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VISCOSITY APPROXIMATION METHOD FOR ATTRACTIVE POINTS OF WIDELY MORE GENERALIZED HYBRID MAPPINGS

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Abstract. In this paper, attractive point problem involving widely more generalized hybrid mappings is studied. Using viscosity approximation method, we establish strong convergence theorem for attractive points of finite families of widely more generalized hybrid mappings which is also a solutions of some variational inequality problems in a real Hilbert space. Our results improve and extend some recent results in the literature.

Keywords. Viscosity approximation method, Attractive Point, Widely more generalized hybrid mappings, Hilbert space.

1. Introduction

Let H be a real Hilbert space and C be a nonempty subset of H. Let T be a mapping from C into H. Recall that a point z is a fixed points of T if $Tz = z$. The set of fixed points of T is denoted by $F(T) = \{z \in C : Tz = z\}$.
A mapping $T : C \to H$ is called

(1) Nonexpansive if
$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{1.1}$$
(2) Nonspreading see [13] if
$$2\|Tx - Ty\|^2 \leq \|x - Ty\|^2 + \|Tx - y\|^2, \quad \forall x, y \in C. \tag{1.2}$$
(3) Hybrid mapping see [22] if
$$3\|Tx - Ty\|^2 \leq \|x - Ty\|^2 + \|Tx - y\|^2 + \|x - y\|^2, \quad \forall x, y \in C. \tag{1.3}$$
(4) Generalized hybrid see [11] if there exist a real numbers α, β. such that
$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2, \quad \forall x, y \in C. \tag{1.4}$$
(5) Super hybrid mapping see [14] if there exist a real numbers α, β and γ such that
$$\begin{aligned}\alpha\|Tx - Ty\|^2 \quad + \quad & (1 - \alpha + \gamma)\|x - Ty\|^2 \leq (\beta + (\beta - \alpha)\gamma)\|Tx - y\|^2 \\ & + (1 - \beta - (\beta - \alpha - 1)\gamma)\|x - y\|^2 + (\alpha - \beta)\gamma\|x - Tx\|^2 \\ & + \gamma\|y - Ty\|^2, \quad \forall x, y \in C. \end{aligned} \tag{1.5}$$

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(6) Nomally generalized hybrid mapping see [23] if there exist a real numbers α, β, γ and δ. such that

$$\alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \leq 0 \quad \forall x, y \in C. \tag{1.6}$$

(7) Further generlized hybrid mapping see [10] if there exist a real numbers $\alpha, \beta, \gamma, \delta$ and ϵ. such that

$$\alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 + \epsilon\|x - Tx\|^2 \leq 0, \quad \forall x, y \in C. \tag{1.7}$$

(8) Widely generalized hybrid mapping see [11] if there exist real numbers $\alpha, \beta, \gamma, \delta, \epsilon$ and ζ such that

$$\begin{aligned} \alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \\ + max\{\epsilon\|x - Tx\|^2, \zeta\|y - Ty\|^2\} \leq 0, \quad \forall x, y \in C. \end{aligned} \tag{1.8}$$

(9) Widely more generalized hybrid mapping see [8] if there exist real numbers $\alpha, \beta, \gamma, \delta, \epsilon, \zeta$ and η such that

$$\begin{aligned} \alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 + \epsilon\|x - Tx\|^2 \\ + \zeta\|y - Ty\|^2 + \eta\|(x - Tx) - (y - Ty)\|^2 \leq 0, \quad \forall x, y \in C. \end{aligned} \tag{1.9}$$

Remark 1.1. (i) An (α, β)- generalized hybrid mapping is nonexpansive if $\alpha = 1$ and $\beta = 0$, it is nonspreading if $\alpha = 2$ and $\beta = 1$, it is hybrid if $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$ for all $x, y \in C$. they proved fixed point theorems for such mappings, see [6, 19].

(ii) An (α, β, γ)- Super hybrid mapping can be reduced to generalized hybrid mapping if $\gamma = 0$. A generalized hybrid mapping with a fixed point is quasi-non expansive however, A super hybrid mapping is not quasi-non expansive generally even if it has a fixed point.

(iii) An $(\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta)$-widely more generalized hybrid mapping can be reduced to widely generalized hybrid mapping if $\eta = 0$. It is Further generlized hybrid mapping if $\zeta = \eta = 0$. It is also Nomally generalized hybrid mapping if $\epsilon = \zeta = \eta = 0$. It is well known that if C is closed and convex then T is quasi-nonexpansive mapping.

A point $z \in H$ is called an attractive point if it satisfies the following

$$\|Tx - z\| \leq \|x - z\| \quad \forall x \in C.$$

Takahashi and Takeuchi [20] introduced the concept attractive point in Hilbert space. They defined and denoted the set of attractive point as follows:

$$A(T) = \{z \in H : \|Tx - z\| \leq \|x - z\| \quad \forall x \in C\}. \tag{1.10}$$

Remark 1.2. From this definition, neither an attractive point is a fixed point nor conversely. However, for a relation between the two, the authors gave some properties of attractive points see [20, 23]. It is easy to see that if T is quasi-nonexpansive, then every fixed point of T is an Attractive point. Thus an attractive point is regarded as a generalization of a fixed point for quasi-nonexpansive mappings.

Basically this concept was introduced to get rid of the hypothesis of closedness and convexity as used in a well-celebrated Baillons nonlinear ergodic theorem in Hilbert space [2]. They also proved an existence theorem for attractive point without convexity in Hilbert space. In Theorem 1.3 below they used so-called generalized hybrid mappings whose class is larger than the class of nonexpansive mappings used in Baillons theorem in which the hypothesis does not require any closedness or convexity assuptions on C.

Theorem 1.3. *Takahashi and Takeuchi* [20]. *Let H be a Hilbert space and C be a nonempty subset of H. Let $T : C \to C$ be a generalized hybrid mapping. Then T has an attractive point if and only if $\exists\ z \in C$ such that $\{T^n z : n = 0, 1, \dots\}$ is bounded.*

Khan [10], introduced the concept of common attractive point for two further generalized hybrid mappings $S, T : C \to C$ with $CAP(S, T)$ as the set of common attractive points of S and T, i.e.

$$CAP(S,T) = \{z \in H : \max(\|Sx - z\|, \|Tx - z\|) \leq \|x - z\| \quad \forall x \in C\},$$

and the common attractive point for finite family of nonlinear mappings is denoted by $A(T_i)$, the set of $\{T_1, T_2, \cdots, T_N\}$ i.e

$$A(T_i) = \{z \in H : \max_{1 \leq i \leq N}(\|T_i x - z\|) \leq \|x - z\| \quad \forall x \in C\}.$$

Remark 1.4. The relationship between the set of common attractive point of finite family and the set of common fixed point of finite family of nonlinear mappings is similar to [23], as well as common attractive point and the common fixed point of nonlinear mappings

$$A(S,T) \cap C = F(T) \cap F(S).$$

Many authors have been working on attractive point in various directions after the publication of Theorem 1.3. Zheng [28] proved weak and strong convergence theorem of attractive points for generalized hybrid mappings in Hilbert space by using the following iterative scheme (1.11) known as Ishikawa iteration [7];

$$\begin{cases} x_1 \in C, \\ y_n = (1 - \alpha_n)x_n + \alpha_n T x_n, \\ x_{n+1} = (1 - \beta_n)x_n + \beta_n T y_n, \end{cases} \tag{1.11}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$ satisfying some condtions. Das and Debata [5], Takahashi and Tamura [24] generalized the Ishikawa iterative algorithm for two nonexpansive and quasi nonexpansive mappings respectively as follows:

$$\begin{cases} x_1 \in C, \\ y_n = (1 - \alpha_n)x_n + \alpha_n T x_n, \\ x_{n+1} = (1 - \beta_n)x_n + \beta_n S y_n, \end{cases} \tag{1.12}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequence in $(0, 1)$. Note that when $S = T$,(1.12) can be reduced to (1.11). it is worthy noting that the approximation of common fixed points of two mappings case has its own importance as there is a direct link with minimization problems; see for example [21] and the references cotained therein.

Thongpaen and Inthakon [25] used the iteration (1.12) to prove a weak convergence theorem for common attractive points of two widely more generalized hybrid mappings in Hilbert space and applied the main result to some common fixed point problems. Furthermore, see [3, 4] and references therein for more results of common attractive points theorems.

Khan [9] employed iterative scheme of Yao and Chen [27] to obtain weak and strong convergence results of the sequence defined by

$$\begin{cases} x_1 \in C, \\ x_{n+1} = \alpha_n x_n + \beta_n S x_n + \gamma_n T x_n, \end{cases} \tag{1.13}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequence in $(0, 1)$, S and T are quasi-asymtotically nonexpansive mappings in uniformly convex Banach space.

Recently, Ali and Ali [1] proved some weak and strong convergence theorems for common fixed points of two generalized nonexpansive mappings using the iteration (1.13) in uniformly convex Banach space.

Very recently, Panadda et al. [18], proved weak and strong convergence theorems for common attractive points of two widely more generalized hybrid mappings without assuming the closedness of the domain. Using iterative (1.13) in Hilbert Space. They obtained strong convergence by imposing compactness assumption on the mappings and using the so called condition A, that is, there exists nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0, f(r) > 0,\ \forall r \in (0, \infty)$ such that either

$$f(d(x, A(S, T))) \leq \|x - Sx\| \quad \text{or} \quad f(d(x, A(S, T))) \leq \|x - Tx\|$$

for all $x \in C$, where $d(x, A(S, T)) = \inf\{\|x - y\| : y \in A(S, T)\}$. Motivated and inspired by the work of Panadda et al. [18] and afore mentioned results, our goal in this paper is to introduce a viscosity approximation method for finite family of widely more generalized hybrid mappings and prove strong convergence theorem to common attractive points of the said mappings which also solve some variational inequality problems in real Hilbert space.

2. Preliminaries

The following notions and results are essentially used in our subsequent discussions.

Lemma 2.1. *see [17]. Let H be a real Hilbert space, let C be nonempty subset of H. And let T be an $(\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into itself which satisfied either of the following conditions:*
(1) $\alpha + \beta + \gamma + \delta \geq 0, \alpha + \gamma > 0, \epsilon + \eta \geq 0, and \zeta + \eta \geq 0;$
(2) $\alpha + \beta + \gamma + \delta \geq 0, \alpha + \beta > 0, \zeta + \eta \geq o, and \epsilon + \eta \geq 0,$

Then, T has an attractive point if and only if there exists $z \in C$ such that $\{T^n z : n = 0, 1, \ldots\}$ is bounded

Remark 2.2. Observed that in Lemma 2.1 the closedness or convexity property is dispensed with.

Lemma 2.3. [16] *Let H be a Hilbert space, and C be a nonempty subset of H. Let $T : C \to H$ be an $(\alpha, \beta, \gamma, \epsilon, \delta, \zeta, \eta)$-widely more generalized hybrid mapping. Suppose that it satisfied either of the following conditions.*

1. $\alpha + \beta + \gamma + \delta \geq 0, \alpha + \gamma > 0$, *and* $\epsilon + \eta \geq 0$
2. $\alpha + \beta + \gamma + \delta \geq 0, \alpha + \beta > 0$, *and* $\zeta + \eta \geq 0$
   if $x_n \rightharpoonup z$ *and* $x_n - Tx_n \to 0$ *then* $z \in A(T)$.

Lemma 2.4. [15] *Let $\{x_n\}$ and $\{w_n\}$ be bounded sequences in a Banach space E and let β_n be a sequence in [0,1] with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose that*

$$x_{n+1} = (1 - \beta_n)w_n + \beta_n x_n \quad \forall n \geq 0$$

and

$$\limsup_{n\to\infty}(\|w_n - w_{n-1}\| - \|x_{n-1} - x_n\|) \leq 0.$$

Then

$$\lim_{n\to\infty} \|w_n - x_n\| = 0.$$

Lemma 2.5. *(see [26]) Suppose that $\{a_n\}$ is a sequence of nonegative real numbers satisfying*

$$a_{n+1} \leq (1-b_n)a_n + b_n\sigma_n, \quad \forall n \geq 0.$$

where $\{b_n\}$ is a sequence in $(0,1)$ and $\{\sigma_n\} \subset \mathbb{R}$ satisfying the following conditions.

i. $\lim_{n\to\infty} b_n = 0$ *and* $\sum_{n=0}^{\infty} b_n = +\infty$,
ii. *either* $\limsup_{n\to\infty} \sigma_n \leq 0$ *or* $\sum_{n=0}^{\infty} |b_n\sigma_n| < +\infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.6. [12] *Let H be a real Hilbert space, let C be a nonempty closed convex subset of H. Suppose $T : C \to H$ is an $(\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta)$-Widely more generalized hybrid mapping such that $F(T) \neq \emptyset$ and satisfying either of the conditions (1) or (2) :*

(1) $\alpha+\beta+\gamma+\delta \geq 0, \alpha+\gamma > 0,$ *and* $\epsilon+\eta \geq 0,$
(2) $\alpha+\beta+\gamma+\delta \geq 0, \alpha+\beta > 0,$ *and* $\zeta+\eta \geq 0.$

Then T is quasi-nonexpansive.

3. Main Results

In this section we introduce a viscosity type iterative scheme for attractive point of finite family of widely more generalized hybrid mapping in Hilbert space as follows.
Let H be a real Hilbert space and C be a nonempty subset of H. Let $T_i : C \to H$. for each $i \in \{1, 2, \cdots, N\}$ are finite family of $(\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta)$-Widely more generalized hybrid mappings with $\bigcap_{i=1}^{N} A(T_i) \neq \emptyset$. Let $\{x_n\}$ be iteratively defined by

$$\begin{cases} x_1 \in C, \\ y_n = \theta_{n,0}x_n + \sum_{i=1}^{N} \theta_{n,i}T_i x_n, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n, \end{cases} \tag{3.1}$$

where $f : C \to C$ is a contraction mapping with constant $\alpha \in [0,1)$ and $\{\theta_{n,0}\}, \{\theta_{n,i}\}, \{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequence in $(0,1)$ such that $\theta_{n,0} + \sum_{i=1}^{N} \theta_{n,i} = 1$ and the following conditions are satisfied;

(i) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,
(ii) $\lim_{n\to\infty} |\theta_{n,0} - \theta_{n-1,0}| = 0$, $\lim_{n\to\infty} |\theta_{n,i} - \theta_{n-1,i}| = 0$, $\lim_{n\to\infty} \theta_{n,i} = 0$,
(iii) $\alpha_n + \beta_n + \gamma_n = 1$, $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} \gamma_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = +\infty$,
(iv) $0 < a \leq \beta_n \leq b < 1$.

Lemma 3.1. *Let H be a Hilbert space and C be a nonempty convex subset of H. Suppose $T_i : C \to H$ for each $i \in \{1, 2, \cdots, N\}$ is finite family of $(\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta)$-Widely more generalized hybrid mappings with $\bigcap_{i=1}^{N} A(T_i) \neq \emptyset$. Let $\{x_n\}$ be defined by (3.1).Then $\{x_n\}, \{y_n\}, \{f(x_n)\}$ and $\{T_i x_n\}$ are Bounded.*

Proof. Let $z \in \bigcap_{i=1}^{N} A(T_i)$, then from the scheme(3.1) and Lemma 2.6, we have

$$\begin{aligned} \|y_n - z\| &= \|\theta_{n,0}x_n + \sum_{i=1}^{N} \theta_{n,i}T_i x_n - z\| \\ &= \|\theta_{n,0}x_n + \sum_{i=1}^{N} \theta_{n,i}T_i x_n - (\theta_{n,0}z + \sum_{i=1}^{N} \theta_{n,i}z)\| \\ &= \|\theta_{n,0}x_n + \sum_{i=1}^{N} \theta_{n,i}T_i x_n - \theta_{n,0}z - \sum_{i=1}^{N} \theta_{n,i}z\| \\ &= \|\theta_{n,0}(x_n - z) + \sum_{i=1}^{N} \theta_{n,i}(T_i x_n - z)\| \end{aligned}$$

$$\begin{aligned}&\leq \theta_{n,0}\|x_n - z\| + \sum_{i=1}^{N}\theta_{n,i}\|x_n - z\|\\ &= \|x_n - z\|.\end{aligned} \tag{3.2}$$

Also from the scheme (3.1) we have,

$$\begin{aligned}\|x_{n+1} - z\| &= \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n - z\|\\ &= \|\alpha_n(f(x_n) - z) + \beta_n(x_n - z) + \gamma_n(y_n - z)\|\\ &\leq \alpha_n\|f(x_n) - z\| + \beta_n\|x_n - z\| + \gamma_n\|y_n - z\|\\ &= \alpha_n\|f(x_n) - f(z) + f(z) - z\| + \beta_n\|x_n - z\| + \gamma_n\|y_n - z\|\\ &\leq \alpha_n\|f(x_n) - f(z)\| + \alpha_n\|f(z) - z\| + \beta_n\|x_n - z\| + \gamma_n\|y_n - z\|.\end{aligned}$$

From (3.2), and contractive property of f we have,

$$\begin{aligned}\|x_{n+1} - z\| &\leq \alpha_n\alpha\|x_n - z\| + \alpha_n\|f(z) - z\| + \beta_n\|x_n - z\| + \gamma_n\|x_n - z\|\\ &= \alpha_n\alpha\|x_n - z\| + \alpha_n\|f(z) - z\| + \beta_n\|x_n - z\| + (1 - \alpha_n - \beta_n)\|x_n - z\|\\ &= \alpha_n\alpha\|x_n - z\| + \alpha_n\|f(z) - z\| + \beta_n\|x_n - z\|\\ &\quad + (1 - \alpha_n)\|x_n - z\| - \beta_n\|x_n - z\|\\ &= \alpha_n\alpha\|x_n - z\| + (1 - \alpha_n)\|x_n - z\| + \alpha_n\|f(z) - z\|\\ &= (1 - \alpha_n(1 - \alpha))\|x_n - z\| + \alpha_n(1 - \alpha)\frac{\|f(z) - z\|}{1 - \alpha}.\end{aligned}$$

Therefore,

$$\begin{aligned}\|x_{n+1} - z\| &\leq (1 - \alpha_n(1 - \alpha))\max\left\{\|x_n - z\|, \tfrac{\|f(z)-z\|}{1-\alpha}\right\} + \alpha_n(1 - \alpha)\max\left\{\|x_n - z\|, \tfrac{\|f(z)-z\|}{1-\alpha}\right\}\\ &\leq \max\left\{\|x_n - z\|, \frac{\|f(z) - z\|}{1 - \alpha}\right\}.\end{aligned}$$

Hence by induction, we have.

$$\|x_n - z\| \leq \max\left\{\|x_0 - z\|, \frac{\|f(z) - z\|}{(1 - \alpha)}\right\}. \tag{3.3}$$

Therefore $\{x_n\}$ is bounded. It follows that $\{y_n\}$, $\{T_i x_n\}$ and $\{f(x_n)\}$ are all bounded. □

Lemma 3.2. *Let $\{x_n\}$ be defined as in Lemma 3.1 and $\{\theta_{n,0}\}, \{\theta_{n,i}\}, \{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0, 1)$ satisfying conditions $(i), (ii)$ and (iii). Then $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$.*

Proof. From the scheme (3.1), we have

$$\begin{aligned}\|y_n - y_{n-1}\| &= \|\theta_{n,0}x_n + \sum_{i=1}^{N}\theta_{n,i}T_i x_n - (\theta_{n-1,0}x_{n-1} + \sum_{i=i}^{N}\theta_{n-1,i}T_i x_{n-1})\|\\ &= \|\theta_{n,0}x_n + \sum_{i=1}^{N}\theta_{n,i}T_i x_n - \theta_{n-1,0}x_{n-1} - \sum_{i=i}^{N}\theta_{n-1,i}T_i x_{n-1}\|\\ &= \|\theta_{n,0}x_n - \theta_{n-1,0}x_{n-1} + \sum_{i=1}^{N}\theta_{n,i}T_i x_n - \sum_{i=i}^{N}\theta_{n-1,i}T_i x_{n-1}\|\\ &\leq \|\theta_{n,0}x_n - \theta_{n-1,0}x_{n-1}\| + \|\sum_{i=1}^{N}\theta_{n,i}T_i x_n - \sum_{i=i}^{N}\theta_{n-1,i}T_i x_{n-1}\|\\ &= \|\theta_{n,0}x_n - \theta_{n,0}x_{n-1} + \theta_{n,0}x_{n-1} - \theta_{n-1,0}x_{n-1}\|\end{aligned}$$

$$
\begin{aligned}
&+ \|\sum_{i=1}^{N}\theta_{n,i}T_ix_n - \sum_{i=1}^{N}\theta_{n,i}x_{n-1} + \sum_{i=1}^{N}\theta_{n,i}x_{n-1} - \sum_{i=i}^{N}\theta_{n-1,i}T_ix_{n-1}\| \\
&\leq \|\theta_{n,0}x_n - \theta_{n,0}x_{n-1}\| + \|\theta_{n,0}x_{n-1} - \theta_{n-1,0}x_{n-1}\| + \|\sum_{i=1}^{N}\theta_{n,i}T_ix_n - \sum_{i=1}^{N}\theta_{n,i}x_{n-1}\| \\
&+ \|\sum_{i=1}^{N}\theta_{n,i}x_{n-1} - \sum_{i=i}^{N}\theta_{n-1,i}T_ix_{n-1}\| \\
&= \theta_{n,0}\|x_n - x_{n-1}\| + |\theta_{n,0} - \theta_{n-1,0}|\|x_{n-1}\| + \|\sum_{i=1}^{N}\theta_{n,i}(T_ix_n - x_{n-1})\| \\
&+ \|\sum_{i=1}^{N}\theta_{n,i}x_{n-1} - \sum_{i=i}^{N}\theta_{n-1,i}T_ix_{n-1}\| \\
&\leq \theta_{n,0}\|x_n - x_{n-1}\| + |\theta_{n,0} - \theta_{n-1,0}|\|x_{n-1}\| + \sum_{i=1}^{N}\theta_{n,i}\|T_ix_n - x_{n-1}\| \\
&+ \|\sum_{i=1}^{N}\theta_{n,i}x_{n-1} - \sum_{i=i}^{N}\theta_{n-1,i}T_ix_{n-1}\| \\
&= \theta_{n,0}\|x_n - x_{n-1}\| + |\theta_{n,0} - \theta_{n-1,0}|\|x_{n-1}\| + \sum_{i=1}^{N}\theta_{n,i}\|T_ix_n - x_n + x_n - x_{n-1}\| \\
&+ \|\sum_{i=1}^{N}\theta_{n,i}x_{n-1} - \sum_{i=1}^{N}\theta_{n,i}T_ix_{n-1} + \sum_{i=1}^{N}\theta_{n,i}T_ix_{n-1} - \sum_{i=i}^{N}\theta_{n-1,i}T_ix_{n-1}\| \\
&\leq \theta_{n,0}\|x_n - x_{n-1}\| + |\theta_{n,0} - \alpha_{n-1,0}|\|x_{n-1}\| + \sum_{i=1}^{N}\alpha_{n,i}\|T_ix_n - x_n\| + \sum_{i=1}^{N}\theta_{n,i}\|x_n - x_{n-1}\| \\
&+ \|\sum_{i=1}^{N}\theta_{n,i}x_{n-1} - \sum_{i=1}^{N}\theta_{n,i}T_ix_{n-1}\| + \|\sum_{i=1}^{N}\theta_{n,i}T_ix_{n-1} - \sum_{i=i}^{N}\theta_{n-1,i}T_ix_{n-1}\| \\
&\leq \|x_n - x_{n-1}\| + |\theta_{n,0} - \theta_{n-1,0}|\|x_{n-1}\| + \sum_{i=1}^{N}\theta_{n,i}\|T_ix_n - x_n\| \\
&+ \sum_{i=1}^{N}\theta_{n,i}\|T_ix_{n-1} - x_{n-1}\| + \sum_{i=1}^{N}|\theta_{n,i} - \theta_{n-1,i}|\|T_ix_{n-1}\|.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\|y_n - y_{n-1}\| \leq \|x_n - x_{n-1}\| + |\theta_{n,0} - \theta_{n-1,0}|\|x_{n-1}\| + \sum_{i=1}^{N}\theta_{n,i}\|T_ix_n - x_n\| \\
+ \sum_{i=1}^{N}\theta_{n,i}\|T_ix_{n-1} - x_{n-1}\| + \sum_{i=1}^{N}|\theta_{n,i} - \theta_{n-1,i}|\|T_ix_{n-1}\|. \quad (3.4)
\end{aligned}
$$

Define w_n by

$$
w_n = \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n}.
$$

Then,

$$\begin{aligned}
w_n - w_{n-1} &= \frac{x_{n+1} - \beta_n x_n}{1-\beta_n} - \left[\frac{x_n - \beta_{n-1}x_{n-1}}{1-\beta_{n-1}}\right] \\
&= \frac{\alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n - \beta_n x_n}{1-\beta_n} \\
&\quad - \left[\frac{\alpha_{n-1} f(x_{n-1}) + \beta_{n-1}x_{n-1} + \gamma_{n-1}y_{n-1} - \beta_{n-1}x_{n-1}}{1-\beta_{n-1}}\right] \\
&= \frac{\alpha_n f(x_n) + \gamma_n y_n}{1-\beta_n} - \left[\frac{\alpha_{n-1} f(x_{n-1}) + \gamma_{n-1}y_{n-1}}{1-\beta_{n-1}}\right] \\
&= \frac{\alpha_n f(x_n) + (1-\alpha_n-\beta_n) y_n}{1-\beta_n} - \left[\frac{\alpha_{n-1} f(x_{n-1}) + (1-\alpha_{n-1}-\beta_{n-1})y_{n-1}}{1-\beta_{n-1}}\right] \\
&= \frac{\alpha_n f(x_n) - \alpha_n y_n + (1-\beta_n) y_n}{1-\beta_n} - \left[\frac{\alpha_{n-1} f(x_{n-1}) - \alpha_{n-1}y_{n-1} + (1-\beta_{n-1})y_{n-1}}{1-\beta_{n-1}}\right] \\
&= \frac{\alpha_n (f(x_n) - y_n) + (1-\beta_n) y_n}{1-\beta_n} - \left[\frac{\alpha_{n-1} (f(x_{n-1}) - y_{n-1}) + (1-\beta_{n-1})y_{n-1}}{1-\beta_{n-1}}\right] \\
&= \frac{\alpha_n}{1-\beta_n}(f(x_n) - y_n) + y_n - \left[\frac{\alpha_{n-1}}{1-\beta_{n-1}}(f(x_{n-1}) - y_{n-1}) + y_{n-1}\right] \\
&= \frac{\alpha_n}{1-\beta_n}(f(x_n) - y_n) + y_n - \frac{\alpha_{n-1}}{1-\beta_{n-1}}(f(x_{n-1}) - y_{n-1}) - y_{n-1} \\
&= \frac{\alpha_n}{1-\beta_n}(f(x_n) - y_n) - \frac{\alpha_{n-1}}{1-\beta_{n-1}}(f(x_{n-1}) - y_{n-1}) + y_n - y_{n-1}
\end{aligned}$$

Therefore

$$\begin{aligned}
\|w_n - w_{n-1}\| &= \|\frac{\alpha_n}{1-\beta_n}(f(x_n) - y_n) - \frac{\alpha_{n-1}}{1-\beta_{n-1}}(f(x_{n-1}) - y_{n-1}) + y_n - y_{n-1}\| \\
&\leq \frac{\alpha_n}{1-\beta_n}\|f(x_n) - y_n\| + \frac{\alpha_{n-1}}{1-\beta_{n-1}}\|f(x_{n-1}) - y_{n-1}\| + \|y_n - y_{n-1}\|.
\end{aligned}$$

Therefore from (3.4), we have.

$$\begin{aligned}
\|w_n - w_{n-1}\| \leq{}& \frac{\alpha_n}{1-\beta_n}\|f(x_n) - y_n\| + \frac{\alpha_{n-1}}{1-\beta_{n-1}}\|f(x_{n-1}) - y_{n-1}\| \\
&+ \|x_n - x_{n-1}\| + |\theta_{n,0} - \theta_{n-1,0}|\|x_{n-1}\| + \sum_{i=1}^{N}\theta_{n,i}\|T_i x_n - x_n\| \\
&+ \sum_{i=1}^{N}\theta_{n,i}\|T_i x_{n-1} - x_{n-1}\| + \sum_{i=1}^{N}|\theta_{n,i} - \theta_{n-1,i}|\|T_i x_{n-1}\|.
\end{aligned}$$

Thus,

$$\begin{aligned}
\|w_n - w_{n-1}\| - \|x_n - x_{n-1}\| \leq{}& \frac{\alpha_n}{1-\beta_n}\|f(x_n) - y_n\| \\
&+|\theta_{n,0} - \theta_{n-1,0}|\|x_{n-1} + \tfrac{\alpha_{n-1}}{1-\beta_{n-1}}\|f(x_{n-1}) - y_{n-1}\|\| \\
&+ \textstyle\sum_{i=1}^{N}\theta_{n,i}\|T_i x_n - x_n\| + \sum_{i=1}^{N}\theta_{n,i}\|T_i x_{n-1} - x_{n-1}\| \\
&+ \textstyle\sum_{i=1}^{N}|\theta_{n,i} - \theta_{n-1,i}|\|T_i x_{n-1}\|.
\end{aligned}$$

By using conditions (i),(ii) and (iii), we have.

$$\limsup_{n\to\infty}(\|w_n - w_{n-1}\| - \|x_n - x_{n-1}\|) \leq 0.$$

By Lemma 2.4, We find that $\lim_{n\to\infty}\|w_n - x_n\| = 0$, since $x_{n+1} = \beta_n x_n + (1-\beta_n)w_n$ and $x_{n+1} - x_n = (1-\beta_n)(w_n - x_n)$ then we have.

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.5}$$

□

Lemma 3.3. *Let $\{x_n\}$ be as in Lemma 3.1 and $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$ satisfying conditions (iii) and (iv). Then $\lim_{n\to\infty}\|x_n - T_i x_n\| = 0$.*

Proof.

$$\begin{aligned}
&\|x_n - T_i x_n\| = \|x_n - x_{n+1} + x_{n+1} - T_i x_n\| \\
&\leq \|x_n - x_{n+1}\| + \|x_{n+1} - T_i x_n\| \\
&= \|x_n - x_{n+1}\| + \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n - T_i x_n\| \\
&= \|x_n - x_{n+1}\| + \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n - (\alpha_n T_i x_n + \beta_n T_i x_n + \gamma_n T_i x_n)\| \\
&= \|x_n - x_{n+1}\| + \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n - \alpha_n T_i x_n - \beta_n T_i x_n - \gamma_n T_i x_n\| \\
&= \|x_n - x_{n+1}\| + \|\alpha_n f(x_n) - \alpha_n T_i x_n + \beta_n x_n - \beta_n T_i x_n + \gamma_n y_n - \gamma_n T_i x_n\| \\
&= \|x_n - x_{n+1}\| + \|\alpha_n (f(x_n) - T_i x_n) + \beta_n (x_n - T_i x_n) + \gamma_n (y_n - T_i x_n)\| \\
&\leq \|x_n - x_{n+1}\| + \alpha_n\|f(x_n) - T_i x_n\| + \beta_n\|x_n - T_i x_n\| + \gamma_n\|y_n - T_i x_n\|.
\end{aligned}$$

Hence,

$$(1-\beta_n)\|x_n - T_i x_n\| \leq \|x_n - x_{n+1}\| + \alpha_n\|f(x_n) - T_i x_n\| + \gamma_n\|y_n - T_i x_n\|.$$

By condition (iii),(iv) and (3.5), we have.

$$\lim_{n\to\infty}\|x_n - T_i x_n\| = 0. \tag{3.6}$$

□

Theorem 3.4. *Let $\{x_n\}$ be as in Lemmas 3.1, 3.2 and 3.3 and $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$ satisfying conditions (iii). Then $\{x_n\}$ converge strongly to a point $z \in \bigcap_{i=1}^N A(T_i)$, which is the unique solution of the following variational inequality.*

$$\langle z - f(z)\ ,\ z_1 - z\rangle \geq 0 \qquad \forall z_1 \in \bigcap_{i=1}^N A(T_i). \tag{3.7}$$

Proof. We start by showing that the solution of the variational inequality in (1.1), is unique. Assume $z, z_1 \in \bigcap_{i=1}^N A(T_i)$ are the solution of the variation inequality (3.7). Then

$$\langle z - f(z)\ ,\ z_1 - z\rangle \geq 0 \quad \textit{and} \quad \langle z_1 - f(z_1)\ ,\ z - z_1\rangle \geq 0.$$

Adding these two relation, we get

$$\langle (z - f(z)) - (z_1 - f(z_1))\ ,\ z - z_1\rangle \leq 0.$$

Therefore,

$$\begin{aligned}
0 &\geq \langle (z - f(z)) - (z_1 - f(z_1))\ ,\ z - z_1 \rangle \\
&= \langle z - f(z) - z_1 + f(z_1)\ ,\ z - z_1 \rangle \\
&= \langle z - z_1 + f(z_1) - f(z)\ ,\ z - z_1 \rangle \\
&= \langle z - z_1\ ,\ z - z_1 \rangle + \langle f(z_1) - f(z)\ ,\ z - z_1 \rangle \\
&\geq \|z - z_1\|^2 + \|f(z_1) - f(z)\| \|z - z_1\| \\
&\geq \|z - z_1\|^2 + \alpha \|z_1 - z\| \|z - z_1\| \\
&= \|z - z_1\|^2 + \alpha \|z - z_1\| \|z - z_1\| \\
&= \|z - z_1\|^2 + \alpha \|z - z_1\|^2 \\
&= (1 + \alpha) \|z - z_1\|^2 \\
&= \|z - z_1\|^2.
\end{aligned}$$

Hence, we obtain $z = z_1$.

Since H is a Hilbert space and $\{x_n\}$ is bounded, then, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup z_1$. Since T is widely more generalized hybrid mapping, It follows from Lemma 2.3 and (3.6), that $z_1 \in \bigcap_{i=1}^{N} A(T_i)$. From (3.7) the following holds:

$$\langle f(z) - z, z_1 - z \rangle \leq 0.$$

We now show

$$\limsup_{n\to\infty} \langle f(z) - z\ ,\ x_n - z \rangle \leq 0.$$

Then by (3.7), we have

$$\begin{aligned}
&\limsup_{n\to\infty} \langle f(z) - z\ ,\ x_n - z \rangle = \limsup_{n\to\infty} \langle f(z) - z\ ,\ x_n - x_{n+1} + x_{n+1} - z \rangle \\
&\leq \limsup_{n\to\infty} \langle f(z) - z\ ,\ x_n - x_{n+1} \rangle + \limsup_{n\to\infty} \langle f(z) - z\ ,\ x_{n+1} - z \rangle \\
&= \lim_{k\to\infty} \langle f(z) - z\ ,\ x_{n_k} - x_{n_k+1} \rangle + \lim_{k\to\infty} \langle f(z) - z\ ,\ x_{n_k+1} - z \rangle \\
&= \langle f(z) - z\ ,\ z_1 - z_1 \rangle + \langle f(z) - z\ ,\ z_1 - z \rangle \\
&= \langle f(z) - z\ ,\ z_1 - z \rangle \leq 0.
\end{aligned}$$

Therefore,

$$\limsup_{n\to\infty} \langle f(z) - z\ ,\ x_n - z \rangle \leq 0. \tag{3.8}$$

Now from the scheme (3.1), we have

$$\begin{aligned}
&\|x_{n+1} - z\|^2 = \langle x_{n+1} - z\ ,\ x_{n+1} - z \rangle \\
&= \langle \alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n - z\ ,\ x_{n+1} - z \rangle \\
&= \langle \alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n - \alpha_n z - \beta_n z - \gamma_n z\ ,\ x_{n+1} - z \rangle \\
&= \langle \alpha_n (f(x_n) - z) + \beta_n (x_n - z) + \gamma_n (y_n - z)\ ,\ x_{n+1} - z \rangle \\
&= \alpha_n \langle f(x_n) - z\ ,\ x_{n+1} - z \rangle + \beta_n \langle x_n - z\ ,\ x_{n+1} - z \rangle + \gamma_n \langle y_n - z\ ,\ x_{n+1} - z \rangle \\
&= \alpha_n \langle f(x_n) - f(z) + f(z) - z\ ,\ x_{n+1} - z \rangle + \beta_n \langle x_n - z\ ,\ x_{n+1} - z \rangle \\
&\ + \gamma_n \langle y_n - z\ ,\ x_{n+1} - z \rangle \\
&= \alpha_n \langle f(x_n) - f(z)\ ,\ x_{n+1} - z \rangle + \alpha_n \langle f(z) - z\ ,\ x_{n+1} - z \rangle \\
&\ + \gamma_n \langle y_n - z\ ,\ x_{n+1} - z \rangle + \beta_n \langle x_n - z\ ,\ x_{n+1} - z \rangle
\end{aligned}$$

$$\begin{aligned}
&\leq \alpha_n|\langle f(x_n) - f(z)\ ,\ x_{n+1} - z\rangle| + \alpha_n\langle f(z) - z\ ,\ x_{n+1} - z\rangle \\
&+ \gamma_n|\langle y_n - z\ ,\ x_{n+1} - z\rangle| + \beta_n|\langle x_n - z\ ,\ x_{n+1} - z\rangle| \\
&\leq \alpha_n\|f(x_n) - f(z)\|\|x_{n+1} - z\| + \alpha_n\langle f(z) - z\ ,\ x_{n+1} - z\rangle \\
&+ \gamma_n\|y_n - z\|\|x_{n+1} - z\| + \beta_n\|x_n - z\|\|x_{n+1} - z\| \\
&\leq \alpha_n\alpha\|x_n - z\|\|x_{n+1} - z\| + \alpha_n\langle f(z) - z\ ,\ x_{n+1} - z\rangle \\
&+ \beta_n\|x_n - z\|\|x_{n+1} - z\| + \gamma_n\|x_n - z\|\|x_{n+1} - z\| \\
&= (\alpha_n\alpha + \beta_n + \gamma_n)\|x_n - z\|\|x_{n+1} - z\| + \alpha_n\langle f(z) - z\ ,\ x_{n+1} - z\rangle \\
&= (\alpha_n\alpha + \beta_n + 1 - \alpha_n - \beta_n)\|x_n - z\|\|x_{n+1} - z\| + \alpha_n\langle f(z) - z\ ,\ x_{n+1} - z\rangle \\
&= (1 - \alpha_n(1 - \alpha))\|x_n - z\|\|x_{n+1} - z\| + \alpha_n\langle f(z) - z\ ,\ x_{n+1} - z\rangle \\
&\leq (1 - \alpha_n(1 - \alpha))\frac{1}{2}(\|x_n - z\|^2 + \|x_{n+1} - z\|^2) + \alpha_n\langle f(z) - z\ ,\ x_{n+1} - z\rangle.
\end{aligned}$$

Therefore,

$$\begin{aligned}
2\|x_{n+1} - z\|^2 &\leq (1 - \alpha_n(1 - \alpha))\|x_n - z\|^2 + (1 - \alpha_n(1 - \alpha))\|x_{n+1} - z\|^2 \\
&+ 2\alpha_n\langle f(z) - z\ ,\ x_{n+1} - z\rangle,
\end{aligned}$$

that is,

$$\begin{aligned}
&(2 - (1 - \alpha_n(1 - \alpha))\|x_{n+1} - z\|^2 \leq (1 - \alpha_n(1 - \alpha))\|x_n - z\|^2 + 2\alpha_n\langle f(z) - z\ ,\ x_{n+1} - z\rangle \\
&\text{i.e., } \|x_{n+1} - z\|^2 \leq \frac{(1 - \alpha_n(1 - \alpha))}{2 - (1 - \alpha_n(1 - \alpha))}\|x_n - z\|^2 + \frac{2\alpha_n}{2 - (1 - \alpha_n(1 - \alpha))}\times \\
&\langle f(z) - z\ ,\ x_{n+1} - z\rangle \\
&= \left(1 - \frac{2\alpha_n(1 - \alpha)}{2 - (1 - \alpha_n(1 - \alpha))}\right)\|x_n - z\|^2 + \frac{2\alpha_n(1 - \alpha)}{2 - (1 - \alpha_n(1 - \alpha))}\times \\
&\frac{\langle f(z) - z\ ,\ x_{n+1} - z\rangle}{1 - \alpha}.
\end{aligned}$$

Let

$$b_n = \frac{2\alpha_n(1 - \alpha)}{2 - (1 - \alpha_n(1 - \alpha))} = \frac{2\alpha_n(1 - \alpha)}{1 + \alpha_n(1 - \alpha)} \in (0, 1),$$

and

$$\sigma_n = \frac{\langle f(z) - z\ ,\ x_{n+1} - z\rangle}{1 - \alpha}.$$

Then,

$$\sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} \frac{2\alpha_n(1 - \alpha)}{1 + \alpha_n(1 - \alpha)} > \sum_{n=0}^{\infty} \frac{2\alpha_n(1 - \alpha)}{2} = \sum_{n=0}^{\infty} \alpha_n(1 - \alpha).$$

By condition (iii), we have

$$\sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} \frac{2\alpha_n(1 - \alpha)}{1 + \alpha_n(1 - \alpha)} = +\infty,$$

and

$$\limsup_{n\to\infty} \sigma_n = \limsup_{n\to\infty} \frac{\langle f(z) - z\ ,\ x_{n+1} - z\rangle}{1 - \alpha}.$$

By (3.8), we have.

$$\limsup_{n\to\infty} \frac{\langle f(z) - z\ ,\ x_{n+1} - z\rangle}{1-\alpha} \leq 0.$$

Therefore by Lemma 2.5, $x_n \to z$. This complete the proof. □

If $N = 1$ then Theorem 3.4 reduces to the following corollary.

Corollary 3.5. *Let H be a real Hilbert space and C be a nonempty convex subset of H. Suppose $T : C \to H$ are $(\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta)$-Widely more generalized hybrid mapping with $A(T) \neq \emptyset$. Let $\{x_n\}$ be defined by*

$$\begin{cases} x_1 \in C, \\ y_n = \theta_n x_n + (1-\theta_n)Tx_n, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n, \end{cases}$$

where $f : C \to C$ is a contraction mapping with constant $\alpha \in [0,1)$,and $\{\theta_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequence in $(0,1)$ for every $n \in \mathbb{N}$. Then the sequence $\{x_n\}$ satisfied the conditions

(i) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,
(ii) $\lim_{n\to\infty} |\theta_n - \theta_{n-1}| = 0$, $\lim_{n\to\infty} \theta_n = 0$,
(iii) $\alpha_n + \beta_n + \gamma_n = 1$, $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} \gamma_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = +\infty$,
(iv) $\beta_n < a \quad i.e \quad \beta_n \in (0,a)$ *for some* $a \in (0,1)$.

Then $\{x_n\}$ converge strongly to the attractive point z of T which is the unique solution of variational inequality (3.7).

By Lemma 3.1, we have the following Corollary

Corollary 3.6. *Let H be a real Hilbert space and C be a nonempty closed and convex subset of H. Suppose $T_i : C \to H$ for each $i \in \{1, 2, \cdots, N\}$ are finite family of $(\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta)$-Widely more generalized hybrid mappings with $\bigcap_{i=1}^{N} Fix(T_i) \neq \emptyset$, α, $\{\theta_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in $(0,1)$ satisfying $(i), (ii), (iii)$ and (iv). Then the sequence $\{x_n\}$ generated by (3.1) converge strongly to the common fixed point z of T_i, which is the unique solution of variational inequality (3.7).*

Also from Remark 1.1(iii), Theorem 3.4 reduces to the following corollary.

Corollary 3.7. *Let H be a real Hilbert space and C be a nonempty convex subset of H. Suppose $T_i : C \to H$ for each $i \in \{1, 2, \cdots, N\}$ are finite family of $(\alpha, \beta, \gamma, \delta, \epsilon)$-Further generalized hybrid mappings with $\bigcap_{i=1}^{N} A(T_i) \neq \emptyset$, α, $\{\theta_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in $(0,1)$ satisfying $(i), (ii), (iii)$ and (iv). Then the sequence $\{x_n\}$ generated by (3.1) converge strongly to the common attractive point z of T_i, which is the unique solution of variational inequality(3.7).*

4. Conclusion

We have studied the convergence of attractive points of finite family of widely more generalized hybrid mappings using viscosity approximation method in the setting of real Hilbert spaces. Our theorem extends the results of Panadda et al. [18] from two to finite family of the said mappings. Strong convergence is established without the so called condition A and compactness assumption on the operators.

Statements and Declarations

The authors declare that they have no conflict of interest, and the manuscript has no associated data.

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