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SOLVING THE SPLIT COMMON NULL POINT PROBLEM VIA GRAPH CONVERGENCE OF MONOTONE MAPPINGS

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Abstract. This paper aims to approximate the solution of the split common null point problem ($\mathrm{S_pCNPP}$) using the graph convergence of monotone mappings. We apply the equivalence between graph convergence of monotone mappings and convergence of the associated resolvent operators to prove that the sequence generated by the proposed algorithm converges to the solution of $\mathrm{S_pCNPP}$. Both weak and strong convergence of the sequence produced by the proposed method are discussed. We validate the main result with the help of numerical examples in finite and infinite dimensional Hilbert spaces.

Keywords. Split common null point problem, Monotone mapping, Graph convergence, Resolvent operator, Fixed point, Strong convergence.

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1. Introduction

Throughout this paper, we use the following notation and symbols:

$\mathrm{VIP_s}$	:	*Variational inequality problems*;
$\mathrm{S_pVIP}$	:	*Split variational inequality problem*;
$\mathrm{S_pCNPP}$	:	*Split common null point problem*;
$\mathrm{S_pMVIP}$	:	*Split monotone variational inclusion problem*;
$\mathbb{N}$	:	*Set of natural numbers*;
$\mathbb{H}, \mathbb{H}_1$ *and* $\mathbb{H}_2$	:	*Real Hilbert spaces*;
$\langle\cdot,\cdot\rangle$ *and* $\|\cdot\|$	:	*Inner product and norm induced by inner product in* $\mathbb{H}$;
$\langle\cdot,\cdot\rangle_1$ *and* $\|\cdot\|_1$	:	*Inner product and norm induced by inner product in* $\mathbb{H}_1$;
$\langle\cdot,\cdot\rangle_2$ *and* $\|\cdot\|_2$	:	*Inner product and norm induced by inner product in* $\mathbb{H}_2$;
$\longrightarrow$	:	*Strong convergence*;
$\rightharpoonup$	:	*Weak convergence.*

Variational inequality problems (VIPs), introduced by Guido Stampacchia, provide a unified framework for modeling optimization, equilibrium, and complementarity problems; see, for example, [9, 16, 22] and references therein. Owing to their broad applicability in economics, engineering, mechanics, transportation, image processing, and network analysis, VIPs have become a central topic in nonlinear analysis and optimization. Over the years, numerous iterative methods have been developed to solve VIPs under various assumptions on the underlying operators, leading to significant advances in both

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theory and computation. Consequently, variational inequality problems continue to attract considerable research attention, with many recent studies focusing on new algorithms, convergence analysis, and practical applications; see, for example, [13, 19, 20, 21, 27] and references therein.

In the magnificent work, Censor et al. [14] brought into existence a new type of variational inequality problem named split variational inequality problem ($\mathrm{S_pVIP}$):

$$\begin{aligned}&\text{Find } x^* \in \mathbb{C}_1 \text{ such that } \langle \varphi_1(x^*), x - x^* \rangle \geq 0, \text{ for all } x \in \mathbb{C}_1,\\ &\text{and such that } y^* = Ax^* \in \mathbb{C}_2 \text{ solves } \langle \varphi_2(y^*), y - y^* \rangle \geq 0, \text{ for all } y \in \mathbb{C}_2,\end{aligned}$$

where $\mathbb{C}_1$ and $\mathbb{C}_2$ are closed, convex subsets of Hilbert spaces $\mathbb{H}_1$ and $\mathbb{H}_2$, respectively; $A : \mathbb{H}_1 \to \mathbb{H}_2$ is a bounded linear operator; $\varphi_1 : \mathbb{H}_1 \to \mathbb{H}_1$ and $\varphi_2 : \mathbb{H}_2 \to \mathbb{H}_2$ are two mappings.

Moudafi [26] extended the split variational inequality problem to split monotone variational inclusion problem ($\mathrm{S_pMVIP}$):

$$\begin{aligned}&\text{Find } x^* \in \mathbb{H}_1 \text{ such that } 0 \in \varphi_1(x^*) + M(x^*),\\ &\text{and such that } y^* = Ax^* \in \mathbb{H}_2 \text{ solves } 0 \in \varphi_2(y^*) + N(y^*),\end{aligned}$$

where $M : \mathbb{H}_1 \to 2^{\mathbb{H}_1}$ and $N : \mathbb{H}_2 \to 2^{\mathbb{H}_2}$ are multivalued mappings on $\mathbb{H}_1$ and $\mathbb{H}_2$ respectively; $A : \mathbb{H}_1 \to \mathbb{H}_2$ is a bounded linear operator; $\varphi_1 : \mathbb{H}_1 \to \mathbb{H}_1$ and $\varphi_2 : \mathbb{H}_2 \to \mathbb{H}_2$ are two single-valued mappings.

Moudafi [26] proposed an iterative algorithm to find the solution of $\mathrm{S_pMVIP}$. Let $\mu > 0$, select an arbitrary starting point $x_0 \in \mathbb{H}_1$. Compute

$$x_{n+1} = U[x_n + \gamma A^*(V - I)Ax_n], \tag{1.1}$$

where $\gamma \in (0, 1/L)$ with L being spectral radius of operator A^*A, A^* is the adjoint operator of A, $U = R_\mu^M(I - \mu\varphi_1)$ and $V = R_\mu^N(I - \mu\varphi_2)$.

If $\varphi_1 = \varphi_2 = 0$, then we have the following split common null point problem ($\mathrm{S_pCNPP}$) for set-valued maximal monotone mappings, introduced and studied by Byrne et al. [12]:

$$\begin{aligned}&\text{Find } x^* \in \mathbb{H}_1 \text{ such that } 0 \in M(x^*),\\ &\text{and such that } y^* = Ax^* \in \mathbb{H}_2 \text{ solves } 0 \in N(y^*).\end{aligned}$$

They proposed the following iterative algorithm to find the solution of $\mathrm{S_pCNPP}$. Let $\mu > 0$, select starting point $x_0 \in \mathbb{H}_1$. Compute

$$x_{n+1} = R_\mu^M[x_n + \gamma A^*(I - R_\mu^N)Ax_n], \tag{1.2}$$

$L = \|A^*A\|$, and $\gamma \in (0, 2/L)$. Byrne implemented the Krasnosel'skii-Mann theorem to prove that the sequence generated by (1.2) converges weakly to the solution of $\mathrm{S_pCNPP}$.

Theorem 1.1. [12] *Let $\mathbb{H}_1$ and $\mathbb{H}_2$ be two real Hilbert spaces. Given two set-valued maximal monotone mappings $M : \mathbb{H}_1 \to 2^{\mathbb{H}_1}$ and $N : \mathbb{H}_2 \to 2^{\mathbb{H}_2}$ and a bounded linear operator $A : \mathbb{H}_1 \to \mathbb{H}_2$, any sequence $\{x_n\}_{n=0}^{\infty}$ generated by Algorithm (1.2) converges weakly to the solution of $\mathrm{S_pCNPP}$ provided the solution exists and $\gamma \in (0, 2/L)$, where $L = \|AA^*\|$.*

In the previous years, numerous iterative methods have been developed to solve split problems, for example, viscosity-type methods [4, 7], Halpern-type method [5, 1], self-adaptive methods [15, 17], extragradient method [18], dynamic techniques [2] and references therein.

The graph convergence of monotone mappings, given by Attouch [8], have also been used to find the solution of variational inclusion problems. Huang [23] introduced and applied the concept of graph convergence of $H(\cdot, \cdot)$-accretive operator to find the solution of variational inclusion problem. Ahmad et al. [3] applied graph convergence of $H(\cdot, \cdot)$-Co-accretive mapping to approximate the solution of system of generalized variational inclusions problems in Banach spaces.

Motivated by the works of Censor et al. [14], Moudafi [26], Byrne et al. [12], Li and Huang [23] and Ahmad et al. [3] in this paper, we are concerned with the study of S_pCNPP for strongly monotone mappings. We prove the strong convergence of the sequence generated by the proposed algorithm applying the graph convergence of maximal monotone and associated resolvent operator convergence in the framework of real Hilbert spaces.

This paper is organized as follows. In Section 2, preliminaries and important definitions are provided. Section 3 describes our iterative method and the main results of the paper including some useful lemmas. To support the theoretical findings, some numerical examples are provided in finite and infinite dimensional spaces in Section 4. Finally, a brief conclusion of the work is discussed in the last section.

2. Preliminaries

In this section some useful definitions and results will be discussed, which will be applied later to achieve our goal.

Definition 2.1. Let $M : \mathbb{H} \to \mathbb{H}$ be a single valued mapping, then

(i) M is said to be monotone if

$$\langle M(x) - M(y), x - y\rangle \geq 0, \ \forall\, x, y \in \mathbb{H}.$$

(ii) M is said to be strongly monotone, if there exists a constant $r > 0$ such that

$$\langle M(x) - M(y), x - y\rangle \geq r\|x - y\|^2, \ \forall\, x, y \in \mathbb{H}.$$

(ii) M is said to be τ-inverse strongly monotone, if there exists a constant $\tau > 0$ such that

$$\langle M(x) - M(y), x - y\rangle \geq \tau\|M(x) - M(y)\|^2, \ \forall\, x, y \in \mathbb{H}.$$

Definition 2.2. Let $M : \mathbb{H} \to 2^{\mathbb{H}}$ be a set-valued mapping, then

(i) the graph of M is defined as

$$Graph(M) = \{(x, y) \in \mathbb{H} \times \mathbb{H} : y \in M(x)\};$$

(ii) the range of M is defined as

$$Range(M) = \{y \in \mathbb{H} : \exists x \in \mathbb{H} : (x, y) \in Graph(M)\};$$

(iii) the inverse of M is defined as

$$M^{-1} = \{(y, x) : y \in M(x)\}.$$

Definition 2.3. A set-valued mapping M is said to be monotone if $\langle u - v, x - y\rangle \geq 0$, for all $u \in M(x), v \in M(y)$. A monotone mapping M is called a maximal monotone if for $x, y \in \mathbb{H}$ and $u \in M(x)$ such that

$$\langle u - v, M(x) - M(y)\rangle \geq 0, \forall (y, v) \in Graph(M)$$

implies that $(x, u) \in Graph(M)$. In other words, we can say M is said to be maximal monotone if there exists no other monotone mapping whose graph properly contains the graph of M.

Definition 2.4. Let $M : \mathbb{H} \to 2^{\mathbb{H}}$ be a set-valued monotone mapping on Hilbert space $\mathbb{H}$, then the resolvent of M is defined by

$$R_\mu^M(x) = [I + \mu M]^{-1}(x),$$

where I is the identity mapping on $\mathbb{H}$ and $\mu > 0$. A resolvent operator of maximal monotone operator is single valued and firmly nonexpansive.

Definition 2.5. A mapping $F : \mathbb{H} \to \mathbb{H}$ is said to be averaged if and only if it can be written as the average of the identity mapping and a nonexpansive mapping, namely $F = (1 - \alpha)I + \alpha S$, where $\alpha \in (0, 1)$ and $S : \mathbb{H} \to \mathbb{H}$ is nonexpansive.

Proposition 2.6. [11, 10]

(i) If $F = (1-\alpha)S+\alpha T$, where $S : \mathbb{H} \to \mathbb{H}$ is averaged, $T : \mathbb{H} \to \mathbb{H}$ is nonexpansive and $\alpha \in (0,1)$, then F is averaged.

(ii) If the mappings $\{\varphi_i\}_{i=1}^{N}$ are averaged and have a nonempty common fixed point, then

$$\bigcap_{i=1}^{N} \mathrm{Fix}(\varphi_i) = \mathrm{Fix}(\varphi_1\varphi_2.....\varphi_N).$$

(iii) If F is τ-inverse strongly monotone mapping, then for $\gamma > 0, \gamma F$ is $\frac{\tau}{\gamma}$-inverse strongly monotone.

(iv) F is averaged if and only if, its complement $I - F$ is τ-inverse strongly monotone for some $\tau > \frac{1}{2}$.

Definition 2.7. [24] Let $M_n, M : \mathbb{H} \to 2^{\mathbb{H}}$ be maximal monotone mappings for $n \in \mathbb{N}$. The sequence of monotone operator $\{M_n\}$ is said to be graph convergence to M, denoted by $M_n \xrightarrow{\mathcal{G}} M$, if for every $(u, v) \in graph(M)$ there exists a sequence $(u_n, v_n) \in graph(M_n)$ such that $u_n \to u,\ v_n \to v$ as $n \to \infty$.

Lemma 2.8. [8] *Let $M_n, M : \mathbb{H} \to 2^{\mathbb{H}}$ be maximal monotone mappings for $n \in \mathbb{N}$. Then $M_n \xrightarrow{\mathcal{G}} M$ if and only if $R_\mu^{M_n}(u) \to R_\mu^M(u)$ for every $u \in \mathbb{H}$ and $\mu > 0$.*

Remark 2.9. [28] If $N^{-1}(0) \neq \emptyset$, then $\langle u - R_\mu^N u, R_{\mu_2}^N u - p\rangle \geq 0$, for each $u \in \mathbb{H}_2$, each $p \in N^{-1}(0)$ and $\mu > 0$.

Lemma 2.10. [29] *Let $\{a_n\}, \{b_n\}$ be two nonnegative sequences of real numbers satisfying*

$$a_{n+1} \leq ka_n + b_n,$$

such that $0 < k < 1,\ b_n \to 0$ as $n \to \infty$. Then $\lim_{n\to\infty} a_n = 0$.

Theorem 2.11. *(Kasnosel'skii-Mann Theorem [25]) Let $F : \mathbb{H} \to \mathbb{H}$ be an averaged and $\mathrm{Fix(F)} \neq \emptyset$. Then for any arbitrary point $z_0 \in \mathbb{H}$, the sequence $\{F^n(z_0)\}$ converges weakly to a fixed point of F.*

3. Main Results

Suppose that ($\mathrm{S_pCNPP}$) is consistent and the solution set is denoted by

$$\Omega = \{x^* \in \mathbb{H}_1 : 0 \in M(x^*) \text{ and } 0 \in N(Ax^*)\}.$$

Remark 3.1. Let M be a maximal monotone mapping then for any $\mu_1 > 0$, it is easy to see that

$$0 \in M(u) \Leftrightarrow u \in \mathrm{Fix}(R_{\mu_1}^M) \Leftrightarrow u = R_{\mu_1}^M(u).$$

Based on Algorithm (1.2), introduced by Byrne [12], we suggest the following iterative algorithm.

Algorithm 3.2. *Let $\mu_1 > 0, \mu_2 > 0$*
Step 0 Choose an arbitrary $u_0 \in \mathbb{H}_1$ and let $n = 0$.
Step 1 Given $u_n \in \mathbb{H}_1$, compute $u_{n+1} \in \mathbb{H}_1$ as

$$u_{n+1} = R_{\mu_1}^{M_n}[u_n - \gamma A^*(I - R_{\mu_2}^{N_n})Au_n], \tag{3.1}$$

Step 2 update $n = n + 1$ and go to Step 1.
where $\gamma \in (0, \frac{1}{L})$ with L being the spectral radius of operator A^*A.

Now, we derive the following result for strongly maximal monotone mappings to obtain the Lipschitz estimate.

Lemma 3.3. *Let M be a strongly monotone mapping with constant r. Then R_μ^M associated with M is $\frac{1}{1+\mu r}$-Lipschitz continuous, that is*

$$\|R_\mu^M(u) - R_\mu^M(v)\| \leq \frac{1}{1+\mu r}\|u-v\|, \ \forall\, u, v \in \mathbb{H} \text{ and } \mu > 0.$$

Proof. Let $u, v \in \mathbb{H}$, then

$$R_\mu^M u = (I+\mu M)^{-1}u \quad \text{and} \quad R_\mu^M v = (I+\mu M)^{-1}v), \tag{3.2}$$

using, (3.2), we obtain

$$\frac{1}{\mu}(u - R_\mu^M u) \in M(R_\mu^M u) \quad \text{and} \quad \frac{1}{\mu}(v - R_\mu^M v) \in M(R_\mu^M v).$$

Since M is r-strongly monotone, we have

$$\begin{aligned} \langle \frac{1}{\mu}\left(u - R_\mu^M u\right) - \frac{1}{\mu}\left(v - R_\mu^M v\right), R_\mu^M u - R_\mu^M v\rangle &\geq r\|R_\mu^M u - R_\mu^M v\|^2 \\ \langle u-v, \left(R_\mu^M u - R_\mu^M v\right)\rangle - \langle R_\mu^M u - R_\mu^M v, R_\mu^M u - R_\mu^M v\rangle &\geq \mu r\|R_\mu^M u - R_\mu^M v\|^2 \end{aligned}$$

Since $\langle x, x\rangle = \|x\|^2$ and $|\langle x, y\rangle| \leq \|x\|\|y\|$, we conclude that

$$\|R_\mu^M u - R_\mu^M v\| \leq \frac{1}{1+\mu r}\|u-v\|,$$

that is, R_μ^M is $\frac{1}{1+\mu r}$-Lipschitz continuous. □

Lemma 3.4. *Let $M_n, M : \mathbb{H} \to 2^{\mathbb{H}}$ are r-strongly maximal monotone mappings for $n \in \mathbb{N}$. Then $M_n \xrightarrow{\mathcal{G}} M$ if and only if $R_\mu^{M_n}(u) \to R_\mu^M(u)$ for every $u \in \mathbb{H}$ and $\mu > 0$.*

Proof. Let $M_n \xrightarrow{\mathcal{G}} M$, we have to show that $R_\mu^{M_n}(u) \to R_\mu^M(u)$ for every $u \in \mathbb{H}$ and $\mu > 0$.
Let $x \in \mathbb{H}$ such that $z_n = R_\mu^{M_n}(x)$ and $z = R_\mu^M(x)$. Then, we obtain

$$(z_n, \frac{1}{\mu}(x - z_n)) \in Graph(M_n) \text{ and } (z, \frac{1}{\mu}(x-z)) \in Graph(M),$$

then there exist $(z_n^{'}, y_n^{'})Graph(M_n)$ such that $z_n^{'} \to z, y_n^{'} \to \frac{1}{\mu}(x-z)$.
Since $y_n^{'} \in M_n(z_n^{'})$, we have

$$\begin{aligned} \mu y_n^{'} &\in \mu M_n(z_n^{'}) \\ z_n^{'} + \mu y_n^{'} &\in [I + \mu M_n](z_n^{'}) \\ R_\mu^{M_n}(z_n^{'} + \mu y_n^{'}) &= z_n^{'}. \end{aligned} \tag{3.3}$$

Since $R_\mu^{M_n}$ is $\frac{1}{1+\mu r}$-Lipschitz continuous, we get

$$\begin{aligned} \|z_n - z\| &\leq \|z_n - z_n^{'}\| + \|z_n^{'} - z\| \\ &\leq \|R_\mu^{M_n}(x) - R_\mu^{M_n}(z_n^{'} + \mu y_n^{'})\| + \|z_n^{'} - z\| \\ &\leq \frac{1}{1+\mu r}\|x - (z_n^{'} + \mu y_n^{'})\| + \|z_n^{'} - z\| \\ &\leq \frac{1}{1+\mu r}\|x - (z_n^{'} + \mu y_n^{'}) - z + z\| + \|z_n^{'} - z\| \\ &\leq \frac{\mu}{1+\mu r}\|\frac{1}{\mu}(x-z) - y_n^{'})\| + \|z_n^{'} - z\| + \|z_n^{'} - z\| \to 0, \quad n \to \infty. \end{aligned}$$

that is $R_\mu^{M_n}(x) \to R_\mu^M(x)$ for every $x \in \mathbb{H}$ and $\mu > 0$. Conversely, let us assume that $R_\mu^{M_n}(x) \to R_\mu^M(u)$ for every $x \in \mathbb{H}$ and $\mu > 0$. Then we have to show that $M_n \xrightarrow{\mathcal{G}} M$. For any $(x, y) \in Graph(M)$, we have

$$x + \mu y \quad \in \quad [I + \mu M](x)$$

and so $x = R_\mu^M(x + \mu y)$ and $x_n = R_\mu^{M_n}(x + \mu y)$. Then we have $\frac{1}{\mu}(x - x_n + \mu y) \in M_n(x_n)$ and let $y_n = \frac{1}{\mu}(x - x_n + \mu y)$. It follows from

$$\begin{aligned} \|y_n - y\| &= \|\frac{1}{\mu}(x - x_n + \mu y) - y\| \\ &\leq \frac{1}{\mu}\|x - x_n\| \to 0, \quad n \to \infty, \end{aligned}$$

that is, $M_n \xrightarrow{\mathcal{G}} M$. This completes the proof. □

Lemma 3.5. *$u \in \mathbb{H}_1$ solves* $\mathrm{S_pCNPP}$ *if and only if*

$$u = R_{\mu_1}^M[u - \gamma A^*(I - R_{\mu_2}^N)Au]. \tag{3.4}$$

Proof. Let u solves $\mathrm{S_pCNPP}$ then $R_{\mu_1}^M u = u$ and $R_{\mu_2}^N Au = Au$, this implies that

$$\begin{aligned} R_{\mu_1}^M[u - \gamma A^*(I - R_{\mu_2}^N)Au] &= R_{\mu_1}^M[u - \gamma A^*(Au - R_{\mu_2}^N Au)] \\ &= R_{\mu_1}^M[u - \gamma A^*(Au - Au)] \\ &= R_{\mu_1}^M[u] \\ &= u. \end{aligned} \tag{3.5}$$

Conversely, let $u = R_{\mu_1}^M[u - \gamma A^*(I - R_{\mu_2}^N)Au]$, this implies that $u = (I + \mu_1 M)^{-1}[u - \gamma A^*(I - R_{\mu_2}^N)Au]$ or $\frac{1}{\mu_1}[(u - \gamma A^*(I - R_{\mu_2}^N)Au) - u] \in M(u)$.
Let $p \in \Omega \neq \emptyset$, that is, $0 \in M(p), 0 \in N(Ap)$, using the monotonicity of M, we have

$$\begin{aligned} \langle \frac{1}{\mu_1}[(u - \gamma A^*(I - R_{\mu_2}^N)Au) - u],\ u - p\rangle_1 &\geq 0, \\ \frac{\gamma}{\mu_1}\langle -A^*(I - R_{\mu_2}^N)Au,\ u - p\rangle_1 &\geq 0, \\ \langle A^*(I - R_{\mu_2}^N)Au,\ u - p\rangle_1 &\leq 0, \\ \langle (I - R_{\mu_2}^N)Au,\ Au - Ap\rangle_2 &\leq 0, \\ \langle Au - R_{\mu_2}^N Au,\ Au - Ap\rangle_2 &\leq 0. \end{aligned} \tag{3.6}$$

On the other hand for $p \in \Omega$ and $u \in \mathbb{H}_1$, using Remark 2.9, we have

$$\langle Au - R_{\mu_2}^N Au,\ R_{\mu_2}^N Au - Ap\rangle_2 \geq 0. \tag{3.7}$$

Adding (3.6) and (3.7), we obtain

$$0 \geq \langle Au - R_{\mu_2}^N Au,\ Au - R_{\mu_2}^N Au\rangle_2 = \|Au - R_{\mu_2}^N Au\|_2^2 \geq 0, \tag{3.8}$$

which implies that $\|Au - R_{\mu_2}^N Au\| = 0$, that is $R_{\mu_2}^N Au = Au$, that is Au solves $0 \in N(Au)$. Put $R_{\mu_2}^N Au = Au$ in (3.5), we have $R_{\mu_1}^M u = u$, u solves $0 \in M(u)$. This completes the proof. □

Theorem 3.6. *Let $\mathbb{H}_1$, $\mathbb{H}_2$ be Hilbert spaces; $M, M_n : \mathbb{H}_1 \to 2^{\mathbb{H}_1}$ be set-valued r_1-strongly maximal monotone mappings and $N, N_n : \mathbb{H}_2 \to 2^{\mathbb{H}_2}$ are set-valued r_2-strongly maximal monotone mappings such that $M_n \xrightarrow{\mathcal{G}} M$ and $N_n \xrightarrow{\mathcal{G}} N$. Let $A : \mathbb{H}_1 \to \mathbb{H}_2$ be a bounded linear operator. Then the sequence $\{x_n\}$ generated by Algorithm 3.2 converges strongly to a solution of* $\mathrm{S_pCNPP}$.

Proof. Define $F : \mathbb{H}_1 \to \mathbb{H}_1$ by

$$F(u) = R_{\mu_1}^{M_n}[u - \gamma A^*(I - R_{\mu_2}^{N_n})Au]. \tag{3.9}$$

We know that $R_{\mu_1}^{M_n}$ is averaged being firmly nonexpansive. Therefore by Proposition 2.6, $I - R_{\mu_1}^{M_n}$ is inverse strongly monotone for some $\alpha > \frac{1}{2}$. Hence $\gamma A^*(I - R_{\mu_2}^{N})$ is $\frac{\alpha}{\gamma L}$-inverse strongly monotone with $0 < \gamma < \frac{1}{L}$ and its complement $I - \gamma A^*(I - R_{\mu_2}^{N})$ is averaged and hence nonexpansive, for more details see [26]. Thus, from Lemma 3.3, we have for all $u, v \in \mathbb{H}_1$, we have

$$\begin{aligned}
\|F(u) - F(v)\|_1 &= \|R_{\mu_1}^{M_n}[u - \gamma A^*(I - R_{\mu_2}^{N_n})Au] - R_{\mu_1}^{M_n}[v - \gamma A^*(I - R_{\mu_2}^{N_n})Av]\|_1 \\
&\leq \frac{1}{1 + r_1\mu_1}\|[I - \gamma A^*(I - R_{\mu_2}^{N_n})A]u - [I - \gamma A^*(I - R_{\mu_2}^{N_n})A]v\|_1 \\
&\leq \frac{1}{1 + r_1\mu_1}\|u - v\|_1.
\end{aligned} \tag{3.10}$$

Since $r_1 > 0, \mu_1 > 0$ then $0 < \frac{1}{1+r_1\mu_1} < 1$, therefore $F = R_{\mu_1}^{M_n}[I - \gamma A^*(I - R_{\mu_2}^{N_n})A]$ is a contraction and has unique fixed point in $\mathbb{H}_1$. Let u be a fixed point of F . Now, we prove that sequence $\{u_n\}$ generated by Algorithm 3.2 converges strongly to u. From Lemma 3.5, we have

$$\begin{aligned}
\|u_{n+1} - u\|_1 &= \|R_{\mu_1}^{M_n}[u_n - \gamma A^*(I - R_{\mu_2}^{N_n})Au_n - R_{\mu_1}^{M}[u - \gamma A^*(I - R_{\mu_2}^{N})Au]\|_1 \\
&= \|R_{\mu_1}^{M_n}[u_n - \gamma A^*(I - R_{\mu_2}^{N_n})Au_n] - R_{\mu_1}^{M_n}[u - \gamma A^*(I - R_{\mu_2}^{N})Au] \\
&\quad + R_{\mu_1}^{M_n}[u - \gamma A^*(I - R_{\mu_2}^{N})Au] - R_{\mu_1}^{M}[u - \gamma A^*(I - R_{\mu_2}^{N})Au]\|_1 \\
&\leq \|R_{\mu_1}^{M_n}[u_n - \gamma A^*(I - R_{\mu_2}^{N_n})Au_n] - R_{\mu_1}^{M_n}[u - \gamma A^*(I - R_{\mu_1}^{N})u]\|_1 \\
&\quad + \|R_{\mu_1}^{M_n}[u - \gamma A^*(I - R_{\mu_2}^{N})Au] - R_{\mu_1}^{M}[u - \gamma A^*(I - R_{\mu_2}^{N})Au]\|_1.
\end{aligned}$$

Since $R_{\mu_1}^{M_n}$ is $\frac{1}{1+r_1\mu_1}$-Lipschitz continuous, we have

$$\begin{aligned}
\|u_{n+1} - u\|_1 &\leq \frac{1}{1 + r_1\mu_1}\|[I - \gamma A^*(I - R_{\mu_2}^{N_n})A]u_n - [I - \gamma A^*(I - R_{\mu_2}^{N})A]u\|_1 \\
&\quad + \|R_{\mu_1}^{M_n}[u - \gamma A^*(I - R_{\mu_2}^{N})Au] - R_{\mu_1}^{M}[u - \gamma A^*(I - R_{\mu_2}^{N})Au]\|_1 \\
&= \frac{1}{1 + r_1\mu_1}\Big(\|[I - \gamma A^*(I - R_{\mu_2}^{N_n})]Au_n - [I - \gamma A^*(I - R_{\mu_2}^{N_n})]_1 Au \\
&\quad + [I - \gamma A^*(I - R_{\mu_2}^{N_n})]Au - [I - \gamma A^*(I - R_{\mu_2}^{N})Au]\|\Big) \\
&\quad + \|R_{\mu_1}^{M_n}[u - \gamma A^*(I - R_{\mu_2}^{N})u] - R_{\mu_1}^{M}[I - \gamma A^*(I - R_{\mu_2}^{N})]Au\|_1 \\
&\leq \frac{1}{1 + \ r_1\mu_1}\Big(\|[I - \gamma A^*(I - R_{\mu_2}^{N_n})A]u_n - [I - \gamma A^*(I - R_{\mu_2}^{N_n})A]u\|_1 \\
&\quad + \|[I - \gamma A^*(I - R_{\mu_2}^{N_n})]Au - [I - \gamma A^*(I - R_{\mu_2}^{N})]Au\|\Big) \\
&\quad + \|R_{\mu_1}^{M_n}[u - \gamma A^*(I - R_{\mu_2}^{N})u] - R_{\mu_1}^{M}[u - \gamma A^*(I - R_{\mu_2}^{N})Au]\|_1.
\end{aligned}$$

Lemma 2.8 states that $M_n \xrightarrow{\mathcal{G}} M$ if and only if $R_{\mu_1}^{M_n}(u) \to R_{\mu_1}^{M}(u)$ for every $u \in \mathbb{H}_1$ and $\mu_1 > 0$. Since $[I - \gamma A^*(I - R_{\mu_2}^{N})A] : \mathbb{H}_1 \to \mathbb{H}_1$ implies that $[I - \gamma A^*(I - R_{\mu_2}^{N})Au] \in \mathbb{H}_1$, $\forall\ u \in \mathbb{H}_1$ and $\mu_2 > 0$ then using Lemma 2.8, we conclude that

$$\|R_{\mu_1}^{M_n}[I - \gamma A^*(I - R_{\mu_2}^{N})Au] - R_{\mu_1}^{M}[I - \gamma A^*(I - R_{\mu_2}^{N})Au]\|_1 \to 0,\ n \to \infty. \tag{3.11}$$

If $R_{\mu_2}^{N_n}(u) \to R_{\mu_2}^{N}(u)$ for every $u \in \mathbb{H}_2$. Since $Au \in \mathbb{H}_2$, for all $u \in \mathbb{H}_1$, we have that

$$\|R_{\mu_2}^{N_n}(Au) - R_{\mu_2}^{N}(Au)\|_2 \to 0,\ \text{as } n \to\ \infty,$$

or

$$\|\big(I - R_{\mu_2}^{N_n}\big)(Au) - \big(I - R_{\mu_2}^{N}\big)(Au)\|_2 \to 0 \text{ as } n \to \infty.$$

Since A is a bounded linear operator then so is A^*. Thus, we have

$$\|[I - \gamma A^*(I - R_{\mu_2}^{N_n})Au] - [I - \gamma A^*(I - R_{\mu_2}^{N})Au]\|_2 \to 0 \text{ as } n \to \infty.$$

Since $[I - \gamma A^*(I - R_{N_n}^{\mu})]$ is nonexpansive being averaged. Thus we have

$$\|u_{n+1} - u\|_1 \quad \leq \quad \frac{1}{1 + r_1\mu_1}\|u_n - u\|_1 + b_n + c_n \tag{3.12}$$

where $b_n = \|[I - \gamma A^*(I - R_{\mu_2}^{N_n})]Au - [I - \gamma A^*(I - R_{\mu_2}^{N})]Au\|$ and $c_n = \|R_{\mu_1}^{M_n}[u - \gamma A^*(I - R_{\mu_2}^{N})u] - R_{\mu_1}^{M}[u - \gamma A^*(I - R_{\mu_2}^{N})u]\|$. Let $d_n = b_n + c_n$, since $b_n \to 0, c_n \to 0$, which implies $d_n \to 0$ as $n \to \infty$, we have

$$\|u_{n+1} - u\|_1 \leq \frac{1}{1 + r_1\mu_1}\|u_n - u\|_1 + d_n,$$

using Lemma 2.10, we conclude that $\|u_{n+1} - u\| \to 0$. This completes the proof. □

Theorem 3.7. *Let $\mathbb{H}_1$, $\mathbb{H}_2$ be Hilbert spaces; $M, M_n : \mathbb{H}_1 \to 2^{\mathbb{H}_1}$ be set-valued r_1-strongly maximal monotone mappings and $N, N_n : \mathbb{H}_2 \to 2^{\mathbb{H}_2}$ are set-valued r_2-strongly maximal monotone mappings such that $M_n \xrightarrow{\mathcal{G}} M$ and $N_n \xrightarrow{\mathcal{G}} N$. Let $A : \mathbb{H}_1 \to \mathbb{H}_2$ be a bounded linear operator. Then the sequence $\{x_n\}$ generated by Algorithm 3.2 weakly converges to a solution of* $\mathrm{S_pCNPP}$.

Proof. It is known to us that the operator $I - \gamma A^*(I - R_{\mu_2}^{N_n})$ is averaged. Since the resolvent of monotone mappings are averaged, then by Lemma 2.6, the composition $R_{\mu_1}^{M_n}[I - \gamma A^*(I - R_{\mu_2}^{N_n})A]$ is averaged. Hence by Theorem 2.11, the sequence obtained by Algorithm 3.2 weakly converges to a fixed point of $R_{\mu_1}^{M_n}[I - \gamma A^*(I - R_{\mu_2}^{N_n})A]$. From Lemma 3.5, we conclude that fixed point of $R_{\mu_1}^{M_n}[I - \gamma A^*(I - R_{\mu_2}^{N_n})A]$ is the solution of $\mathrm{S_pCNPP}$.

□

4. Numerical Example

Example 4.1. (Finite dimension) Let $\mathbb{H}_1 = \mathbb{H}_2 = \mathbb{R}^2$ endowed with the inner product $\langle u, v\rangle = u_1v_1 + u_2v_2 \; \forall \; u = (u_1, u_2), v = (v_1, v_2) \in \mathbb{R}^2$ and corresponding norm $\|u\| = \sqrt{u_1^2 + u_2^2}, \; \forall \; (u_1, u_2) \in \mathbb{R}^2$. We define the monotone operators M and M_n as follows:

$$M(u_1,\ u_2) = \big(u_1,\ \frac{u_1}{2}\big), \quad M_n(u_1,\ u_2) = \big(u_1,\ \frac{u_2}{2} + \frac{1}{n^2}\big).$$

It can be easily seen that the mappings M and M_n are $\frac{1}{2}$-strongly monotone. The resolvents of operators M and M_n with parameter μ_1 are given below:

$$R_{\mu_1}^{M}(u_1,\ u_2) = \left(\frac{u_1}{1 + \mu_1},\ \frac{2u_2}{2 + \mu_1}\right)$$

and

$$R_{\mu_1}^{M_n}(u_1,\ u_2) = \left(\frac{u_1}{1 + \mu_1},\ \frac{u_2}{2 + \mu_1} - \frac{2\mu_1}{(2 + \mu_1)n^2}\right).$$

The mappings N and N_n are defined as :

$$N(u_1,\ u_2) = \big(u_1,\ 2u_2\big) \text{ and } N_n(u_1,\ u_2) = \big(u_1 + \frac{1}{n},\ 2u_2\big),$$

which are 2-strongly monotone. The resolvents of operators N and N_n with parameter μ_2 are given below:

$$R^N_{\mu_1}(u_1,\ u_2) = \left(\frac{u_1}{1+\mu_2},\ \frac{u_2}{1+2\mu_2}\right)$$

and

$$R^{N_n}_{\mu_1}(u_1,\ u_2) = \left(\frac{u_1}{1+\mu_2} - \frac{\mu_2}{(1+\mu_2)n},\ \frac{u_2}{1+2\mu_2}\right).$$

Let the bounded linear operator A is defined by $A(u_1, u_2) = (\frac{u_1}{2},\ \frac{u_2}{2})$. Then for $A^*A = \frac{1}{4}I, L = \frac{1}{4}, \gamma = \frac{1}{4} \in (0, 4)$, $\mu_1 = \frac{1}{2}$, $\mu_2 = \frac{1}{3}$. We plot the $\|u_{n_1} - u\|$ for three different initial values:
Case $(A_1) : u_0 = (11, 21)^T$,
Case $(A_2) : u_0 = (1, -10)^T$,
Case $(A_3) : u_0 = (-0.5, 2.5)^T$.
It is noted that the sequence $\{u_n\}$ converges to the solution $\{0\}$ without any hurdle as shown in Figure 1.

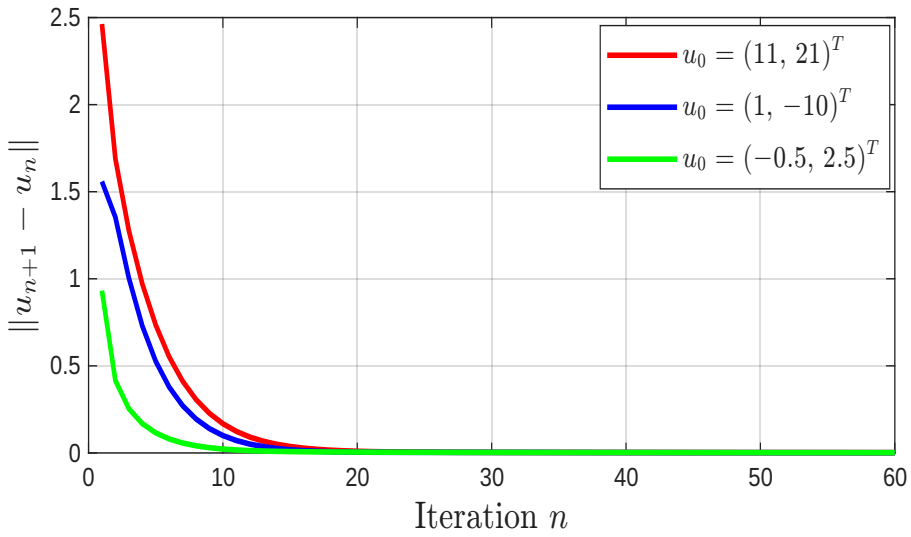


FIGURE 1. Convergence of $\{u_n\}$ for three initial values in $\mathbb{R}^2$

Example 4.2. (Infinite dimension) Let $\mathbb{H}_1 = \mathbb{H}_2 = L^2([0, 1])$ endowed with the inner product

$$\langle f, g\rangle = \int_0^1 f(t)g(t)\,dt, \quad \forall f, g \in L^2([0, 1]),$$

and the corresponding norm

$$\|f\| = \left(\int_0^1 |f(t)|^2 dt\right)^{1/2}.$$

We define the operators M and M_n by

$$M(f)(t) = a(t)f(t), \quad M_n(f)(t) = a(t)f(t) + \frac{1}{n^2},$$

where

$$a(t) = \begin{cases} 1, & t \in [0, \frac{1}{2}], \\ \frac{1}{2}, & t \in (\frac{1}{2}, 1]. \end{cases}$$

The resolvents of M and M_n with parameter $\mu_1 > 0$ are given by

$$R^M_{\mu_1}(f)(t) = \frac{f(t)}{1+\mu_1 a(t)}, \quad R^{M_n}_{\mu_1}(f)(t) = \frac{f(t)-\mu_1/n^2}{1+\mu_1 a(t)}.$$

The operators N and N_n are defined by $N(f)(t) = b(t)f(t), \quad N_n(f)(t) = b(t)f(t) + \frac{1}{n}$, where

$$b(t) = \begin{cases} 1, & t \in [0, \frac{1}{2}], \\ 2, & t \in (\frac{1}{2}, 1]. \end{cases}$$

The resolvents of N and N_n with parameter $\mu_2 > 0$ are

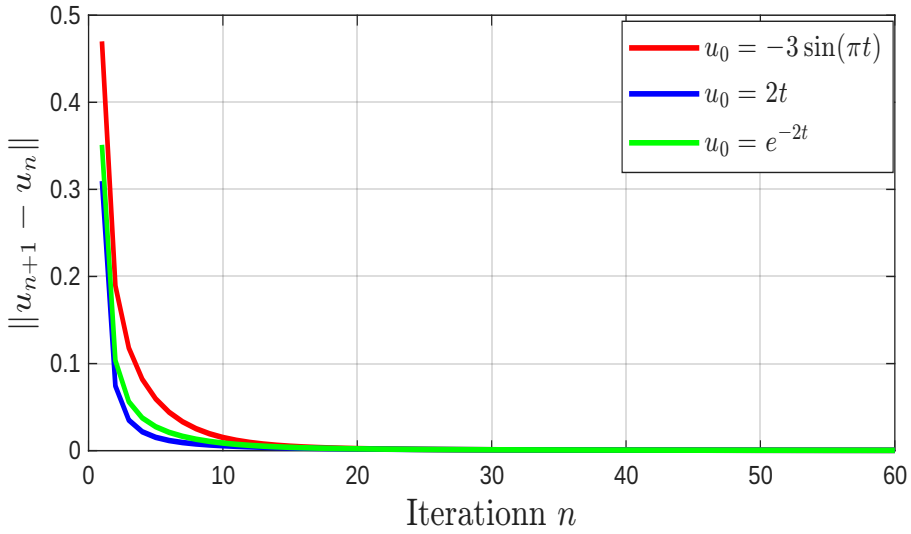


FIGURE 2. Convergence of $\{u_n\}$ for three initial values in $L^2[0,1]$

$$R^N_{\mu_2}(f)(t) = \frac{f(t)}{1+\mu_2 b(t)}, \quad R^{N_n}_{\mu_2}(f)(t) = \frac{f(t)-\mu_2/n}{1+\mu_2 b(t)}.$$

Let $A : L^2([0,1]) \to L^2([0,1])$ be defined by

$$A(f)(t) = \frac{1}{2}f(t).$$

Then $A^* = A$ and $A^*A = \frac{1}{4}I$, so the spectral radius is $L = \frac{1}{4}$. Hence, we choose $\gamma = \frac{1}{4} \in (0,4)$, $\mu_1 = 0.5$ and $\mu_2 = 0.25$. The convergence of $\{u_n\}$ is illustrated in Figure 2 using the following three initial values.
Case $(B_1) : u_0 = -3\sin(\pi t)$,
Case $(B_2) : u_0 = 2t$,
Case $(B_3) : u_0 = \exp(-2t)$.

Example 4.3. (Finite dimension) Let $H_1 = H_2 = \mathbb{R}^5$ equipped with the inner product $\langle u, v\rangle = u^T v$ and the norm is $\|u\|^2 = \sum_{n=1}^{5} |u_n|^2$ for all $u, v \in \mathbb{R}^5$. Consider the bounded linear operator

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix},$$

we can easily calculate that the spectral radius is $L = 9$. We choose $\gamma = \frac{1}{2}$, $\mu_1 = \frac{1}{3}$ and $\mu_2 = \frac{1}{4}$. Define the monotone operators $M, N : \mathbb{R}^5 \to \mathbb{R}^5$ by

$$M(u) = Du, \qquad N(v) = Ev,$$

where

$$D = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 6 \end{pmatrix}, \qquad E = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}.$$

For each $n \geq 0$, define

$$M_n(u) = \left(D + \frac{1}{n+2} I\right) u, \qquad N_n(v) = \left(E + \frac{1}{2(n+2)} I\right) v.$$

We have $M_n \xrightarrow{\mathcal{G}} M$, and $N_n \xrightarrow{\mathcal{G}} N$. The corresponding resolvent operators are

$$R_{\mu_1}^{M_n} = \left(I + \mu_1 \left(D + \frac{1}{n+2} I\right)\right)^{-1}, \quad \text{and} \quad R_{\mu_2}^{N_n} = \left(I + \mu_2 \left(E + \frac{1}{2(n+2)} I\right)\right)^{-1}.$$

Since both D and E are symmetric positive definite matrices, the operators M and N are strongly monotone; Hence, the split common null point problem admits the unique solution $u^* = (0, 0, 0, 0, 0)^T$. Algorithm 3.2 is implemented with the parameters specified above and the following three initial values:

Case (C_1) : $u_0 = (5, 4, 3, 2, 1)^T$,
Case (C_2) : $u_0 = (-3, 2, -1, 4, -2)^T$,
Case (C_3) : $u_0 = (10, -8, 6, -4, 2)^T$.

The convergence behavior of Algorithm 3.2 is illustrated by plotting the sequence $\{\|u_{n+1} - u_n\|\}$ against the iteration number for the above three initial values in Figure 3. In all cases, the generated sequence converges to the unique solution $u^* = (0, 0, 0, 0, 0,)^T$, demonstrating the convergence and stability of the proposed algorithm.

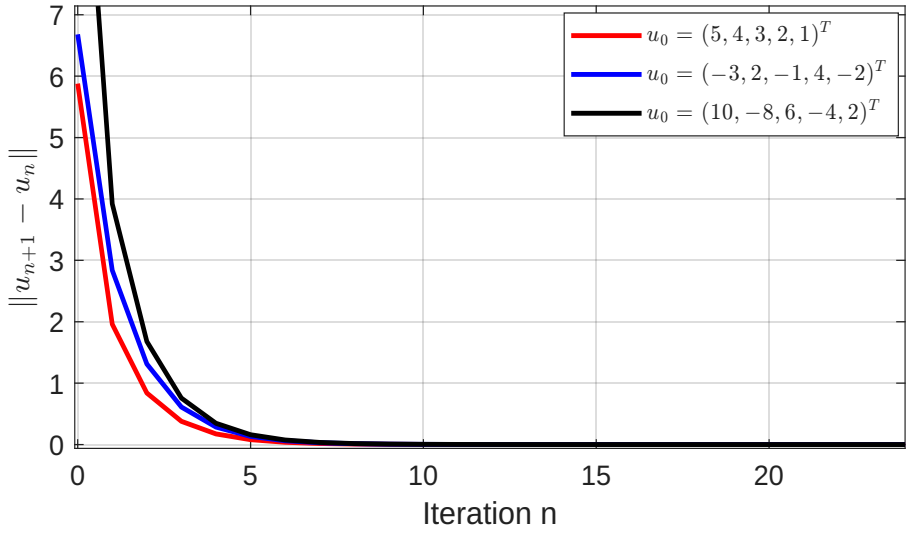


FIGURE 3. Convergence of $\{u_n\}$ for three initial values in $\mathbb{R}^5$

5. Conclusion

In this paper, we deal with the split common null point problem for set-valued strongly maximal monotone mappings. We proposed an iterative algorithm and proved the convergence of the algorithm using the concept of graph convergence of the associated resolvent operators. We proved that the sequence generated by the proposed algorithm converges strongly to the solution of $\mathrm{S_pCNPP}$. Theoretical ideas have been elaborated with the help of numerical examples in finite and infinite dimensional Hilbert spaces demonstrating the effectiveness of the proposed algorithm.

We note that the idea of graph convergence for approximating solutions of the $\mathrm{S_pCNPP}$ deserves further investigation in the context of generalized monotone mappings to solve split $\mathrm{S_pCNPP}$ and $\mathrm{S_pMVIP}$ in the setting of Hilbert and Banach spaces.

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