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APPROXIMATION OF FIXED POINTS OF $S-$GENERALIZED ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

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Abstract. In this paper, a new class of mapping that unifies various classes of mappings associated with the class of asymptotically nonexpansive mappings is introduced. In addition, an iterative technique for approximation of fixed points of this class of mappings is introduced and studied in the setting of uniformly convex real Banach space. Moreover, Demiclosedness principle for the class of mapping under study is proved; in addition, weak and strong convergence theorems are obtained. The theorems obtained augment, generalize, improve and unify several results that are recently announced. The method of proof used is of independent interest.

Keywords. $S-$generalized asymptotically nonexpansive mappings, Iterative algorithm, Fixed point theory, Weak and strong onvergence theorems, Uniformly convex real Banach spaces.

1. Introduction

A lot of work cum research had been carried out (in recent past) on several classes of mappings that are intimateely connected with the class of nonexpansive mappings and asymptotically nonexpansive mappings (see, for example, Goebel and Kirk [12], Bruck et al. [4], Sahu [20], Alber et al. [1], Rhoades and Temir [15], Chidume, Ofoedu and Zegeye [11], Mukhamedov and Saburov [14], Ofoedu and Madu [16] and the references therein).

Motivated by the reseacrh of the authors mentioned above, it is our aim in this paper to study an approximation method for approximate solution of nonlinear equations involving a new class of S-generalized asymptoitcally nonexpansive mappings in the setting of uniformly convex real Banach spaces. Demiclossedness principle for this class of mappings is obtained; weak and strong convergence theorems are established under some mild conditions on iterative parameters. The results obtained augment and unify several results in the literature.

This paper is organized as follows: in Section 2, preliminaries and a clear problem statement are provided; several definitions and explanation of concepts are presented. Demiclosedness Principle for the new class of mapping introduced is proved. Several Lemmas that aided the establishment of the main results obtained in this paper are presented. In Section 3, the main results of this research are presented, and this section is broken into four subsections for sequential flow of the results obtained. Sections 4 shall take care of conclusion; followed by Declarations of conflict of interest and acknowledgments.

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2. Preliminaries

In order to put things in the right perspective, we shall commence with the following definitions and explanation of terms and concepts that shall be encountered in the sequel:

Let E be a real normed linear space, let $T : D(T) \subseteq E \to R(T) \subseteq E$ be a mapping, a point $x \in D(T)$ is called a fixed point of T if and only if $Tx = x$. The set of all fixed points of a mapping T is denoted by $F(T)$. Thus, $F(T) = \{u \in D(T) : Tu = u\}$

A mapping $T : D(T) \subseteq E \to R(T) \subseteq E$ is called a **contraction** if and only if there exists a constant $k \in [0, 1)$ such that for all $x, y \in D(T)$,

$$\|Tx - Ty\| \leq k\|x - y\|.$$

A mapping $T : D(T) \subseteq E \to R(T) \subseteq E$ is called **nonexpansive** if and only if for all $x, y \in D(T)$,

$$\|Tx - Ty\| \leq \|x - y\|.$$

It is well known that every contraction is nonexpansive, but the converse is however not the case.

A mapping $T : D(T) \subseteq E \to R(T) \subseteq E$ is said to be **Lipschitz** if and only if there exists a constant $L > 0$ such that for all $x, y \in D(T)$,

$$\|Tx - Ty\| \leq L\|x - y\|.$$

It is easy to see that every nonexpansive mapping is Lipschitz with Lipschitz constant $L = 1$. Some authors usually refer to Lipschitz mappings as L-Lipschitzian mappings (see, for example, Goebel and Kirk [12], Chidume and Zegeye [11], Ofoedu [15].

A mapping $T : D(T) \subseteq E \to R(T) \subseteq E$ is said to be **uniformly L-Lipschitzian** if and only if there exists a constant $L > 0$ such that for all $x, y \in D(T), \forall\, n \geq 1$,

$$\|T^n x - T^n y\| \leq L\|x - y\|.$$

A mapping $T : D(T) \subseteq E \to R(T) \subseteq E$ is said to be **asymptotically nonexpansive** [12] if there exists a sequence $\{k_n\}_{n\geq 1} \subset [0, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that for all $x, y \in D(T), \forall\, n \geq 1$

$$\|T^n x - T^n y\| \leq k_n\|x - y\|.$$

Every asymptotically nonexpansive mapping is uniformly L- Lipschitzian thus, L- Lipschitzian and continuous. Every nonexpansive mapping is asymptotically nonexpansive.

The following example shows that the class of asymptotically nonexpansive mappings is larger than that of nonexpansive mappings:

Example 2.1. (Goebel and Kirk, [12]). Let C denote the unit ball of the space $l_2 := \{x = (x_1, x_2, x_3, \cdots) : x_i \in \mathbb{R}, \sum\limits_{n=1}^{\infty} |x_i|^2 < \infty\}$ endowed with the norm , $\|.\|_{l2}$ giving by $\|x\|_{l2} = (\sum\limits_{n=1}^{\infty} |x_i|^2)^{\frac{1}{2}}$ and let $T : C \to C$ be defined by $T(x_1, x_2, x_3, \cdots) = (0, x_1^2, a_2x_2, a_3x_3, \cdots)$ for all $(x_1, x_2, x_3, \cdots) \in C$, where $\{a_i\}_i \geq 1$ is a sequence of numbers in $(0, 1)$ and $\prod\limits_{i=2}^{\infty} a_i = \frac{1}{2}$. Then, T is a Lipschitz mapping, asymptotically nonexpansive but is not nonexpansive.

A mapping $T : D(T) \subseteq E \to R(T) \subseteq E$ is called **nearly Lipschitzian** [20] if $\forall n \in \mathbb{N}$, there exist $a_n, k_n \in [0, \infty)$ with $\lim_{n\to\infty} a_n = 0$ such that $\forall x, y \in D(T), \forall n \in \mathbb{N}$,

$$\|T^n x - T^n y\| \leq k_n\big(\|x - y\| + a_n\big). \tag{2.1}$$

Now, define

$$\eta(T^n) := \sup\big(\frac{\|T^nx - T^ny\|}{\|x-y\| + a_n} : x, y \in K, x \neq y\big) \tag{2.2}$$

Observe that for any sequence $\{k_n\}_{n\geq 1}$ satisfying (2.1), $\eta(T^n) \leq k_n$ for all $n \in \mathbb{N}$ and that

$$\|T^nx - T^ny\| \leq \eta(T^n)\big(\|x-y\| + a_n\big)\, \forall x, y \in K, n \in \mathbb{N} \tag{2.3}$$

$\eta(T^n)$ is called the **nearly Lipschitzian** constant. A nearly Lipschitzian mapping T is called

(i) nearly contraction if $\eta(T^n) < 1$ for all $n \in \mathbb{N}$.
(ii) nearly nonexpansive if $\eta(T^n) = 1$ for all $n \in \mathbb{N}$.
(iii) nearly asymptotically nonexpansive if $\eta(T^n) \geq 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \eta(T^n) = 1$.
(iv) nearly uniform L-Lipschitzian if there exists $L \geq 0$ such that $\forall\, n \in \mathbb{N}$, $\eta(T^n) \leq L$.
(v) nearly uniform K-contraction if there exists $k \in [0, \infty)$ such that $\forall\, n \in \mathbb{N}$, $\eta(T^n) \leq k$.

Remark 2.1. If $D(T)$ is a bounded domain of an asymptotically nonexpansive mapping T, then T is nearly nonexpansive. In fact, for all $x, y \in D(T)$ and $n \in \mathbb{N}$, we have that

$$\begin{aligned}\|T^nx - T^ny\| &\leq (1+\mu_n)\|x-y\| \\ &= \|x-y\| + u_n\|x-y\| \\ &\leq \|x-y\| + diam\Big(D(T)\Big)\mu_n\end{aligned}$$

A mapping $T : D(T) \subseteq E \to R(T) \subseteq E$ is said to be **asymptotically nonexpansive in the intermediate sense** [4] if it is continuous and the following inequality holds:

$$\limsup_{n\to\infty} \sup_{x,y\in D(T)} (\|T^nx - T^ny\| - \|x-y\|) \leq 0. \tag{2.4}$$

Remark 2.2. Observe that if we define

$$a_n := \sup_{x,y\in D(T)} (\|T^nx - T^ny\| - \|x-y\|), \sigma_n := \max\{0, a_n\}$$

then, $\sigma_n \to 0$ as $n \to \infty$ and (2.4) reduces to

$$\|T^nx - T^ny\| \leq \|x-y\| + \sigma_n, \forall x, y \in D(T), n \geq 1,$$

which gives us nearly asymptotically nonexpansive mapping with constant sequence $\{k_n\}_{n\geq 1} = \{1\}_{n\geq 1}$.

Remark 2.3. If $D(T)$ is a bounded domain of a nearly asymptotically nonexpansive mapping T, then T is asymptotically nonexpansive in the intermediate sense. To see this, let T be a nearly asymptotically nonexpansive mapping with a bounded domain $D(T)$. Then,$\forall\, x, y \in D(T), n \in \mathbb{N}$

$$\|T^nx - T^ny\| \leq \eta(T^n)\big(\|x-y\| + a_n\big),$$

which implies that $\forall n \geq 1$,

$$\sup_{x,y\in K} (\|T^nx - T^ny\| - \|x-y\|) \leq (\eta(T^n) - 1)diam(K) + \eta(T^n)a_n,$$

Hence,

$$\limsup_{n\to\infty} \sup_{x,y\in K} (\|T^nx - T^ny\| - \|x-y\|) \leq 0.$$

Furthermore, we easily observe that every nearly nonexpansive mapping is nearly asymptotically nonexpansive with $\eta(T^n) \equiv 1$ for all $n \in \mathbb{N}$.We observe from Remarks (2.1) and (2.3) that the classes of nearly nonexpansive mappings and nearly asymptotically nonexpansive mappings are intermediate classes between the class of asymptotically nonexpansive mappings and that of asymptotically nonexpansive in the intermediate sense mappings.

A mapping $T : D(T) \subseteq E \to R(T) \subseteq E$ is called **total asymptotically nonexpansive** [15] if and only if there exists a sequences $\{\mu_n\}_{n\geq 1}, \{\iota_n\}_{n\geq 1} \subset [0,\infty)$ with $\lim_{n\to\infty} \mu_n = 0 = \lim_{n\to\infty} \iota_n$ and a strictly increasing continuous function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that for all $x, y \in D(T), n \geq 1$, we have

$$\|T^n x - T^n y\| \leq \|x - y\| + \mu_n \phi(\|x - y\|) + \iota_n \tag{2.5}$$

If in (2.5), $\iota_n = 0$ for all $n \geq 1$,and $\phi : [0,\infty) \to [0,\infty)$ is the identity map, then the total asymptotically nonexpansive mappings coincide with asymptotically nonexpansive mappings.

If $\mu_n = 0$ and $\iota_n = 0$ for all $n \geq 1$, we obtain from (2.5) the class of mappings that includes the class of nonexpansive mappings, that is, we obtain the class of mappings satisfying

$$\|T^n x - T^n y\| \leq \|x - y\|$$

As observed by Chidume and Ofoedu [10], if ϕ is identity mapping, then (2.5) reduces to

$$\|T^n x - T^n x\| \leq (1 + \mu_n)(\|x - y\|) + \iota_n \tag{2.6}$$

Shahzad and Zegeye [24] called any mappings satisfying (2.6) generalized asymptotically nonexpansive mapping.Thus, the class of generalized asymptotically nonexpansive mapping is a subclass of total asymptotically nonexpansive mappings with $\phi(t) = t$. If $\forall\, t \in [0,\infty)$, $\phi(t) = 0$, then (2.5) reduces to

$$\|T^n x - T^n x\| \leq \|x - y\| + \iota_n \tag{2.7}$$

So that the class of total asymptotically nonexpansive mappings include the class of asymptotically nonexpansive mappings in the intermediate sense.

A mapping $T : D(T) \subseteq E \to R(T) \subseteq E$ is called **total asymptotically weakly contractive** [20] if and only if there exists sequences $\{\mu_n\}_{n\geq 1}, \{\iota_n\}_{n\geq 1} \subset [0,\infty)$ with $\lim_{n\to\infty} \mu_n = 0 = \lim_{n\to\infty} \iota_n$ and a strictly increasing continuous functions $\Phi, \Psi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0 = \Psi(0)$ such that for all $x, y \in D(T), n \geq 1$, we have

$$\|T^n x - T^n y\| \leq \|x - y\| + \mu_n \Phi(\|x - y\|) - \Psi\big(\|x - y\|\big) + \iota_n \tag{2.8}$$

Let $T : D(T) \subseteq E \to R(T) \subseteq E$ and $I : D(I) \subseteq E \to R(I) \subseteq E$ be two mappings. The map T is called **I-nonexpansive** [15] if and only if $D(T) \cap D(I) \neq \emptyset$ and $\forall\, x, y \in D(T) \cap D(S)$,

$$\|Tx - Ty\| \leq \|Ix - Iy\|.$$

Let $T : D(T) \subseteq E \to R(T) \subseteq E$ and $I : D(I) \subseteq E \to R(I) \subseteq E$ be two mappings. The map T is called **asymptotically I- nonexpansive** [23] if and only if $D(T) \cap D(I) \neq \emptyset$ and there exists a sequence $\{\mu_n\}_{n=1}^{\infty} \subset [0,\infty)$ with $\lim_{n\to\infty} \mu_n = 0$ such that for all $x, y \in D(T) \cap D(I), \forall\, n \geq 1$, we have,

$$\|T^n x - T^n y\| \leq (1 + \mu_n)\|I^n x - I^n y\|.$$

Let $T : D(T) \subseteq E \to R(T) \subseteq E$ and $I : D(I) \subseteq E \to R(I) \subseteq E$ be two mappings. The map T is called **total asymptotically I-nonexpansive** [14] if and only if $D(T) \cap D(S) \neq \emptyset$ and there exists a sequences $\{\mu_n\}_{n\geq 1}, \{\iota_n\}_{n\geq 1} \subset [0,\infty)$ with $\lim_{n\to\infty} \mu_n = 0 = \lim_{n\to\infty} \iota_n$ and a strictly increasing continuous function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that for all $x, y \in D(T), \forall\, n \geq 1$, we have,

$$\|T^n x - T^n x\| \leq \|I^n x - I^n y\| + \mu_n \phi(\|I^n x - I^n y\|) + \iota_n$$

We now define a new class of mapping studied in this paper.

Let $T : D(T) \subseteq E \to R(T) \subseteq E$ and $S : D(S) \subseteq E \to R(S) \subseteq E$ be two mappings. The map T is called $S-$generalized asymptotically nonexpansive if and only if $D(T) \cap D(S) \neq \emptyset$ and there exists real sequences $\{\mu_n\}_{n=1}^{\infty}$ and $\{\lambda_n\}_{n=1}^{\infty}$ in $[0,+\infty)$ with $\lim_{n\to\infty}\mu_n = 0 = \lim_{n\to\infty}\lambda_n$ such that $\forall x, y \in D(T)\cap D(S),\ \forall n\in\mathbb{N}$,

$$\|T^n x - T^n y\| \le \|x-y\| + \mu_n\|S^n x - S^n y\| + \lambda_n \tag{2.9}$$

Remark 2.1. If $S = I$, the identity map on $D(T)\cap D(S)$, then (2.9) reduces to

$$\|T^n x - T^n y\| \le (1+\mu_n)\|x-y\| + \lambda_n$$

which is the class of generalized asymptotically nonexpansive mappings studied by Zegeye and Shahzad [24].

Definition 2.1. *Two mappings $T, S : C \to C$ are said to satisfy condition (B) if there is a nondecreasing continuous function $f : [0,+\infty) \to [0,+\infty)$ with $f(0) = 0,\ f(r) > 0,\ for\ all\ r\ \in [0,+\infty)$ such that $\frac{1}{2}(\|x-Tx\| + \|x-Sx\|) \ge f(d(x,F))$ for all $x \in C$, where $d(x,F) = \inf\{\|x-p\| : p \in F = F(T)\cap F(S)\}$.*

Let C be a nonempty subset of a Banach space E. Let T be an $S-$generalized asymptotically nonexpansive and S a generalized asymptotically nonexpansive self-mappings of C. Let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ in $[0,1]$. Let $x_1 \in C$, then the sequence $\{x_n\}_{n=1}^{\infty}$ is generated as follows:

$$\begin{aligned} y_n &= (1-\beta_n)x_n + \beta_n T^n x_n \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n S^n y_n,\ n\ge 1 \end{aligned} \tag{2.10}$$

The aim of this paper is to prove the weak and strong convergence of explicit iterative sequence $\{x_n\}_{n=1}^{\infty}$ defined by (2.10) to a fixed point $S-$generalized asymptotically nonexpansive mappings in Banach space.

Definition 2.2. *A mapping $T : D(T) \subset E \to R(T) \subset E$ is said to be demiclosed at u_0 if for any sequence $\{x_n\}_{n=1}^{\infty}$ in $D(T)$ such that $\{x_n\}_{n=1}^{\infty}$ converges weakly to $x_0 \in D(T)$ and $\{T(x_n)\}_{n=1}^{\infty}$ converges strongly to u_0, then $Tx_0 = u_0$. Thus, if $u_0 = 0$, we say that T is demiclosed at 0.*

Definition 2.3. *Let $T : D(T) \subset E \to R(T) \subset E$ be a mapping. A sequence $\{x_n\}_{n=1}^{\infty}$ in $D(T)$ is called an approximate fixed point sequence of the operator T if and only if*

$$\lim_{n\to\infty}\|x_n - Tx_n\| = 0.$$

Definition 2.4. *A function $f : D(f) \subset E \to R(f) \subset E$ is said to be lower semicontinuous at $x_0 \in D(f)$ if and only if for any sequence $\{x_n\}_{n=1}^{\infty}$ in $D(f)$ that converges to x_0, we have that*

$$f(x_0) \le \liminf_{n\to\infty} f(x_n)$$

Definition 2.5. *A function f is said to be weakly lower semicontinuous at $u_0 \in D(f)$ for any sequence $\{u_n\}_{n=1}^{\infty}$ in $D(f)$ that converges weakly to u_0, then*

$$f(u_0) \le \liminf_{n\to\infty} f(u_n)$$

Definition 2.6. *Recall that a Banach space E is said to satisfy Opial's condition [7] if, for each sequence $\{x_n\}_{n=1}^{\infty}$ in E, the condition $x_n \rightharpoonup x$ implies that either*

$$\liminf_{n\to\infty}\|x_n - x\| < \liminf_{n\to\infty}\|x_n - y\|\ \forall\ x, y \in E \text{ with } y \ne x$$

or

$$\limsup_{n\to\infty}\|x_n - x\| < \limsup_{n\to\infty}\|x_n - y\|\ \forall\ x, y \in E \text{ with } y \ne x$$

Definition 2.7. *A Banach space E is said to satisfy the Generalized Gossez–Lami Dozo property (GGLD-property) if*

$$\limsup_{n\to\infty}\|x_n\| < \limsup_{m\to\infty}\limsup_{n\to\infty}\|x_m - x_n\|$$

whenever $\{x_n\}$ *is a weak null sequence which is not norm convergent.*

Notation: If a sequence $\{x_n\}_{n=1}^{\infty}$ converges weakly to x^* in E, we write $x_n \rightharpoonup x^*$ as $n \to \infty$. If $\{x_n\}_{n=1}^{\infty}$ converges strongly to $x_0 \in E$, we write $x_n \to x_0$ as $n \to \infty$
In order to prove the main results of this paper, we need the following Lemmas:

Lemma 2.1. ([12]). Let $\{a_n\}, \{b_n\}$ *and* $\{\sigma_n\}$ be sequences of nonnegative real sequences satisfying the following conditions: $\forall n \geq 1,\ a_{n+1} \leq (1+\sigma_n)a_n + b_n$. If $\sum_{n=1}^{\infty}\sigma_n < +\infty$ and $\sum_{n=1}^{\infty} b_n < +\infty$, then $\lim_{n\to\infty} a_n$ exists.

Lemma 2.2. ([9]). Let C be a nonempty closed bounded convex subset of a uniformly convex Banach space E and $\{\alpha_n\}$ a sequence in $[\delta, 1-\delta]$, for some $\delta \in (0,1)$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in C such that $\limsup_{n\to\infty}\|x_n\| \leq k$,
$\limsup_{n\to\infty}\|y_n\| \leq k$, and $\limsup_{n\to\infty}\|\alpha x_n + (1-\alpha)y_n\| = k$ holds for some $k \geq 0$.
Then $\lim_{n\to\infty}\|x_n - y_n\| = 0$.

Lemma 2.3. Let C be a nonempty closed and convex subset of a reflexive real Banach space E with Opial's condition. Let $T : C \to C$ be a generalized asymptotically nonexpansive mapping. Suppose $\{x_n\}_{n=1}^{\infty}$ is a sequence in C such that $x_n \rightharpoonup x^*$ *for some* $x^* \in C$ *and* $\forall\ m \in \mathbb{N}$ $x_n - T^m x_n \to 0$ *as* $n \to \infty$, then $T^n x^* \rightharpoonup x^*$.

Proof. Since T is generalized asymptotically nonexpansive mapping, then there exists sequences $\{\mu_k\}_{k=1}^{\infty}$ and $\{l_k\}_{k=1}^{\infty}$ with $\lim_{k\to\infty}\mu_k = \lim_{k\to\infty} l_k = 0$ such that $\forall\ k \in \mathbb{N}$,

$$\|T^k x_n - T^k x^*\| \leq \|x_n - x^*\| + \mu_k\|x_n - x^*\| + l_k. \tag{2.11}$$

Since $\{x_n\}_{n=1}^{\infty}$ converges weakly to x^*, then $\{x_n\}_{n=1}^{\infty}$ is bounded; thus there exists $M_0 \geq 0$ such that $\forall\ n \in \mathbb{N},\ \|x_n - x^*\| \leq M_0$. So, we obtain from (2.11) that

$$\|T^k x_n - T^k x^*\| \leq \|x_n - x^*\| + \mu_k M_0 + l_k. \tag{2.12}$$

Since $\lim_{k\to\infty}\mu_k = \lim_{k\to\infty} l_k = 0$, then $\forall\ \varepsilon > 0,\ \exists\ k_\varepsilon \in \mathbb{N}$ such that $\forall\ k \geq k_\varepsilon,\ \mu_k M_0 + l_k < \varepsilon$.
So, we obtain from (2.12) that $\forall\ k \geq k_\varepsilon$,

$$\|T^k x_n - T^k x^*\| \leq \|x_n - x^*\| + \varepsilon \tag{2.13}$$

Now, consider the map $f : E \to \mathbb{R}$ defined $\forall\ x \in E$ *by* $f(x) = \limsup_{n\to\infty}\|x_n - x\|$. If we assume for contradiction that $\{T^n x^*\}_{n=1}^{\infty}$ is not weakly convergent to x^*, then there exists a subsequence $\{T^{n_j}x^*\}_{j=1}^{\infty}$ *of* $\{T^n x^*\}_{n=1}^{\infty}$ such that $T^{n_j}x^* \rightharpoonup y \neq x^*$ *as* $j \to \infty$. Since E satisfies Opial's condition, we obtain that $f(x^*) < f(y)$.
Thus, there exists $\varepsilon_0 > 0$ such that $0 < \varepsilon_0 < \frac{1}{2}(f(y) - f(x^*))$. For this $\varepsilon_0 > 0$, we obtain from (2.13) that there exists $k_0 \in \mathbb{N}$ such that $\forall\ k \geq k_0$,

$$\|T^k x_n - T^k x^*\| \leq \|x_n - x^*\| + \varepsilon_0. \tag{2.14}$$

By weakly-lower semi-continuity of f, we obtain that $f(y) \leq \liminf_{j\to\infty} f(T^{n_j}x^*)$.
Thus, there exists $j_0 \in \mathbb{N}$ such that $n_{j_0} \geq k_0$ and $f(y) < f(T^{n_{j_0}}x^*) + \varepsilon_0$.

But then

$$\begin{aligned} f(y) &< f(T^{n_{j_0}}x^*) + \varepsilon_0 \\ &= \limsup_{n\to\infty} \|x_n - T^{n_{j_0}}x^*\| + \varepsilon_0 \\ &\leq \limsup_{n\to\infty} (\|x_n - T^{n_{j_0}}x_n\| + \|T^{n_{j_0}}x_n - T^{n_{j_0}}x^*\|) + \varepsilon_0 \\ &\leq \limsup_{n\to\infty} \|x_n - T^{n_{j_0}}x_n\| + \limsup_{n\to\infty} \|T^{n_{j_0}}x_n - T^{n_{j_0}}x^*\| + \varepsilon_0 \end{aligned} \tag{2.15}$$

Using the fact that $\lim_{n\to\infty} \|x_n - T^{n_{j_0}}x_n\| = 0$, we obtain from (2.15) and (2.14) that

$$\begin{aligned} f(y) &< \limsup_{n\to\infty} \|T^{n_{j_0}}x_n - T^{n_{j_0}}x^*\| + \varepsilon_0 \\ &\leq \limsup_{n\to\infty} (\|x_n - x^*\| + \varepsilon_0) + \varepsilon_0 \\ &\leq \limsup_{n\to\infty} \|x_n - x^*\| + 2\varepsilon_0 \\ &= f(x^*) + 2\varepsilon_0. \end{aligned} \tag{2.16}$$

But $0 < \varepsilon_0 < \frac{1}{2}(f(y) - f(x^*))$. Thus, $2\varepsilon_0 < f(y) - f(x^*)$. So, we obtain from (2.16) that

$$\begin{aligned} f(y) &< f(x^*) + 2\varepsilon_0 \\ &< f(x^*) + f(y) - f(x^*) \\ &= f(y), \end{aligned}$$

a contradiction. Thus, the conclusion of Lemma (2.3) holds. This completes the proof. □

Lemma 2.4. Let E be a reflexive real Banach space with $GGLD$ and Opial's condition. Suppose that $C, T, \{x_n\}_{n=1}^{\infty}$ *and* x^* are as in Lemma 2.3, then $T^n x^* \to x^*$ *as* $n \to \infty$

Proof. Observe that from Lemma 2.3, $T^n x^* \rightharpoonup x^*$ *as* $n \to \infty$. Suppose for contradiction that $T^n x^* \not\to x^*$ *as* $n \to \infty$, then $\alpha_0 := \liminf_{n\to} \|T^n x^* - x^*\| > 0$; and by $GGLD$, $0 < \alpha_0 < \limsup_{n\to\infty} \limsup_{k\to\infty} \|T^n x^* - T^k x^*\|$
Thus, there exists $\varepsilon_1 > 0$ such that

$$\alpha_0 + \varepsilon_1 < \limsup_{n\to\infty} \limsup_{k\to\infty} \|T^n x^* - T^k x^*\|. \tag{2.17}$$

Moreover, since T is generalized asymptotically nonexpansive mapping and $\{T^n x^*\}_{n=1}^{\infty}$ is bounded, there exists $M_1 \geq 0$ and $N_0 \in \mathbb{N}$ such that $\forall\, n \in \mathbb{N}$, $\|T^n x^* - x^*\| \leq M_1$ and $\forall\, m \geq N_0$,

$$\|T^m x^* - T^m(T^n x^*)\| \leq \|x^* - T^n x^*\| + \frac{\varepsilon_1}{2}. \tag{2.18}$$

From (2.17), we can choose $m^* > N_0$ and a strictly increasing sequence $\{k_j\}_{j=0}^{\infty}$ *in* $\mathbb{N}$ such that $\alpha_0 + \varepsilon_1 < \|T^{m^*}x^* - T^{k_j}x^*\| = \|T^{m^*}x^* - T^{m^*}(T^{k_j - m^*}x^*)\|$, so that using (2.18), we obtain that $\forall\, j \geq 0$

$$\begin{aligned} \alpha_0 + \varepsilon_1 &< \|T^{m^*}x^* - T^{m^*}(T^{k_j - m*}x^*)\| \\ &\leq \|x^* - T^{k_j - m^*}x^*\| + \frac{\varepsilon_1}{2}. \end{aligned}$$

Thus,

$$\begin{aligned} \alpha_0 + \varepsilon_1 &\leq \limsup_{j\to\infty} \|x^* - T^{k_j - m^*}x^*\| + \frac{\varepsilon_1}{2} \\ &\leq \limsup_{n\to\infty} \|x^* - T^n x^*\| + \frac{\varepsilon_1}{2} \\ &= \alpha_0 + \frac{\varepsilon_1}{2}, \end{aligned}$$

a contradiction. Hence, $T^n x^* \to x^* \; as \; n \to \infty$. □

Lemma 2.5. Suppose the condition of Lemma 2.4 hold and suppose that $\exists \; n_0 \in \mathbb{N}$ such that T^{n_0} is continuous at x^*, then $Tx^* = x^*$.

Proof. Since $\exists \; n_0 \in \mathbb{N}$ such that T^{n_0} is continuous at x^*, then T^{n_0+1} is also continuous at x^*. From Lemma 2.4 we know that $\lim_{n\to\infty} T^n x^* = x^*$. Thus, $\lim_{n\to\infty} T^{n_0+n}x^* = x^* = \lim_{n\to\infty} T^{n_0+1+n}x^*$. This implies that

$$\lim_{n\to\infty} T^{n_0}(T^n x^*) = x^*$$

and

$$\lim_{n\to\infty} T^{n_0+1}(T^n x^*) = x^*,$$

so that by continuity of T^{n_0} and T^{n_0+1}, we obtain that $T^{n_0}(\lim_{n\to\infty} T^n x^*) = x^* = T^{n_0+1}(\lim_{n\to\infty} T^n x^*)$. Thus, $T^{n_0}x^* = x^* = T^{n_0+1}x^*$. This implies that $Tx^* = x^*$. This completes the proof. □

Lemma 2.6. (Demiclosedness Principles) Let E be a reflexive real Banach space with GGLD and Opial's condition. Let C be a nonempty closed convex subset of E. Let $T : C \to C$ be a uniformly continuous generalized asymptotically nonexpansive mapping, then $I - T$ is demiclosed at 0, where I is the identity mapping on E.

Proof. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in C such that $x_n \rightharpoonup x^* (for \; some \; x^* \in \; C) \; and$
$|x_n - Tx_n\| \to 0 \; as \; n \to \infty$. We show that $x^* - Tx^* = 0$. Since T is uniformly continuous, it follows from the fact that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ that for each $m \; \in \mathbb{N}, \; x_n - T^m x_n \to 0 \; as \; n \to \infty$.
But by Lemma 2.3, $x_n \rightharpoonup x^* \; and \; x_n - T^m x_n \to 0 \; as \; n \to \infty$ implies that $T^n x^* \rightharpoonup x^* \; as \; n \to \infty$.
Thus, by Lemma 2.4, we obtain that $T^n x^* \to x^* \; as \; n \to \infty$.
Hence, by Lemma 2.5, we obtain that $x^* = Tx^* \iff x^* - Tx^* = 0$.
So, $I - T$ is demiclosed at 0. This completes the proof. □

Lemma 2.7. Let E be a reflexive real Banach space with GGLD and Opial's condition. Let C be a nonempty closed convex subset of E. Let $T : C \to C$ be a uniformly continuous $S-$generalized asymptotically nonexpansive mapping. Suppose that S is generalized asymptotically nonexpansive mapping, then $I - T$ is demiclosed at 0.

Proof. Since T is S-generalized asymptotically nonexpansive mapping, then there exists two real sequences $\{\mu_n\}_{n=1}^{\infty} \; and \; \{l_n\}_{n=1}^{\infty}$ with $\lim_{n\to\infty} \mu_n = \lim_{n\to\infty} l_n = 0$ such that $\forall \; x, y \in C, \; \forall \; n \in \mathbb{N}$

$$\|T^n x - T^n y\| \leq \|x - y\| + \mu_n \|S^n x - S^n y\| + l_n; \tag{2.19}$$

and since S is generalized asymptotically nonexpansive mapping, there exists two sequences $\{\mu'_n\}_{n=1}^{\infty}$ and $\{l'_n\}_{n=1}^{\infty}$ with $\lim_{n\to\infty} \mu'_n = 0 = \lim_{n\to\infty} l'_n$ such that $\forall \; x, y \; \in C, \; \forall \; n \in \mathbb{N}$

$$\|S^n x - S^n y\| \leq \|x - y\| + \mu'_n \|x - y\| + l'_n \tag{2.20}$$

Using (2.20) in (2.20), we obtain that $\forall\ x, y \in C,\ \forall\ n \in \mathbb{N}$,

$$\begin{aligned} \|T^n x - T^n y\| &\leq \|x-y\| + \mu_n(\|x-y\| + \mu_n'\|x-y\| + l_n') + l_n \\ &= \|x-y\| + (\mu_n + \mu_n\mu_n')\|x-y\| + \mu_n l_n' + l_n \\ &= \|x-y\| + \alpha_n\|x-y\| + \theta_n, \end{aligned}$$

where $\alpha_n = \mu_n + \mu_n\mu_n' \to 0\ and\ \theta_n = \mu_n l_n' + l_n \to 0\ as\ n \to \infty$. Thus, T is uniformly continuous generalized asymptotically nonexpansive mapping.
Thus, by Lemma 2.6, $I - T$ is demiclosed at 0. This completes the proof. □

Lemma 2.8. Let C be a nonempty subset of a real normed linear space E. Let $T, S : C \to C$ be two continupus mappings, then $F(T) \cap F(S)$ is closed.
Proof:

$$F = F(T) \cap F(S) = \{x \in E : Tx = x, Sx = x\}$$

If $F = \emptyset$, then we are done since $\emptyset$ is closed. If $F \neq \emptyset$, then for any sequence $\{x_n\}_{n=1}^{\infty} \in F, T(x_n) = x_n$ and $S(x_n) = x_n$. Thus, if $x_n \to x^*$ as $n \to \infty$. then, $\lim_{n\to\infty} x_n = x^*$.

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} T(x_n)$$

and

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} S(x_n)$$

By continuity of T and S, we obtain that

$$\lim_{n\to\infty} x_n = T\big(\lim_{n\to\infty} x_n\big)$$

and

$$\lim_{n\to\infty} x_n = S\big(\lim_{n\to\infty} x_n\big)$$

So,

$$x^* = T(x^*) = S(x^*).$$

This implies that $x^* \in F$. Hence, $F(T) \cap F(S)$ is closed.

Lemma 2.9. Let E be a real normed linear space. Let $S : D(S) \subset E \to R(S) \subset E$ be a total asymptotically nonexpansive mapping with sequences $\{\mu_n\}_{n=1}^{\infty}$,$\{\iota_n\}_{n=1}^{\infty}$ and gauge function $\varphi : [0, \infty) \to [0, \infty)$. Suppose that there exists constants $M_0 > 0, M_1 > 0$ such that $\forall\, t > M_0,\ \varphi(t) \leq M_1 t$, then S is a generalized asymptotically nonexpansive mapping.

Proof: Since S is total asymptotically nonexpansive with sequences $\{\mu_n\}_{n=1}^{\infty}$,$\{\iota_n\}_{n=1}^{\infty}$ and $\varphi : [0, \infty) \to [0, \infty)$, then $\forall\, x, y \in D(S), \forall\, n \in \mathbb{N}$,

$$\|S^n x - S^n y\| \leq \|x-y\| + \mu_n\varphi\big(\|x-y\|\big) + \iota_n.$$

Since φ is continuous, then $\exists\, M_2 > 0$ such that $\forall\, t \in [0, M_0],\ \varphi(t) \leq M_2$; and since $\forall\, t > M_0,\ \varphi(t) \leq M_1 t,$ then $\forall\, t \in [0, \infty), \varphi(t) \leq M_2 + M_1 t$. This implies that $\forall x, y \in D(S)$,

$$\varphi\big(\|x-y\|\big) \leq M_2 + M_1\|x-y\|$$

Thus, $\forall\, n \in \mathbb{N}, \forall\, x, y \in D(S)$,

$$\begin{aligned}\|S^n x - S^n y\| &\le \|x-y\| + \mu_n \varphi\big(\|x-y\|\big) + \iota_n.\\ &\le \|x-y\| + \mu_n\big[M_2 + M_1\|x-y\|\big] + \iota_n\\ &= \|x-y\| + \mu_n M_1\|x-y\| + \mu_n M_2 + \iota_n\\ &= (1+\mu_n M_1)\|x-y\| + \mu_n M_2 + \iota_n\\ &= (1+\sigma_n)\|x-y\| + \theta_n\end{aligned}$$

where $\sigma_n = \mu_n M_1 \to 0$ and $\theta_n = \mu_n M_2 + \iota_n \to 0$ as $n \to \infty$. Thus, S is generalized asymptotically nonexpansive mapping.

Lemma 2.10. Let E be a real Banach space and C be a nonempty subset of E. Let $S : C \to C$ be a totally asymptotically $I-$ nonexpansive mapping with sequences $\{\mu_n\}_{n=1}^{\infty}$,$\{\iota_n\}_{n=1}^{\infty}$ with a gauge function $\phi : [0,\infty) \to [0,\infty)$ and $I : C \to C$ be a total asymptotically nonexpansive mapping with sequences $\{\mu_n^*\}_{n=1}^{\infty}$,$\{\iota_n^*\}_{n=1}^{\infty}$ with a gauge function $\Psi : [0,\infty) \to [0,\infty)$. Suppose that there exists constants $M_1 > 0$,$M_1^* > 0$, $M_2 > 0$,$M_2^* > 0$, such that $\forall\, q > M_1$, $\Phi(q) \le M_1^* q$,and $\forall\, t > M_2$, $\Psi(t) \le M_2^* t$ then S is a generalized asymptotically nonexpansive mapping.

Proof: Since I is total asymptotically nonexpansive with sequences $\{\mu_n^*\}_{n=1}^{\infty}$,$\{\iota_n^*\}_{n=1}^{\infty}$ and $\Psi : [0,\infty) \to [0,\infty)$, then $\forall\, x, y \in C, \forall\, n \in \mathbb{N}$,

$$\|I^n x - I^n y\| \le \|x-y\| + \mu_n^* \Psi\big(\|x-y\|\big) + \iota_n^*.$$

Since Ψ is continuous, then $\exists\, M_0 > 0$ such that $\forall\, t \in [0, M_2]$, $\Psi(t) \le M_0$; and since $\forall\, t > M_2$, $\Psi(t) \le M_2^* t$, then $\forall\, t \in [0,\infty)$, $\Psi(t) \le M_0 + M_2^* t$. This implies that $\forall x, y \in C$,

$$\Psi\big(\|x-y\|\big) \le M_0 + M_2^*\|x-y\|$$

Thus, $\forall\, n \in \mathbb{N}, \forall\, x, y \in C$,

$$\begin{aligned}\|I^n x - I^n y\| &\le \|x-y\| + \mu_n^* \Psi\big(\|x-y\|\big) + \iota_n^*.\\ &\le \|x-y\| + \mu_n^*\big[M_0 + M_2^*\|x-y\|\big] + \iota_n^*\\ &= \|x-y\| + \mu_n^* M_2^*\|x-y\| + \mu_n^* M_0 + \iota_n^*\\ &= (1+\mu_n^* M_2^*)\|x-y\| + \mu_n^* M_0 + \iota_n^*\end{aligned}$$

$$\therefore \|I^n x - I^n y\| \le (1+\mu_n^* M_2^*)\|x-y\| + \mu_n^* M_0 + \iota_n^* \tag{2.21}$$

Also, since S is total asymptotically I-nonexpansive mapping with sequences $\{\mu_n\}_{n=1}^{\infty}$,$\{\iota_n\}_{n=1}^{\infty}$ and $\Phi : [0,\infty) \to [0,\infty)$, then $\forall\, x, y \in K, \forall\, n \in \mathbb{N}$,

$$\|S^n x - S^n y\| \le \|I^n x - I^n y\| + \mu_n \Phi\big(\|I^n x - I^n y\|\big) + \iota_n.$$

Since Φ is continuous, then $\exists\, M_0^* > 0$ such that $\forall\, q \in [0, M_1]$, $\Phi(q) \le M_0^*$; and since $\forall\, q > M_1$, $\Phi(q) \le M_1^* q$, then $\forall\, q \in [0,\infty)$, $\Phi(q) \le M_0^* + M_1^* q$. This implies that $\forall\, x, y \in C$,

$$\Phi\big(\|I^n x - I^n y\|\big) \le M_0^* + M_1^*\|I^n x - I^n y\|$$

Thus, $\forall\, n \in \mathbb{N}, \forall\, x, y \in K$,

$$\begin{aligned}
\|S^n x - S^n y\| &\leq \|I^n x - I^n y\| + \mu_n \Phi\big(\|I^n x - I^n y\|\big) + \iota_n. \\
&\leq \|I^n x - I^n y\| + \mu_n\big(M_0^* + M_1^*\|I^n x - I^n y\|\big) + \iota_n. \\
&\leq (1 + \mu_n^* M_2^*)\|x - y\| + \mu_n^* M_0 + \iota_n^* + \mu_n\big[M_1 + M_1^*(1 + \mu_n^* M_2^*)\|x - y\| \\
&\quad + \mu_n^* M_0 + \iota_n^*\big] + \iota_n. \\
&= \|x - y\| + \big[\mu_n^* M_2^* + \mu_n M_0^*(1 + \mu_n^* M_2^*)\big]\|x - y\| + \mu_n^* M_0 + \iota_n^* + \mu_n \mu_n^* M_0 \\
&\quad + \mu_n \iota_n^* + \mu_n M_0^* + \iota_n. \\
&= \big[1 + \mu_n^* M_2^* + \mu_n M_0^*(1 + \mu_n^* M_2^*)\big]\|x - y\| + \mu_n^* M_0 + \iota_n^* + \mu_n \mu_n^* M_0 \\
&\quad + \mu_n \iota_n^* + \mu_n M_0^* + \iota_n. \\
&= (1 + \alpha_n)\|x - y\| + \theta_n
\end{aligned}$$

where $\alpha_n = \mu_n^* M_2^* + \mu_n M_0^*(1 + \mu_n^* M_2^*) \to 0$ and $\theta_n = \mu_n^* M_0 + \iota_n^* + \mu_n \mu_n^* M_0 + \mu_n \iota_n^* + \mu_n M_0^* + \iota_n \to 0$ as $n \to \infty$. Thus, S is generalized asymptotically nonexpansive mapping.

Lemma 2.11. Let E be a real normed linear space. Let $S : D(S) \subset E \to R(S) \subset E$ be a total asymptotically weakly contractive mapping with sequences $\{\mu_n\}_{n=1}^{\infty}$,$\{\iota_n\}_{n=1}^{\infty}$ and gauge functions $\Phi, \Psi : [0, \infty) \to [0, \infty)$. Suppose that there exists constants $M_0 > 0, M_1 > 0$ such that $\forall\, t > M_0,\ \Psi(t) \leq M_1 t$, then S is a generalized asymptotically nonexpansive mapping.

Proof: Since S is total asymptotically weakly contractive mapping with sequences $\{\mu_n\}_{n=1}^{\infty}$,$\{\iota_n\}_{n=1}^{\infty}$ and $\Phi, \Psi : [0, \infty) \to [0, \infty)$, then $\forall\, x, y \in K, \forall\, n \in \mathbb{N}$,

$$\|S^n x - S^n y\| \leq \|x - y\| + \mu_n \Phi\big(\|x - y\|\big) - \Psi\big(\|x - y\|\big) + \iota_n.$$

Since Φ is continuous, then $\exists\, M_2 > 0$ such that $\forall\, t \in [0, M_0],\ \Phi(t) \leq M_2$; and since $\forall\, t > M_0,\ \Phi(t) \leq M_1 t$, then $\forall\, t \in [0, \infty), \Phi(t) \leq M_2 + M_1 t$. This implies that $\forall x, y \in D(S)$,

$$\Phi\big(\|x - y\|\big) \leq M_2 + M_1\|x - y\|$$

Thus, $\forall\, n \in \mathbb{N}, \forall\, x, y \in D(S)$,

$$\begin{aligned}
\|S^n x - S^n y\| &\leq \|x - y\| + \mu_n \Phi\big(\|x - y\|\big) - \Psi\big(\|x - y\|\big) + \iota_n. \\
&\leq \|x - y\| + \mu_n\big[M_2 + M_1\|x - y\|\big] - \Psi\big(\|x - y\|\big) + \iota_n \\
&\leq \|x - y\| + \mu_n\big[M_2 + M_1\|x - y\|\big] + \iota_n \\
&= \|x - y\| + \mu_n M_1\|x - y\| + \mu_n M_2 + \iota_n \\
&= (1 + \mu_n M_1)\|x - y\| + \mu_n M_2 + \iota_n \\
&= (1 + \sigma_n)\|x - y\| + \theta_n
\end{aligned}$$

where $\sigma_n = \mu_n M_1 \to 0$ and $\theta_n = \mu_n M_2 + \iota_n \to 0$ as $n \to \infty$. Thus, S is generalized asymptotically nonexpansive mapping. This completes the proof.

3. MAIN RESULTS

The main results of this paper are now presented as follows:

3.1. **Necessary and sufficient convergence results.**

Theorem 3.1. *Let E be a uniformly convex real Banach space, C be a nonempty closed convex subset of E, $T : C \to C$ an S-generalized asymptotically nonexpansive mapping with sequences $\{\mu_n\}_{n=1}^{\infty},\ \{\lambda_n\}_{n=1}^{\infty} \subseteq$*

$[0,+\infty)$ *such that* $\sum_{n=1}^{\infty}\mu_n < +\infty,\ \sum_{n=1}^{\infty}\lambda_n < +\infty$. *Suppose that S is generalized asymptotically nonexpansive mapping with sequences* $\{\mu'_n\}_{n=1}^{\infty}$ *and* $\{\lambda'_n\}_{n=1}^{\infty}$ *such that* $\sum_{n=1}^{\infty}\mu'_n < +\infty,\ \sum_{n=1}^{\infty}\lambda'_n < +\infty$ *and that* $F = F(T)\cap F(S) \neq \emptyset$. *Let* $\{\alpha_n\}_{n=1}^{\infty},\ \{\beta_n\}_{n=1}^{\infty} \subseteq [\delta, 1-\delta]$ $for\ some$ $\delta \in (0,1)$, *then iterative sequence* $\{x_n\}_{n=1}^{\infty}$ *generated by (2.10) converges strongly to some point in F if and only if* $\liminf_{n\to\infty} d(x_n, F) = 0$.

Proof. Let $p \in F = F(T)\cap F(S)$, then

$$\begin{aligned}\|x_{n+1}-p\| &= \|(1-\alpha_n)x_n+\alpha_n S^n y_n - p\|\\ &\leq (1-\alpha_n)\|x_n-p\|+\alpha_n\|S^n y_n-p\|\\ &\leq (1-\alpha_n)\|x_n-p\|+\alpha_n\big[\|y_n-p\|+\mu'_n\|y_n-p\|+\lambda'_n\big]\\ &= (1-\alpha_n)\|x_n-p\|+\alpha_n\big[(1+\mu'_n)\|y_n-p\|+\lambda'_n\big] \end{aligned} \tag{3.1}$$

$$\begin{aligned}\|y_n-p\| &= \|(1-\beta_n)x_n+\beta_n T^n x_n - p\|\\ &\leq (1-\beta_n)\|x_n-p\|+\beta_n\|T^n x_n-p\|\\ &\leq (1-\beta_n)\|x_n-p\|+\beta_n\big[\|x_n-p\|+\mu_n\|S^n x_n-p\|+\lambda_n\big]\\ &\leq (1-\beta_n)\|x_n-p\|+\\ &\quad \beta_n\big[\|x_n-p\|+\mu_n(\|x_n-p\|+\mu'_n\|x_n-p\|+\lambda'_n)+\lambda_n\big]\\ &= (1-\beta_n)\|x_n-p\|+\\ &\quad \beta_n\big[\|x_n-p\|+\mu_n\|x_n-p\|+\mu_n\mu'_n\|x_n-p\|+\mu_n\lambda'_n+\lambda_n\big]\\ &= (1+\beta_n\mu_n+\beta_n\mu_n\mu'_n)\|x_n-p\|+\beta_n(\mu_n\lambda'_n+\lambda_n) \end{aligned} \tag{3.2}$$

Using (3.2) in (3.1) gives

$$\begin{aligned}\|x_{n+1}-p\| &\leq (1-\alpha_n)\|x_n-p\|+\\ &\quad \alpha_n\big[(1+\mu'_n)\{(1+\beta_n(\mu_n+\mu_n\mu'_n))\|x_n-p\|+\beta_n(\mu_n\lambda'_n+\lambda_n)\}+\lambda'_n\big]\\ &= [1+\alpha_n\left(\mu'_n+\beta_n(\mu_n+\mu_n\mu'_n)\right)]\|x_n-p\|+\alpha_n[\beta_n(\mu_n\lambda'_n+\lambda_n)+\lambda'_n]\\ &= (1+\rho_n)\|x_n-p\|+\sigma_n \end{aligned} \tag{3.3}$$

where $\rho_n := \alpha_n\left(\mu'_n+\beta_n(\mu_n+\mu_n\mu'_n)\right),\ \sigma_n := \alpha_n[\beta_n(\mu_n\lambda'_n+\lambda_n)+\lambda'_n]$.

Since $\forall\ n \in \mathbb{N},\ \ \alpha_n, \beta_n \in [\delta, 1-\delta]$ for some $\delta \in (0,1)$ and $\sum_{n=1}^{\infty}\mu_n < \infty,\ \sum_{n=1}^{\infty} l_n < \infty,\ \sum_{n=1}^{\infty}\mu'_n < \infty,\ \sum_{n=1}^{\infty} l'_n < \infty$, it then implies that $\sum_{n=1}^{\infty}\rho_n < \infty$ and $\sum_{n=1}^{\infty}\sigma_n < \infty$. Since by (3.3) we have that $\forall\ n \in \mathbb{N}$,

$$\|x_{n+1}-p\| \leq (1+\rho_n)\|x_n-p\|+\sigma_n, \tag{3.4}$$

then by Lemma 2.7, we obtain that $\lim_{n\to\infty}\|x_n-p\|$ exists. $\forall\, p \in F$. Moreover, it is easy to see from (3.4) that

$$d(x_{n+1}, F) \leq (1+\rho_n)d(x_n - F)+\sigma_n.$$

Thus, by Lemma 2.7, $\lim_{n\to\infty} d(x_n, F)$ exists.

Now, suppose $\exists\ p^* \in F$ such that $x_n \to p^*\ as\ n\to\infty$ then $\|x_n-p^*\| \to 0\ as\ n\to\infty$. But

$$0 \leq d(x_n, F) = \inf_{p\in F}\|x_n-p\| \leq \|x_n-p^*\|$$

So, by sandwish theorem, $\lim_{n\to\infty} d(x_n, F) = 0 = \liminf_{n\to\infty} d(x_n, F)$.
On the other hand, if $\liminf_{n\to\infty} d(x_n, F) = 0$, then $\exists$ a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that $\lim_{k\to\infty} d(x_{n_k}, F) = 0$. But $\lim_{n\to\infty} d(x_n, F)\}$ exists. Thus, $\lim_{k\to\infty} d(x_{n_k}, F) = 0 \implies \lim_{n\to\infty} d(x_n, F) = 0$.
Next, we show that the sequence $\{x_n\}_{n=1}^{\infty}$ is Cauchy. Observe that from (3.4) $\forall\, p \in F,\ \forall\, n, m \in \mathbb{N}$,

$$\begin{aligned}
\|x_{n+m} - p\| &\leq (1+\rho_{n+m-1})\|x_{n+m-1} - p\| + \sigma_{n+m-1} \\
&\leq (1+\rho_{n+m-1})\big((1+\rho_{n+m-2})\|x_{n+m-2} - p\|\sigma_{n+m-2}\big) \\
&\quad +\sigma_{n+m-1} \\
&= (1+\rho_{n+m-1})(1+\rho_{n+m-2})\|x_{n+m-2} - p\| \\
&\quad +(1+\rho_{n+m-1})\rho_{n+m-2} + \rho_{n+m-1} \\
&\vdots \\
&\leq \prod_{j=n}^{n+m-1}(1+\rho_j)\|x_n - p\| + \left(\sum_{j=n}^{n+m-1}\sigma_j\right)\prod_{j=n}^{n+m-1}(1+\rho_j) \\
&\leq \exp\left(\sum_{j=n}^{n+m-1}\rho_j\right)\|x_n - p\| + \left(\sum_{j=n}^{n+m-1}\sigma_j\right)\exp\left(\sum_{j=n}^{n+m-1}\rho_j\right) \\
&\leq M\|x_n - p\| + \left(\sum_{j=n}^{n+m-1}\sigma_j\right)\exp\left(\sum_{j=n}^{n+m-1}\rho_j\right)
\end{aligned}$$

$$\therefore\ \|x_{n+m} - p\| \leq M\|x_n - p\| + \left(\sum_{j=n}^{n+m-1}\sigma_j\right)\exp\left(\sum_{j=n}^{n+m-1}\rho_j\right) \tag{3.5}$$

Since $\sum_{n=1}^{\infty}\sigma_n < \infty$, and $\sum_{n=1}^{\infty}\rho_n < \infty$, then $\forall\, \varepsilon > 0,\ \exists\, n_\varepsilon \in \mathbb{N}$ such that $\forall\, n \geq n_\varepsilon$,

$$\exp\left(\sum_{j=n}^{\infty}\rho_j\right)\sum_{j=n}^{\infty}\sigma_j < \frac{\varepsilon}{3},$$

and since $\lim_{n\to\infty} d(x_n, F) = 0,\ \exists\, n'_\varepsilon \in \mathbb{N}$ such that $\forall\, n \geq n'_\varepsilon,\ d(x_n, F) < \frac{\varepsilon}{3(M+1)}$.
Thus, $\forall\ n \geq n'_\varepsilon,\ \inf_{p\in F} d(x_n, p) < \dfrac{\varepsilon}{3(M+1)}$. But by definition of $\inf_{p\in F} d(x_n, p)$, we obtain that $\forall\, \varepsilon > 0,\ \exists\, p_\varepsilon \in F$ such that

$$\inf_{p\in F} d(x_n, p) \leq d(x_n, p_\varepsilon) < \inf_{p\in F} d(x_n, p) + \frac{\varepsilon}{3(M+1)}$$

Setting $N_\varepsilon = \max\{n_\varepsilon, n'_\varepsilon\} \in \mathbb{N}$, we obtain that $\forall\, n \geq N_\varepsilon$,

$$\begin{aligned}
\|x_n - p_\varepsilon\| = d(x_n, p_\varepsilon) &< \inf_{p\in F} d(x_n, p) + \frac{\varepsilon}{3(M+1)} \\
&< \frac{\varepsilon}{3(M+1)} + \frac{\varepsilon}{3(M+1)}
\end{aligned}$$

So, $\forall\, n \geq N_\varepsilon,\ \forall m \geq 1$, we obtain using (3.5)that

$$\begin{aligned}\|x_{n+m}-x_n\| &\leq \|x_{n+m}-p_\varepsilon\|+\|x_n-p_\varepsilon\| \\ &\leq M\|x_n-p_\varepsilon\|+\left(\sum_{j=n}^{\infty}\sigma_j\right)\exp\left(\sum_{j=n}^{\infty}\rho_j\right)+\|x_n-p_\varepsilon\| \\ &= (M+1)\|x_n-p_\varepsilon\|+\left(\sum_{j=n}^{\infty}\sigma_j\right)\exp\left(\sum_{j=n}^{\infty}\rho_j\right) \\ &< (M+1)\left(\frac{2\varepsilon}{3(M+1)}\right)+\frac{\varepsilon}{3}=\varepsilon.\end{aligned}$$

So $\forall n \geq N_\varepsilon,\ \forall\, m \in \mathbb{N}$,

$$\|x_{n+m}-x_n\| < \varepsilon.$$

Thus, the sequence $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in C and since E is complete and C is closed subset of E, $\exists\, p^* \in C$ such that $x_n \to\ p^*\ as\ n \to \infty$. Since $\lim_{n\to\infty} d(x_n, F) = 0$, then $d(p^*, F) = 0$. Thus, by Lemma(2.8) F is closed, thus $p^* \in F = F(T) \cap F(S)$. This completes the proof. □

Corollary 3.1. *Let E be a real Banach Space, C be a nonempty closed convex subset of E, T be S-generalized asymptotically nonexpansive self-mappings of C with sequence $\{\mu'_n\}_{n\geq1}$ and $\{\iota'_n\}_{n\geq1}$ such that $\sum_{n=1}^{\infty}\mu'_n < \infty$ and $\sum_{n=1}^{\infty}\iota'_n < \infty$. Suppose that S is a total asymptotically nonexpansive self mapping of K with sequences $\{\mu_n\}_{n=1}^{\infty}$,$\{\iota_n\}_{n=1}^{\infty}$ and gauge function $\varphi : [0,\infty) \to [0,\infty)$. Suppose that there exists constants $M_0 > 0, M_1 > 0$ such that $\forall\, t > M_0,\ \varphi(t) \leq M_1 t$, and that $F := F(T)\cap F(S) \neq \emptyset$. Then, the explicitly iterative sequence $\{x_n\}$ defined by (2.10) converges to some element of $F := F(T) \cap F(S)$.*

Proof. By Lemma 2.9; S is a generalized asymptotically nonexpansive mapping. By Theorem 3.1, the result follows. □

Corollary 3.2. *Let E be a real Banach Space, K be a nonempty closed convex subset of E, T be S-generalized asymptotically nonexpansive self-mappings of K with sequence $\{\mu'_n\}_{n\geq1}$ and $\{\iota'_n\}_{n\geq1}$ such that $\sum_{n=1}^{\infty}\mu'_n < \infty$ and $\sum_{n=1}^{\infty}\iota'_n < \infty$. Suppose that S is a totally asymptotically I- nonexpansive self mappings of K with sequences $\{\mu_n\}_{n=1}^{\infty}$,$\{\iota_n\}_{n=1}^{\infty}$ with a gauge function $\phi : [0,\infty) \to [0,\infty)$ where $I : C \to C$ be a total asymptotically nonexpansive mapping with sequences $\{\mu^*_n\}_{n=1}^{\infty}$,$\{\iota^*_n\}_{n=1}^{\infty}$ with a gauge function $\Psi : [0,\infty) \to [0,\infty)$.Suppose that there exists constants $M_1 > 0$,$M_1^* > 0$, $M_2 > 0$,$M_2^* > 0$, such that $\forall\, q > M_1,\ \Phi(q) \leq M_1^* q$,and $\forall\, t > M_2,\ \Psi(t) \leq M_2^* t$ and that $F := F(T) \cap F(S) \neq \emptyset$. Then, the explicitly iterative sequence $\{x_n\}$ defined by (2.10) converges to some element of $F := F(T) \cap F(S)$.*

Proof. By Lemma 2.10; S is a generalized asymptotically nonexpansive mapping. By Theorem 3.1, the result follows. □

Corollary 3.3. *Let E be a real Banach Space, K be a nonempty closed convex subset of E, T be S-generalized asymptotically nonexpansive self-mappings of K with sequence $\{\mu_n\}_{n\geq1}$ and $\{\iota_n\}_{n\geq1}$ such that $\sum_{n=1}^{\infty}\mu_n < \infty$ and $\sum_{n=1}^{\infty}\iota_n < \infty$. Suppose that S is a total asymptotically weakly contractive self mappings of K with sequences $\{\mu'_n\}_{n=1}^{\infty}$,$\{\iota'_n\}_{n=1}^{\infty}$ and gauge functions $\Phi, \Psi : [0,\infty) \to [0,\infty)$. Suppose that there exists constants $M_0 > 0, M_1 > 0$ such that $\forall\, t > M_0,\ \Psi(t) \leq M_1 t$, and that $F := F(T) \cap F(S) \neq \emptyset$. Then, the explicitly iterative sequence $\{x_n\}$ defined by (2.10) converges to some element of $F := F(T) \cap F(S)$.*

Proof. By Lemma 2.11; S is a generalized asymptotically nonexpansive mapping. By Theorem 3.1, the result follows. □

3.2. **Approximate fixed point sequence.**

Theorem 3.2. *Let E be a uniformly convex real Banach space, C be a nonempty closed subset of E, $T : C \to C$ a uniformly continuous S-generalized asymptotically nonexpansive mapping with sequences $\{\mu_n\}_{n=1}^{\infty}$, $\{l_n\}_{n=1}^{\infty} \subseteq [0, +\infty)$ such that $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} l_n < \infty$. Suppose that $S : K \to K$ is a uniformly continuous generalized asymptotically nonexpansive mapping with $\sum_{n=1}^{\infty} \mu'_n < \infty$, $\sum_{n=1}^{\infty} l'_n < \infty$ and that $F = F(T) \cap F(S) \neq \emptyset$. Let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty} \subseteq [\delta, 1-\delta]$ $for\ some$ $\delta \in (0,1)$. Suppose that for any given $x \in K$, the sequence $\{x_n\}_{n=1}^{\infty}$ is generated by (2.10) then $\{x_n\}_{n=1}^{\infty}$ - is an approximate fixed point sequence of T and S; that is ,*

$$\lim_{n\to\infty} \|Tx_n - x_n\| = \lim_{n\to\infty} \|Sx_n - x_n\| = 0.$$

Proof. By Theorem 3.1, for any $p \in F = F(T) \cap F(S)$, $\lim_{n\to\infty} \|x_n - p\|$ exists. Let $\lim_{n\to\infty} \|x_n - p\| = d$. If $d = 0$, by uniform continuity of T and S, the proof is complete.
Now, suppose $d > 0$, then

$$\begin{aligned}
\|y_n - p\| &= \|(1-\beta_n)x_n + \beta_n T^n x_n - p\| \\
&\le (1-\beta_n)\|x_n - p\| + \beta_n \|T^n x_n - p\| \\
&\le (1-\beta_n)\|x_n - p\| + \beta_n \left(\|x_n - p\| + \mu_n \|S^n x_n - p\| + l_n\right) \\
&\le (1-\beta_n)\|x_n - p\| + \\
&\quad \beta_n\|x_n - p\| + \beta_n\mu_n \left(\|x_n - p\| + \mu'_n\|x_n - p\| + l'_n\right) + \beta_n l_n \\
&= (1-\beta_n)\|x_n - p\| + \beta_n\|x_n - p\| \\
&\quad + \beta_n\mu_n\|x_n - p\| + \beta_n\mu_n\mu'_n\|x_n - p\| + \beta_n\mu_n l'_n + \beta_n l_n \\
&= \|x_n - p\| + \beta_n\mu_n\|x_n - p\| + \beta_n\mu_n\mu'_n\|x_n - p\| + \beta_n\mu_n l'_n \\
&\quad + \beta_n l_n
\end{aligned}$$

This implies that

$$\|y_n - p\| \le \left[1 + \beta_n(\mu_n + \mu_n\mu'_n)\right]\|x_n - p\| + \beta_n\mu_n l'_n + \beta_n l_n \tag{3.6}$$

Taking $\limsup$ on both sides in (3.6), we have

$$\limsup_{n\to\infty} \|y_n - p\| \le d \tag{3.7}$$

Since S is generalized asymptotically nonexpansive self-mapping on C, we obtain that,

$$\begin{aligned}
\|S^n y_n - p\| &\le \|y_n - p\| + \mu'_n\|y_n - p\| + l'_n \\
&= (1 + \mu'_n)\|y_n - p\| + l'_n.
\end{aligned}$$

Taking $\limsup_{n\to\infty}$ and using (3.7) gives $\limsup_{n\to\infty} \|S^n y_n - p\| \le d$.
Moreover, since $\lim_{n\to\infty} \|x_n - p\| = d$ it follows that

$$\lim_{n\to\infty} \|x_{n+1} - p\| = d = \limsup_{n\to\infty} \|x_{n+1} - p\|.$$

But, $\limsup_{n\to\infty} \|x_{n+1} - p\| = d$ means that
$\limsup_{n\to\infty} \|(1-\alpha_n)x_n + \alpha_n S^n y_n - p\| = \limsup_{n\to\infty} \|(1-\alpha_n)(x_n - p) + \alpha_n(Sy_n - p)\| = d$. It therefore

follows from Lemma 2.2 that

$$\lim_{n\to\infty} \|S^n y_n - x_n\| = 0. \tag{3.8}$$

Furthermore,

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|\alpha_n(S^n y_n - x_n)\| \\ &\leq \alpha_n \|S^n y_n - x_n\|. \end{aligned} \tag{3.9}$$

Thus using (3.8) in (3.9) we obtain that

$$\lim_{n\to\infty} \|x_{n=1} - x_n\| = 0. \tag{3.10}$$

Next,

$$\begin{aligned} \|x_n - p\| &\leq \|x_n - S^n y_n\| + \|S^n y_n - p\| \\ &\leq \|x_n - S^n y_n\| + \|y_n - p\| + \mu_n'\|y_n - p\| + l_n' \\ &= \|x_n - S^n y_n\| + (1 + \mu_n')\|y_n - p\| + l_n'. \end{aligned} \tag{3.11}$$

Using (3.11) and (3.7), we obtain that

$$\begin{aligned} d = \liminf_{n\to\infty} \|x_n - p\| &\leq \liminf_{n\to\infty} \|y_n - p\| \\ &\leq \limsup_{n\to\infty} \|y_n - p\| \leq d. \end{aligned}$$

Thus, $\lim_{n\to\infty} \|y_n - p\| = d$.
Besides,

$$\begin{aligned} \|T^n x_n - p\| &\leq \|x_n - p\| + \mu_n \|S^n x_n - p\| + l_n \\ &\leq \|x_n - p\| + \mu_n(\|x_n - p\| + \mu_n'\|x_n - p\| + l_n') + l_n \\ &= (1 + \mu_n(1 + \mu_n'))\|x_n - p\| + \mu_n l_n' + l_n \end{aligned}$$

This implies that

$$\|T^n x_n - p\| \leq (1 + \mu_n(1 + \mu_n'))\|x_n - p\| + \mu_n l_n' + l_n. \tag{3.12}$$

So, we obtain from (3.12) that

$$\limsup_{n\to\infty} \|T^n x_n - p\| \leq d.$$

Since,

$$\limsup_{n\to\infty} \|\beta_n(T^n x_n - p) + (1 - \beta_n)(x_n - p)\| = \limsup_{n\to\infty} \|y_n - p\| = \lim_{n\to\infty} \|y_n - p\| = d.$$

We obtain by Lemma (2.2) that

$$\lim_{n\to\infty} \|T^n x_n - x_n\| = 0. \tag{3.13}$$

In addition,

$$\begin{aligned}
\|S^n x_n - x_n\| &= \|S^n x_n + S^n y_n - S^n y_n - x_n\| \\
&\leq \|S^n x_n - S^n y_n\| + \|S^n y_n - x_n\| \\
&\leq \|x_n - y_n\| + \mu'_n\|x_n - y_n\| + l'_n + \|S^n y_n - x_n\| \\
&= (1+\mu'_n)\|x_n - y_n\| + \|S^n y_n - x_n\| + l'_n \\
&= (1+\mu'_n)\|x_n - [(1-\beta_n)x_n + \beta_n T^n x_n]\| + \|S^n y_n - x_n\| + l'_n \\
&= (1+\mu'_n)\|\beta_n(T^n x_n - x_n)\| + \|S^n y_n - x_n\| + l'_n \\
&= (1+\mu'_n)\beta_n\|T^n x_n - x_n\| + \|S^n y_n - x_n\| + l'_n
\end{aligned}$$

$$\therefore\ \|S^n x_n - x_n\| \leq (1+\mu'_n)\beta_n\|T^n x_n - x_n\| + \|S^n y_n - x_n\| + l'_n \tag{3.14}$$

Thus from (3.8) ,(3.13) and (3.14), we obtain that

$$\lim_{n\to\infty}\|S^n x_n - x_n\| = 0. \tag{3.15}$$

We now show that

$$\lim_{n\to\infty}\|Tx_n - x_n\| = \lim_{n\to\infty}\|Sx_n - x_n\| = 0.$$

But since

$$\begin{aligned}
\|S^{n-1}x_n - x_n\| &= \|S^{n-1}x_n - S^{n-1}x_{n-1} + S^{n-1}x_{n-1} - x_{n-1} + x_{n-1} - x_n\| \\
&\leq \|S^{n-1}x_n - S^{n-1}x_{n-1}\| + \|S^{n-1}x_{n-1} - x_{n-1}\| + \|x_{n-1} - x_n\| \\
&\leq (1+\mu_{n-1})\|x_n - x_{n-1}\| + l_{n-1} + \|S^{n-1}x_{n-1} - x_{n-1}\| + \|x_{n-1} - x_n\|,
\end{aligned}$$

we obtain from (3.10) and (3.15) that

$$\lim_{n\to\infty}\|S^{n-1}x_n - x_n\| = 0 \tag{3.16}$$

Thus,

$$\begin{aligned}
\|x_n - Sx_n\| &\leq \|x_n - S^n x_n\| + \|S^n x_n - Sx_n\| \\
&= \|x_n - S^n x_n\| + \|S\big(S^{n-1}x_n\big) - Sx_n\|.
\end{aligned} \tag{3.17}$$

Since S is uniformly continuous we obtain using (3.15) and (3.16) in (3.17) that

$$\lim_{n\to\infty}\|x_n - Sx_n\| = 0. \tag{3.18}$$

Also,

$$\begin{aligned}
\|T^{n-1}x_n - x_n\| &= \|T^{n-1}x_n - T^{n-1}x_{n-1} + T^{n-1}x_{n-1} - x_{n-1} + x_{n-1} - x_n\| \\
&\leq \|T^{n-1}x_n - T^{n-1}x_{n-1}\| + \|T^{n-1}x_{n-1} - x_{n-1}\| + \|x_{n-1} - x_n\| \\
&\leq \|x_n - x_{n-1}\| + \mu_{n-1}\|S^{n-1}x_n - S^{n-1}x_{n-1}\| + l_{n-1} \\
&\quad + \|T^{n-1}x_{n-1} - x_{n-1}\| + \|x_{n-1} - x_n\| \\
&\leq \|x_n - x_{n-1}\| + \mu_{n-1}\left((1+\mu'_{n-1})\|x_n - x_{n-1} + l'_{n-1}\right) \\
&\quad + \|T^{n-1}x_{n-1} - x_{n-1}\| + \|x_{n-1} - x_n\| + l^{n-1} \\
&= \left[1 + \mu_{n-1}(1+\mu'_{n-1})\right]\|x_n - x_{n-1}\| + \|T^{n-1}x_{n-1} - x_{n-1}\| \\
&\quad + \|x_{n-1} + x_n\| + \mu_{n-1}l'_{n-1} + l_{n-1}
\end{aligned} \tag{3.19}$$

Using (3.10) and (3.13) in (3.19) we have that

$$\lim_{n\to\infty}\|T^{n-1}x_n - x_n\| = 0. \tag{3.20}$$

But,

$$\begin{aligned} \|x_n - Tx_n\| &= \|x_n - T^n x_n + T^n x_n - Tx_n\| \\ &\leq \|x_n - T^n x_n\| + \|T^n x_n - Tx_n\| \\ &= \|x_n - T^n x_n\| + \|T(T^{n-1}x_n) - Tx_n\| \end{aligned} \tag{3.21}$$

Again since T is uniformly continuous, then using (3.13) and (3.20) in (3.21) we obtain that $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$.
Hence , $\lim_{n\to\infty} \|Tx_n - x_n\| = 0 = \lim_{n\to\infty} \|Sx_n - x_n\|$. This completes the proof. □

3.3. **Weak convergence results.**

Theorem 3.3. *Let E be a uniformly convex real Banach space, C be a nonempty closed convex subset of E, $T : C \to C$ be a uniformly continuous S-generalized asymptotically nonexpansive mapping with sequences $\{\mu_n\}_{n=1}^{\infty}, \{l_n\}_{n=1}^{\infty} \subseteq [0, +\infty)$ such that $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} l_n < \infty$. Suppose that S is a uniformly continuous generalized asymptotically nonexpansive mapping with with sequences $\{\mu'_n\}_{n=1}^{\infty}, \{l'_n\}_{n=1}^{\infty} \subseteq [0, +\infty)$ such that $\sum_{n=1}^{\infty} \mu'_n < \infty$, $\sum_{n=1}^{\infty} l'_n < \infty$ and that $F = F(T) \cap F(S) \neq \emptyset$. Let $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty} \subseteq [\delta, 1-\delta]$ $for\ some$ $\delta \in (0,1)$. Then the sequence $\{x_n\}_{n=1}^{\infty}$ generated by (2.10) converges weakly to a common fixed point of T and S.*

Proof. As in the proof of Theorem 3.1, it follows that $p \in F$, $\lim_{n\infty} \|x_n - p\|$ exists and so the sequence $\{x_n\}_{n=1}^{\infty}$ is bounded. Since E is uniformly convex, and thus reflexive, there exists a subsequence $\{x_{\sigma(n)}\}_{n=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that $\{x_{\sigma(n)}\}_{n=1}^{\infty} \rightharpoonup p^*$ as $n \to \infty$ for some $p^* \in E$. From Theorem 3.2 we have that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = \lim_{n\to\infty} \|x_n - Sx_n\| = 0.$$

Thus, by Lemma 2.7, we know that $p^* \in F = F(T) \cap F(S)$. If $F = F(T) \cap F(S)$ is a singleton, then the proof is complete. If $F = F(T) \cap F(S)$ is not a singleton, we claim that p^* is unique. If not, let $q^* \in E$, $q^* \neq p^*$ be another weakly limit point of $\{x_n\}_{n=1}^{\infty}$, then there exists another subsequence $\{x_{\gamma(n)}\}_{n=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that$\{x_{\gamma(n)}\}_{n=1}^{\infty} \rightharpoonup q^*$ as $n \to \infty$ for some $q^* \in E$. By Theorem (3.2) and lemma (2.7) guarantees that $q^* \in F(T)$, $q^* \in F(S)$.Thus, $q^* \in F$. Since $p^* \neq q^*$ and E satisfies Opial condition, then

$$\begin{aligned} \lim_{n\to\infty} \|x_n - p^*\| &= \liminf_{n\to\infty} \|x_n - p^*\| \\ &= \liminf_{n\to\infty} \|x_{\sigma(n)} - p^*\| \\ &< \liminf_{n\to\infty} \|x_{\sigma(n)} - q^*\| \\ &= \lim_{n\to\infty} \|x_{\sigma(n)} - q^*\| \\ &= \lim_{n\to\infty} \|x_{\gamma(n)} - q^*\| \\ &= \liminf_{n\to\infty} \|x_{\gamma(n)} - q^*\| \\ &< \liminf_{n\to\infty} \|x_{\gamma(n)} - p^*\| \\ . &= \lim_{n\to\infty} \|x_n - p^*\|, \end{aligned}$$

This is a contradiction. Thus,$q^* = p^*$. Hence, $\{x_n\}$ converges weakly to an element of $F = F(T) \cap F(S)$. This completes the proof. □

Corollary 3.4. *Let E be a uniformly convex real Banach Space, K be a nonempty closed convex subset of E, $T : C \to C$ be uniformly continuous S- generalized asymptotically nonexpansive mapping with sequences $\{\mu_n\}_{n\geq 1}$, $\{\iota_n\}_{n\geq 1} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty} \mu_n < \infty$ and $\sum_{n=1}^{\infty} \iota_n < \infty$. Suppose that S is a uniformly continuous total asymptotically nonexpansive mapping of C with sequences $\{\mu'_n\}_{n=1}^{\infty}$,$\{\iota'_n\}_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} \mu'_n < \infty$ and $\sum_{n=1}^{\infty} \iota'_n < \infty$ and gauge function $\varphi : [0,\infty) \to [0,\infty)$. Suppose that there exists constants $M_0 > 0, M_1 > 0$ such that $\forall\, t > M_0$, $\varphi(t) \leq M_1 t$, and that $F := F(T) \cap F(S) \neq \emptyset$.Then, the sequence $\{x_n\}_{n=1}^{\infty}$ generated by (2.10) converges weakly to a common fixed point of T and S.*

Proof. By Lemma 2.9; S is a generalized asymptotically nonexpansive mapping. Thus, by Theorem 3.3, the result follows. □

Corollary 3.5. *Let E be a uniformly convex real Banach Space, C be a nonempty closed convex subset of E, $T : C \to C$ be uniformly continuous S- generalized asymptotically nonexpansive mapping with sequences $\{\mu_n\}_{n\geq 1}$, $\{\iota_n\}_{n\geq 1} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty} \mu_n < \infty$ and $\sum_{n=1}^{\infty} \iota_n < \infty$. Suppose that S is a uniformly continuous total asymptotically I nonexpansive self mapping of C with sequences $\{\mu'_n\}_{n=1}^{\infty}$,$\{\iota'_n\}_{n=1}^{\infty}$, a gauge function $\phi : [0,\infty) \to [0,\infty)$ such that $\sum_{n=1}^{\infty} \mu'_n < \infty$ and $\sum_{n=1}^{\infty} \iota'_n < \infty$ where $I : C \to C$ is a total asymptotically nonexpansive self mapping with sequences $\{\mu^*_n\}_{n=1}^{\infty}$,$\{\iota^*_n\}_{n=1}^{\infty}$ with a gauge function $\Psi : [0,\infty) \to [0,\infty)$ such that $\sum_{n=1}^{\infty} \mu^*_n < \infty$ and $\sum_{n=1}^{\infty} \iota^*_n < \infty$.Suppose that there exists constants $M_1 > 0$,$M^*_1 > 0$, $M_2 > 0$,$M^*_2 > 0$, such that $\forall\, q > M_1$, $\Phi(q) \leq M^*_1 q$,and $\forall\, t > M_2$, $\Psi(t) \leq M^*_2 t$ and that $F := F(T) \cap F(S) \neq \emptyset$, then, the sequence $\{x_n\}_{n=1}^{\infty}$ generated by (2.10) converges weakly to a common fixed point of T and S.*

Proof. S is a generalized asymptotically nonexpansive mapping by Lemma 2.10 and by Theorem 3.3, the result follows. □

Corollary 3.6. *Let E be a uniformly convex real Banach Space, C be a nonempty closed convex subset of E, $T : C \to C$ be uniformly continuous S- generalized asymptotically nonexpansive mapping with sequences $\{\mu_n\}_{n\geq 1}$, $\{\iota_n\}_{n\geq 1} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty} \mu_n < \infty$ and $\sum_{n=1}^{\infty} \iota_n < \infty$. Suppose that S is a uniformly continuous total asymptotically weakly contractive mapping of C with sequences $\{\mu'_n\}_{n=1}^{\infty}$,$\{\iota'_n\}_{n=1}^{\infty}$ and gauge functions $\Phi, \Psi : [0,\infty) \to [0,\infty)$ such that $\sum_{n=1}^{\infty} \mu'_n < \infty$ and $\sum_{n=1}^{\infty} \iota'_n < \infty$. Suppose that there exists constants $M_0 > 0, M_1 > 0$ such that $\forall\, t > M_0$, $\Psi(t) \leq M_1 t$, and that $F := F(T) \cap F(S) \neq \emptyset$ .Then, the sequence $\{x_n\}_{n=1}^{\infty}$ generated by(2.10) converges weakly to a common fixed point of T and S.*

Proof. By Lemma 2.11; S is a generalized asymptotically nonexpansive mapping. Thus, by Theorem 3.3, the result follows. □

3.4. **Strong convergence results.**

Theorem 3.4. *Let E be a uniformly convex Banach Space, C be a nonempty closed convex subset of E, T be a uniformly continuous $S-$generalized asymptotically nonexpansive self-mmapings of C with sequences μ_n , $\iota_n \subset [0,\infty)$ such that $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \iota_n < \infty$. Suppose S is a uniformly continuous generalized asymptotically nonexpansive mappings of C with sequences μ'_n, $l'_n \subset [0,\infty)$ such that*

$\sum_{n=1}^{\infty}\mu'_n < \infty, \sum_{n=1}^{\infty}\iota'_n < \infty$. *Suppose that* T *and*S *satisfy condition (B)and* $F = F(T) \cap F(S) \neq \emptyset$. *Let* $\{\alpha_n\}$ *and* $\{\beta_n\} \subset [\delta, 1-\delta]$ *for some* $\delta \in (0,1)$. *Then the explicit iterative sequence* $\{x_n\}$ *defined by (2.10) converges strongly to a common fixed point of* T *and* S.

Proof. Since T and S satisfy condition (B), we obtain that there exists a nondecreasing continuous fuction $f : [0, +\infty) \to [0, +\infty)$ with $f(0) = 0$ such that $\forall\, n \in \mathbb{N}$

$$\frac{1}{2}\big(\|x_n - Tx_n\| + \|x_n - Sx_n\|\big) \geq f\big(x_n, F)\big) \geq 0.$$

Since by Theorem 3.2, $\lim_{n\to\infty}\|x_n - Tx_n\| = 0 = \lim_{n\to\infty}\|x_n - Sx_n\|$, then we obtain by Sandwich Theorem that $\lim_{n\to\infty} f\big(d(x_n, F)\big) = 0$. Since f is continuous, we obtain that $\lim_{n\to\infty}(d(x_n, F) = 0$. Thus, by Theorem 3.1, this implies that $\{x_n\}$ is convergent. This completes the proof. □

Corollary 3.7. *Let* E *be a uniformly convex real Banach Space,* C *be a nonempty closed convex subset of* E, $T : C \to C$ *be* S*- generalized asymptotically nonexpansive self- mappings of* C *with sequences* $\{\mu'_n\}_{n\geq 1}$, $\{\iota'_n\}_{n\geq 1} \subset [0,\infty)$ *such that* $\sum_{n=1}^{\infty}\mu'_n < \infty$ *and* $\sum_{n=1}^{\infty}\iota'_n < \infty$. *Suppose that* S *is a uniformly continous total asymptotically nonexpansive self mappings of* C *with sequences* $\{\mu_n\}_{n=1}^{\infty}$,$\{\iota_n\}_{n=1}^{\infty}$ *and gauge function* $\varphi : [0,\infty) \to [0,\infty)$ *such that* $\sum_{n=1}^{\infty}\mu_n < \infty$ *and* $\sum_{n=1}^{\infty}\iota_n < \infty$. . *Suppose that there exists constants* $M_0 > 0, M_1 > 0$ *such that* $\forall\, t > M_0$, $\varphi(t) \leq M_1 t$. *Suppose that* $F(T) \cap F(S) \neq \emptyset$ *and that* T *and* S *satisfy condition* (B), *then the explicit iterative sequence* $\{x_n\}_{n=1}^{\infty}$ *defined by 2.10 converges strongly to a common fixed point of* T *and* S.

Proof. By Lemma 2.9; S is a generalized asymptotically nonexpansive mapping, and by Theorem 3.4, the result follows. □

Corollary 3.8. *Let* E *be a uniformly convex real Banach Space,* C *be a nonempty closed convex subset of* E, $T : C \to C$ *be uniformly continuous* S*- generalized asymptotically nonexpansive self- mappings of* C *with sequences* $\{\mu'_n\}_{n\geq 1}$, $\{\iota'_n\}_{n\geq 1} \subset [0,\infty)$ *such that* $\sum_{n=1}^{\infty}\mu'_n < \infty$ *and* $\sum_{n=1}^{\infty}\iota'_n < \infty$. *Suppose that* S *is a uniformly continuous total asymptotical* $I-$ *nonexpansive self mappings of* C *with sequences* $\{\mu_n\}_{n=1}^{\infty}$,$\{\iota_n\}_{n=1}^{\infty}$ *with a gauge function* $\phi : [0,\infty) \to [0,\infty)$ *such that* $\sum_{n=1}^{\infty}\mu_n < \infty$, $\sum_{n=1}^{\infty}\iota_n < \infty$. *and* $I : C \to C$ *be a total asymptotically nonexpansive mapping with sequences* $\{\mu^*_n\}_{n=1}^{\infty}$,$\{\iota^*_n\}_{n=1}^{\infty}$ *with a gauge function* $\Psi : [0,\infty) \to [0,\infty)$ *such that* $\sum_{n=1}^{\infty}\mu^*_n < \infty$ *and* $\sum_{n=1}^{\infty}\iota^*_n < \infty$. *Suppose that there exists constants* $M_1 > 0$,$M^*_1 > 0$, $M_2 > 0$,$M^*_2 > 0$, *such that* $\forall\, q > M_1$, $\Phi(q) \leq M^*_1 q$,*and* $\forall\, t > M_2$, $\Psi(t) \leq M^*_2 t$. *Suppose further that and* $F(T) \cap F(S) \neq \emptyset$ *and that* T *and* S *satisfy condition* (B), *then the explicit iterative sequence* $\{x_n\}_{n=1}^{\infty}$ *defined by 2.10 converges strongly to a common fixed point of* T *and* S.

Proof. By Lemma 2.10; S is a generalized asymptotically nonexpansive mapping and by Theorem 3.4, the result follows. □

Corollary 3.9. *Let* E *be a uniformly convex real Banach Space,* C *be a nonempty closed convex subset of* E, $T : C \to C$ *be a uniformly- continous* S*- generalized asymptotically nonexpansive self- mappings of* C *with sequences* $\{\mu_n\}_{n\geq 1}$, $\{\iota_n\}_{n\geq 1} \subset [0,\infty)$ *such that* $\sum_{n=1}^{\infty}\mu_n < \infty$ *and* $\sum_{n=1}^{\infty}\iota_n < \infty$. *Suppose that* S *is a uniformly continuous total asymptotically weakly contractive self mappings of* C *with sequences* $\{\mu'_n\}_{n=1}^{\infty}$

,$\{\iota'_n\}_{n=1}^{\infty}$ and gauge functions $\Phi, \Psi : [0, \infty) \to [0, \infty)$ such that $\sum_{n=1}^{\infty} \mu'_n < \infty$ and $\sum_{n=1}^{\infty} \iota'_n < \infty$.. Suppose that there exists constants $M_0 > 0, M_1 > 0$ such that $\forall\, t > M_0$, $\Psi(t) \leq M_1 t$, Suppose further that $F(T) \cap F(S) \neq \emptyset$and that T and S satisfy condition (B). Then,the explicit iterative sequence $\{x_n\}_{n=1}^{\infty}$ defined by (2.10) converges strongly to a common fixed point of T and S.

Proof. S is a generalized asymptotically nonexpansive mapping by Lemma 2.11 and by Theorem 3.4, the result follows. □

4. Conclusion

It is of interest to note here that extension of the results obtained in this paper to finite families of classes of S-generalized asymptotically nonexpansive mappings leads to no further generalization since the method of proof displayed in this paper carries of to finite family of Mappings. Moreover, addition of error terms to the iterative algorithm studied in this paper leads to no further generalization.

Statements and Declarations

The authors declare that they have no conflict of interest, and the manuscript has no associated data.

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