{\Omega(\mu)}[0]\| + \|\Pi_{\Omega(\mu)}[0] - g(\mu)\|\} \\
&\leq \lambda\{\delta\|\{g(\mu) - \rho(\frac{2w\mu}{w+\mu})\| + \|\Pi_{\Omega(\mu)}[g(\mu)] - \Pi_{\Omega(\mu)}[0]\| + \|\Pi_{\Omega(\mu)}[0] - g(u)\|\} \\
&\leq \lambda\{(\rho + 2 + 2\zeta)\|u\| + \|\Pi_{\Omega(\mu)}[0]\|\}.
\end{aligned}$$

Then

$$\begin{aligned}
\|\mu(t)\| &\leq \|\mu_0\| + \int_{t_0}^{t} \|\mu(s)\| ds \\
&\leq (\|\mu_0\| + k_1(t - t_0)) + k_2 \int_{t_0}^{t} \|\mu(s)\| ds,
\end{aligned}$$

where $k_1 = \lambda\|\Pi_{\Omega(\mu)}[0]\|$ and $k_2 = \delta\lambda(\rho + 2 + 2\zeta)$. Hence by the Gronwall Lemma 5.1, we have

$$\|\mu(t)\| \leq \{\|u_0\| + k_1(t - t_0)\}e^{k_2(t-t_0)}, \quad t \in [t_0, T_1).$$

This shows that the solution is bounded on $[t_0, T_1)$. So $T_1 = \infty$. □

Theorem 5.2. *If the operator $g : \mathcal{H} \longrightarrow \mathcal{H}$ is strongly monotone with constant $\sigma > 0$ and $\zeta > 0$, then the dynamical system (5.3) converges globally exponentially to the unique solution of the general inverse harmonic-like quasi variational inequality (2.2).*

Proof. Since the operator g is Lipschitz continuous, it follows from Theorem 5.1 that the dynamical system (5.3) has unique solution $\mu(t)$ over $[t_0, T_1)$ for any fixed $\mu_0 \in H$. Let $\mu(t)$ be a solution of the initial value problem (5.3). For a given $\mu^* \in H$ satisfying (2.2), consider the Lyapunov function

$$L(\mu) = \lambda\|\mu(t) - \mu^*\|^2, \quad u(t) \in \Omega(\mu). \tag{5.5}$$

From (5.3) and (5.5), we have

$$\begin{aligned}
\frac{dL}{dt} &= 2\lambda\langle\mu(t)-\mu^*, \frac{d\mu}{dt}\rangle \\
&= 2\lambda\langle\mu(t)-\mu^*, \Pi_{\Omega(\mu)}[g(\mu(t))-\rho(\frac{2w\mu(t)}{w+\mu(t)})]-g(\mu(t))\rangle \\
&= 2\lambda\langle\mu(t)-\mu^*, \Pi_{\Omega(\mu)}[g(\mu(t))-\rho(\frac{2w\mu(t)}{w+\mu(t)})]-g(\mu^*(t))+g(\mu^*)-g(\mu(t))\rangle \\
&= -2\lambda\langle\mu(t)-\mu^*, g(\mu(t))-g(\mu^*(t))\rangle \\
&\quad +2\lambda\langle\mu(t)-\mu^*, \Pi_{\Omega(\mu)}[g(\mu(t))-\rho(\frac{2w\mu(t)}{w+\mu(t)})]-g(\mu^*(t))\rangle \\
&\leq -2\lambda\langle\rho(\mu(t)-\mu^*), g(\mu(t))-g(\mu^*(t))\rangle \\
&\quad +2\lambda\langle\mu(t)-\mu^*(t), \Pi_{\Omega(\mu)}[g(\mu(t))-\rho(\frac{2w\mu(t)}{w+\mu(t)})]-\Pi_{\Omega(\mu)}[g(\mu^*(t))-\rho(\frac{2w\mu^*(t)}{w+\mu^*(t)})]\rangle, \\
&\leq -2\lambda\sigma\|\mu(t)-\mu^*(t)\|^2+\lambda\|g(\mu(t))-g(\mu^*t)\|^2 \\
&\quad +\lambda\|\Pi_{\Omega(\mu)}[\mu(t)-\rho(\frac{2w\mu(t)}{w+\mu(t)})]-\Pi_{\Omega(\mu)}[g(\mu^*(t))-\rho(\frac{2w\mu^*(t)}{w+\mu^*(t)})]\|^2 \qquad (5.6)
\end{aligned}$$

Using the Lipschitz continuity of the operator g, we have

$$\begin{aligned}
\|\Pi_{\Omega(\mu)}[g(\mu)-\rho\mu]-\Pi_{\Omega(\mu)}[g(\mu^*)-\rho\mu^*]\| &\leq \delta\|g(\mu)-g(\mu^*)-\rho(\mu-\mu^*)\| \\
&\leq \delta(\zeta+\rho)\|\mu-\mu^*\|. \qquad (5.7)
\end{aligned}$$

From (5.6) and (5.7), we have

$$\frac{d}{dt}\|\mu(t)-\mu^*\| \leq 2\xi\lambda\|\mu(t)-\mu^*\|,$$

where

$$\xi = (\delta(1+\rho)-2\sigma).$$

Thus, for $\lambda = -\lambda_1$, where λ_1 is a positive constant, we have

$$\|\mu(t)-\mu^*\| \leq \|\mu(t_0)-\mu^*\|e^{-\xi\lambda_1(t-t_0)},$$

which shows that the trajectory of the solution of the dynamical system (5.3) converges globally exponentially to the unique solution of the inverse harmonic-like quasi variational inequality (2.2). □

We use the dynamical system (5.3) to suggest some iterative for solving the inverse harmonic-like quasi variational inequalities (2.2). These methods can be viewed in the sense of Korpelevich [22] and Noor [37, 38] involving the double projection.

For simplicity, we take $\lambda = 1$. Thus the dynamical system(5.3) becomes

$$\frac{d\mu}{dt}+g(\mu) = \Pi_{\Omega(\mu)}\big[g(\mu)-\rho(\frac{2w\mu}{w+\mu(t)})\big], \quad \mu(t_0)=\alpha. \qquad (5.8)$$

The forward difference scheme is used to construct the implicit iterative method. Discretizing (5.8), we have

$$\frac{\mu_{n+1}-\mu_n}{h}+g(\mu_n) = \Pi_{\Omega(\mu_n)}[g(\mu_n)-\rho(\frac{2w\mu_{n+1}}{w+\mu_{n+1}})], \qquad (5.9)$$

where h is the step size.
Now, we can suggest the following implicit iterative method for solving the inverse harmonic-like quasi variational inequality (2.2).

For a given μ_0, compute μ_{n+1} by the iterative scheme

$$\mu_{n+1} = \mu_n - g(\mu_n) + \Pi_{\Omega(\mu_{n+1})}\left[g(\mu_n)-\rho(\frac{2w\mu_{n+1}}{w+\mu_{n+1}})-\frac{\mu_{n+1}-\mu_n}{h}\right],$$

This is an implicit method, which is equivalent to the following two-step method. Discretizing (5.8), we have

For a given μ_0, compute μ_{n+1} by the iterative scheme

$$\begin{aligned} y_n &= \mu_n - g(\mu_n) + \Pi_{\Omega(\mu_n)}[g(\mu_n) - \rho(\frac{2w\mu_n}{w+\mu_n})] \\ \mu_{n+1} &= \mu_n - g(\mu_n) + \Pi_{\Omega(y_n)}\big[g(\mu_n) - \rho(\frac{y_n}{w+y_n}) - \frac{y_n-\mu_n}{h}\big], \end{aligned}$$

$$\frac{\mu_{n+1}-\mu_n}{h} = \mu_n - g(\mu_n) + \Pi_{\Omega(\mu_{n+1})}[g(\mu_{n+1}) - \rho\mu_{n+1}], \tag{5.10}$$

where h is the step size.
This helps us to suggest the following implicit iterative method for solving the problem (2.2). Discretizing (5.8),

For a given μ_0, compute μ_{n+1} by the iterative scheme

$$\begin{aligned} \omega_n &= \mu_n - g(\mu_n) + \Pi_{\Omega(\mu_n)}[g(\mu_n) - \rho(\frac{2w\mu_n}{w+\mu_n})] \\ \mu_{n+1} &= \mu_n - g(\mu_n) + \Pi_{\Omega(\omega_n)}\left[g(\omega_n) - \rho(\frac{2w\omega_n}{w+\omega_n})\right]. \end{aligned}$$

we propose another implicit iterative method.

$$\frac{\mu_{n+1}-\mu_n}{h} + g(\mu_n) = \Pi_{\Omega(\mu_{n+1})}[g(\mu_n) - \rho(\frac{2w\mu_{n+1}}{w+\mu_{n+1}})],$$

where h is the step size.

For $h = 1$, we can suggest an implicit iterative method for solving the problem (2.2).

For a given μ_0, compute μ_{n+1} by the iterative scheme

$$\mu_{n+1} = \mu_n - g(\mu_n) + \Pi_{\Omega(\mu_{n+1})}[g(\mu_n) - \rho(\frac{2w\mu_{n+1}}{w+\mu_{n+1}})].$$

From (5.8), we have

$$\frac{d\mu}{dt} + g(\mu) = \Pi_{\Omega((1-\alpha)\mu \quad +\alpha\mu)}\left[g((1-\alpha)\mu + \alpha\mu) - \rho(\frac{2w((1-\alpha)\mu+\alpha\mu))}{w+(1-\alpha)\mu+\mu})\right],$$

where $\alpha \in [0,1]$ is a constant.
Discretization (5.11) and taking $h = 1$, we have

$$\mu_{n+1} = \mu_n - g(\mu_n) + \Pi_{\Omega((1-\alpha)\mu_n+\alpha\mu_{n-1})}\left[g\big((1-\alpha)\mu_n + \alpha\mu_{n-1}\big) - \rho(\frac{2w(1-\alpha)\mu_n + \alpha\mu_{n-1}}{w+(1-\alpha)\mu_n+\mu_{n+1}})\right], \tag{5.11}$$

which is an inertial type iterative method for solving the inverse harmonic-like quasi variational inequality (2.2).
Using the predictor-corrector techniques, we have We now introduce the second order dynamical system as-

For a given μ_0, μ_1, compute μ_{n+1} by the iterative schemes

$$\begin{aligned} \omega_n &= (1-\alpha)\mu_n + \alpha\mu_{n-1}, \quad n = 1, 2, \dots \\ \mu_{n+1} &= \mu_n - g(\mu_n) + \Pi_{\omega_n}\big[g(\omega_n) - \rho\omega_n\big], \end{aligned}$$

which is known as the inertial two-step iterative method.

sociated with the inverse harmonic-like quasi variational inequality (2.2). To be more precise, we consider the problem of finding $\mu \in \mathtt{H}$ such that

$$\gamma\frac{d^2\mu}{dx^2} + \frac{d\mu}{dx} = \lambda\{\Pi_{\Omega(\mu)}[g(\mu) - \rho(\frac{2w\mu}{w+\mu})] - g(\mu)\}, \quad \mu(a) = \alpha, \quad \mu(b) = \beta, \tag{5.12}$$

where $\gamma > 0, \lambda > 0$ and $\rho > 0$ are constants. We would like to emphasize that the problem (5.12) is indeed a second order boundary vale problem. In a similar way, we can define the second order initial value problem associated with the dynamical system.

The equilibrium point of the dynamical system (5.12) is defined as follows.

Definition 5.4. An element $\mu \in \mathcal{H}$, is an equilibrium point of the dynamical system (5.12), if,

$$\gamma\frac{d^2\mu}{dx^2} + \frac{d\mu}{dx} = 0.$$

Thus it is clear that $\mu \in \mathcal{H}$ is a solution of the inverse quasi harmonic-like variational inequality (2.2), if and only if, $\mu \in \mathcal{H}$ is an equilibrium point.
From (5.12), we have

$$g(\mu) = \Pi_{\Omega(\mu)}[g(\mu) - \rho(\frac{2w\mu}{w+\mu})].$$

Thus, we can rewrite (5.12) as follows:

$$g(\mu) = \Pi_{\Omega(\mu)}\big[g(\mu) - \rho(\frac{2w\mu}{w+\mu}) + \gamma\frac{d^2\mu}{dx^2} + \frac{d\mu}{dx}\big]. \tag{5.13}$$

For $\lambda = 1$, the problem (5.12) is equivalent to finding $\mu \in \Omega$ such that

$$\gamma\ddot{\mu} + \dot{\mu} + g(\mu) = \Pi_{\Omega(\mu)}\big[g(\mu) - \rho(\frac{2w\mu}{w+\mu})\big], \quad \mu(a) = \alpha, \quad \mu(b) = \beta. \tag{5.14}$$

The problem (5.14) is called the second dynamical system, which is in fact a second order boundary value problem. This interlink among various fields of mathematical and engineering sciences is fruitful in developing implementable numerical methods for finding the approximate solutions of the inverse harmonic-like quasi variational inequalities. Consequently, one can explore the ideas and techniques of the differential equations to suggest and propose hybrid proximal point methods for solving the inverse harmonic-like quasi variational inequalities and related optimization problems.
We discretize the second-order dynamical systems (5.14) using central finite difference and backward difference schemes to have

$$\gamma\frac{\mu_{n+1} - 2\mu_n + \mu_{n-1}}{h^2} + \frac{\mu_n - \mu_{n-1}}{h} + g(\mu_n) = P_{\Omega(\mu_n)}\left[\mu_n - \rho(\frac{2w\mu_{n+1}}{w+\mu_{n+1}})\right], \tag{5.15}$$

where h is the step size.
If $\gamma = 1, h = 1$, then, from equation(5.15) we have which is the extragradient method for solving the inverse

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For a given μ_0, compute μ_{n+1} by the iterative scheme

$$\mu_{n+1} = \mu_n + g(\mu_n) + \Pi_{\Omega(\mu_n)}[g(\mu_n) - \rho(\frac{2w\mu_{n+1}}{w+\mu_{n+1}})],$$

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harmonic-like quasi variational inequalities (2.2). Algorithm 5 is an implicit method. To implement the implicit method, we use the predictor-corrector technique to suggest the method.

For given μ_0, μ_1, compute μ_{n+1} by the iterative scheme

$$\begin{aligned} y_n &= (1-\theta_n)\mu_n + \theta_n\mu_{n-1} \\ \mu_{n+1} &= \mu_n - g(\mu_n) + \Pi_{\Omega(\mu_n)}[g(\mu_n) - \rho(\frac{2wy_n}{w+y_n})], \end{aligned}$$

---

is called the two-step inertial iterative method, where $\theta_n \in [0,1]$ is a constant.
which is also called the double projection method for solving the inverse harmonic-like quasi variational in-

For given μ_0, μ_1, compute μ_{n+1} by the iterative scheme

$$\begin{aligned} y_n &= (1-\theta_n)\mu_n + \theta_n\mu_{n-1} \\ \mu_{n+1} &= \mu_n - g(\mu_n) + \Pi_{\Omega(y_n)}[g(y_n) - \rho(\frac{2wy_n}{w+y_n})], \end{aligned}$$

equalities (2.2).

We discretize the second-order dynamical systems (5.3) using central finite difference and backward difference schemes to have

$$\gamma\frac{\mu_{n+1}-2\mu_n+\mu_{n-1}}{h^2} + \frac{\mu_n-\mu_{n-1}}{h} + g(\mu_{n+1}) = \Pi_{\Omega(\mu_{n+1})}\big[g(\mu_n) - \rho(\frac{2w\mu_{n+1}}{w+\mu_{n+1}})\big],$$

where h is the step size.
Using this discrete form, we can suggest the following an iterative method for solving the inverse harmonic-like quasi variational inequalities (2.2). Algorithm 5 is called the hybrid inertial proximal method for solving the

For given μ_0, μ_1, compute μ_{n+1} by the iterative scheme

$$\mu_{n+1} = \mu_n - g(\mu_{n+1}) + \Pi_{\Omega(\mu_n)}\big[g(\mu_{n+1}) - \rho(\frac{2w\mu_{n+1}}{w+\mu_{n+1}}) - \gamma\frac{\mu_{n+1}-2\mu_n+\mu_{n-1}}{h^2} + \frac{\mu_n-\mu_{n-1}}{h}\big].$$

inverse harmonic-like quasi variational inequalities and related optimization problems. This is a new proposed method.
Note that, for $\gamma = 0$, Algorithm 5 reduces to the following iterative method. which is called the inertial double

For given μ_0, μ_1, compute μ_{n+1} by the iterative scheme

$$\mu_{n+1} = \mu_n - g(\mu_{n+1}) + \Pi_{\Omega(\mu_{n+1})}\big[g(\mu_{n+1}) - \rho(\frac{2w\mu_{n+1}}{w+\mu_{n+1}}) + \frac{\mu_n-\mu_{n-1}}{h}\big],$$

projection method.

We now consider the third order dynamical systems associated with the general inverse harmonic-like quasi variational inequalities of the type (2.2). To be more precise, we consider the problem of finding $\mu \in \mathcal{H}$, such that

$$\gamma\frac{d^3\mu}{dt^3} + \zeta\frac{d^2\mu}{dt^2} + \xi\frac{d\mu}{dt} + g(\mu) = \Pi_{\Omega(\mu)}[g(\mu) - \rho(\frac{2w\mu}{w+\mu})], \quad u(a) = \alpha, \dot{\mu}(a) = \beta, \dot{\mu}(b) = 0 \tag{5.16}$$

where $\gamma > 0, \zeta, \xi$ and $\rho > 0$ are constants. Problem (5.16) is called third order dynamical system associated with inverse harmonic-like quasi variational inequalities (2.2).
The equilibrium point of the dynamical system (5.16) is defined as follows.

Definition 5.5. An element $\mu \in \mathcal{H}$, is an equilibrium point of the dynamical system (5.12), if,

$$\gamma\frac{d^3\mu}{dt^3} + \zeta\frac{d^2\mu}{dt^2} + \xi\frac{d\mu}{dt} = 0.$$

Thus it is clear that $\mu \in \mathcal{H}$ is a solution of the general inverse harmonic-like quasi variational inequality (2.2), if and only if, $\mu \in \mathcal{H}$ is an equilibrium point.

Consequently, the problem (5.3) can be equivalent written as

$$g(\mu) = \Pi_{\Omega(\mu)}\left[g(\mu) - \rho(\frac{2w\mu}{w+\mu}) + \gamma\frac{d^3\mu}{dt^3} + \zeta\frac{d^2\mu}{dt^2} + \xi\frac{d\mu}{dt}\right]. \tag{5.17}$$

We discretize the third-order dynamical systems (5.16) using central finite difference and backward difference schemes to have

$$\gamma\frac{u_{n+2}-2u_{n+1}+2u_{n-1}-u_{n-2}}{2h^3}+\zeta\frac{u_{n+1}-2u_n+u_{n-1}}{h^2}$$
$$+\xi\frac{3\mu_n-4\mu_{n-1}+\mu_{n-2}}{2h}+g(\mu_n)=\Pi_{\Omega(\mu_n)}[g(\mu_n)-\rho(\frac{2w(\mu_{n+1}}{w+\mu_{n+1}}))], \tag{5.18}$$

where h is the step size.
If $\gamma=1,h=1,\zeta=1,\xi=1$, then, from equation(5.18) after adjustment, we have which is an inertial type

For a given μ_0,μ_1, compute u_{n+1} by the iterative scheme

$$u_{n+1} = \mu_n-g(\mu_n)+\Pi_{\Omega(\mu_n)}\big[g(\mu_n)-\rho(\frac{2w\mu_{n+1}}{w+\mu_{n+1}})+\frac{\mu_{n-1}-3\mu_n}{2}\big], \quad n=0,1,2,\ldots$$

hybrid iterative methods for solving the inverse harmonic-like quasi variational inequalities (2.2).

Remark 5.6. For appropriate and suitable choice of the operator g, convex-valued set, parameters and the spaces, one can suggest a wide class of implicit, explicit and inertial type methods for solving inverse harmonic-like quasi variational inequalities and related optimization problems. Using the techniques and ideas of Noor et al [45, 46], one can discuss the convergence analysis of the proposed methods.

6. Change of Variable Method

In this section, we consider the change of variable method for solving inverse harmonic-like harmonic-like quasi variational inequalities (2.2). This technique is mainly due to Noor [34] and Noor et al.[42, 55]. For the sake of completeness and to convey the main idea, we include some details.
We note that the complementarity problem (2.6) can be rewritten in the following form:

$$\zeta=g(\mu)\in\Omega(\mu),\quad \nu=\mathcal{T}(\frac{2w\mu}{w+\mu})\in\Omega^*(\mu),\quad \langle(\frac{2w\mu}{w+\mu}),g(\mu)\rangle=0. \tag{6.1}$$

which is useful in developing a fixed point formulation.
It is well known that, for $z\in\mathcal{H}$, we have

$$z=\Pi_\Omega(\mu)z+\Pi_{-\Omega^*(\mu)}(\mu)z=\Pi_{\Omega(\mu)}z+\Pi_{\Omega(\mu)^*}(-z). \tag{6.2}$$

Following the idea of Noor [34] Noor et al.[42, 55], we consider the following change of variables

$$g(\mu)=\frac{|z|+z}{2}=z^+=\Pi_{\Omega(\mu)}(z) \tag{6.3}$$

and

$$\nu = \frac{|z|-z}{2\rho}=\rho^{-1}z^-. \tag{6.4}$$

From (6.2), (6.3) and (6.4), we have

$$\mu = g^{-1}\Pi_{\Omega(\mu)}z \tag{6.5}$$
$$z = z^+-z^-=\Pi_{\Omega(\mu)}z+\Pi_{\omega(\mu)}(-z)$$
$$= g(\mu)-\rho\mathcal{T}g^{-1}\Pi_{\Omega(\mu)}z=g(\mu)-\rho(\frac{2w\mu}{w+\mu}). \tag{6.6}$$

Combining (6.5) and (6.6), we obtain

$$g(\mu) = \Pi_{\Omega(\mu)}[g(\mu)-\rho(\frac{2w\mu}{w+\mu})]. \tag{6.7}$$

Thus, we have shown that the complementarity problem (2.6) is equivalent to the fixed point problem (6.7). This implies that $\mu \in \Omega$ is the solution of the general harmonic-like quasi variational inequality (2.2). That is, $\mu \in \Omega(\mu)$ satisfies the inequality

$$\langle (\frac{2w\mu}{w+\mu}), g(\nu) - g(\mu) \rangle \geq 0, \quad \forall w, \nu \in \Omega(\mu).$$

This can be rewritten equivalently as finding $\mu \in \Omega$ such that

$$\langle (\frac{2w\mu}{w+\mu}) + g(\mu) - g(\mu), g(\nu) - g(\mu \rangle \geq 0, \quad \forall w, \nu \in \Omega(\mu),$$

which is the general harmonic-like quasi variational inequality (2.2).

In recent years, this technique have been used to develop modulus based methods for solving the system of absolute value equations, which is another area in numerical analysis and optimization. This approach can be extended for solving the mixed variational inequalities, which needs further research efforts.

7. Generalizations and Applications

We would like to mention that some of the results obtained and presented in this paper can be extended for more general inverse harmonic quasi variational inequalities. To be more precise, for guven operators $g, h : \mathcal{H} \longrightarrow \mathcal{H}$, we consider the problem of finding $\mu \in \Omega(\mu)$, such that

$$\langle \rho(\frac{2w\mu}{w+\mu}), g(\nu) - h(\mu) \rangle \geq 0, \quad \forall w, \nu \in \Omega(\mu), \tag{7.1}$$

where ρ is a constant, is called the extened general inverse harmonic-like quasi variational inequality.

For $h = g$, the problem (7.1) reduces to finding $\mu \in \Omega(\mu)$, such that

$$\langle \rho(\frac{2w\mu}{w+\mu}), g(\nu) - g(\mu) \rangle \geq 0, \quad \forall w, \nu \in \Omega(\mu), \tag{7.2}$$

which is the exactly inverse harmonic quasi variational inequality (2.2). We would like to mention that one can obtain various classes of inverse quasi variational inequalities for appropriate and suitable choices of the operators g, h, and convex-valued set $\Omega(\mu)$.

Using Lemma 3.1, one can prove that the problem (7.2) is equivalent to finding $\mu \in \Omega(\mu)$ such that

$$h(u) = \Pi_{\Omega(\mu)}[g(\mu) - \rho(\frac{2w\mu}{w+\mu})] \tag{7.3}$$

which can be written as

$$\mu = \mu - h(\mu) + \Pi_{\Omega(\mu)}[g(\mu) - \rho(\frac{2w\mu}{w+\mu})].$$

Thus one can consider the operator Φ associated with the problem (7.2) as

$$\Phi(\mu) = \mu - h(\mu) + \Pi_{\Omega(\mu)}[g(\mu) - \rho(\frac{2w\mu}{w+\mu})], \tag{7.4}$$

which can be used to discuss the uniqueness of the solution and to propose iterative methods for the problem (7.2).

From (7.1) and (7.4, it follows that the extended general inverse harmoni-like quasi variational inequalities are equivalent to the fixed problems. Consequently, all results obtained for the problem (2.2) continue to hold for the problem (7.1) with suitable modifications and adjustments. Applying the technique and idea of this paper, similar results can be established for solving system of inverse harmonic-like quasi variational inequalities considered with appropriate modifications. The development of efficient implementable numerical methods for solving the general inverse harmonic-like quasi variational inequalities and non optimization problems requires further efforts.

8. Conclusion

In this paper, some new classes of inverse harmonic-like quasi variational inequalities are introduced and studied. We have established the equivalence between the fixed point problem and the inverse harmonic-like quasi inequalities. This alternative formulation is used to suggest some new multi step multi-step iterative methods for solving the inverse harmonic-like quasi variational inequalities. These new methods include extragradient methods, modified double projection methods and inertial type methods, which are suggested using the techniques of projection method, auxiliary techniques and dynamical systems. Convergence analysis of the proposed method is discussed for monotone operators. It is an open problem to compare these proposed methods with other methods. Sensitivity analysis is also investigated for inverse quasi variational inequalities using the equivalent fixed point approach. Applying the technique and ideas of Ashish et. al. [2, 3, 4], Cho et al. [9], Kwuni et al. [23], Negi et al [28] and Roy et al. [59], one can explore the properties of Julia set and Mandelbrot set in fractal geometry, data science, chaos theory, coding, number theory, spectral geometry, dynamical systems, complex analysis, nonlinear programming, graphics and computer aided design. This is an open problem, which deserves further research efforts. Applications of the fuzzy set theory [36], stochastic [5, 6], data science [28], quantum calculus, fractal, logistic map [65], fractional and random traffic equilibrium [62, 63] can be found in many branches of mathematical and engineering sciences including artificial intelligence, financial mathematics, computer science, control engineering, management science, operations research, green energy [27] and variational inequalities. One may explore these aspects for the inverse harmonic-like quasi variational inequality and its variant forms.

Statements and Declaratins

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