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APPROXIMATION OF SOLUTIONS OF NONLINEAR PROBLEMS IN HYPERBOLIC SPACES

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ABSTRACT. In this paper, we introduce a new iterative algorithm for approximating a common element of the set of solutions of an attractive point of further 2-generalized hybrid mapping, equilibrium problem and a common zero of a finite family of monotone operators in hyperbolic spaces. We establish strong convergence theorem under suitable assumptions, and also give numerical example to support our main result. Our results generalize and improve many recent results in the literature.

Keywords. Hyperbolic space, Attractive point, Equilibrium.

1. INTRODUCTION

The concept of attractive points was originally studiied in Hilbert space by Takahashi and Takeuchi [31]. The introduction was motivated basically to get rid of the closedness and convexity hypotheses imposed on the nonempty subset $C \subset H$ in a celebrated Bailon's [4] nonlinear ergodic theorem.

Let C be a nonempty subset of a metric space X and let $T : C \longrightarrow X$ be a nonlinear mapping. We denote the set of attractive point of T by $A(T)$ and defined by

$$A(T) = \{u \in X : d(Tv, u) \leq d(v, u), \forall v \in C\}$$

Recall that a mapping $T : C \longrightarrow X$ is said to be (α, β)-generalized hybrid [20] if there exists $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha d^2(Tx, Ty) + (1 - \alpha)d^2(x, Ty) \leq \beta d^2(Tx, y) + (1 - \beta)d^2(x, y), \forall x, y \in C.$$

Observe that mapping T reduces to a nonexpansive mapping if $\alpha = 1$ and $\beta = 0$. i.e.,

$$d(Tx, Ty) \leq d(x, y), \ \forall x, y \in C.$$

If $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$, then it is said to be hybrid [21, 30], i.e.,

$$3d^2(Tx, Ty) \leq d^2(x, y) + d^2(Tx, y) + d^2(Ty, x), \ \forall x, y \in C.$$

It is also said to be nonspreading [21, 22] if $\alpha = 2$ and $\beta = 1$. i.e.,

$$2d^2(Tx, Ty) \leq d^2(Tx, y) + d^2(Ty, x), \ \forall x, y \in C.$$

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Let's recall that a mapping T is said to be normally generalized hybrid [34] if there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

(a) $\alpha + \beta + \gamma + \delta \geq 0$
(b) $\alpha + \beta > 0$ or $\alpha + \gamma > 0$, and

$$\alpha d^2(Tx, Ty) + \beta d^2(x, Ty) + \gamma d^2(Tx, y) + \delta d^2(x, y) \leq 0, \ \forall x, y \in C.$$

To generalize the class of normally generalized hybrid mapping, the class of normally 2-generalized hybrid and further generalized hybrid were introduced. A mapping T is said to be

(i) normally 2-generalized hybrid [23] if there exist $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3 \in \mathbb{R}$ such that
 (a) $\sum_{i=1}^{3}(\alpha_i + \beta_i) \geq 0$
 (b) $\sum_{i=1}^{3} \alpha_i > 0$, and
 (c) $\alpha_1 d^2(T^2x, Ty) + \alpha_2 d^2(Tx, Ty) + \alpha_3 d^2(x, Ty) + \beta_1 d^2(T^2x, y) + \beta_2 d^2(Tx, y) + \beta_3 d^2(x, y) \leq 0, \forall x, y \in C$.
(ii) further generalized hybrid [16] if there exists $\alpha, \beta, \gamma, \delta, \epsilon \in \mathbb{R}$ such that
 (a) $\alpha + \beta + \gamma + \delta \geq 0, \epsilon \geq 0$.
 (b) $\alpha + \beta > 0$ or $\alpha + \gamma > 0$ and
 (c) $\alpha d^2(Tx, Ty) + \beta d(x, Ty) + \gamma d^2(Tx, y) + \delta d(x, y) + \epsilon d(x, Tx) \leq 0, \forall x, y \in C$.

Convergence theorems for attractive points of the above mentioned generalized nonlinear mappings have been studied in Hilbert spaces by various authors; see for example, [24, 16, 32, 34]. In 2008, Safeer Hussain Khan [16] proposed a mann iterative scheme that converges weakly to a common attractive point of two further generalized hybrid mappings in Hilbert spaces. Kondo and Takahashi [24] constructed a Halpern's type iterative scheme that converges strongly to an attractive point of normally 2-generalized hybrid mappings,also in Hilbert spaces.

Let (X, d) be a metric space and $x, y \in X$. Let $d(x, y) = l$. An isometry $c : [0, l] \longrightarrow X$ satisfying $c(0) = x$ and $c(l) = y$ is called a geodesic path joining x to y. A geodesic segment between x and y is the image of a geodesic path joining x to y, which is denoted by [x,y] when it is unique. A geodesic space is a metric space (X, d) in which every two points of X are joined by a geodesic segment. A metric space in which every two points of the space are joined by only one geodesic segment is referred to as uniquely geodesic space. Let X be a uniquely geodesic space and $(1-t)xoplusty$ denote the unique point z of the geodesic segment joining x to y for each $x, yinX$ such that $d(z, x) = td(x, y)$ and $d(z, y) = (1-t)d(x, y)$. Set $[x, y] := \{(1-t)x \oplus ty : t \in [0, 1]\}$.Then a subset $C \subset X$ is said to be convex if $[x, y]subsetC$ for all $x, y \in C$.

A geodesic space (X, d) is a CAT(0) space if and only if it satisfies the (CN) inequality,[7] i.e., If x,y,z are points in X and q is the midpoint of the segment [y,z], then

$$d^2(x, q) \leq \frac{1}{2}d^2(x, y) + \frac{1}{2}d^2(x, z) - \frac{1}{4}d^2(y, z).$$

A complete CAT(0) space is known as Hadamard space. Examples include, Hilbert spaces, the Hilbert ball, Euclidean space $\mathbb{R}^n$, $\mathbb{R} - trees$ [29]

A geodesic space (X, d) is called a hyperbolic space [1]. if for any $x, y, z \in X$,

$$d(\frac{1}{2}z \oplus \frac{1}{2}x, \frac{1}{2}z \oplus \frac{1}{2}y) \leq \frac{1}{2}d(x, y).$$

Equivalently [1], a hyperbolic space is a geodesic space (X, d) that satisfies

$$d(\alpha x \oplus (1-\alpha)y, \alpha w \oplus (1-\alpha)z) \leq \alpha d(x, w) + (1-\alpha)d(y, z),$$

for all $x, y, z, w \in X, \alpha \in (0, 1)$. The class of hyperbolic spaces include the normed spaces, CAT(0) spaces and some others.

Let (X,d) be a metric space and $\{x_n\}_{n=1}^{\infty}$ be any bounded sequence in X. For $x \in X$, set $r(x,\{x_n\}) := \limsup_{n \to \infty} d(x_n, x)$, then

- the asymptotic radius of the sequence $\{x_n\} \subseteq X$ denoted by $r(\{x_n\})$ is defined by

$$r(\{x_n\}) = \inf_{x \in X} r(\{x_n\}, x).$$

- the asymptotic center of $\{x_n\} \subseteq X$ is a set

$$A(\{x_n\}) = \{z \in X : r(z,\{x_n\}) = r(\{x_n\})\}.$$

In a Complete $CAT(0)$ space,it is known that $A(\{x_n\})$ consists of exactly one point. A sequence $\{x_n\} \subseteq X$ is said to Δ-converge to x if every subsequence $\{x_{n_k}\}$ of $\{x_n\}$ satisfies the condition that

$$A(\{x_{n_k}\}) = \{x\}.$$

That is to say a sequence $\{x_n\} \subseteq X$ Δ-converges to a point $x \in X$ if x is the unique asymptotic center of $\{x_{n_k}\}$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and this is written as $\Delta - \lim_{n \to \infty} x_n = x$.

Berg and Nikolev [5] introduced the notion of quasilinearization in $CAT(0)$ spaces. Let X be a $CAT(0)$ space and $(a,b) \in X \times X$. Then quasilinearization is a map $\langle , \rangle : (X \times X) \times (X \times X) \to \mathbb{R}$ defined by

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2}d^2(a,d) + \frac{1}{2}d^2(b,c) - \frac{1}{2}d^2(a,c) - \frac{1}{2}d^2(b,d), \forall a,b,c,d \in X.$$

It can easily be checked that $\langle \overrightarrow{ab}, \overrightarrow{ab} \rangle = d^2(a,b)$, $\langle \overrightarrow{ba}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle$, $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{ae}, \overrightarrow{cd} \rangle + \langle \overrightarrow{eb}, \overrightarrow{cd} \rangle$ and $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{cd}, \overrightarrow{ab} \rangle$ $\forall a,b,c,d,e \in X$. We say that the spacce X satisfies Chauchy-Schwart inequality if

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \leq d(a,b)d(c,d), \ \forall a,b,c,d \in X.$$

Kakavandi and Amini [3] introduced the concept of duality in a complete $CAT(0)$ space X based on the work of Berg and Nikolaev [5].

Consider the map $H : \mathbb{R} \times X \times X \to C(X)$ defined by

$$H(t,a,b)(x) = t\langle \overrightarrow{ab}, \overrightarrow{ax} \rangle, \ t \in \mathbb{R}, a,b,x \in X,$$

where $C(X,\mathbb{R})$ is a space of all continuous real-valued functions on X. Then the Cauchy-Schwarz inequality implies that the map $H(t,a,b)$ is a Lipschitz map with Lipschitz semi-norm $L(H(t,a,b)) = td(a,b), \forall\, t \in \mathbb{R}\; and\; a,b \in X$, where $L(\varphi) = \sup\{\frac{\varphi(x)-\varphi(y)}{d(x,y)} : x,\ y \in X, x \neq y\}$ is the semi-norm for any function $\varphi : X \to \mathbb{R}$.

Define a map M on $\mathbb{R} \times X \times X$ by

$$M((t,a,b),(s,c,d)) = L(H(t,a,b) - H(s,c,d)), \forall\, t,s \in \mathbb{R}, a,b,c,d \in X.$$

Clearly $\hat{D}$ is a pseudometric.

A relation $\sim$ on $\mathbb{R} \times X \times X$ defined by $(t,a,b) \sim (s,c,d)$ if $M((t,a,b),(s,c,d)) = 0$ is an equivalence relation, where the equivalence class of (t,a,b) is given as

$$[t\overrightarrow{ab}] = \{s\overrightarrow{cd} : t\langle \overrightarrow{ab}, \overrightarrow{xy} \rangle = s\langle \overrightarrow{cd}, \overrightarrow{xy} \rangle, x,y \in X\}.$$

We denote by $X^* := \{[t\overrightarrow{ab}] : (t,a,b) \in \mathbb{R} \times X \times X\}$ the set of all equivalence classes of (t,a,b). This together with the metric $\hat{D}$ on X^* is called the dual space of (X,d).

The concept of attractive points of a nonlinear map T was first studied in the setting of $CAT(0)$ spaces by Kunwai, Kaewkhao and Inthakon [25]. In 2015, Kaekhaon, Inthakon and Kunwai [14] proved the Δ-convergence of a Mann-type scheme to a point in the set of attractive points of normally generalized hybrid mappings. Also,Cuntavepanit and Phuengrattana [10] studied the class of further generalized hybrid mappings in Hadamard spaces. They established the demiclosed principle and proved the Δ-convergence for attractive points.

Recently, Ali and Yusuf [2] introduced a further 2-generalized hybrid mapping, which includes normally 2-generalized hybrid and further generalized hybrid mappings as special cases in a complete CAT(0) space. They constructed the below Halpern's type iterative scheme for finding an element in the set of attractive point of such mapping.

$$\begin{cases} y_n = \alpha_n x_n \oplus \beta_n T x_n \oplus \gamma_n T^2 x_n \\ x_{n+1} = \delta_n u \oplus (1-\delta_n) y_n. \end{cases} \tag{1.1}$$

The new generalized nonlinear map is defined below as;
Let X be a complete CAT(0) space and let C be a nonempty subset of X. A mapping $T : C \longrightarrow C$ is said to be further 2-generalized hybrid if there exists $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \epsilon_1, \epsilon_2 \in \mathbb{R}$ such that

(i) $\sum_{i=1}^{3}(\alpha_i + \beta_i) \geq 0$, $\epsilon_1, \epsilon_2 \geq 0$.
(ii) $\sum_{i=1}^{3} \alpha_i > 0$
(iii) $\alpha_1 d^2(T^2x, Ty) + \alpha_2 d^2(Tx, Ty) + \alpha_3 d^2(x, Ty) + \beta_1 d^2(T^2x, y) + \beta_2 d^2(Tx, y) + \beta_3 d^2(x, y) + \epsilon_1 d^2(x, T^2x) + \epsilon_2 d^2(x, Tx) \leq 0, \forall x, y \in C$.

Remark 1.1. If $\alpha_1 = \beta_1 = \epsilon_2 = 0$, then the mapping is reduced to further generalized hybrid mapping. Also, the mapping is reduced to a normally 2-generalized hybrid mapping if $\epsilon_1 = \epsilon_2 = 0$.

Let C be a nonempty closed convex subset of a hyperbolic space X and $f : C \times C \longrightarrow \mathbb{R}$ be a bifunction. The equilibrium problem for a bifunction f is to find;

$$x^* \in C \text{ such that } f(x^*, z) \geq 0, \forall z \in C. \tag{1.2}$$

The set of solutions of (1.2) is denoted by $EP(f, C)$. This problem was originally studied in [6] which includes, as a special cases, many important Mathematical problems such as optimization problems, variational inequality problems, saddle point problems and other problems of interest in many applications.

Methods of solving Equilibrium problems and their generalizations have been a very important tool for solving problems arising in the areas of linear or nonlinear programming, variational inequalities, optimization problems, fixed point problems and so on. It has been widely applied to physics, structural analysis, management sciences, e.t.c., see for example [3, 9, 13, 26, 27]. Various methods have been used to study equilibrium problems, one of such methods is the proximal point algorithm which was used in [17] to study the existence of solutions of equilibrium problems. Other methods include the extragradient method which was introduced in [28] by Quoc et al. in the setting of Hilbert spaces. They studied the following scheme;

$$\begin{cases} z_n \in \text{Argmin}_{z\in C}\{f(x_n, z) + \frac{1}{2\lambda_n}\|z - x_n\|^2\}, \\ x_{n+1} \in \text{Argmin}_{z\in C}\{f(z_n, z) + \frac{1}{2\lambda_n}\|z - x_n\|^2\}. \end{cases} \tag{1.3}$$

and they established weak convergence of the sequence $\{x_n\}$ generated by (1.3) to a solution of some equilibrium problem. In recent time, several authors have extended the notion of equilibrium to Hadamard spaces.

Khatibzadeh and Mohebbi [18] studied both $\Delta-$convergence and strong convergence of a sequence generated by the Extragradient Method for pseudo-monotone equilibrium problems in a complete $CAT(0)$ space.

Let X be a hyperbolic space with dual X^* and let $A : X \longrightarrow 2^{X^*}$ be a multivalued operator with domain $D(A) := \{x \in X : Ax \neq \emptyset\}$, range $R(A) := \bigcup_{x\in X} Ax$, $A^{-1}(x^*) = \{x \in X : x^* \in Ax\}$ and graph $gra(A) := \{(x, x^*) \in X \times X^* : x \in D(A), x^* \in Ax\}$. Also, let X be a Hadamard space with dual X^*. The multivalued operator $A : X \longrightarrow 2^{X^*}$ is said to be monotone if the inequality $\langle x^* - y^*, \overrightarrow{yx}\rangle \geq 0$ holds for every $(x, x^*), (y, y^*) \in gra(A)$. [19].

A monotone operator $A : X \longrightarrow 2^{X^*}$ is maximal if there exists no monotone operator $B : X \longrightarrow 2^{X^*}$ such that $gra(B)$ properly contains $gra(A)$ (thai is, for any $(y, y^*) \in X \times X^*$, the inequality $\langle x^* - y^*, \overrightarrow{yx}\rangle \geq 0$ for all $(x, x^*) \in gra(A)$ implies that $y^* \in Ay$).

The resolvent of a multivalued operator $A : X \longrightarrow 2^{X^*}$ of order $\lambda > 0$ is the multivalued mapping $J_\lambda^A : X \longrightarrow 2^X$ defined by $J_\lambda^A(x) := \{z \in X : [\frac{1}{\lambda}\overrightarrow{zx}] \in Az\}$. Monotone operator A is said to satisfy range condition if for every $\lambda > 0$, $D(J_\lambda^A) = X$, where $D(J_\lambda^A)$ is the domain of J_λ^A.

Let $A : X \longrightarrow 2^{X^*}$ be a monotone operator. A monotone inclusion problem is a problem of the form:

$$\text{find } \ x \in D(A) \ \text{ such that } \ 0 \in Ax. \tag{1.4}$$

The solution set of equation (1.4) is denoted by $A^{-1}(0)$ [26].

One of the most important problems in monotone operator theory is approximating a zero of a monotone operator. Martinet [26] introduced one of the most popular methods for approximating a zero of a monotone operator in Hilbert spaces that is called the proximal point algorithm. Recently, Khatibzadeh and Ranjbar [19] generalized monotone operators and their resolvents to Hadamard spaces by using the duality theory.

Very recently, Moharami and Eskandani [27] proposed the following extragradient type algorithm for finding a common element of the set of solutions of an equilibrium problem for a single bifunction f and a common zero of a finite family of monotone operators $A_1, A_2, \cdots, A_N$ in Hadamard spaces;

$$\begin{cases} w_n = J_{\beta_n^N}^{A_N} \circ J_{\beta_n^{N-1}}^{A_{N-1}} \circ \cdots \circ J_{\beta_n^1}^{A_1} x_n, \\ y_n = \underset{y \in K}{\text{argmin}}\{f(w_n, y) + \frac{1}{2\lambda_n} d^2(w_n, y)\}, \\ r_n = \underset{y \in K}{\text{argmin}}\{f(y_n, y) + \frac{1}{2\lambda_n} d^2(w_n, y)\}, \\ x_{n+1} = \alpha_n w \oplus (1 - \alpha_n) r_n, \end{cases} \tag{1.5}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\lambda_n\}$ are sequences satisfying some conditions. They proved strong convergence theorem of the sequence $\{x_n\}$ generated by the above scheme.

In this article, motivated and inspired by the result of Moharami and Eskandani [27], and the result of Ali and Yusuf [2], we propose an iterative algorithm for finding a common element of the set of solutions of an attractive point problem of further 2-generalized hybrid mapping, equilibrium problem and a common zero of a finite family of monotone operators in hyperbolic spaces. The strong convergence theorem is established under suitable assumptions.We also give numerical example to support our main result.

2. Preliminaries

The following notions and results are very vital in our subsequent discussion.

Definition 2.1. ([18]) A function $f : X \to (-\infty, +\infty]$ is called

i) convex if

$$f((1 - \sigma)x \oplus \sigma y) \leq (1 - \sigma)f(x) + \sigma f(y)\ \forall x, y \in X \text{ and } \sigma \in [0, 1].$$

ii) strictly convex if

$$f((1 - \sigma)x \oplus \sigma y) < (1 - \sigma)f(x) + \sigma f(y)\ \forall x, y \in X\ x \neq y \text{ and } \sigma \in [0, 1].$$

Remark 2.2. Observed that if f is strictly convex, then the minimizer of f is unique.

Definition 2.3. Let X be a hyperbolic space and $g : D(g) \subseteq X \to \mathbb{R}$ be a function (D(g) denotes the domain of g). Then g is said to be $\Delta-$upper semicontinuous at some point $x_0 \in D(g)$ if

$$g(x_0) \geq \limsup g(x_n)$$

for every sequence $\{x_n\} \subseteq D(g)$ satisfying the condition that $\Delta - \lim_{n\to\infty} x_n = x_0$. We say that g is $\Delta-$upper semicontinuous on $D(g)$ if it is $\Delta-$upper semicontinuous at every point in $D(g)$.

Definition 2.4. Let X be a hyperbolic space and $h : D(h) \subseteq X \to \mathbb{R}$ be a function (D(h) denotes the domain of h). Then h is said to be $\Delta-$lower semicontinuous at some point $x_0 \in D(h)$ if

$$h(x_0) \leq \limsup h(x_n)$$

for every sequence $\{x_n\} \subseteq D(h)$ satisfying the condition that $\Delta - \lim_{n\to\infty} x_n = x_0$. We say that h is $\Delta-$lower semicontinuous on $D(h)$ if it is $\Delta-$lower semicontinuous at every point in $D(h)$.

Definition 2.5. [27] Let X be a hyperbolic space. A bifunction $f : X \times X \to \mathbb{R}$ is said to be monotone and pseudo-monotone if for every $x, y \in X$,

$$f(x,y) + f(y,x) \leq 0 \; and \; f(x,y) \geq 0 \text{ implies } f(y,x) \leq 0.$$

respectively.

In this paper, f is assumed to satisfy the following conditions;

$B_1 :$ $f(x,.) : X \to \mathbb{R}$ is convex and lower semicontinuous for all $x \in X$.
$B_2 :$ $f(.,y) : X \to \mathbb{R}$ is $\Delta-$upper semicontinuous for all $y \in X$.
$B_3 :$ f is Lipschitz-type continuous, that is there exist two positive constant c_1 and c_2 such that

$$f(x,y) + f(y,z) \geq f(x,z) - c_1 d^2(x,y) - c_2 d^2(y,z), \; \forall x,y,z \in X.$$

$B_4 :$ f is pseudo-monotone.

Definition 2.6. [15] A hyperbolic space (X, d) is said to satisfy the $\mathbb{S}$ property if for any $(x,y) \in X \times X$, there exists a point y_x such that $[\overrightarrow{xy}] = [\overrightarrow{y_x x}]$.

Definition 2.7. [15] A hyperbolic space (X, d) is said to satisfy $(\overline{Q}_4)$ condition if for any $x, y, p, q \in X, d(p,x) < d(x,q) \quad and \quad d(p,y) < d(y,q) \quad imply \quad d(p,m) \leq d(m,q), \quad \forall m \in [x,y]$.

Definition 2.8. [33] Let l^∞ be the Banach space of bounded sequences with supremum norm and $\mu : l^\infty \to \mathbb{R}$ be a bounded and linear functional on l^∞. Let $\mu(f)(or \mu_n(x_n))$ denotes the value of μ at $f = (x_1, x_2, x_3, \ldots) \in l^\infty$. A mean μ_n is a linear functional defined on l^∞ satisfying $\mu_n(e) = ||\mu_n|| = 1$, where $e = (1,1,1,\ldots)$. And a Banach limit on l^∞ is a mean μ_n such that $\mu_n(x_{n+1}) = \mu_n(x_n)$.

Lemma 2.9. [11] *Let $(X, \ d)$ be a complete $CAT(0)$ space, $r, x, y, v \in X$ and $t \in (0,1)$. Then,*

i. $d(tx \oplus (1-t)y, v) \leq td(x,v) + (1-t)d(y,v)$,
ii. $d^2(tx \oplus (1-t)y, v) \leq td^2(x,v) + (1-t)d^2(y,v)$,
iii. $d(tx \oplus (1-t)y, tr \oplus (1-t)zv \leq td(x,r) + (1-t)d(y,v)$,
iv. $d^2(tx \oplus (1-t)y, v) \leq td^2(x,v) + (1-t)d^2(y,v) - t(1-t)d^2(x,y)$.

Lemma 2.10. [15] *Let X be a complete $CAT(0)$ space that satisfies the $\mathbb{S}$ property. Let $\{x_n\}$ be a sequence in X and $x \in X$. Then $\{x_n\}$ Δ converges to x iff $\limsup_{n\to\infty} \langle \overrightarrow{x_n x}, \overrightarrow{yx} \rangle = 0 \; \forall y \in X$.*

Let $x_1, x_2, \ldots, x_n$ be points in $CAT(0)$ spaces. For $\lambda_1, \lambda_2, \ldots, \lambda_n \in (0,1)$ with $\sum_{i=1}^n \lambda_i = 1$, we write

$$\bigoplus_{i=1}^n \lambda_i x_i = (1-\lambda_n)\Big(\frac{\lambda_1}{1-\lambda_n} x_1 \oplus \frac{\lambda_2}{1-\lambda_n} x_2 \oplus \cdots \oplus \frac{\lambda_{n-1}}{1-\lambda_n} x_{n-1}\Big) \oplus \lambda_n x_n,$$

where the definition of $\oplus$ is an ordered one in the sense that it depends on the order of points $x_1, x_2, \ldots, x_n$.

The notation above was introduced by Dompongsa, Kaewkhao and Panyanak [12] in $CAT(0)$ spaces.

Lemma 2.11. [8] *Let (X, d) be a complete $CAT(0)$ space and $x, y \in X$ $t_i \in (0,1)$. Then*

$$d^2(\oplus_{i=0}^{n} t_i x_i, y) \leq \sum_{i=0}^{n} t_i d^2(x_i, y) - t_i t_j d^2(x_i, x_j)$$

where $i, j \in \{0, 1, \ldots, n\}$ and $\sum_{i=0}^{n} t_i = 1$

Lemma 2.12. [1] *Every bounded sequence in a complete $CAT(0)$ space has a Δ-convergent subsequence.*

Lemma 2.13. [14, 25] *Let (X, d) be a $CAT(0)$ space and K be a nonempty subset of X.Let $T : K \to K$ be a mapping. Let $\{x_n\}$ be a bounded sequence in K such that $\lim_{n \to \infty} d(x_n, Tx_n) = 0$. Then*

(i) *The sequences $\{d(x_n, y)\}$ and $\{d(Tx_n, y)\}$ are bounded for all $y \in C$*
(ii) *$\mu_n d(x_n, y) = \mu_n d(Tx_n, y)$ for any Banach limit μ_n on l^∞.*

Lemma 2.14. [14] *Let (X,d) be a complete $CAT(0)$ space satisfying the $\overline{Q}_4$ condition. Let C be a nonempty subset of X and $T : C \longrightarrow X$ be any map. Then A(T) is closed and convex.*

Lemma 2.15. [11] *Let C be a closed and convex subset of a complete $CAT(0)$ space X, $T : C \longrightarrow C$ be a nonexpansive mapping and $\{x_n\}$ be a bounded sequence in C such that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ and x_n Δ-converges to x. Then x=Tx.*

Lemma 2.16. [27] *If a bifunction f satisfies conditions B_1, B_2 and B_4, then EP(f,C) is closed and convex.*

Lemma 2.17. [35] *Let $\{b_n\}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ be a sequence of real numbers in (0,1) with $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\{t_n\}$ be a sequence of real numbers. Suppose that*

$$b_{n+1} \leq (1 - \alpha_n) b_n + \alpha_n t_n, \ \forall n \geq 0.$$

If $\limsup_{k\to\infty} t_{n_k} \leq 0$, then, for every subsequence $\{b_{n_k}\}$ of $\{b_n\}$ satisfying $\liminf_{k\to\infty}(b_{n_{k+1}} - b_{n_k}) \geq 0$, it holds that $\lim_{n\to\infty} b_n = 0$.

Lemma 2.18. [2] *Let (X, d) be a complete $CAT(0)$ space which satisfies the $(\mathbb{S})$ property and let C be a nonempty subset of X. Let $T : C \longrightarrow C$ be a further 2-generalized hybrid mapping. Let $\{x_n\}$ be a bounded sequence in K that Δ converges to x and $d(x_n, Tx_n) \to 0$, $d(x_n, T^2 x_n) \to 0$ as $n \to \infty$. Then $z \in A(T)$.*

Theorem 2.19. [19] *Let X be a CAT(0) space with dual X^* and let $A : X \longrightarrow 2^{X^*}$ be a multivalued mapping. Then*

(i) *For any $\lambda > 0$, $R(J_\lambda^A) \subset D(A)$, $F(J_\lambda) = A^{-1}(0)$.*
(ii) *If A is monotone, then J_λ^A is a single-valued on its domain and*

$$d^2(J_\lambda^A x, J_\lambda^A y) \leq \langle \overrightarrow{J_\lambda^A x J_\lambda^A y}, \overrightarrow{xy} \rangle, \forall x, y \in D(J_\lambda^A),$$

in particular J_λ^A is a nonexpansive mapping.
(iii) *If A is monotone and $0 < \lambda \leq \mu$, then $d^2(J_\lambda^A x, J_\mu^A x) \leq \frac{\mu-\lambda}{\mu+\lambda} d^2(x, J_\mu^A x)$, which implies that $d(x, J_\lambda^A x) \leq 2d(x, J_\mu^A x)$.*

Remark 2.20. It is well known that if T is a nonexpansive mapping on a subset C of a $CAT(0)$ space X, then $F(T)$ is closed and convex. Thus, if A is monotone operator on a $CAT(0)$ space X, then by parts (i) and (ii) of theorem 2.14, $A^{-1}(0)$ is closed and convex. Also by using part (iii) of the same theorem for all u $\in F(J_\lambda^A)$ and $x \in D(J_\lambda^A)$, we have

$$d^2(J_\lambda^A x, x) \leq d^2(u, x) - d^2(u, J_\lambda^A x). \tag{2.1}$$

Remark 2.21. Observe that Lemmas 2.9, 2.10, 2.11, 2.12, 2.14, 2.18 and Theorem 2.19 holds true also in the setting of hyperbolic spaces.

3. Main Results

In this section, we study the strong convergence of the following iterative scheme. Let $u, x_1 \in X$ and

$$\begin{cases} v_n = J^{A_N}_{\sigma_n^N} \circ J^{A_{N-1}}_{\sigma_n^{N-1}} \circ \cdots \circ J^{A_1}_{\sigma_n^1} x_n, \\ w_n = \underset{y \in C}{\text{argmin}}\{f(v_n, y) + \frac{1}{2\lambda_n} d^2(v_n, y)\}, \\ z_n = \underset{y \in C}{\text{argmin}}\{f(w_n, y) + \frac{1}{2\lambda_n} d^2(v_n, y)\}, \\ y_n = \alpha_n z_n \oplus \beta_n T z_n \oplus \gamma_n T^2 z_n, \\ x_{n+1} = \delta_n u \oplus (1-\delta_n) y_n, \end{cases} \tag{3.1}$$

Where, $0 < \alpha \leq \lambda_n \leq \beta < min\{\frac{1}{2c_1}, \frac{1}{2c_2}\}$, $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset [a, b] \subset (0, 1)$ with $\alpha_n + \beta_n + \gamma_n = 1$, $\delta_n \in (0, 1)$, $\lim_{n \to \infty} \delta_n = 0$, $\sum_{n=0}^{\infty} \delta_n = \infty$, $\{\sigma_n^i\} \subset (0, \infty)$ and $\liminf_{n\to\infty} \sigma_n^i > 0$, for $i = 1, 2, \ldots, N$.

Lemma 3.1. *If $\{v_n\}, \{w_n\}, \{z_n\}$ be sequences defined in algorithm (3.1) and $x^* \in A(T) \cap EP(f, C) \cap \bigcap_{i=1}^N A_i^{-1}(0)$, then*

(i) $d^2(z_n, x^*) \leq d^2(v_n, x^*) - (1 - 2c_1\lambda_n) d^2(v_n, w_n) - (1 - 2c_2\lambda_n) d^2(w_n, z_n)$.
(ii) $f(w_n, z_n) \leq \frac{1}{2\lambda_n}\{d^2(v_n, x^*) - d^2(v_n, z_n) - d^2(z_n, x^*)\}$
(iii) $(\frac{1}{2\lambda_n} - c_1) d^2(v_n, w_n) + (\frac{1}{2\lambda_n} - c_2) d^2(w_n, z_n) - \frac{1}{2\lambda_n} d^2(v_n, z_n) \leq f(w_n, z_n)$.

Proof. The proof is similar to the proof of Lemma 3.1 in [27]. □

Theorem 3.2. *Let (X,d) be a hyperbolic space satisfying the $\mathbb{S}$ property and $\overline{Q}_4$ condition, and C be a nonempty, convex and closed subset of X. Let $T : C \longrightarrow C$ be a further 2-generalized hybrid mapping, $f : X \times X \to \mathbb{R}$ be a bifunction satisfying condition B_1, B_2, B_3 and B_4. Let $A_i : X \longrightarrow 2^{X^*}$, i=1,2,...,N be N multi-valued monotone operators satisfying the range condition. If $\Omega = A(T) \cap EP(f, C) \cap \bigcap_{i=1}^N A_i^{-1}(0) \neq \emptyset$. Then the sequence $\{x_n\}$ generated by algorithm (3.1) converges strongly to $x^* = P_\Omega(u)$.*

Proof. From Lemma 2.14, Remark 2.20 and Lemma 2.16, it follows that $\Omega = A(T) \cap EP(f, C) \cap \bigcap_{i=1}^N A_i^{-1}(0)$ is closed and convex, and so $x^* = P_\Omega(u)$ is well defined.

Let $x^* = P_\Omega(u) \in \Omega$. From nonexpansivity of $J^{A_i}_{\sigma_n^i}$, we have

$$\begin{aligned} d(v_n, x^*) &= d(J^{A_N}_{\sigma_n^N} \circ J^{A_{N-1}}_{\sigma_n^{N-1}} \circ \cdots \circ J^{A_1}_{\sigma_n^1} x_n, x^*) \\ &\leq d(J^{A_{N-1}}_{\sigma_n^{N-1}} \circ \cdots \circ J^{A_1}_{\sigma_n^1} x_n, x^*) \\ &\vdots \\ &\leq d(J^{A_1}_{\sigma_n^1} x_n, x^*) \\ &\leq d(x_n, x^*) \end{aligned} \tag{3.2}$$

Using Lemma 3.1(i), we have

$$d(z_n, x^*) \leq d(v_n, x^*) \leq d(x_n, x^*) \tag{3.3}$$

Again, using Lemma 2.9 (i) and (3.3), we have

$$\begin{aligned} d(y_n, x^*) &= d(\alpha_n z_n \oplus \beta_n T z_n \oplus \gamma_n T^2 z_n, x^*) \\ &\leq \alpha_n d(z_n, x^*) + \beta_n d(T z_n, x^*) + \gamma_n d(T^2 z_n, x^*) \\ &\leq \alpha_n d(z_n, x^*) + \beta_n d(z_n, x^*) + \gamma_n d(z_n, x^*) \\ &= d(z_n, x^*) \\ &\leq d(x_n, x^*). \end{aligned} \tag{3.4}$$

Also using (3.4),

$$\begin{aligned} d(x_{n+1}, x^*) &= d(\delta_n u \oplus (1-\delta_n) y_n, x^*) \\ &\leq \delta_n d(u, x^*) + (1-\delta_n) d(y_n, x^*) \\ &\leq \delta_n d(u, x^*) + (1-\delta_n) d(x_n, x^*) \\ &\leq max\{d(u, x^*), d(x_n, x^*)\} \\ &\leq max\{d(u, x^*), max\{d(u, x^*), d(x_{n-1}, x^*)\}\} \\ &= max\{d(u, x^*), d(x_{n-1}, x^*)\} \\ &\leq max\{d(u, x^*), max\{d(u, x^*), d(x_{n-2}, x^*)\}\} \\ &= max\{d(u, x^*, d(x_{n-2}, x^*)\} \\ &\vdots \\ &\leq max\{d(u, x^*), d(x_1, x^*)\} \end{aligned}$$

This implies that the sequence $\{d(x_n, x^*)\}$ is bounded and consequently $\{x_n\}$, $\{T^n x_n\}$, $\{v_n\}$, $\{z_n\}$ and $\{y_n\}$ are bounded.

Now, from the scheme (3.1), (3.3) and Lemma 2.11, we have

$$\begin{aligned} d^2(y_n, x^*) &= d^2(\alpha_n z_n \oplus \beta_n T z_n \oplus \gamma_n T^2 z_n, x^*) \\ &\leq \alpha_n d^2(z_n, x^*) + \beta_n d^2(T z_n, x^*) + \gamma_n d^2(T^2 z_n, x^*) - \alpha_n \beta_n d^2(z_n, T z_n) \\ &\leq \alpha_n d^2(z_n, x^*) + \beta_n d^2(z_n, x^*) + \gamma_n d^2(z_n, x^*) - \alpha_n \beta_n d^2(z_n, T z_n) \\ &= d^2(z_n, x^*) - \alpha_n \beta_n d^2(z_n, T z_n) \\ &\leq d^2(x_n, x^*) - \alpha_n \beta_n d^2(z_n, T z_n). \end{aligned}$$

Thus,

$$d^2(y_n, x^*) \leq d^2(x_n, x^*) - \alpha_n \beta_n d^2(z_n, T z_n). \tag{3.5}$$

Similarly,

$$d^2(y_n, x^*) \leq d^2(x_n, x^*) - \alpha_n \gamma_n d^2(z_n, T^2 z_n). \tag{3.6}$$

On the other hand, using (3.5) and Lemma 2.9 (iv), we have

$$\begin{aligned} d^2(x_{n+1}, x^*) &= d^2(\delta_n u \oplus (1-\delta_n) y_n, x^*) \\ &\leq \delta_n d^2(u, x^*) + (1-\delta_n) d^2(y_n, x^*) - \delta_n (1-\delta_n) d^2(u, y_n) \\ &\leq \delta_n d^2(u, x^*) + (1-\delta_n)[d^2(x_n, x^*) - \alpha_n \beta_n d^2(z_n, T z_n)] - \delta_n (1-\delta_n) d^2(u, y_n) \\ &= (1-\delta_n) d^2(x_n, x^*) + \delta_n [d^2(u, x^*) - (1-\delta_n) d^2(u, y_n) - \frac{\alpha_n \beta_n}{\delta_n}(1-\delta_n) \\ &\quad d^2(z_n, T z_n)]. \end{aligned}$$

Now we show that $d^2(x_n, x^*) \to 0$. To do this using Lemma 2.17, it is sufficient to show that:

$$\limsup_{k\to\infty}(d^2(u,x^*) - (1-\delta_{n_k})d^2(u,y_{n_k}) - \frac{(1-\delta_{n_k})\alpha_{n_k}\beta_{n_k}}{\delta_{n_k}}d^2(z_{n_k}, Tz_{n_k})) \le 0,$$

for every subsequence $\{d^2(x_{n_k}, x^*)\}$ of $\{d^2(x_n, x^*)\}$ that satisfies,

$$\liminf_{k\to\infty}(d^2(x_{n_{k+1}}, x^*) - d^2(x_{n_k}, x^*)) \ge 0. \tag{3.7}$$

Now, suppose $\{d^2(x_{n_k}, x^*)\}$ is a subsequence of $\{d^2(x_n, x^*)\}$ that satisfies (3.7). Then we have,

$$\begin{aligned}
0 &\le \liminf_{k\to\infty}(d^2(x_{n_{k+1}}, x^*) - d^2(x_{n_k}, x^*)) \\
&\le \liminf_{k\to\infty}(\delta_{n_k}d^2(u,x^*) + (1-\delta_{n_k})d^2(y_{n_k}, x^*) \\
&\quad -\delta_{n_k}(1-\delta_{n_k})d^2(u,y_{n_k}) - d^2(x_{n_k}, x^*)) \\
&\le \liminf_{k\to\infty}(\delta_{n_k}d^2(u,x^*) + (1-\delta_{n_k})d^2(y_{n_k}, x^*) - d^2(x_{n_k}, x^*)) \\
&\le \liminf_{k\to\infty}[\delta_{n_k}(d^2(u,x^*) - d^2(y_{n_k}, x^*)) + d^2(y_{n_k}, x^*) - d^2(x_{n_k}, x^*)] \\
&\le \limsup_{k\to\infty}(\delta_{n_k}d^2(u,x^*) - d^2(y_{n_k}, x^*)) + \liminf_{k\to\infty}(d^2(y_{n_k}, x^*) - d^2(x_{n_k}, x^*)) \\
&= \liminf_{k\to\infty}(d^2(y_{n_k}, x^*) - d^2(x_{n_k}, x^*)) \\
&\le \limsup_{k\to\infty}(d^2(y_{n_k}, x^*) - d^2(x_{n_k}, x^*)) \\
&\le 0.
\end{aligned}$$

In conclusion, $\lim_{k\to\infty}(d^2(y_{n_k}, x^*) - d^2(x_{n_k}, x^*)) = 0$. Since $\{y_{n_k}\}$ is bounded, then there exists a subsequence $\{y_{n_{k_\epsilon}}\}$ of $\{y_{n_k}\}$ such that $\Delta - \lim_{\epsilon\to\infty} y_{n_{k_\epsilon}} = p \in C$, therefore we have

$$\begin{aligned}
&\limsup_{k\to\infty}(d^2(u,x^*) - (1-\delta_{n_k})d^2(u,y_{n_k}) - \frac{(1-\delta_{n_k})\alpha_{n_k}\beta_{n_k}}{\delta_{n_k}}d^2(z_{n_k}, Tz_{n_k})) \\
&\qquad \le \limsup_{k\to\infty}(d^2(u,x^*) - (1-\delta_{n_k})d^2(u,y_{n_k}) + \alpha_{n_k}\beta_{n_k}d^2(z_{n_k}, Tz_{n_k})) \\
&\qquad = \lim_{\epsilon\to\infty}(d^2(u,x^*) - (1-\delta_{n_{k_\epsilon}})d^2(u,y_{n_{k_\epsilon}}) + \alpha_{n_{k_\epsilon}}\beta_{n_{k_\epsilon}}d^2(z_{n_{k_\epsilon}}, Tz_{n_{k_\epsilon}}))
\end{aligned}$$

Since $d^2(u,.)$ is Δ-lower semicontinuous, we have

$$\begin{aligned}
&\limsup_{k\to\infty}(d^2(u,x^*) - (1-\delta_{n_k})d^2(u,y_{n_k}) - \frac{(1-\delta_{n_k})\alpha_{n_k}\beta_{n_k}}{\delta_{n_k}}d^2(z_{n_k}, Tz_{n_k})) \\
&\qquad \le \limsup_{k\to\infty}(d^2(u,x^*) - (1-\delta_{n_k})d^2(u,y_{n_k}) + \alpha_{n_k}\beta_{n_k}d^2(z_{n_k}, Tz_{n_k})) \\
&\qquad = \lim_{\epsilon\to\infty}(d^2(u,x^*) - (1-\delta_{n_{k_\epsilon}})d^2(u,y_{n_{k_\epsilon}}) + \alpha_{n_{k_\epsilon}}\beta_{n_{k_\epsilon}}d^2(z_{n_{k_\epsilon}}, Tz_{n_{k_\epsilon}})) \\
&\qquad \le d^2(u,x^*) - d^2(u,p)
\end{aligned} \tag{3.8}$$

Hence, it now remains to prove that

$$d(u,x^*) \le d(u,p).$$

Now, from the boundedness of $\{z_n\}$, there exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that $\Delta - \lim_{k\to\infty} z_{n_k} = p$. Thus, from (3.5) and (3.6), we have that

$$\alpha_{n_k}\beta_{n_k}d^2(z_{n_k}, Tz_{n_k}) \le -(d^2(y_{n_k}, x^*) - d^2(x_{n_k}, x^*))$$

and

$$\alpha_{n_k}\gamma_{n_k}d^2(z_{n_k}, T^2z_{n_k}) \le -(d^2(y_{n_k}, x^*) - d^2(x_{n_k}, x^*))$$

which implies that $\alpha_{n_k}\beta_{n_k}d^2(z_{n_k}, Tz_{n_k}) \to 0$
and $\alpha_{n_k}\gamma_{n_k}d^2(z_{n_k}, T^2z_{n_k}) \to 0$ as $k \to \infty$.
So, $\alpha_{n_k}, \beta_{n_k}, \gamma_{n_k} \in [a, b] \subset (0, 1)$, $\alpha_{n_k}\beta_{n_k}d^2(z_{n_k}, Tz_{n_k}) \to 0$
and $\alpha_{n_k}\gamma_{n_k}d^2(z_{n_k}, T^2z_{n_k}) \to 0$ as $k \to \infty$ imlies that

$$\lim_{k\to\infty} d(z_{n_k}, Tz_{n_k}) = 0 \; and \; \lim_{k\to\infty} d(z_{n_k}, T^2z_{n_k}) = 0 \tag{3.9}$$

Then by Lemma 2.18, we have that $p \in A(T)$.
Let $U_n^1 = J_{\sigma_n^1}^{A_1}x_n$, $U_n^2 = J_{\sigma_n^2}^{A_2}U_n^1$,..., $U_n^{N-1} = J_{\sigma_n^{N-1}}^{A_{N-1}}U_n^{N-2}$,
$U_n = U_n^N = J_{\sigma_n^N}^{A_N}U_n^{N-1}$, $U_n^0 = x_n$.Thus $v_n = U_n$. Therefore, (3.3) gives

$$d^2(U_n, x^*) - d^2(x_n, x^*) \le 0$$

Hence,

$$\limsup_{n\to\infty}(d^2(U_n, x^*) - d^2(x_n, x^*)) \le 0 \tag{3.10}$$

Now, from (3.1) and (3.3)

$$\begin{aligned} d(y_n, x^*) &= d(\alpha_n z_n \oplus \beta_n Tz_n \oplus \gamma_n T^2 z_n, x^*) \\ &\le \alpha_n d(z_n, x^*) + \beta_n d(Tz_n, x^*) + \gamma_n d(T^2 z_n, x^*) \\ &\le \alpha_n d(z_n, x^*) + \beta_n d(z_n, x^*) + \gamma_n d(z_n, x^*) \\ &= d(z_n, x^*) \\ &\le d(v_n, x^*). \end{aligned} \tag{3.11}$$

(3.1) and (3.11) gives,

$$\begin{aligned} d^2(x_{n+1}, x^*) &= d^2(\delta_n u \oplus (1-\delta_n)y_n, x^*) \\ &\le \delta_n d^2(u, x^*) + (1-\delta_n)d^2(y_n, x^*) - \delta_n(1-\delta_n)d^2(u, y_n) \\ &\le \delta_n d^2(u, x^*) - \delta_n d^2(y_n, x^*) - \delta_n(1-\delta_n)d^2(y_n, u) + d^2(y_n, x^*) \\ &\le \delta_n d^2(u, x^*) - \delta_n d^2(y_n, x^*) - \delta_n(1-\delta_n)d^2(y_n, u) + d^2(v_n, x^*). \end{aligned}$$

Therefore,

$$\begin{aligned} d^2(x_{n_k+1}, x^*) - d^2(x_{n_k}, x^*) \le\; & \delta_{n_k}(d^2(u, x^*) - d^2(y_{n_k}, x^*) - (1-\delta_{n_k})d^2(y_{n_k}, u)) \\ & + d^2(v_{n_k}, x^*) - d^2(x_{n_k}, x^*). \end{aligned} \tag{3.12}$$

Since $\lim_{k\to\infty} \delta_{n_k} = 0$, then by using (3.7) and (3.12), we have

$$0 \le \liminf_{k\to\infty}(d^2(U_{n_k}, x^*) - d^2(x_{n_k}, x^*)). \tag{3.13}$$

Using (3.10) and (3.13) we get,

$$\lim_{k\to\infty}(d^2(U_{n_k}, x^*) - d^2(x_{n_k}, x^*)) = 0. \tag{3.14}$$

By applying (2.1), we obtain

$$\begin{aligned} d^2(U_{n_k},x^*) &\le d^2(U^{N-1}_{n_k},x^*) - d^2(U^{N-1}_{n_k},U_{n_k}) \\ &\le d^2(U^{N-2}_{n_k},x^*) - d^2(U^{N-2}_{n_k},U^{N-1}_{n_k}) - d^2(U^{N-1}_{n_k},U_{n_k}) \\ &\le d^2(U^{N-3}_{n_k},x^*) - d^2(U^{N-3}_{n_k},U^{N-2}_{n_k}) - d^2(U^{N-2}_{n_k},U^{N-1}_{n_k}) \\ &\quad -d^2(U^{N-1}_{n_k},U_{n_k}) \\ &\vdots \\ &\le d^2(x_{n_k},x^*) - d^2(U^1_{n_k},x_{n_k}) - d^2(U^2_{n_k},U^1_{n_k}) - \dots \\ &\quad -d^2(U^N_{n_k},U^{N-1}_{n_k}). \end{aligned} \tag{3.15}$$

From (3.15) we have,

$$0 \le d^2(U^1_{n_k},x^*) \le d^2(x_{n_k},x^*) - d^2(U_{n_k},x^*).$$

Using (3.14), we have

$$\lim_{k\to\infty} d^2(U^1_{n_k},x_{n_k}) = 0. \tag{3.16}$$

Similarly, we have,

$$\lim_{k\to\infty} d^2(U^2_{n_k},U^1_{n_k}) = \lim_{k\to\infty} d^2(U^3_{n_k},U^2_{n_k}) = \dots = \lim_{k\to\infty} d^2(U_{n_k},U^{N-1}_{n_k}) = 0. \tag{3.17}$$

It follows from (3.16) and (3.17) that

$$\lim_{k\to\infty} d(U^2_{n_k},x_{n_k}) \le \lim_{k\to\infty} (d(U^2_{n_k},U^1_{n_k}) + \lim_{k\to\infty} d(U^1_{n_k},x_{n_k})) = 0.$$

Hence,

$$\lim_{k\to\infty} d(U^2_{n_k},x_{n_k}) = 0.$$

Consequently, we have

$$\lim_{k\to\infty} d(U^3_{n_k},x_{n_k}) = \lim_{k\to\infty} d(U^4_{n_k},x_{n_k}) = \dots = \lim_{k\to\infty} d(U^N_{n_k},x_{n_k}) = 0. \tag{3.18}$$

Now,

$$\lim_{k\to\infty} d(J^{A_1}_{\sigma^1_{n_k}} x_{n_k},x_{n_k}) = \lim_{k\to\infty} d(U^1_{n_k},x_{n_k}) = 0.$$

Also, from nonexpansivity of $J^{A_i}_{\sigma_{n_k}}$ and (3.18), we have

$$\begin{aligned} d(J^{A_2}_{\sigma^2_{n_k}} x_{n_k},x_{n_k}) &\le d(J^{A_2}_{\sigma^2_{n_k}} x_{n_k}, J^{A_2}_{\sigma^2_{n_k}} U^1_{n_k}) + d(J^{A_2}_{\sigma^2_{n_k}} U^1_{n_k},x_{n_k}) \\ &\le d(x_{n_k},U^1_{n_k}) + d(U^2_{n_k},x_{n_k}) \to 0 \; as \; k\to\infty. \end{aligned}$$

Therefore, $\lim_{k\to\infty} d(J^{A_2}_{\sigma^2_{n_k}} x_{n_k},x_{n_k}) = 0.$
Again,

$$\begin{aligned} d(J^{A_3}_{\sigma^3_{n_k}} x_{n_k},x_{n_k}) &\le d(J^{A_3}_{\sigma^3_{n_k}} x_{n_k}, J^{A_3}_{\sigma^3_{n_k}} U^2_{n_k}) + d(J^{A_3}_{\sigma^3_{n_k}} U^2_{n_k},x_{n_k}) \\ &\le d(x_{n_k},U^2_{n_k}) + d(U^3_{n_k},x_{n_k}) \to 0 \; as \; k\to\infty. \end{aligned}$$

Hence $\lim_{k\to\infty} d(J^{A_3}_{\sigma^3_{n_k}} x_{n_k},x_{n_k}) = 0.$
So continuing in this fashion, we obtain

$$\lim_{k\to\infty} d(J^{A_4}_{\sigma^4_{n_k}} x_{n_k},x_{n_k}) = \lim_{k\to\infty} d(J^{A_5}_{\sigma^5_{n_k}} x_{n_k},x_{n_k}) = \dots = \lim_{k\to\infty} d(J^{A_N}_{\sigma^N_{n_k}} x_{n_k},x_{n_k}) = 0.$$

Hence,

$$d(J^{A_i}_{\sigma} x_{n_k},x_{n_k}) \to 0 \; as \; k\to\infty, \; \forall i = 1,2,3,\dots,N. \tag{3.19}$$

Let $\{x_{n_{k_\epsilon}}\}$ be a subsequence of $\{x_{n_k}\}$ such that $\Delta - \lim_{\epsilon\to\infty} x_{n_{k_\epsilon}} = p$. By Lemma 2.15, (3.19) and theorem 2.19 (i), we get $p \in A_i^{-1}(0)$. So, $p \in \bigcap_{i=1}^N A_i^{-1}(0)$. Since $\liminf_{n\to\infty}(1-2c_i\lambda_n) > 0$ for i=1,2, and using Lemma 3.1 (i), we have

$$\lim_{k\to\infty} d^2(v_{n_k}, w_{n_k}) = \lim_{k\to\infty} d^2(w_{n_k}, z_{n_k}) = \lim_{k\to\infty} d^2(v_{n_k}, z_{n_k}) = 0. \tag{3.20}$$

Using Lemma 3.1(ii),(iii), and (3.20), we have

$$\lim_{k\to\infty} f(w_{n_k}, z_{n_k}) = 0. \tag{3.21}$$

Now assume that $t = \eta z_n \oplus (1-\eta)y$, where $0 < \eta < 1$ and $y \in C$, then by condition B_1 we have,

$$\begin{aligned} f(w_n, z_n) + \frac{1}{2\lambda_n} d^2(v_n, z_n) &\leq f(w_n, t) + \frac{1}{2\lambda_n} d^2(v_n, t) \\ &= f(w_n, \eta z_n \oplus (1-\eta)y) + \frac{1}{2\lambda_n} d^2(v_n, \eta z_n \oplus (1-\eta)y) \\ &\leq \eta f(w_n, z_n) + (1-\eta) f(w_n, y) \\ &\quad + \frac{1}{2\lambda_n}[\eta d^2(v_n, z_n) + (1-\eta) d^2(v_n, y) - \eta(1-\eta) d^2(z_n, y)]. \end{aligned}$$

Therefore,

$$\begin{aligned} (1-\eta) f(w_n, z_n) &- (1-\eta) f(w_n, y) \leq \frac{1}{2\lambda_n}[(1-\eta) d^2(v_n, y) \\ &- (1-\eta) d^2(v_n, z_n) - \eta(1-\eta) d^2(z_n, y)]. \end{aligned}$$

So,

$$f(w_n, z_n) - f(w_n, y) \leq \frac{1}{2\lambda_n}[d^2(v_n, y) - d^2(v_n, z_n) - \eta d^2(z_n, y)].$$

Now, if $\eta \to 1^-$, we obtain

$$\frac{1}{2\lambda_n}[d^2(v_n, z_n) + d^2(z_n, y) - d^2(v_n, y)] \leq f(w_n, y) - f(w_n, z_n).$$

Assume $d(v_n, z_n) \geq 1$. It can easily be seen that

$$\frac{-1}{2\lambda_n} d(v_n, z_n)[d(z_n, y) + d(v_n, y)] \leq f(w_n, y) - f(w_n, z_n). \tag{3.22}$$

Now replacing n with n_{k_ϵ} in (3.22), taking limsup and using (3.20) and (3.21), since $\Delta - \lim_{\epsilon\to\infty} w_{n_{k_\epsilon}} = p$, then using B_2 we have

$$0 \leq \limsup_{\epsilon\to 0} f(w_{n_{k_\epsilon}}, y) \leq f(p, y),\ \forall y \in C.$$

Therefore $p \in EP(f, C)$. Hence $p \in \Omega = A(T) \cap EP(f, C) \cap \bigcap_{i=1}^N A_i^{-1}(0)$.
Since $x^* = P_\Omega u$, we have

$$d(u, x^*) \leq d(u, p).$$

Using (3.8), we get

$$\limsup_{k\to\infty}(d^2(u, x^*) - (1-\delta_{n_k}) d^2(u, y_{n_k}) - \frac{(1-\delta_{n_k})\alpha_{n_k}\beta_{n_k}}{\delta_{n_k}} d^2(x_{n_k}, Tx_{n_k})) \leq 0.$$

Thus, using Lemma 2.17, we get $x_n \to x^*$. This completes the proof. □

Since the identity map $I : C \longrightarrow C$ defined by $Ix = x$ is a further 2-generalized hybrid mapping with $\alpha_1 = 1, \beta_3 = -1,$ *and* $\alpha_1 = \alpha_3 = \beta_1 = \beta_2 = \epsilon_1 = \epsilon_2 = 0$ in definition 1.7, then If we set $T = I$ in (3.1), it reduces to the result of Moharami and Eskandani [27].

Corollary 3.3. *Let (X,d) be a hyperbolic space satisfying the* $\mathbb{S}$ *property and* $\overline{Q}_4$ *condition. Let C be a nonempty, convex and closed subset of X. Let f be a bifunction satisfying condition* B_1, B_2, B_3 *and* B_4*. Let* $A_i : X \longrightarrow 2^{X^*}, i = 1, 2, 3 \dots N$ *be N multi-valued monotone operators that satisfy the range condition. If* $\Omega = A(T) \cap EP(f, C) \cap \bigcap_{i=1}^{N} A_i^{-1}(0) \neq \emptyset$*. Then the sequence* $\{x_n\}$ *generated by*

$$\begin{cases} v_n = J_{\sigma_n^N}^{A_N} \circ J_{\sigma_n^{N-1}}^{A_{N-1}} \circ \cdots \circ J_{\sigma_n^1}^{A_1} x_n, \\ w_n = \underset{y \in C}{argmin}\{f(v_n, y) + \frac{1}{2\lambda_n} d^2(v_n, y)\}, \\ z_n = \underset{y \in C}{argmin}\{f(w_n, y) + \frac{1}{2\lambda_n} d^2(v_n, y)\}, \\ x_{n+1} = \delta_n u \oplus (1 - \delta_n) z_n, \end{cases} \tag{3.23}$$

converges strongly to $x^* = P_\Omega(u)$*.*

Corollary 3.4. *Let (X,d) be a hyperbolic space satisfying the* $\mathbb{S}$ *property and* $\overline{Q}_4$ *condition. Let C be a nonempty, convex and closed subset of X and* $T : C \longrightarrow C$ *be a further generalized hybrid mapping. Let f be a bifunction satisfying condition* B_1, B_2, B_3 *and* B_4*. Let* $A_i : X \longrightarrow 2^{X^*}, i = 1, 2, 3 \dots N$ *be N multi-valued monotone operators that satisfy the range condition. If* $\Omega = A(T) \cap EP(f, C) \cap \bigcap_{i=1}^{N} A_i^{-1}(0) \neq \emptyset$*. Then the sequence* $\{x_n\}$ *generated by algorithm (3.1) converges strongly to* $x^* = P_\Omega(u)$*.*

Proof. Since a furthet 2-generalized hybrid mapping reduces to a further generalized hybrid mapping if $\alpha_1 = \beta_1 = \epsilon_2 = 0$. It follows from theorem 3.4 that the sequence $\{x_n\}$ converges strongly to $x^* = P_\Omega u$. This completes the proof. □

Corollary 3.5. *Let (X,d) be a hyperbolic space satisfying the* $\mathbb{S}$ *property and* $\overline{Q}_4$ *condition. Let C be a nonempty, convex and closed subset of X and* $T : C \longrightarrow C$ *be a normally 2-generalized hybrid mapping. Let f be a bifunction satisfying condition* $B_1, B_2, B_3 and B_4$*. Let* $A_1, A_2, \dots, A_N : X \longrightarrow 2^{X^*}$ *be N multi-valued monotone operators that satisfy the range condition. If* $\Omega = A(T) \cap EP(f, C) \cap \bigcap_{i=1}^{N} A_i^{-1}(0) \neq \emptyset$*. Then the sequence* $\{x_n\}$ *generated by algorithm (3.1) converges strongly to* $x^* = P_\Omega(u)$*.*

Proof. Since if $\epsilon_1 = \epsilon_2 = 0$, a further 2-generalized hybrid mapping is reduced to a normally 2-generalized hybrid mapping, then from theorem 3.4, we see that $\{x_n\}$ converges strongly to $z = P_\Omega u$. □

4. NUMERICAL EXAMPLE

In this section, we provide a numerical example to validate our obtained results in a hyperbolic space.

Example 4.1. Let $X = \mathbb{R}$ with the usual metric and $C = [-7, 7]$. Then $\mathbb{R}$ is a hyperbolic space satisfying the $\mathbb{S}$ property and the Q_4 condition, and C is a nonempty, closed and convex subset of $X = \mathbb{R}$.

Now, we define $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ by $f(x, y) = y^2 + 6xy - 7x^2$. It is very easy to see that $0 \in EP(f, C)$ and f satisfies conditions $B_1 \; and \; B_2$. Also, f satisfies condition B_3 with $c_1 = c_2 = 3$, and also satisfies condition B_4.
Indeed, for B_3, let $x, y, z \in X = \mathbb{R}$,

$$\begin{aligned} f(x, y) + f(y, z) &= y^2 + 6xy - 7x^2 + z^2 + 6yz - 7y^2 \\ &= z^2 + 6xy - 7x^2 + 6yz - 6y^2 \\ &= f(x, z) - 6xz + 6xy + 6yz - 6y^2 \\ &= f(x, z) - 3(y - x)^2 - 3(z - y)^2 + 3(z - x)^2 \\ &= f(x, z) - 3d^2(x, y) - 3d^2(y, z) \end{aligned}$$

For B_4, Let $x, y \in X = \mathbb{R}$ and $f(x,y) \geq 0$. We show $f(y,x) = x^2 + 6xy - 7y^2 \leq 0$.
Now

$$\begin{aligned} f(x,y) \geq 0 \quad &\Longrightarrow \quad y^2 + 6xy - 7x^2 \geq 0 \\ &\Longrightarrow \quad -y^2 - 6xy + 7x^2 \leq 0 \\ &\Longrightarrow \quad x^2 + 6xy - 7y^2 \leq -6y^2 - 6x^2 + 12xy \\ &\Longrightarrow \quad x^2 + 6xy - 7y^2 \leq -6(x-y)^2 \leq 0 \\ &\Longrightarrow \quad f(y,x) \leq 0. \end{aligned}$$

Also, for N=2, i.e.; i=1,2, we define $A_i : \mathbb{R} \longrightarrow \mathbb{R}$ by $A_1(x) = 3x$ and $A_2(x) = 5x$.
A_1 and A_2 are monotone operators.
Indeed, for $x, y \in \mathbb{R}$,

$$(A_1(x) - A_1(y))(x-y) = (3x - 3y)(x-y) = 3(x-y)^2 \geq 0$$

and

$$(A_2(x) - A_2(y))(x-y) = (5x - 5y)(x-y) = 5(x-y)^2 \geq 0$$

Now,

$$\begin{aligned} J^{A_1}_{\sigma^1_n} = y \quad &\Leftrightarrow \quad \frac{1}{\sigma^1_n}(x-y) \in A_1 y \\ &\Leftrightarrow \quad (x-y) \in \sigma^1_n A_1 y \\ &\Leftrightarrow \quad x \in \sigma^1_n A_1 y + y \\ &\Leftrightarrow \quad (I + \sigma^1_n A_1) y = x \\ &\Leftrightarrow \quad y + 3\sigma^1_n y = x \\ &\Leftrightarrow \quad y = \frac{x}{1 + 3\sigma^1_n} \end{aligned}$$

i.e., $J^{A_1}_{\sigma^1_n}(x) = \frac{x}{1+3\sigma^1_n}$
Similarly, $J^{A_2}_{\sigma^2_n}(x) = \frac{x}{1+3\sigma^1_n}$
Thus,

$$\begin{aligned} J^{A_2}_{\sigma^2_n}(J^{A_1}_{\sigma^1_n}(x) \quad &= \quad J^{A_2}_{\sigma^2_n}(\frac{x}{1+3\sigma^1_n} \\ &= \quad \frac{\frac{x}{1+3\sigma^1_n}}{1+5\sigma^2_n} \\ &= \quad \frac{x}{(1+3\sigma^1_n)(1+5\sigma^2_n)} \end{aligned}$$

Define also a map $T : C \longrightarrow C$ by $Tx = \frac{x}{3}$.Then T is further 2-generalized hybrid mapping with $\alpha_2 = 3, \beta_3 = -2, \alpha_1 = \alpha_3 = \beta_1 = \beta_2 = \epsilon_1 = \epsilon_2 = 0$ and $A(T) = \{0\}$. Indeed, let $x, y \in C = [-7, 7]$, for $\alpha_2 = 3, \beta_3 = -2, \alpha_1 = \alpha_3 = \beta_1 = \beta_2 = \epsilon_1 = \epsilon_2 = 0$, we have

(i) $\sum_{i=1}^{3}(\alpha_i + \beta_i) = 3 - 2 = 1 > 0$
(ii) $\sum_{i=1}^{3} \alpha_i = 3 > 0$, and
(iii) $\alpha_1 d^2(T^2x, Ty) + \alpha_2 d^2(Tx, Ty) + \alpha_3 d^2(x, Ty) + \beta_1 d^2(T^2x, y) + \beta_2 d^2(Tx, y) + \beta_3 d^2(x, y) + \epsilon_1 d^2(x, T^2x) + \epsilon_2 d^2(x, Tx) =$
$3|\frac{x}{3} - \frac{y}{3}|^2 - 2|x-y|^2 \leq 0$

Again, for $z = 0 \in [-7, 7]$, we have
$|z - Tx| = |0 - \frac{x}{3}| \leq |0 - x| = |z - x|$ and for $z \neq 0 \in [-7, 7]$, we have

$|z - Tx| = |z - \frac{x}{3}| = |3z - x| \nleq |z - x|$.
Therefore, 0 is the only attractive point of the map T.

Thus our proposed algorithm (3.1) takes the following form;

$$\begin{cases} v_n = \frac{x}{(1+3\sigma_n^1)(1+5\sigma_n^2)}, \\ w_n = \frac{(1-6\lambda_n)}{(2\lambda_n+1)} v_n, \\ z_n = \frac{v_n - 6\lambda_n w_n}{(2\lambda_n+1)}, \\ y_n = \alpha_n z_n \oplus \beta_n T z_n \oplus \gamma_n T^2 z_n, \\ x_{n+1} = \delta_n u \oplus (1-\delta_n) y_n, \end{cases} \tag{4.1}$$

Set $\lambda_n = \frac{1}{n+7}$, $\delta_n = \frac{1}{n}$ and $\alpha_n = \beta_n = \gamma_n = \frac{1}{3}, \forall n \in \mathbb{N}$. It can be observed that all assumptions of Theorem 3.2 are clearly satisfied. Let $\{x_n\}$ be a sequence generated by algorithm (4.1).

Case 1: x_1 =0.05; u=-8.5; $\sigma_n^1 = \sigma_n^2$=0.005;

Case 2: x_1 =-0.05; u=4.0; $\sigma_n^1 = \sigma_n^2$=0.001.

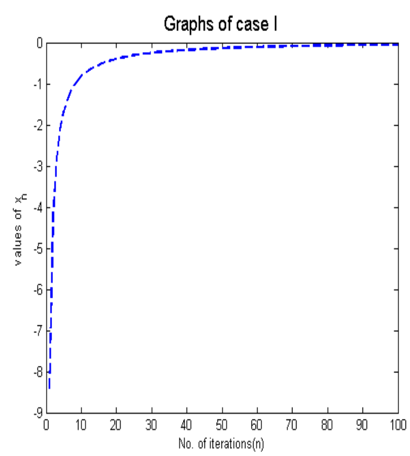

FIGURE 1. The graph of sequence $\{x_n\}$ generated by algorithm (4.1) versus number of iterations (Case 1).

5. CONCLUSION

Our result improve the results of Quoc et al [28], Moharami and Eskandani [27] and Ali and Yusuf [2] in the following sense

(i) From weak convergence in [28] to strong convergence and extending the result from equilibrium problem to attractive point, zero and equilibrium problems.
(ii) Approximate solutions of attractive point, zero and equilibrium problems against solution of equilibrium and zero problems in [27]
(iii) Finds common element in the solution set of attractive point, equilibrium and zero problem unlike in [2] where they find an attractive point of a further 2-generalized hybrid mapping only.

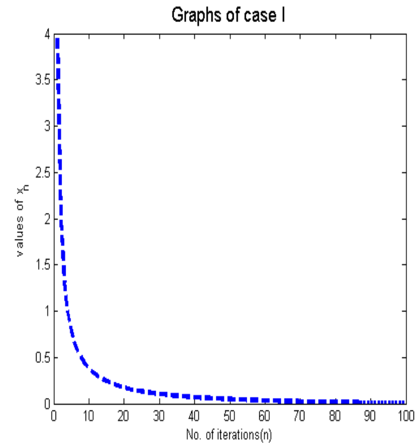

FIGURE 2. The graph of sequence $\{x_n\}$ generated by algorithm (4.1) versus number of iterations (Case 2).

STATEMENTS AND DECLARATIONS

The authors declare that they have no conflict of interest, and the manuscript has no associated data.

REFERENCES

[1] B. Ali. Convergence theorems for finite families of total asymptotically nonexpansive mappings in hyperbolic spaces. *Fixed point theory and Applications*, 2016:24, 2016.

[2] B. Ali. and L. Y. Haruna. Attractive point approximation of further 2-generalized hybrid mappings in $CAT(0)$ spaces. *Journal of Nonlinear and Variational Analysis*, 3(3):235-246, 2019.

[3] A. Amini and B. A. Kakavandi. Duality and subdifferential for convex functions in complete $CAT(0)$ metric spaces. *Nonlinear Analysis: Theory, Methods & Applications*, 73(10):3450-3455, 2010.

[4] J. B. Baillon. Un theoreme de type ergodique pour less contractions nonlinears dans un espaces de Hilbert. *Comptes rendus de l'Académie des Sciences, Series A-B*, 280:1511-1541, 1975.

[5] I. D. Berg and I. D. Nikolaev. Quasilinearization and curvature of Alexandrov spaces. *Geometriae Dedicata*, 133: 195-218, 2008.

[6] E. Blum and W. Oettli. From optimazation and variational inequalities to equilibrium problems. *Mathematics Student* 63:123-145, 1994.

[7] M. Bridson and A. Haefliger. *A Metric Space of Non-Positive Curvature*, Springer, Berlin, 1999.

[8] C. E. Chidume, A. U. Bello, and P. Ndambomve. Strong and $Delta$-convergence theorems for common fixed points of a finite family of multi-valued demicontractive mappings in $CAT(0)$ spaces. *Abstract and Applied Analysis*, 2014(1):805168, 2014.

[9] P. L. Combettes and S. A. Hirstoaga. Equilibrium programming in Hilbert spaces. *Journal of Nonlinear and Convex Analysis*, 6(1):117-136, 2005.

[10] A. Cuntavepanit and W. Phuengrattana. Iterative approximation of attractive points of further generalized hybrid mappings in Hadamard spaces. *Fixed Point Theory and Applications*, 2019:3, 2019.

[11] S. Dhompongsa and B. Panyanak. On Δ-convergence theorems in CAT(0) spaces. *Computers & Mathematics with Applications*, 56(10):2572-2579, 2008.

[12] S. Dhompongsa, A. Kaewkhao, and B. Panyanak. On Kirk's strong convergence theorems for multivalued nonexpansive mappings on $CAT(0)$ spaces. *Nonlinear Analysis: Theory, Methods & Applications*, 75(2):459-468, 2012.

[13] G. Z. Eskandani, M. Raeisi, and T. M. Rassias. A hybrid extragradient method for solving pseudo-monotone euilibrium problems using Bregman distance. *Journal of Fixed Point Theory and Applications*, 20:132, 2018.

[14] A. Kaewkhao, W. Inthakon, and K. Kunwai. Attractive points and convergence theorems for normally generalized hybrid mappings in $CAT(0)$ spaces. *Fixed Point Theory and Applications*, 2015:96, 2015.

[15] B. A. Kakavandi. Weak topologies in complete $CAT(0)$ metric spaces. *Proceedings of the American Mathematical Society*, 141(3):1029-1039, 2013.

[16] S. H. Khan. Iterative approximations of common attractive points of further generalized hybrid mappings. *Fixed Point Theory and Applications*, 2018:8 , 2018.

[17] H. Khatibzadeh and V. Mohebbi. Monotone and pseudo-monotone equilibrium problems in Hadamard spaces. *Journal of the Australian Mathematical Society*, 110(2):220-242, 2021.

[18] H. Khatibzadeh and V. Mohebbi. Approximating solutions of equilibrium problems in Hadamard spaces. *Miskolc Mathematical Notes*, 20(1):281-297, 2019.

[19] H. Khatibzadeh and S. Ranjbar. Monotone operators and the proximal point algorithm in complete $CAT(0)$ metric spaces. *Journal of the Australian Mathematical Society*, 103(1):70-90, 2017.

[20] P. Kocourek, W. Takahashi, and J.-C. Yao. Fixed points and weak convergence theorems for generalized hybrid mappings in Hilbert spaces. *Taiwanese Journal of Mathematics*, 14(6):2497 - 2511, 2010.

[21] F. Kohsaka and W. Takahashi. Fixed points theorems for a class of nonlinear mappings related to a maximal monotone operatos in Banach spaces. *Archiv der Mathematik*, 91:166-177, 2008.

[22] F. Kohsaka and W. Takahashi. Existence and approximation of firmly nonexpansive type mappings in Banach spaces. *SIAM Journal on Optimization*, 19(2):824-835, 2008.

[23] A. Konda and W. Takahashi. Attractive point and weak convergence theorems for normally N-generalized hybrid mappings in Hilbert spaces. *Linear Nonlinear Analysis*, 3(2):297-310, 2017.

[24] A. Kondo and W. Takahashi. Strong convergence theorems of Halpern's type for normally 2-generalized hybrid mappinggs in Hilbert spaces. *Journal of Nonlinear and Convex Analysis*, 19(14):617-631, 2018.

[25] K. Kunwai, A. Kaewkhao, and W. Inthakon. Properties of attractive point in CAT(0) spaces. *Thai Journal of Mathematics*, 13(1):109-121, 2015.

[26] B. Martinet. Regularisation dinequations variationelles par approximations successive. *Recherche Opérationnelle*, 4:154-159, 1970.

[27] R. Moharami and G. Z. Eskandani. An extragradient algorithm for solving equilibrium problem and zero point problem in Hadamard spaces. *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas*, 114:152, 2020.

[28] X. Qin, Y. J. Cho, and S. M. Kang. Convergence theorems of common elements for equilibrium problems and fixed point problems in Banach spaces. *Journal of Computational and Applied Mathematics*, 25(1):20-30, 2009.

[29] S. Reich and I. Shafrir. Nonexpansive iterations in Hyperbolic spaces. *Nonlinear analysis: theory, methods & applications*, 15(6):537-558, 1990.

[30] W. Takahashi. Fixed point theorems for new nonlinear mappings in Hilbert spaces. *Journal of Nonlinear and Convex Analysis*, 11:79-88, 2010.

[31] W. Takahashi and Y. Takeuchi. Nonlinear ergodic theorem without convexity for generalized hybrid mappings in Hilbert spasces. *Journal of Nonlinear and Convex Analysis*, 12:399-406, 2011.

[32] W. Takahashi, N.-C. Wong, and J.-C. Yao. Attractive points and Halpern type strong convergence theorems in Hilbert spaces. *Journal of Fixed Point Theory and Applications*, 17:301-311, 2015.

[33] W. Takahashi. *Nonlinear Functional Analysis-Fixed Point Theory and its Application*. Yokohama Publishers, Yokohama, 2000.

[34] W. Takahashi, N.-C. Wong, and J.-C. Yao. Attractive point and weak convergence theorem for new generalized hybrid mappings in Hilbert spaces. *Journal of Nonlinear and Convex Analysis*, 13(4):745-757, 2012.

[35] H. K. Xu. Another control condition in an iterative method for nonexpansive mappings. *Bulletin of the Australian Mathematical Society*, 65(1):109-113, 2002.