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ANALYSIS OF RELAXED INERTIAL METHOD FOR NON-CONVEX MIXED VARIATIONAL INEQUALITIES

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Abstract. This paper extends inertial forward-backward-forward splitting method already studied by several authors for solving convex variational inequalities to solving non-convex mixed variational inequalities and obtain appropriate convergence results under some conditions. Next, we propose another inertial forward-backward-forward splitting method for which the inertial factor θ is chosen in $[0, 1]$ with $\theta = 1$ possible. As far as we know, the choice $\theta = 1$ has not been considered before in the literature for inertial forward-backward-forward splitting method for solving non-convex mixed variational inequalities. Numerical illustrations are given to confirm the theoretical analysis.

Keywords. Non-convex variational inequalities, inertial extrapolation, forward-reflected-backward splitting, proximal point algorithms.

1. Introduction

Suppose $\mathcal{C}$ is a non-empty subset of $\mathbb{R}^n$, $\mathcal{F} : \mathbb{R}^n \to \mathbb{R}^n$, a given operator and $h : \mathcal{C} \to \mathbb{R}$ a real-valued function. A mixed varitional inequality problem or variational inequality of the second kind (MVI, for short) is defined by

$$\text{find } \bar{x} \in \mathcal{C} : \ \langle \mathcal{F}(\bar{x}), y - \bar{x} \rangle + h(y) - h(\bar{x}) \geq 0, \ \ \forall\, y \in \mathcal{C}. \tag{1.1}$$

Let us denote the set of solutions of MVI (1.1) by Ω. Problems arising from continuous optimization and variational analysis like minimization problems, linear complementary problems, vector optimization problems or variational inequalities which are applied in economics, engineering, physics, mechanics and electronics (see [7, 8, 9, 10, 19, 27, 28, 29, 31] among others) are special cases of MVI (1.1). Furthermore, MVI (1.1) can be seen as a reformulation of a Walrasian equilibrium model or of an oligopolistic equilibrium model (see, for example, [19, Section 2]), where $\mathcal{F}$ in MVI (1.1) stands for the demand and h stands for supply (see also [26]). Furthermore, in [9, 10], MVI (1.1) is applied to electrical circuits. Also, in [31] a dual variational formulation for strain of an elastoplasticity model with hardening is reformulated as MVI (1.1) and in [23] the frictional contact of an elastic cylinder with a rigid obstacle in the antiplane framework is recast as MVI (1.1).

We observe that if $h \equiv 0$ in MVI (1.1), then MVI (1.1) becomes the variational inequality problem (see [7, 8, 18]). Also, if $\mathcal{F} \equiv 0$ in MVI (1.1), then MVI (1.1) reduces to the constrained optimization problem of minimizing h over $\mathcal{C}$. Note that MVI (1.1) is a convex mixed variational inequality problem when h is convex, for which one can find various (proximal point type) algorithms in the literature to solve it, see, for instance, [4, 21, 32, 34, 37, 38, 39]. However, when h in MVI (1.1) is non-convex, MVI (1.1) becomes a non-convex mixed variational inequality problem and it is harder to solve. This is because

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the solution set may be empty even when K is a compact and convex set (see [25, Example 3.1], [16, page 127] and [12] for more details).

Recently, existence results for MVI (1.1) involving quasi-convex functions are obtained in [15, 16] and this raises the quest for numerical iterative methods to solve non-convex MVI (1.1). Iterative methods for solving convex MVI (1.1) have been obtained and studied in [27, 28, 29], where the involved h is continuous and the methods proposed are either implicit methods or only conceptual methods with no numerical implementations. The proposed methods in [27, 28, 29] also involve inner loops or two forward steps per iteration. Our approach in this paper is to propose and study a numerical method to solve non-convex MVI (1.1) for which h is a non-continuous non-convex function which further extends the results on minimization of quasi-convex functions obtained in [20, 30].

Quite recently, Grad and Lara [12] applied the Malitsky's Golden Ratio Algorithm to solve non-convex MVI (1.1). They proposed the following method:

Algorithm 1.1. *Golden Ratio Algorithm (GRA)*

(1) *Choose $u^0, u^1 \in \mathcal{C}$ such that $u^0 \neq u^1$, let $\phi = \frac{1+\sqrt{5}}{2}$, $z^0 = u^1$ and $k = 0$.*
(2) *If $u^{n+1} = u^n = z^n$, then STOP: $u^n \in \Omega$. Otherwise, go to Step 3.*
(3) *Take $n = n + 1$, and*

$$\begin{aligned} z^n &= (1 - \frac{1}{\phi})u^n + \frac{1}{\phi} z^{n-1}, \\ u^{n+1} &= \text{Prox}_{h+\iota_{\mathcal{C}}}\Big(u^n - \frac{1}{\alpha}\mathcal{F}u^n\Big), \end{aligned} \tag{1.2}$$

and go to Step 2.

Grad and Lara [12] proved that the sequences $\{u^n\}$ and $\{z^n\}$ generated by Algorithm 1.1 converge to a solution of non-convex MVI (1.1). As far as we know, the Algorithm 1.1 proposed in [12] is the first iterative method in the literature to solve non-convex MVI (1.1).

In 2022, Shehu et al. [17] studied the follwing forward-reflected-backward iterative method to solve the non-convex MVI (1.1).

Algorithm 1.2. *Forward-Reflected-Backward Method*

(1) *Choose $u^0, u^1 \in \mathbb{R}^n$ and set $n = 1$.*
(2) *Given u^{n-1} and u^n, compute u^{n+1} as follows:*

$$u^{n+1} = \text{Prox}_{h+\iota_{\mathcal{C}}}\Big(u^n - \frac{1}{\alpha}\Big(2\mathcal{F}u^n - \mathcal{F}u^{n-1}\Big)\Big). \tag{1.3}$$

If $u^{n+1} = u^n = u^{n-1}$, then STOP: $u^n \in \Omega$.
(3) *Set $n \leftarrow n + 1$, and go to Step 2.*

Convergence results and numerical experiments are given for this proposed method under some appropriate conditions.

In this paper, we apply the forward-backward-forward splitting method with inertial extrapolation step to solve non-convex mixed variational inequalities. Our results extend the usage of forward-backward-forward splitting method from convex variational inequality problem already studied in [3, 35, 36] to non-convex MVI (1.1)

2. Preliminaries

The indicator function $\iota_{\mathcal{C}}$ of a nonempty set $\mathcal{C} \subseteq \mathbb{R}^n$ is defined by

$$\iota_{\mathcal{C}}(x) := \begin{cases} 0 & \text{if } x \in \mathcal{C} \\ +\infty & \text{otherwise.} \end{cases}$$

Given any $x, y, z \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$, we have

$$\langle x - z, y - x\rangle = \frac{1}{2}\|z - y\|^2 - \frac{1}{2}\|x - z\|^2 - \frac{1}{2}\|y - x\|^2, \tag{2.1}$$

$$\|\alpha x + \beta y\|^2 \quad = \quad \alpha(\alpha+\beta)\|x\|^2 + \beta(\alpha+\beta)\|y\|^2 - \alpha\beta\|x - y\|^2 \tag{2.2}$$

and

$$\|\beta x + (1-\beta)y\|^2 = \beta\|x\|^2 + (1-\beta)\|y\|^2 - \beta(1-\beta)\|x - y\|^2. \tag{2.3}$$

Given any function $h : \mathbb{R}^n \to \bar{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$, the effective domain of h is defined by $\mathrm{dom}h := \{x \in \mathbb{R}^n : h(x) < +\infty\}$. It is said that h is a proper function if $h(x) > -\infty$ for every $x \in \mathbb{R}^n$ and dom h is nonempty (clearly, $h(x) = +\infty$ for every $x \notin \mathrm{dom}h$). By $\arg\min_{\mathbb{R}^n} h$ we mean the set of all minimizers of h in this paper.

A function h with a convex domain is said to be

(a) convex if, given any $x, y \in \mathrm{dom}h$, then

$$h(\lambda x + (1-\lambda)y) \le \lambda h(x) + (1-\lambda)h(y) \quad \forall\, \lambda \in [0,1]; \tag{2.4}$$

(b) quasi-convex if, given any $x, y \in \mathrm{dom}h$, then

$$h(\lambda x + (1-\lambda)y) \le \max\{h(x), h(y)\} \;\; \forall\, \lambda \in [0,1]. \tag{2.5}$$

Clearly, every convex function is quasi-convex. However, the converse fails. Take, for example, $h : \mathbb{R} \to \mathbb{R}$ with $h(x) = x^3$, is quasi-convex but not convex. Also, we have

$$\begin{aligned} h \;\text{ is convex} \quad &\Longleftrightarrow \quad \text{epi}\, h \text{ is a convex set;} \\ h \;\text{ is quasi-convex} \quad &\Longleftrightarrow \quad S_\lambda(h) \text{ is a convex set, for all } \lambda \in \mathbb{R}, \end{aligned}$$

where epi $h := \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : h(x) \le t\}$ is the epigraph of h and $S_\lambda(h) := \{x \in \mathbb{R}^n : h(x) \le \lambda\}$ its sublevel set at the height $\lambda \in \mathbb{R}$.

The proximity operator $\mathrm{Prox}_{\gamma h} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ of a function $h : \mathbb{R}^n \to \bar{\mathbb{R}}$ at $x \in \mathbb{R}^n$ with parameter $\gamma > 0$ is defined by

$$\mathrm{Prox}_{\gamma h}(x) := \arg\min_{y \in \mathbb{R}^n} \left\{ h(y) + \frac{1}{2\gamma}\|y - x\|^2 \right\}. \tag{2.6}$$

Suppose $\mathcal{C}$ is a closed and convex subset of $\mathbb{R}^n$ with $h : \mathcal{C} \to \bar{\mathbb{R}}$ with $\mathcal{C} \cap \mathrm{dom}h \neq \emptyset$ a proper function, and $f : \mathcal{C} \times \mathcal{C} \to \mathbb{R}$ be a real-valued bifunction. We say that f is

(a) monotone on $\mathcal{C}$, if for every $x, y \in \mathcal{C}$

$$f(x,y) + f(y,x) \le 0; \tag{2.7}$$

(b) h-pseudomonotone on $\mathcal{C}$, if for every $x, y \in \mathcal{C}$

$$f(x,y) + h(y) - h(x) \ge 0 \Longrightarrow f(y,x) + h(x) - h(y) \le 0. \tag{2.8}$$

From the above definitions, we see that every monotone bifunction is h-pseudomonotone, but the converse statement is not true in general. Also, if $h \equiv 0$, then h-pseudomonotonicity becomes pseudomonotonicity [13].

In [11], a new class of generalized convex functions (which includes quasiconvex functions and weakly convex functions) was introduced.

Definition 2.1. Let $\mathcal{C}$ be a closed set in $\mathbb{R}^n$ and $h : \mathbb{R}^n \to \bar{\mathbb{R}}$ be a proper function such that $\mathcal{C} \cap \mathrm{dom} h \neq \emptyset$. We say that h is prox-convex on $\mathcal{C}$ (with prox-convex value α) if there exists $\alpha > 0$ such that for every $z \in \mathcal{C}$, $\mathrm{Prox}_{h+\iota_{\mathcal{C}}}(z) \neq \emptyset$, and

$$\bar{x} \in \mathrm{Prox}_{h+\iota_{\mathcal{C}}}(z) \Longrightarrow h(\bar{x}) - h(x) \leq \alpha \langle \bar{x} - z, x - \bar{x} \rangle, \ \forall x \in \mathcal{C}. \tag{2.9}$$

The article [11] gave some properties of prox-convex functions. We list some of them here for the sake of completeness.

Lemma 2.2. [22] *Let $\{\varphi^n\}$, $\{\delta^n\}$ and $\{\theta^n\}$ be sequence in $[0, +\infty)$ such that*

$$\varphi^{n+1} \leq \varphi^n + \theta^n(\varphi^n - \varphi^{n-1}) + \delta^n, \ \ \forall n \geq 1, \ \sum_{n=1}^{\infty} \delta^n < +\infty$$

and there exists a real number θ with $0 \leq \theta^n \leq \theta < 1$ for all $n \in \mathbb{N}$. Then the following hold:

(a) $\sum_{n=1}^{\infty} [\varphi^n - \varphi^{n-1}]_+ < \infty$ *where* $[t]_+ := \max\{t, 0\}$;
(b) *there exists* $\varphi^* \in [0, +\infty)$ *such that* $\lim_{n\to\infty} \varphi^n = \varphi^*$.

3. Proposed Method

We give the following assumptions in order to obtain our convergence analysis.

Assumption 3.1. (A1) *$\mathcal{F}$ is an L-Lipschitz-continuous operator on $\mathcal{C}$, where $L > 0$; that is, there exists $L > 0$ such that $\|\mathcal{F}x - \mathcal{F}y\| \leq L\|x - y\|$ for all $x, y \in \mathcal{C}$;*
(A2) *h is a lower semicontinuous prox-convex function on $\mathcal{C}$ with prox-convex value $\alpha > 0$;*
(A3) *$\mathcal{F}$ and h satisfy the following generalized monotonicity condition on $\mathcal{C}$ (cf. [12, 21, 33])*

$$\langle \mathcal{F}(y), y - \bar{x} \rangle + h(y) - h(\bar{x}) \geq 0, \ \ \forall\, y \in K, \ \forall\, \bar{x} \in \Omega; \tag{3.1}$$

(A4) $\Omega \neq \emptyset$;
(A5) $\alpha > L$.

Algorithm 3.2. *Inertial Non-convex Modified Forward-Backward-Forward Method*

(1) Choose $\theta \in [0, 1)$ and $0 < \rho < \frac{2}{1+L/\alpha}$. Let $u^0, u^1 \in \mathbb{R}^n$ be a given starting point. Set $n := 1$.
(2) Given u^{n-1} and u^n, compute u^{n+1} as follows:

$$\begin{cases} z^n = u^n + \theta(u^n - u^{n-1}) \\ y^n = \mathit{Prox}_{h+i_{\mathcal{C}}}(z^n - \frac{1}{\alpha}\mathcal{F}(z^n)), \\ u^{n+1} = (1-\rho)z^n + \rho(y^n + \frac{1}{\alpha}(\mathcal{F}(z^n) - \mathcal{F}(y^n))) \end{cases} \tag{3.2}$$

(3) Set $n \leftarrow n + 1$ and go to **Step 2.**

Lemma 3.3. *Suppose the Assumptions 3.1 (A1)-(A5) are satisfied and let $\bar{x} \in \Omega$. Suppose $\{u^n\}$ is generated by Algorithm 3.2. Then the following holds:*

$$\|u^{n+1} - \bar{x}\|^2 \leq \|z^n - \bar{x}\|^2 - \left(\frac{2}{\rho\left(1 + \frac{L}{\alpha}\right)} - 1 \right) \|u^{n+1} - z^n\|^2.$$

Proof. From

$$y^n = \mathrm{Prox}_{h+i_{\mathcal{C}}}(z^n - \frac{1}{\alpha}\mathcal{F}(z^n))$$

we obtain using (2.9) that $\forall y \in \mathcal{C}$

$$h(y^n) - h(y) \leq \alpha \langle y^n - z^n + \frac{1}{\alpha}\mathcal{F}(z^n), y - y^n \rangle \tag{3.3}$$

Thus,

$$0 \leq \alpha\langle y^n - z^n + \frac{1}{\alpha}\mathcal{F}(z^n), y - y^n\rangle + h(y) - h(y^n) \;\; \forall y \in \mathcal{C}.$$

Replacing $y = \bar{x} \in \Omega$, we have from the last inequality

$$0 \leq \alpha < y^n - z^n + \frac{1}{\alpha}\mathcal{F}(z^n), \bar{x} - y^n\rangle + h(\bar{x}) - h(y^n). \tag{3.4}$$

Since $\mathcal{F}$ satisfied Assumption 3.1 (A3), we obtain

$$\langle \mathcal{F}(y), y - \bar{x}\rangle + h(y) - h(\bar{x}) \geq 0, \;\; \forall y \in \mathcal{C}.$$

In particular

$$\langle \mathcal{F}(y^n), y - \bar{x}\rangle + h(y) - h(\bar{x}) \geq 0. \tag{3.5}$$

Combining (3.4) and (3.5), we get

$$-\langle \mathcal{F}(y^n), \bar{x} - y^n\rangle + h(y) - h(\bar{x}) + \alpha\langle y^n - z^n + \tfrac{1}{\alpha}\mathcal{F}(z^n), \bar{x} - y^n\rangle + h(\bar{x}) - h(y_n) \geq 0.$$

That is

$$\alpha\langle y^n - z^n + \frac{1}{\alpha}\mathcal{F}(z^n) - \mathcal{F}(y^n), \bar{x} - y^n\rangle \geq 0. \tag{3.6}$$

Letting

$$v^n := y^n + \frac{1}{\alpha}(\mathcal{F}(z^n) - \mathcal{F}(y^n)) \;\; \forall n \neq 1.$$

We have from (3.6) that

$$\alpha\langle v^n - z^n, \bar{x} - y^n\rangle \geq 0.$$

This further implies that

$$\begin{aligned}
\langle v^n - \bar{x}, v^n - z^n\rangle &\leq \langle v^n - y^n, v^n - z^n\rangle \\
&= \|v^n - z^n\|^2 + \langle z^n - y^n, v^n - z^n\rangle \\
&= \|v^n - z^n\|^2 + \langle z^n - y^n, y^n + \frac{1}{\alpha}\left(\mathcal{F}(z^n) - \mathcal{F}(y^n)\right) - z^n\rangle \\
&= \|v^n - z^n\|^2 - \|z^n - y^n\|^2 + \frac{1}{\alpha}\langle z^n - y^n, \mathcal{F}(z^n) - \mathcal{F}(y^n)\rangle.
\end{aligned} \tag{3.7}$$

Also, we obtain from (2.1) that

$$2\langle v^n - \bar{x}, v^n - z^n\rangle = \|v^n - \bar{x}\|^2 - \|z^n - \bar{x}\|^2 + \|v^n - z^n\|^2. \tag{3.8}$$

If we combine (3.7) and (3.8), we get

$$\begin{aligned}
\|v^n - \bar{x}\|^2 \;\; &\leq \;\; \|z^n - \bar{x}\|^2 + \|v^n - z^n\|^2 - 2\|z^n - y^n\|^2 \\
&\quad + \frac{2}{\alpha}\langle z^n - y^n, \mathcal{F}(z^n) - \mathcal{F}(y^n)\rangle.
\end{aligned} \tag{3.9}$$

Using the fact that $\mathcal{F}$ is L-Lipschitz continuous, we get

$$\begin{aligned}
\|v^n - z^n\|^2 &= \|y^n + \frac{1}{\alpha}\left(\mathcal{F}(z^n) - \mathcal{F}(y^n)\right) - z^n\|^2 \\
&= \|y^n - z^n\|^2 + \frac{2}{\alpha}\langle y^n - z^n, \mathcal{F}(z^n) - \mathcal{F}(y^n)\rangle + \frac{1}{\alpha^2}\|\mathcal{F}(z^n) - \mathcal{F}(y^n)\|^2 \\
&\leq \|y^n - z^n\|^2 + \frac{2}{\alpha}\langle y^n - z^n, \mathcal{F}(z^n) - \mathcal{F}(y^n)\rangle + \frac{L^2}{\alpha^2}\|z^n - y^n\|^2.
\end{aligned} \tag{3.10}$$

We obtain from (3.9) and (3.10) that

$$\|v^n - \bar{x}\|^2 \leq \|z^n - \bar{x}\|^2 - \left(1 - \frac{L^2}{\alpha^2}\right)\|y^n - z^n\|^2. \tag{3.11}$$

Consequently,

$$\begin{aligned} \|u^{n+1}-\bar{x}\|^2 &= \|\rho(v^n-\bar{x})+(1-\rho)(z^n-\bar{x})\|^2 \\ &= \rho\|v^n-\bar{x}\|^2+(1-\rho)\|z^n-\bar{x}\|^2-\rho(1-\rho)\|v^n-z^n\|^2. \end{aligned} \tag{3.12}$$

Using (3.11) and (3.12) we have

$$\begin{aligned} \|u^{n+1}-\bar{x}\|^2 \leq& \rho\|z^n-\bar{x}\|^2-\rho\left(1-\frac{L^2}{\alpha^2}\right)\|y^n-z^n\|^2-(1-\rho)\|z^n-\bar{x}\|^2-\rho(1-\rho)\|v^n-z^n\|^2 \\ =&\|z^n-\bar{x}\|^2-\rho\left(1-\frac{L^2}{\alpha^2}\right)\|y^n-z^n\|^2-(1-\rho)\|z^n-\bar{x}\|^2-\rho(1-\rho)\|v^n-z^n\|^2. \end{aligned} \tag{3.13}$$

From the definition of $\{u^{n+1}\}$, we obtain

$$\|v^n-z^n\|^2=\frac{1}{\rho^2}\|u^{n+1}-z^n\|^2 \tag{3.14}$$

Putting (3.14) in (3.13) gives

$$\|u^{n+1}-\bar{x}\|^2 \leq \|z^n-\bar{x}\|^2-\rho\left(1-\frac{L^2}{\alpha^2}\right)\|y^n-z^n\|^2-\frac{1-\rho}{\rho}\|u^{n+1}-z^n\|^2 \tag{3.15}$$

By Algorithm (3.2) and using the fact that $\mathcal{F}$ is Lipschitz continuous, we obtain

$$\begin{aligned} \frac{1}{\rho}\|u^{n+1}-z^n\| &= \|v^n-z^n\| \leq \|v^n-y^n\|+\|y^n-z^n\| \\ &= \frac{1}{\alpha}\|\mathcal{F}(z^n)-\mathcal{F}(y^n)\|\|y^n-z^n\| \\ &\leq \frac{L}{\alpha}\|y^n-z^n\|+\|y^n-z^n\| \\ &= \left(1+\frac{L}{\alpha}\right)\|y^n-z^n\|. \end{aligned}$$

Therefore

$$-\|y^n-z^n\|^2 \leq -\frac{1}{\rho^2\left(1+\frac{L}{\alpha}\right)^2}\|u^{n+1}-z^n\|^2. \tag{3.16}$$

Substituting (3.16) into (3.14), we get

$$\begin{aligned} \|u^{n+1}-\bar{x}\|^2 &\leq \|z^n-\bar{x}\|^2-\left[\frac{\left(1-\frac{L^2}{\alpha^2}\right)}{\rho^2\left(1+\frac{L}{\alpha}\right)^2}+\frac{1-\rho}{\rho}\right]\|u^{n+1}-z^n\|^2 \\ &= \|z^n-\bar{x}\|^2-\left(\frac{2}{\rho\left(1+\frac{L}{\alpha}\right)}-1\right)\|u^{n+1}-z^n\|^2. \end{aligned} \tag{3.17}$$

This completes the proof. □

Lemma 3.4. *Suppose Assumption 3.1 is fulfilled. Then $\{u^n\}$ generated by Algorithm 3.2 above is bounded.*

Proof. Let $\bar{x}\in\Omega$, By (2.2) and Algorithm 3.2, we get

$$\begin{aligned} \|z^n-\bar{x}\|^2 &= \|u^n+\theta(u^n-u^{n-1})-\bar{x}\|^2 \\ &= \|(1+\theta)(u^n-\bar{x})-\theta(u^{n-1}-\bar{x})\|^2 \\ &= (1+\theta)\|u^n-\bar{x}\|^2-\theta\|u^{n-1}-\bar{x}\|^2+\theta(1+\theta)\|u^n-u^{n-1}\|^2. \end{aligned} \tag{3.18}$$

Observe that

$$\begin{aligned}\|u^{n+1}-z^n\|^2 &= \|(u^{n+1}-u^n)-\theta(u^n-u^{n-1})\|^2\\ &= \|u^{n+1}-u^n\|^2+\theta^2\|u^n-u^{n-1}\|^2-2\theta\langle u^{n+1}-u^n,u^n-u^{n-1}\rangle\\ &\geq (1-\theta)\|u^{n+1}-u^n\|^2+(\theta^2-\theta)\|u^n-u^{n-1}\|^2. \end{aligned} \tag{3.19}$$

Using (3.18) and (3.19) in (3.17), we get

$$\begin{aligned}\|u^{n+1}-\bar{x}\|^2 &\leq (1+\theta)\|u^n-\bar{x}\|^2-\theta\|u^{n-1}-\bar{x}\|^2+\theta(1+\theta)\|u^n-u^{n-1}\|^2\\ &\quad -(1-\theta)\left[\frac{2}{\rho\,(1+L/\alpha)}-1\right]\|u^{n+1}-u^n\|^2\\ &\quad -(\theta^2-\theta)\left[\frac{2}{\rho\,(1+L/\alpha)}-1\right]\|u^n-u^{n-1}\|^2. \end{aligned} \tag{3.20}$$

Rearranging gives us

$$\begin{aligned}&\|u^{n+1}-\bar{x}\|^2-\theta\|u^n-\bar{x}\|^2+\theta\,[1+\theta-(\theta-1)]\left[\frac{2}{\rho\,(1+L/\alpha)}-1\right]\|u^n-u^{n-1}\|^2\\ &\leq \|u^n-\bar{x}\|^2-\theta\|u^{n-1}-\bar{x}\|^2+\theta\,[1+\theta-(\theta-1)]\left[\frac{2}{\rho\,(1+L/\alpha)}-1\right]\|u^{n+1}-u^n\|^2\\ &\quad -\theta\,[1+\theta-(\theta-1)]\left[\frac{2}{\rho\,(1+L/\alpha)}-1\right]\|u^{n+1}-u^n\|^2\\ &\quad -(1-\theta)\left[\frac{2}{\rho\,(1+L/\alpha)}-1\right]\|u^{n+1}-u^n\|^2. \end{aligned} \tag{3.21}$$

Define

$$\Upsilon_n := \|u^n-\bar{x}\|^2-\theta\|u^{n-1}-\bar{x}\|^2+\theta\,[1+\theta-(\theta-1)]\left[\frac{2}{\rho\,(1+L/\alpha)}-1\right]\|u^n-u^{n-1}\|^2$$

and

$$\sigma := (1-\theta)\left[\frac{2}{\rho\,(1+L/\alpha)}-1\right]-\theta\,[1+\theta-(\theta-1)]\left[\frac{2}{\rho\,(1+L/\alpha)}-1\right].$$

From (3.21), we obtain

$$\Upsilon_{n+1}-\Upsilon_n\leq -\sigma\|u^{n+1}-u^n\|^2. \tag{3.22}$$

By the condition $\theta\in[0,1)$ $\alpha>L$ and $0<\rho<\frac{2}{1+L/\alpha}$, we get that $\sigma>0$. Therefore $\{\Upsilon_n\}$ is non-increasing. Similarly,

$$\begin{aligned}\Upsilon_n &:= \|u^n-\bar{x}\|^2-\theta\|u^{n-1}-\bar{x}\|^2+\theta\,[1+\theta-(\theta-1)]\left[\frac{2}{\rho\,(1+L/\alpha)}-1\right]\|u^n-u^{n-1}\|^2\\ &\geq \|u^n-\bar{x}\|^2-\theta\|u^{n-1}-\bar{x}\|^2. \end{aligned} \tag{3.23}$$

Fromn(3.23), we have

$$\begin{aligned}\|u^n-\bar{x}\|^2 &\leq \theta\|u^{n-1}-\bar{x}\|^2+\Upsilon_n\\ &\leq \theta\|u^{n-1}-\bar{x}\|^2+\Upsilon_1\\ &\ \vdots\\ &\leq \theta^n\|u^0-\bar{x}\|^2+(1+\cdots+\theta^{n-1})\Upsilon_1\\ &\leq \theta^n\|u^0-\bar{x}\|^2+\frac{\Upsilon_1}{1-\theta}. \end{aligned} \tag{3.24}$$

Now that

$$\begin{aligned}\Upsilon_{n+1} &= \|u^{n+1}-\bar{x}\|^2-\theta\|u^n-\bar{x}\|^2\theta\left[1+\theta-(\theta-1)\right]\left[\frac{2}{\rho\left(1+L/\alpha\right)}-1\right]\|u^{n+1}-u^n\|^2 \\ &\geq -\theta\|u^n-\bar{x}\|^2\end{aligned}$$

and this means from (3.24) that

$$\begin{aligned}-\Upsilon_{n+1} &\leq \theta\|u^n-\bar{x}\|^2 \\ &\vdots \\ &\leq \theta^{n+1}\|u^0-\bar{x}\|^2+\frac{\theta\Upsilon_1}{1-\theta}. \end{aligned} \tag{3.25}$$

By (3.22) and (3.25), we get

$$\sigma\sum_{n=1}^{k}\|u^{n+1}-u^n\|^2 \leq \Upsilon_1-\Upsilon_{k+1}\leq\theta^{k+1}\|u^0-\bar{x}\|^2+\frac{\Upsilon_1}{1-\theta}.$$

This implies

$$\sum_{n=1}^{\infty}\|u^{n+1}-u^n\|^2\leq\frac{\Upsilon_1}{\sigma(1-\theta)}<+\infty. \tag{3.26}$$

Therefore

$$\lim_{n\to\infty}\|u^{n+1}-u^n\|=0 \tag{3.27}$$

and

$$\lim_{n\to\infty}\|\mathcal{F}u^{n+1}-\mathcal{F}u^n\|=0 \ \ \text{(by the fact that } \mathcal{F} \text{ is Lipschtz continuous)}.$$

From Algorithm 3.2, we get

$$\|z^n-u^n\|=\theta\|u^n-u^{n-1}\|\to 0,\ \ n\to\infty. \tag{3.28}$$

Also

$$\|u^{n+1}-z^n\|\leq\|z^n-u^n\|+\|u^{n+1}-u^n\|\to 0,\ \ n\to\infty$$

and

$$\|v^n-z^n\|=\frac{1}{\rho}\|u^{n+1}-z^n\|\to 0,\ \ n\to\infty. \tag{3.29}$$

From (3.20), we get

$$\begin{aligned}\|u^{n+1}-\bar{x}\|^2 &\leq (1+\theta)\|u^n-\bar{x}\|^2-\theta\|u^{n-1}-\bar{x}\|^2 \\ &\quad +\theta\left[1+\theta-(\theta-1)\left(\frac{2}{\rho\left(1+L/\alpha\right)}-1\right)\right]\|u^n-u^{n-1}\|^2. \end{aligned} \tag{3.30}$$

Using Lemma 2.2 in (3.30) (noting (3.26)), we get

$$\lim_{n\to\infty}\|u^n-\bar{x}\|=\ell<+\infty.$$

Hence $\{\|u^n-\bar{x}\|\}$ is bounded. Therefore $\{u^n\}$ is bounded. It is clear to see that the boundedness of $\{u^n\}$ implies that $\{v^n\}$, $\{z^n\}$ and $\{y^n\}$ are bounded. □

Our global convergence result for Algorithm 3.2 is given next.

Theorem 3.5. *Suppose that Assumption 3.1 are satisfied, then $\{u^n\}$ generated by Algorithm 3.2 converges to a solution of MVI (1.1).*

Proof. By Lemma 3.4, we have that $\{u^n\}$ is bounded. Let v^* be an accumulating point of $\{u^n\}$. By (3.3), we have $\forall y \in \mathcal{C}$

$$h(y^n) - h(y) \leq \alpha\langle y^n - z^n + \frac{1}{\alpha}\mathcal{F}(z^n), y - y^n\rangle. \tag{3.31}$$

From (3.11)

$$\begin{aligned}\left(1 - \frac{L^2}{\alpha^2}\right)\|y^n - z^n\|^2 &\leq \|z^n - \bar{x}\|^2 - \|v^n - \bar{x}\|^2\\ &= (\|z^n - \bar{x}\| - \|v^n - \bar{x}\|)(\|z^n - \bar{x}\| + \|v^n - \bar{x}\|)\\ &\leq \mathcal{M}(\|z^n - \bar{x}\| - \|v^n - \bar{x}\|)\\ &\leq \mathcal{M}\|z^n - v^n\|,\end{aligned}$$

where $\mathcal{M} := \sup_{n\geq 1}(\|z^n - \bar{x}\| + \|v^n - \bar{x}\|) < \infty$. Since both $\{v^n\}$ and $\{z^n\}$ are bounded By Assumption 3.1 $(A5)$ and (3.29), one derives that

$$\lim_{n\to\infty}\|y^n - z^n\| = 0 \tag{3.32}$$

This implies that

$$\|u^{n+1} - y^n\| \leq \|y^n - z^n\| + \|u^{n+1} - z^n\| \to 0, \quad n \to \infty. \tag{3.33}$$

Because v^* is an accumulating point of $\{u^n\}$, by (3.28) and (3.32), it is also accumulating point of $\{y^n\}$ and of $\{z^n\}$. Also h is lower semicontinuous and $\mathcal{F}$ is L-Lipschitz continuous. By taking the limit in (3.31) (passing to subsequence if neccessary), we have noting (3.32),(3.33) and (3.27) that

$$h(v^*) - h(y) \leq \langle\mathcal{F}(v^*), y - v^*\rangle \ \ \forall\, y \in \mathcal{C}.$$

Thus, $v^* \in \Omega$. Hence every cluster point (accumulation point) of $\{u^n\}$ is a solution to MVI. □

4. Second Proposed Method

Our aim in this section is to modify Algorithm 3.2 such that the upper condition $\theta^n \leq \theta < 1$, $\forall n \geq 1$ in (3.2) can be extended to $0 \leq \theta^n \leq 1$, $\forall n \geq 1$. We obtain the global convergence result of our proposed algorithm under standard assumptions. The proposed iterative method is given below:

Algorithm 4.1. *(1) Choose $\theta \in [0,1]$ and $0 < \rho < \frac{1}{2}$. Let $u^0, u^1 \in \mathbb{R}^n$ be a given starting point. Set $n := 1$.*

(2) Given u^{n-1} and u^n, compute u^{n+1} as follows:

$$\begin{cases} z^n = u^n + \theta(u^n - u^{n-1})\\ y^n = Prox_{h+i_{\mathcal{C}}}(z^n - \frac{1}{\alpha}\mathcal{F}(z^n)),\\ u^{n+1} = (1-\rho)u^n + \rho(y^n + \frac{1}{\alpha}(\mathcal{F}(z^n) - \mathcal{F}(y^n)))\end{cases} \tag{4.1}$$

(3) Set $n \leftarrow n+1$ and go to **Step 2.**

Remark 4.2. Suppose $\theta = 1$ and $\rho = 0$ in Algorithm 3.2. Then $z^n = 2u^n - u^{n-1}$ is the reflected step. Therefore, our Algorithm (3.2) covers the reflected version Algorithm 1.2 studied by Shehu et al. in [17] for solving Non-convex MVI (1.1).

Using Algorithm 4.1, we obtain the following results.

Lemma 4.3. *Suppose Assumption 3.1 is fulfilled. Then $\{u^n\}$ generated by Algorithm 4.1 above is bounded.*

Proof. Let

$$v^n := y^n + \frac{1}{\alpha}(\mathcal{F}(z^n) - \mathcal{F}(y^n)), \quad \forall n \neq 1.$$

From (3.11), we get

$$\begin{aligned} \|v^n - \bar{x}\|^2 &\leq \|z^n - \bar{x}\|^2 - \left(1 - \frac{L^2}{\alpha^2}\right)\|y^n - z^n\|^2 \\ &\leq \|z^n - \bar{x}\|^2. \end{aligned} \tag{4.2}$$

Let $\bar{x} \in \Omega$. Using Algorithm 4.1, we get

$$\begin{aligned} \|u^{n+1} - \bar{x}\|^2 &= \|(1-\rho)(u^n - \bar{x}) + \rho(v^n - \bar{x})\|^2 \\ &= (1-\rho)\|u^n - \bar{x}\|^2 + \alpha\|v^n - \bar{x}\|^2 - \rho(1-\rho)\|u^n - v^n\|^2, \end{aligned} \tag{4.3}$$

which by (4.2) implies that

$$\|u^{n+1} - \bar{x}\|^2 \leq (1-\rho)\|u^n - \bar{x}\|^2 + \rho\|z^n - \bar{x}\|^2 - \rho(1-\rho)\|u^n - v^n\|^2. \tag{4.4}$$

Note that

$$u^{n+1} = (1-\rho)u^n + \rho v^n$$

and this implies

$$v^n - u^n = \frac{1}{\rho}(u^{n+1} - u^n), \ \forall n. \tag{4.5}$$

Using (4.4) and (4.5), we get

$$\|u^{n+1} - \bar{x}\|^2 \leq (1-\rho)\|u^n - \bar{x}\|^2 + \rho\|z^n - \bar{x}\|^2 - \frac{(1-\rho)}{\rho}\|u^{n+1} - u^n\|^2. \tag{4.6}$$

Also, by (3.18) and (4.6), we get

$$\begin{aligned} \|u^{n+1} - \bar{x}\|^2 &\leq (1-\rho)\|u^n - \bar{x}\|^2 + \rho(1+\theta)\|u^n - \bar{x}\|^2 - \rho\theta\|u^{n-1} - \bar{x}\|^2 \\ &\quad +\rho\theta(1+\theta)\|u^n - u^{n-1}\|^2 - \frac{1-\rho}{\rho}\|u^{n+1} - u^n\|^2 \\ &= (1+\rho\theta)\|u^n - \bar{x}\|^2 - \rho\theta\|u^{n-1} - \bar{x}\|^2 + \rho\theta(1+\theta)\|u^n - u^{n-1}\|^2 \\ &\quad -\frac{1-\rho}{\rho}\|u^{n+1} - u^n\|^2. \end{aligned} \tag{4.7}$$

Define

$$\Upsilon_n := \|u^n - \bar{x}\|^2 - \rho\theta\|u^{n-1} - \bar{x}\|^2 + \rho\theta(1+\theta)\|u^n - u^{n-1}\|^2, \ n \geq 1.$$

Then we have by (4.7) that

$$\begin{aligned} \Upsilon_{n+1} - \Upsilon_n &= \|u^{n+1} - \bar{x}\|^2 - (1+\rho\theta)\|u^n - \bar{x}\|^2 + \rho\theta\|u^{n-1} - \bar{x}\|^2 \\ &\quad +\rho\theta(1+\theta)\|u^{n+1} - u^n\|^2 - \rho\theta(1+\theta)\|u^n - u^{n-1}\|^2 \\ &\leq -\frac{1-\rho}{\rho}\|u^{n+1} - u^n\|^2 + \rho\theta(1+\theta)\|u^{n+1} - u^n\|^2 \\ &= -\left(\frac{1-\rho}{\rho} - \rho\theta(1+\theta)\right)\|u^{n+1} - u^n\|^2. \end{aligned} \tag{4.8}$$

By the condition that $\rho \in (0, \frac{1}{2})$ and $\theta \in [0, 1]$, we obtain

$$\sigma := \left(\frac{1-\rho}{\rho} - \rho\theta(1+\theta)\right) > 0. \tag{4.9}$$

By (3.21), we have

$$\Upsilon_{n+1} - \Upsilon_n \leq -\sigma\|u^{n+1} - u^n\|^2 \tag{4.10}$$

Therefore, $\{\Upsilon_n\}$ is non-increasing. Similarly,

$$\begin{aligned}\Upsilon_n &= \|u^n-\bar{x}\|^2-\rho\theta\|u^{n-1}-\bar{x}\|^2+\rho\theta(1+\theta)\|u^n-u^{n-1}\|^2\\ &\geq \|u^n-\bar{x}\|^2-\rho\theta\|u^{n-1}-\bar{x}\|^2. \qquad (4.11)\end{aligned}$$

Define $\epsilon:=\rho\theta<1$. From (4.11), we have

$$\begin{aligned}\|u^n-\bar{x}\|^2 &\leq \rho\theta\|u^{n-1}-\bar{x}\|^2+\Upsilon_n\\ &\leq \epsilon\|u^{n-1}-\bar{x}\|^2+\Upsilon_1\\ &\vdots\\ &\leq \epsilon^n\|u^0-\bar{x}\|^2+(1+\cdots+\epsilon^{n-1})\Upsilon_1\\ &\leq \epsilon^n\|u^0-\bar{x}\|^2+\frac{\Upsilon_1}{1-\epsilon}. \qquad (4.12)\end{aligned}$$

Note that

$$\Upsilon_{n+1} = \|u^{n+1}-\bar{x}\|^2-\rho\theta\|u^n-\bar{x}\|^2+\rho\theta(1+\theta)\|u^{n+1}-u^n\|^2\geq-\rho\theta\|u^n-\bar{x}\|^2$$

and this means from (4.13) that

$$\begin{aligned}-\Upsilon_{n+1} &\leq \rho\theta\|u^n-\bar{x}\|^2\\ &= \epsilon\|u^n-\bar{x}\|^2\\ &\vdots\\ &\leq \epsilon^{n+1}\|u^0-\bar{x}\|^2+\frac{\epsilon\Upsilon_1}{1-\epsilon}. \qquad (4.13)\end{aligned}$$

By (4.10) and (4.13), we get

$$\sigma\sum_{n=1}^{k}\|u^{n+1}-u^n\|^2 \leq \Upsilon_1-\Upsilon_{k+1}\leq\epsilon^{k+1}\|u^0-\bar{x}\|^2+\frac{\Upsilon_1}{1-\rho}.$$

This implies

$$\sum_{n=1}^{\infty}\|u^{n+1}-u^n\|^2\leq\frac{\Upsilon_1}{\sigma(1-\epsilon)}<+\infty. \qquad (4.14)$$

Therefore

$$\lim_{n\to\infty}\|u^{n+1}-u^n\|=0. \qquad (4.15)$$

From Algorithm 4.1, we get

$$\|z^n-u^n\|=\theta\|u^n-u^{n-1}\|\to 0,\ \ n\to+\infty. \qquad (4.16)$$

Also

$$\|u^{n+1}-z^n\|\leq\|z^n-u^n\|+\|u^{n+1}-u^n\|\to 0,\ \ n\to\infty$$

and

$$\|v^n-z^n\|=\frac{1}{\rho}\|u^{n+1}-z^n\|\to 0,\ \ n\to\infty. \qquad (4.17)$$

From (3.20), we get

$$\|u^{n+1}-\bar{x}\|^2\leq(1+\rho\theta)\|u^n-\bar{x}\|^2-\rho\theta\|u^{n-1}-\bar{x}\|^2+2\|u^n-u^{n-1}\|^2. \qquad (4.18)$$

Using Lemma 3.4 in (4.18) (noting (4.14)), we get

$$\lim_{n\to\infty}\|u^n-\bar{x}\|^2=\ell<\infty.$$

Hence, $\{\|u^n - \bar{x}\|\}$ is bounded. Therefore $\{u^n\}$ is bounded. $\square$

Theorem 4.4. *The sequence $\{u^n\}$ generated by Algorithm 4.1 converges globaly to a point in Ω when Assumption 3.1 is satisfied.*

Proof. Following same line of arguments as in Theorem 3.5, we obtain the desired conclusion. $\square$

5. Numerical Experiments

In this section, we provide computational experiments by comparing our proposed Algorithm 3.2 with the existing state-of-the-art Algorithm 1.1 (proposed in [12]) and Algorithm 1.2 proposed in [17] using test examples below. Numerical experiments were carried out on MATLAB R2015a version. All programs were run on a 64-bit OS PC with Intel(R) Core(TM) i7-3540M CPU @ 1.00GHz 1.19 GHz and 3GB RAM. All figures were plotted using the $\log\log$ plot command.

5.1. **Application in Oligopolistic Equilibrium.** The Nash-Cournot oligopolistic market equilibrium assumes that there are n companies producing a common homogeneous commodity (see, for example, [19, 26]). For $i \in \{1, ..., n\}$, company i has strategy set $D_i \subseteq \mathbb{R}_+$, a cost function φ defined on the strategy set $D = \Pi_{i=1}^n D_i$ of the model and a profit function f_i that is usually defined as price times production minus costs (of producing the considered production). Each company is interested in maximizing its profit by choosing the corresponding production level under knowledge on demand of the market and of production of the competition (seen as input parameters). A commonly used solution notion in this model is the celebrated Nash equilibrium. A point (strategy) $\bar{x} = (\bar{x}_1, ..., \bar{x}_n)^T \in D$ is said to be a Nash equilibrium point of this Nash-Cournot oligopolistic market model if $f_i(\bar{x}) \geq f_i(\bar{x}[x_i]) \ \forall x_i \in D_i, \quad i = 1, ..., n$, where the vector $\bar{x}[x_i]$ is obtained from $\bar{x}$ by replacing $\bar{x}_i$ with x_i. Using similar ideas in [12, 19, 26], the problem of determining a Nash equilibrium of a Nash-Cournot oligopolistic market situation can be recast into a mixed variational inequality of type (1.1), where the involved operator captures various parameters and additional information and the function h is the sum of the cost function of the considered companies, each of them depending on a different variable that represents the corresponding production.

Following the oligopolistic equilibrium model in [26], let us we consider a mixed variational inequality corresponding to Nash-Cournot oligopolistic market equilibrium model with 5 companies, whose cost functions are defined as

$$\begin{cases} \varphi_1 : [0,2] \to \mathbb{R}, & \varphi_1(x) = -x^2 - x, \\ \varphi_2 : \mathbb{R} \to \mathbb{R}, & \varphi_2(x) = x^2 \\ \varphi_3 : [1,2] \to \mathbb{R}, & \varphi_3(x) = 5x + \ln(1+10x), \\ \varphi_4 : \mathbb{R} \to \mathbb{R}, & \varphi_4(x) = \begin{cases} \frac{x^2}{2}, & \text{if} \quad |x| \leq 1, \\ |x| - \frac{1}{2}, & \text{otherwise}, \end{cases} \\ \varphi_5 : [0,2] \to \mathbb{R}, & \varphi_5(x) = 8 - x^3. \end{cases} \tag{5.1}$$

As explained in [12], the cost functions φ_1, φ_3 and φ_5 are prox-convex while φ_2 and φ_4 are convex. Functions φ_1 and φ_3 were employed in the similar application considered in [26] (following the more applied paper [2]) as (DC) cost functions for oligopolistic equilibrium problems, while φ_4 is the Huber loss function (cf. [14]) that is used in various contexts in economics and machine learning for quantifying costs. The model also includes functions with negative values such as φ_1 that quantify the possibility of having negative costs, for instance in case of a surplus of energy on the market (as discussed in the recent works [1, 6]). As done in [12], we consider the involved convex cost functions defined over the whole space.

In our numerical experiments for solving MVI (1.1) (which we denote as (OMP) in this case as called in [12]), $\mathcal{F}(x) = Ax, \ x \in \mathbb{R}^5$ with $A \in \mathbb{R}^{5\times 5}$ a real symmetric positive semidefinite matrix of norm

1 (for simplicity) and $h(x_1, ..., x_5) = \sum_{i=1}^{5} \varphi_i(x_i)$. In each experiment, the matrix A is randomly generated and scaled in order to have $L = 1$. The proximity operator of (the separable function) h has as components the ones of the involved functions, which are known (cf. [5, 12, 24]). Also, we consider the following two cases of initial points:

Case A: $u^0 = (1, 23, 1.4, 39, 1)^T, u^1 = (0, 32, 1.8, 22, 0)^T$.

Case B: $u^0 = (0.1, 2, 2, 2, 0.1)^T, u^1 = (0, 0, 1.9, 0, 0)^T$.

TABLE 1. Computational Results

No. of Iterations	Parameters	$\|x_{n+1} - x_n\|_2$	Time(secs)
50	$\alpha = 2; \theta = 0.05; \rho = 1.3$	$7.91e - 08$	1.5858
	$\alpha = 4; \theta = 0.05; \rho = 1.3$	$1.08e - 08$	1.3530
100	$\alpha = 2; \theta = 0.05; \rho = 1.3$	$6.55e - 09$	2.9380
	$\alpha = 4; \theta = 0.05; \rho = 1.3$	$6.18e - 09$	2.8734

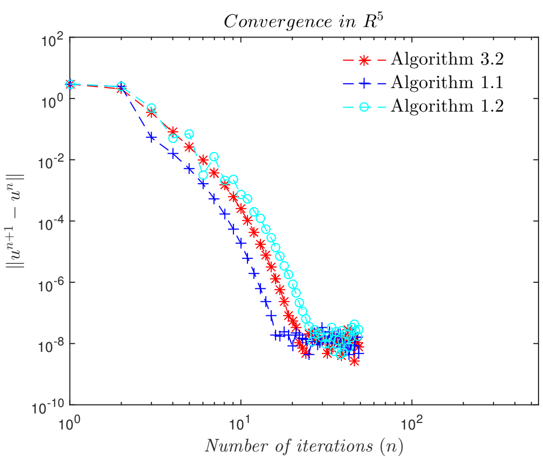

FIGURE 1. $\alpha = 2$, $\theta = 0.005$ and $\rho = 1.3$

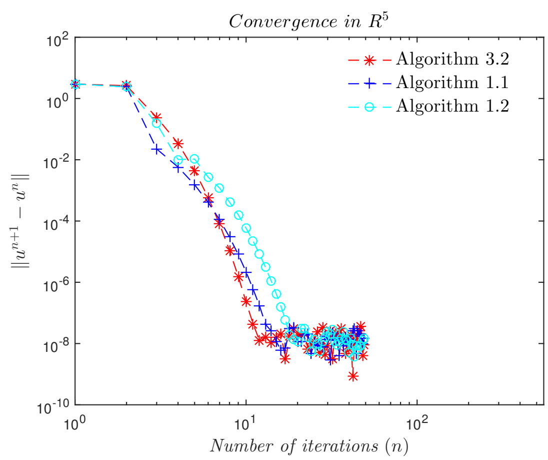

FIGURE 2. $\alpha = 4$, $\theta = 0.005$ and $\rho = 1.3$

6. FINAL REMARK

In this paper we showed that forward-backward-forward splitting method with inertial extrapolation can be adapted to solve non-convex mixed variational inequalities. Global convergence results of the sequence of iterates generated by the proposed method is given and some numerical illustrations are also given. It can be easily seen from the graphs that Algorithm (3.1) performs better in the long run with a better choice of alpha. It can also be observed that appropiate choice of alpha speeds up the convergence. Finally, One of our proposed methods contains the possibility of $\theta = 1$ which is not covered before in the literature.

STATEMENTS AND DECLARATIONS

The authors declare that they have no conflict of interest, and the manuscript has no associated data.

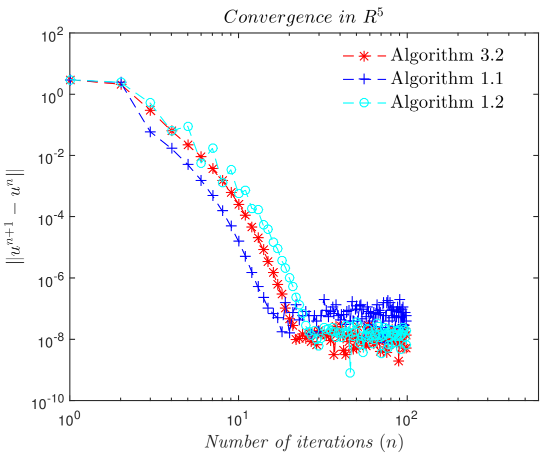

FIGURE 3. $\alpha = 2$, $\theta = 0.005$ and $\rho = 1.3$

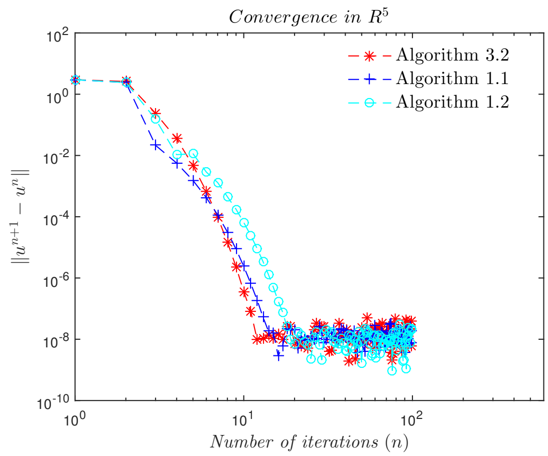

FIGURE 4. $\alpha = 4$, $\theta = 0.005$ and $\rho = 1.3$

REFERENCES

[1] B. Aust and A. Horsch. Negative market prices on power exchanges: evidence and policy implications from Germany. *The Electricity Journal*, 33(3):106716, 2020.

[2] G. Bigi and M. Passacantando. Differentiated oligopolistic markets with concave cost functions via Ky Fan inequalities. *Decisions in Economics and Finance* , 40:63-79, 2017.

[3] R. I. Bot, E. R. Csetnek, and P. T. Vuong The forward-backward-forward method from continuous and discrete perspective for pseudo-monotone variational inequalities in Hilbert spaces. *European Journal of Operational Research,* 287(1):49-60, 2020.

[4] C. Chen, S. Ma, and J. Yang. A general inertial proximal point algorithm for mixed variational inequality problem. *SIAM Journal on Optimization,* 25(4):2120-2142, 2015.

[5] P. L. Combettes and J.-C. Pesquest. Proximal thresholding algorithm for minimization over orthonormal bases. *SIAM Journal on Optimization,* 18(4):1351-1376, 2008.

[6] S. Corbet, J. W. Goodell, and S. Günay. Co-movements and spillovers of oil and renewable firms under extreme conditions: new evidence from negative WTI prices during COVID-19. *Energy Economics,* 92:104978, 2020.

[7] F. Facchinei and J.-S Pang. *Finite-Dimensional Variational Inequalities and Complementarity Problems*, vol. I. Springer, New York, 2003.

[8] F. Facchinei and J.-S Pang. *Finite-Dimensional Variational Inequalities and Complementarity Problems*, vol. II. Springer, New York, 2003.

[9] D. Goeleven. *Complementarity and Variational Inequalities in Electronics*. Academic Press, London, 2017.

[10] D. Goeleven. Existence and uniqueness for a linear mixed variational inequality arising in electrical circuits with transistors. *Journal of Optimization Theory and Applications,* 138:397-406, 2008.

[11] S.-M. Grad and F. Lara. An extension of the proximal point algorithm beyond convexity. *Journal of Global Optimization,* 82: 313-329, 2022.

[12] S.-M. Grad and F. Lara. Solving mixed variational inequalities beyond convexity. *Journal of Optimization Theory and Applications,* 190:565-580, 2021.

[13] N. Hadjisavvas, S. Komlosi, and S. Schaible. *Handbook of Generalized Convexity and Generalized Monotonicity*. Springer, Boston, 2005.

[14] P. J. Huber. Robust estimation of a location parameter. *The Annals of Mathematical Statistics,* 35(1):73-101, 1964.

[15] A. Iusem and F. Lara. Optimality conditions for vector equilibrium problems with applications. *Journal of Optimization Theory and Applications,* 180: 187-206, 2019.

[16] A. Iusem and F. Lara. Existence results for noncoercive mixed variational inequalities in finite dimensional spaces. *Journal of Optimization Theory and Applications,* 183:122-138, 2019.

[17] C. Izuchukwu, C. C. Okeke, and Y. Shehu. Extension of forward-reflected-backward method to non-convex mixed variational inequalities. *Journal of Global Optimization,* 86:123-140, 2023.

[18] D. Kinderlehrer and G. Stampacchia. *An Introduction to Variational Inequalities and Their Applications.* Academic Press, New York (1980).

[19] I. Konnov and E. O. Volotskaya. Mixed variational inequalities and economic equilibrium problems. *Journal of Applied Mathematics,* 2:289-314, 2002.

[20] N. Langenberg and R. Tichatschke. Interior proximal methods for quasiconvex optimization. *Journal of Global Optimization,* 52:641-661, 2012.

[21] Y. Malitsky. Golden ratio algorithms for variational inequalities. *Mathematical Programming,* 184:383-410, 2020.

[22] P-E. Mingé. Convergence theorems for inertial KM-type algorithms. *Journal of Computational and Applied Mathematics,* 219(1):223–236, 2008.

[23] A. Matei and M. Sofonea. Solvability and optimization for a class of mixed variational problems. *Optimization*, 69(5):1097-1116, 2020.

[24] J.-J Moreau. Proximité et dualité dans un espace hilbertien. *Bulletin de la S. M. F,* 93: 273-299, 1965.

[25] L. D. Muu, V. H. Nguyen, and N. V. Quy. On Nash–Cournot oligopolistic market equilibrium models with concave cost functions. *Journal of Global Optimization ,* 41: 351-364, 2008.

[26] L. D. Muu, and N. V. Quy. Global optimization from concave minimization to concave mixed variational inequality. *Acta Mathematica Vietnamica,* 45: 449-462, 2020.

[27] M. A. Noor. Proximal methods for mixed variational inequalities. *Journal of Optimization Theory and Applications,* 115: 447-452, 2002.

[28] M. A. Noor and Z. Huang. Some proximal methods for solving mixed variational inequalities. *Journal of Applied Mathematics,* 2012:610852, 2012.

[29] M. A. Noor, K. I. Noor, S. Zainab, and E. Al-Said. Proximal algorithms for solving mixed bifunction variational inequalities. *International Journal of Physical Sciences,* 6(17): 4203-4207, 2011.

[30] E. A. Papa Quiroz, L. Mallma Ramirez, and P. R. Oliveira. An inexact proximal method for quasiconvex minimization. *European Journal of Operational Research,* 246(3): 721-729, 2015.

[31] L. Prigozhin. Variational inequalities in critical-state problems. *Physica D: Nonlinear Phenomena,* 197(3-4): 197-210, 2004.

[32] T. D. Quoc, L. D. Muu, and N. V. Hien, Extragradient algorithms extended to equilibrium problems. *Optimization,* 57: 749-766, 2008.

[33] M. Solodov and B. F. Svaiter. A new projection method for variational inequality problems. *SIAM Journal on Control and Optimization,* 37(3): 765-776, 1999.

[34] B. S. Thakur and S. Varghese. Approximate solvability of general strongly mixed variational inequalities. *Tbilisi Mathematical Journal,* 6: 13-20, 2013.

[35] D. V. Thong and P. T. Vuong. Modified Tseng's extragradient methods for solving pseudo-monotone variational inequalities. *Optimization,* 68: 2203–2222, 2009.

[36] D. V. Thong and D. Van Hieu. Modified Tseng's extragradient algorithms for variational inequality problems. *Journal of Fixed Point Theory and Applications,* 20:152, 2018.

[37] Z. Wang, Z. Cheng, Y. Xiao, and C. Zhang. A new projection-type method for solving multi-valued mixed variational inequalities without monotonicity. *Applicable Analysis,* 99(9):1453-1466, 2020.

[38] F.-Q. Xia, and N.-J. Huang. An inexact hybrid projection-proximal point algorithm for solving generalized mixed variational inequalities. *Computers & Mathematics with Applications,* 62(12): 4596-4604, 2011.

[39] F.-Q. Xia and Y.-Z. Zou. A projective splitting algorithm for solving generalized mixed variational inequalities. *Journal of Inequalities and Applications,* 2011:27, 2011.