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# PARALLEL COMPOSITE-TYPE EXTRAGRADIENT IMPLICIT METHOD FOR A SYSTEM OF VARIATIONAL INCLUSIONS WITH THE COMMON FIXED-POINT CONSTRAINT OF PSEUDOCONTRACTIVE MAPPINGS

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Abstract. In a uniformly convex and $q$-uniformly smooth Banach space with $q \in (1, 2]$, let the VI indicate a variational inclusion for two accretive operators and let the CFPP denote a common fixed point problem of a countable family of $\ell$-uniformly Lipschitzian pseudocontractive mappings. In this paper, we introduce a parallel composite-type extragradient implicit method for solving a general system of variational inclusions (GSVI) with the VI and CFPP constraints. We then prove the strong convergence of the suggested algorithm to a solution of the GSVI with the VI and CFPP constraints under some appropriate assumptions. As applications, we apply our main result to the variational inequality problem (VIP), split feasibility problem (SFP) and LASSO problem in Hilbert spaces.

**Keywords.** Parallel composite-type extragradient implicit method; General system of variational inclusions; Variational inclusion; Common fixed point problem; Strong convergence; Banach space.



## 1. Introduction

Let $H$ be a real Hilbert space, whose inner product and induced norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let $\emptyset \neq C \subset H$ be a closed convex set. We denote by $P_C$ the metric projection from $H$ onto $C$. Given a mapping $A : C \to H$. Consider the classical variational inequality problem (VIP) of finding a point $x^* \in C$ s.t. $\langle Ax^*, y - x^* \rangle \geq 0 \ \forall y \in C$. We denote by VI$(C, A)$ the solution set of the VIP. Up to now, Korpelevich's extragradient method [32] has been one of the most popular methods for solving the VIP. It is worth mentioning that if VI$(C, A) \neq \emptyset$, this method has only weak convergence, and only requires that the mapping $A$ is monotone and Lipschitz continuous. To the most of our knowledge, Korpelevich's extragradient method has been improved and modified in various ways so that some new iterative methods happen to solve the VIP and related optimization problems; see e.g., [4, 6, 8, 14, 15, 19, 20, 24, 26, 27, 28, 30, 31, 34, 35, 36, 37, 38, 39, 40, 41] and references therein, to name but a few.

Assume that the operators $A : C \to H$ and $B : D(B) \subset C \to H$ are $\alpha$-inverse-strongly monotone and maximal monotone, respectively. Consider the variational inclusion (VI) of finding a point $x^* \in C$ s.t. $0 \in (A + B)x^*$. In order to solve the FPP of nonexpansive mapping $S : C \to C$ and the VI for both monotone mappings $A, B$, Takahashi et al. [11] suggested a Mann-type Halpern iterative method, i.e.,

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for any given $x_1 = x \in C$, $\{x_j\}$ is the sequence generated by

$$x_{j+1} = \beta_j x_j + (1-\beta_j)S(\alpha_j x + (1-\alpha_j)J^B_{\lambda_j}(x_j - \lambda_j A x_j)) \quad \forall j \geq 1, \tag{1.1}$$

where $\{\lambda_j\} \subset (0, 2\alpha)$ and $\{\alpha_j\}, \{\beta_j\} \subset (0,1)$. They proved the strong convergence of $\{x_j\}$ to a point of $\mathrm{Fix}(S) \cap (A+B)^{-1}0$ under some mild conditions.

Recently, Abdou et al. [22] suggested a parallel algorithm, i.e., for any given $x_0 \in C$, $\{x_j\}$ is the sequence generated by

$$x_{j+1} = (1-\zeta)Sx_j + \zeta J^B_{\lambda_j}(\alpha_j \gamma f(x_j) + (1-\alpha_j)x_j - \lambda_j A x_j) \quad \forall j \geq 0, \tag{1.2}$$

where $S, A, B$ are the same as above, $\zeta \in (0,1)$, $\{\lambda_j\} \subset (0, 2\alpha)$ and $\{\alpha_j\} \subset (0,1)$. They proved strong convergence of $\{x_j\}$ to a point of $\mathrm{Fix}(S) \cap (A+B)^{-1}0$ under some appropriate conditions. In the practical life, many mathematical models have been formulated as the VI. Without question, many researchers have presented and developed a great number of iterative methods for solving the VI in various approaches; see e.g., [4, 11, 15, 17, 19, 22, 27, 28] and the references therein. Due to the importance and interesting of the VI, many mathematicians are now interested in finding a common solution of the VI and FPP.

For $q \in (1,2]$, let $E$ be a uniformly convex and $q$-uniformly smooth Banach space with $q$-uniform smoothness coefficient $\kappa_q$. Suppose that $f : E \to E$ is a $\rho$-contraction and $S : E \to E$ is a nonexpansive mapping. Let $A : E \to E$ be an $\alpha$-inverse-strongly accretive mapping of order $q$ and $B : E \to 2^E$ be an $m$-accretive operator. Very recently, Sunthrayuth and Cholamjiak [15] proposed a modified viscosity-type extragradient method for the FPP of $S$ and the VI of finding $x^* \in E$ s.t. $0 \in (A+B)x^*$, i.e., for any given $x_0 \in E$, $\{x_j\}$ is the sequence generated by

$$\begin{cases} y_j = J^B_{\lambda_j}(x_j - \lambda_j A x_j), \\ z_j = J^B_{\lambda_j}(x_j - \lambda_j A y_j + r_j(y_j - x_j)), \\ x_{j+1} = \alpha_j f(x_j) + \beta_j x_j + \gamma_j S z_j \quad \forall j \geq 0, \end{cases} \tag{1.3}$$

where $J^B_{\lambda_j} = (I + \lambda_j B)^{-1}$, $\{r_j\}, \{\alpha_j\}, \{\beta_j\}, \{\gamma_j\} \subset (0,1)$ and $\{\lambda_j\} \subset (0,\infty)$ are such that: (i) $\alpha_j + \beta_j + \gamma_j = 1$; (ii) $\lim_{j\to\infty} \alpha_j = 0$, $\sum_{j=1}^\infty \alpha_j = \infty$; (iii) $\{\beta_j\} \subset [a,b] \subset (0,1)$; and (iv) $0 < \lambda \leq \lambda_j < \lambda_j / r_j \leq \mu < (\alpha q/\kappa_q)^{1/(q-1)}$, $0 < r \leq r_j < 1$. They proved the strong convergence of $\{x_j\}$ to a point of $\mathrm{Fix}(S) \cap (A+B)^{-1}0$, which solves a certain VIP.

Furthermore, suppose that $J : E \to 2^{E^*}$ is the normalized duality mapping from $E$ into $2^{E^*}$ defined by $J(x) = \{\phi \in E^* : \langle x, \phi \rangle = \|x\|^2 = \|\phi\|^2\}$ $\forall x \in E$, where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing between $E$ and $E^*$. It is known that if $E$ is smooth then $J$ is single-valued. Let $C$ be a nonempty closed convex subset of a smooth Banach space $E$. Let $A_1, A_2 : C \to E$ and $B_1, B_2 : C \to 2^E$ be nonlinear mappings with $B_i x \neq \emptyset$ $\forall x \in C, i = 1,2$. Consider the general system of variational inclusions (GSVI) of finding $(x^*, y^*) \in C \times C$ s.t.

$$\begin{cases} 0 \in \zeta_1(A_1 y^* + B_1 x^*) + x^* - y^*, \\ 0 \in \zeta_2(A_2 x^* + B_2 y^*) + y^* - x^*, \end{cases} \tag{1.4}$$

where $\zeta_i$ is a positive constant for $i = 1,2$. It is known that problem (1.4) has been transformed into a fixed point problem in the following way.

**Lemma 1.1.** *(see [13, Lemma 2]). Assume that $B_1, B_2 : C \to 2^E$ are both $m$-accretive operators and $A_1, A_2 : C \to E$ are both operators. For given $x^*, y^* \in C$, $(x^*, y^*)$ is a solution of problem (1.4) if and only if $x^* \in \mathrm{Fix}(G)$, where $\mathrm{Fix}(G)$ is the fixed point set of the mapping $G := J^{B_1}_{\zeta_1}(I - \zeta_1 A_1) J^{B_2}_{\zeta_2}(I - \zeta_2 A_2)$, and $y^* = J^{B_2}_{\zeta_2}(I - \zeta_2 A_2)x^*$.*

Suppose that $E$ is a uniformly convex and 2-uniformly smooth Banach space with 2-uniform smoothness coefficient $\kappa_2$. Let $B_1, B_2 : C \to 2^E$ be both $m$-accretive operators and $A_i : C \to E$ $(i = 1, 2)$ be $\zeta_i$-inverse-strongly accretive operator. Let $f : C \to C$ be a contraction with constant $\delta \in [0, 1)$. Let $V : C \to C$ be a nonexpansive operator and $T : C \to C$ be a $\lambda$-strict pseudocontraction. Very recently, using Lemma 1.1, Ceng et al. [13] suggested a composite viscosity implicit rule for solving the GSVI (1.4) with the FPP constraint of $T$, i.e., for any given $x_0 \in C$, the sequence $\{x_j\}$ is generated by

$$\begin{cases} y_j = J_{\zeta_2}^{B_2}(x_j - \zeta_2 A_2 x_j), \\ x_j = \alpha_j f(x_{j-1}) + \delta_j x_{j-1} + \beta_j V x_{j-1} + \gamma_j[\mu S x_j + (1-\mu) J_{\zeta_1}^{B_1}(y_j - \zeta_1 A_1 y_j)] \quad \forall j \geq 1 \end{cases}$$

where $\mu \in (0, 1)$, $S := (1 - \alpha)I + \alpha T$ with $0 < \alpha < \min\{1, \frac{2\lambda}{\kappa_2}\}$, and the sequences $\{\alpha_j\}, \{\delta_j\}, \{\beta_j\}, \{\gamma_j\} \subset (0, 1)$ are such that (i) $\alpha_j + \delta_j + \beta_j + \gamma_j = 1$ $\forall j \geq 1$; (ii) $\lim_{j\to\infty} \alpha_j = 0$, $\lim_{j\to\infty} \frac{\beta_j}{\alpha_j} = 0$; (iii) $\lim_{j\to\infty} \gamma_j = 1$; (iv) $\sum_{j=0}^{\infty} \alpha_j = \infty$. They proved that $\{x_j\}$ converges strongly to a point of $\mathrm{Fix}(G) \cap \mathrm{Fix}(T)$, which solves a certain VIP.

In addition, assume that $\{\mu_j\} \subset (0, \frac{1}{L})$, $\{\lambda_j\} \subset (0, 2\alpha]$ and $\{\alpha_j\}, \{\hat{\alpha}_j\} \subset (0, 1]$ with $\alpha_j + \hat{\alpha}_j \leq 1$. Ceng et al. [4] introduced a Mann-type hybrid extragradient algorithm, i.e., for any initial $u_0 = u \in C$, $\{u_j\}$ is the sequence generated by

$$\begin{cases} y_j = P_C(u_j - \mu_j \mathcal{A} u_j), \\ v_j = P_C(u_j - \mu_j \mathcal{A} y_j), \\ \hat{v}_j = J_{\lambda_j}^{B}(v_j - \lambda_j A v_j), \\ z_j = (1 - \alpha_j - \hat{\alpha}_j) u_j + \alpha_j \hat{v}_j + \hat{\alpha}_j S \hat{v}_j, \\ u_{j+1} = P_{C_j \cap Q_j} u \quad \forall j \geq 0, \end{cases}$$

where $C_j = \{x \in C : \|z_j - x\| \leq \|u_j - x\|\}$, $Q_j = \{x \in C : \langle u_j - x, u - u_j \rangle \geq 0\}$, $J_{\lambda_j}^B = (I + \lambda_j B)^{-1}$, $\mathcal{A} : C \to H$ is a monotone and $L$-Lipschitzian mapping, $A : C \to H$ is an $\alpha$-inverse-strongly monotone mapping, $B$ is a maximal monotone mapping with $D(B) = C$ and $S : C \to C$ is a nonexpansive mapping. They proved strong convergence of $\{u_j\}$ to the point $P_\Omega u$ in $\Omega = \mathrm{Fix}(S) \cap (A + B)^{-1}0 \cap \mathrm{VI}(C, \mathcal{A})$ under some mild conditions.

In a uniformly convex and $q$-uniformly smooth Banach space with $q \in (1, 2]$, let the VI indicate a variational inclusion for two accretive operators and let the CFPP denote a common fixed point problem of a countable family of $\ell$-uniformly Lipschitzian pseudocontractive mappings. In this paper, we introduce a parallel composite-type extragradient implicit method for solving the GSVI (1.4) with the VI and CFPP constraints. We then prove the strong convergence of the suggested algorithm to a solution of the GSVI (1.4) with the VI and CFPP constraints under some appropriate assumptions. As applications, we apply our main result to the variational inequality problem (VIP), split feasibility problem (SFP) and LASSO problem in Hilbert spaces. Our results improve and extend the corresponding results in Abdou et al. [22], Sunthrayuth and Cholamjiak [15], and Ceng et al. [13] to a certain extent.

## 2. Preliminaries

Let $E$ be a real Banach space with the dual $E^*$, and $\emptyset \neq C \subset E$ be a closed convex set. For convenience, we shall use the following symbols: $x_n \to x$ (resp., $x_n \rightharpoonup x$) indicates the strong (resp., weak) convergence of the sequence $\{x_n\}$ to $x$. Given a self-mapping $T$ on $C$. We use the symbols $\mathbf{R}$ and $\mathrm{Fix}(T)$ to denote the set of all real numbers and the fixed point set of $T$, respectively. Recall that $T$ is called a nonexpansive mapping if $\|Tx - Ty\| \leq \|x - y\|$ $\forall x, y \in C$. A mapping $f : C \to C$ is called a contraction if $\exists \varrho \in [0, 1)$ s.t. $\|f(x) - f(y)\| \leq \varrho \|x - y\|$ $\forall x, y \in C$. Also, recall that the normalized duality mapping $J$ defined by

$$J(x) = \{\phi \in E^* : \langle x, \phi \rangle = \|x\|^2 = \|\phi\|^2\} \quad \forall x \in E. \tag{2.1}$$

is the one from $E$ into the family of nonempty (by Hahn-Banach's theorem) weak* compact subsets of $E^*$, satisfying $J(\tau u) = \tau J(u)$ and $J(-u) = -J(u)$ for all $\tau > 0$ and $u \in E$.

The modulus of convexity of $E$ is the function $\delta_E : (0, 2] \to [0, 1]$ defined by

$$\delta_E(\epsilon) = \inf\{1 - \frac{\|x + y\|}{2} : x, y \in E,\ \|x\| = \|y\| = 1,\ \|x - y\| \geq \epsilon\}.$$

The modulus of smoothness of $E$ is the function $\rho_E : \mathbf{R}_+ := [0, \infty) \to \mathbf{R}_+$ defined by

$$\rho_E(\tau) = \sup\{\frac{\|x + \tau y\| + \|x - \tau y\|}{2} - 1 : x, y \in E,\ \|x\| = \|y\| = 1\}.$$

A Banach space $E$ is said to be uniformly convex if $\delta_E(\epsilon) > 0\ \forall \epsilon \in (0, 2]$. It is said to be uniformly smooth if $\lim_{\tau \to 0^+} \frac{\rho_E(\tau)}{\tau} = 0$. Also, it is said to be $q$-uniformly smooth with $q > 1$ if $\exists c > 0$ s.t. $\rho_E(t) \leq ct^q\ \forall t > 0$. If $E$ is $q$-uniformly smooth, then $q \leq 2$ and $E$ is also uniformly smooth and if $E$ is uniformly convex, then $E$ is also reflexive and strictly convex. It is known that Hilbert space $H$ is 2-uniformly smooth. Further, sequence space $\ell_p$ and Lebesgue space $L_p$ are $\min\{p, 2\}$-uniformly smooth for every $p > 1$ [33].

Let $q > 1$. The generalized duality mapping $J_q : E \to 2^{E^*}$ is defined by

$$J_q(x) = \{\phi \in E^* : \langle x, \phi\rangle = \|x\|^q,\ \|\phi\| = \|x\|^{q-1}\}, \tag{2.2}$$

where $\langle \cdot, \cdot\rangle$ denotes the generalized duality pairing between $E$ and $E^*$. In particular, if $q = 2$, then $J_2 = J$ is the normalized duality mapping of $E$. It is known that $J_q(x) = \|x\|^{q-2}J(x)\ \forall x \neq 0$ and that $J_q$ is the subdifferential of the functional $\frac{1}{q}\|\cdot\|^q$. If $E$ is uniformly smooth, the generalized duality mapping $J_q$ is one-to-one and single-valued. Furthermore, $J_q$ satisfies $J_q = J_p^{-1}$, where $J_p$ is the generalized duality mapping of $E^*$ with $\frac{1}{p} + \frac{1}{q} = 1$. Note that no Banach space is $q$-uniformly smooth for $q > 2$; see [18] for more details. Let $q > 1$ and $E$ be a real normed space with the generalized duality mapping $J_q$. Then the following inequality is an immediate consequence of the subdifferential inequality of the functional $\frac{1}{q}\|\cdot\|^q$:

$$\|x + y\|^q \leq \|x\|^q + q\langle y, j_q(x + y)\rangle \quad \forall x, y \in E,\ j_q(x + y) \in J_q(x + y). \tag{2.3}$$

**Proposition 2.1.** *(see [33]). Let $q \in (1, 2]$ a fixed real number and let $E$ be $q$-uniformly smooth. Then $\|x + y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + \kappa_q\|y\|^q\ \forall x, y \in E$, where $\kappa_q$ is the $q$-uniform smoothness coefficient of $E$.*

Recall that a mapping $T : C \to C$ is called pseudocontractive if for each $x, y \in C$, there exists $j(x - y) \in J(x - y)$ such that $\langle Tx - Ty, j(x - y)\rangle \leq \|x - y\|^2$. Also, it is called strongly pseudocontractive if for each $x, y \in C$, there exists $j(x - y) \in J(x - y)$ such that $\langle Tx - Ty, j(x - y)\rangle \leq \alpha\|x - y\|^2$ for some $\alpha \in (0, 1)$. We will use the following concept in the sequel.

**Definition 2.2.** Let $\{S_n\}_{n=0}^{\infty}$ be a sequence of continuous pseudocontractive self-mappings on $C$. Then $\{S_n\}_{n=0}^{\infty}$ is said to be a countable family of $\ell$-uniformly Lipschitzian pseudocontractive self-mappings on $C$ if there exists a constant $\ell > 0$ such that each $S_n$ is $\ell$-Lipschitz continuous.

**Lemma 2.3.** *(see [10]). Let $\{S_n\}_{n=0}^{\infty}$ be a sequence of self-mappings on $C$ such that $\sum_{n=1}^{\infty} \sup_{x \in C} \|S_n x - S_{n-1}x\| < \infty$. Then for each $y \in C$, $\{S_n y\}$ converges strongly to some point of $C$. Moreover, let $S$ be a self-mapping on $C$ defined by $Sy = \lim_{n\to\infty} S_n y\ \forall y \in C$. Then $\lim_{n\to\infty} \sup_{x\in C} \|S_n x - Sx\| = 0$.*

The following lemma can be obtained from the result in [33].

**Lemma 2.4.** *Let $q > 1$ and $r > 0$ be two fixed real numbers and let $E$ be uniformly convex. Then there exist strictly increasing, continuous and convex functions $g, h : \mathbf{R}_+ \to \mathbf{R}_+$ with $g(0) = 0$ and $h(0) = 0$ such that*

(a) $\|\mu x + (1 - \mu)y\|^q \leq \mu\|x\|^q + (1 - \mu)\|y\|^q - \mu(1 - \mu)g(\|x - y\|)$ with $\mu \in [0, 1]$;

(b) $h(\|x-y\|) \le \|x\|^q - q\langle x, j_q(y)\rangle + (q-1)\|y\|^q$
*for all* $x, y \in B_r$ *and* $j_q(y) \in J_q(y)$, *where* $B_r := \{x \in E : \|x\| \le r\}$.

The following lemma is an analogue of Lemma 2.4 (a).

**Lemma 2.5.** *Let* $q > 1$ *and* $r > 0$ *be two fixed real numbers and let* $E$ *be uniformly convex. Then there exists a strictly increasing, continuous and convex function* $g : \mathbf{R}_+ \to \mathbf{R}_+$ *with* $g(0) = 0$ *such that* $\|\lambda x + \mu y + \nu z\|^q \le \lambda\|x\|^q + \mu\|y\|^q + \nu\|z\|^q - \lambda\mu g(\|x-y\|)$ *for all* $x, y, z \in B_r$ *and* $\lambda, \mu, \nu \in [0,1]$ *with* $\lambda + \mu + \nu = 1$.

**Proposition 2.6.** *(see [25]). Let* $\emptyset \ne C \subset E$ *be a closed convex set. If* $T : C \to C$ *is a continuous and strong pseudocontraction mapping, then* $T$ *has a unique fixed point in* $C$.

Let $D$ be a subset of $C$ and let $\Pi$ be a mapping of $C$ into $D$. Then $\Pi$ is said to be sunny if $\Pi[\Pi(x) + t(x - \Pi(x))] = \Pi(x)$, whenever $\Pi(x) + t(x - \Pi(x)) \in C$ for $x \in C$ and $t \ge 0$. A mapping $\Pi$ of $C$ into itself is called a retraction if $\Pi^2 = \Pi$. If a mapping $\Pi$ of $C$ into itself is a retraction, then $\Pi(z) = z$ for each $z \in R(\Pi)$, where $R(\Pi)$ is the range of $\Pi$. A subset $D$ of $C$ is called a sunny nonexpansive retract of $C$ if there exists a sunny nonexpansive retraction from $C$ onto $D$. In terms of [23], we know that if $E$ is smooth and $\Pi$ is a retraction of $C$ onto $D$, then the following statements are equivalent:

(i) $\Pi$ is sunny and nonexpansive;
(ii) $\|\Pi(x) - \Pi(y)\|^2 \le \langle x - y, J(\Pi(x) - \Pi(y))\rangle\ \forall x, y \in C$;
(iii) $\langle x - \Pi(x), J(y - \Pi(x))\rangle \le 0\ \forall x \in C, y \in D$.

Let $B : C \to 2^E$ be a set-valued operator with $Bx \ne \emptyset\ \forall x \in C$. Let $q > 1$. An operator $B$ is said to be accretive if for each $x, y \in C$, $\exists j_q(x-y) \in J_q(x-y)$ s.t. $\langle u - v, j_q(x-y)\rangle \ge 0\ \forall u \in Bx, v \in By$. An accretive operator $B$ is said to be $\alpha$-inverse-strongly accretive of order $q$ if for each $x, y \in C$, $\exists j_q(x-y) \in J_q(x-y)$ s.t. $\langle u - v, j_q(x-y)\rangle \ge \alpha\|u - v\|^q\ \forall u \in Bx, v \in By$ for some $\alpha > 0$. If $E = H$ a Hilbert space, then $B$ is called $\alpha$-inverse-strongly monotone. An accretive operator $B$ is said to be $m$-accretive if $(I + \lambda B)C = E$ for all $\lambda > 0$. For an accretive operator $B$, we define the mapping $J_\lambda^B : (I + \lambda B)C \to C$ by $J_\lambda^B = (I + \lambda B)^{-1}$ for each $\lambda > 0$. Such $J_\lambda^B$ is called the resolvent of $B$ for $\lambda > 0$.

**Lemma 2.7.** *(see [17, 19]). Let* $B : C \to 2^E$ *be an* $m$*-accretive operator. Then the following statements hold:*

(i) *the resolvent identity:* $J_\lambda^B x = J_\mu^B(\frac{\mu}{\lambda}x + (1 - \frac{\mu}{\lambda})J_\lambda^B x)\ \forall \lambda, \mu > 0,\ x \in E$;
(ii) *if* $J_\lambda^B$ *is a resolvent of* $B$ *for* $\lambda > 0$, *then* $J_\lambda^B$ *is a firmly nonexpansive mapping with* $\mathrm{Fix}(J_\lambda^B) = B^{-1}0$, *where* $B^{-1}0 = \{x \in C : 0 \in Bx\}$;
(iii) *if* $E = H$ *a Hilbert space,* $B$ *is maximal monotone.*

Let $A : C \to E$ be an $\alpha$-inverse-strongly accretive mapping of order $q$ and $B : C \to 2^E$ be an $m$-accretive operator. In the sequel, we will use the notation $T_\lambda := J_\lambda^B(I - \lambda A) = (I + \lambda B)^{-1}(I - \lambda A)\ \forall \lambda > 0$.

**Proposition 2.8.** *(see [17]). The following statements hold:*

(i) $\mathrm{Fix}(T_\lambda) = (A + B)^{-1}0\ \forall \lambda > 0$;
(ii) $\|y - T_\lambda y\| \le 2\|y - T_r y\|$ *for* $0 < \lambda \le r$ *and* $y \in C$.

**Proposition 2.9.** *(see [36]). Let* $E$ *be uniformly smooth,* $T : C \to C$ *be a nonexpansive mapping with* $\mathrm{Fix}(T) \ne \emptyset$ *and* $f : C \to C$ *be a fixed contraction. For each* $t \in (0,1)$, *let* $z_t \in C$ *be the unique fixed point of the contraction* $C \ni z \mapsto tf(z) + (1-t)Tz$ *on* $C$, *i.e.,* $z_t = tf(z_t) + (1-t)Tz_t$. *Then* $\{z_t\}$ *converges strongly to a fixed point* $x^* \in \mathrm{Fix}(T)$, *which solves the VIP:* $\langle (I - f)x^*, J(x^* - x)\rangle \le 0\ \forall x \in \mathrm{Fix}(T)$.

**Proposition 2.10.** *(see [17]). Let $E$ be $q$-uniformly smooth with $q \in (1,2]$. Suppose that $A : C \to E$ is an $\alpha$-inverse-strongly accretive mapping of order $q$. Then, for any given $\lambda \geq 0$,*

$$\|(I-\lambda A)x-(I-\lambda A)y\|^q \leq \|x-y\|^q - \lambda(\alpha q - \kappa_q \lambda^{q-1})\|Ax-Ay\|^q \quad \forall x,y \in C,$$

*where $\kappa_q > 0$ is the $q$-uniform smoothness coefficient of $E$. In particular, if $0 \leq \lambda \leq (\frac{\alpha q}{\kappa_q})^{\frac{1}{q-1}}$, then $I-\lambda A$ is nonexpansive.*

**Lemma 2.11.** *(see [13]). Let $E$ be $q$-uniformly smooth with $q \in (1,2]$. Let $B_1, B_2 : C \to 2^E$ be two $m$-accretive operators and $A_i : C \to E$ $(i=1,2)$ be $\sigma_i$-inverse-strongly accretive mapping of order $q$. Define an operator $G : C \to C$ by $G := J^{B_1}_{\zeta_1}(I-\zeta_1 A_1)J^{B_2}_{\zeta_2}(I-\zeta_2 A_2)$. If $0 \leq \zeta_i \leq (\frac{\sigma_i q}{\kappa_q})^{\frac{1}{q-1}}$ $(i=1,2)$, then $G$ is nonexpansive.*

**Lemma 2.12.** *(see [2]). Let $E$ be smooth, $A : C \to E$ be accretive and $\Pi_C$ be a sunny nonexpansive retraction from $E$ onto $C$. Then $\mathrm{VI}(C,A) = \mathrm{Fix}(\Pi_C(I-\lambda A))$ $\forall \lambda > 0$, where $\mathrm{VI}(C,A)$ is the solution set of the VIP of finding $z \in C$ s.t. $\langle Az, J(z-y)\rangle \leq 0$ $\forall y \in C$.*

Recall that if $E = H$ a Hilbert space, then the sunny nonexpansive retraction $\Pi_C$ from $E$ onto $C$ coincides with the metric projection $P_C$ from $H$ onto $C$. Moreover, if $E$ is uniformly smooth and $T$ is a nonexpansive self-mapping on $C$ with $\mathrm{Fix}(T) \neq \emptyset$, then $\mathrm{Fix}(T)$ is a sunny nonexpansive retract from $E$ onto $C$ [29]. By Lemma 2.12 we know that, $x^* \in \mathrm{Fix}(T)$ solves the VIP in Proposition 2.9 if and only if $x^*$ solves the fixed point equation $x^* = \Pi_{\mathrm{Fix}(T)} f(x^*)$.

**Lemma 2.13.** *(see [16]). Let $\{\Gamma_n\}$ be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence $\{\Gamma_{n_i}\}$ of $\{\Gamma_n\}$ which satisfies $\Gamma_{n_i} < \Gamma_{n_i+1}$ for each integer $i \geq 1$. Define the sequence $\{\tau(n)\}_{n \geq n_0}$ of integers as follows:*

$$\tau(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\},$$

*where integer $n_0 \geq 1$ such that $\{k \leq n_0 : \Gamma_k < \Gamma_{k+1}\} \neq \emptyset$. Then, the following hold:*

(i) $\tau(n_0) \leq \tau(n_0+1) \leq \cdots$ *and* $\tau(n) \to \infty$;

(ii) $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ *and* $\Gamma_n \leq \Gamma_{\tau(n)+1}$ $\forall n \geq n_0$.

**Lemma 2.14.** *(see [1]). Let $E$ be strictly convex, and $\{S_n\}_{n=0}^\infty$ be a sequence of nonexpansive mappings on $C$. Suppose that $\bigcap_{n=0}^\infty \mathrm{Fix}(S_n)$ is nonempty. Let $\{\lambda_n\}$ be a sequence of positive numbers with $\sum_{n=0}^\infty \lambda_n = 1$. Then a mapping $S$ on $C$ defined by $Sx = \sum_{n=0}^\infty \lambda_n S_n x$ $\forall x \in C$ is defined well, nonexpansive and $\mathrm{Fix}(S) = \bigcap_{n=0}^\infty \mathrm{Fix}(S_n)$ holds.*

**Lemma 2.15.** *(see [36]). Let $\{a_n\}$ be a sequence in $[0,\infty)$ such that $a_{n+1} \leq (1-s_n)a_n + s_n \nu_n$ $\forall n \geq 0$, where $\{s_n\}$ and $\{\nu_n\}$ satisfy the conditions: (i) $\{s_n\} \subset [0,1]$, $\sum_{n=0}^\infty s_n = \infty$; (ii) $\limsup_{n\to\infty} \nu_n \leq 0$ or $\sum_{n=0}^\infty |s_n \nu_n| < \infty$. Then $\lim_{n\to\infty} a_n = 0$.*

## 3. Main Results

Throughout this paper, suppose that $C$ is a nonempty closed convex subset of a uniformly convex and $q$-uniformly smooth Banach space $E$ with $q \in (1,2]$. Let $B_1, B_2 : C \to 2^E$ be both $m$-accretive operators and $A_i : C \to E$ be $\sigma_i$-inverse-strongly accretive mapping of order $q$ for $i = 1,2$. Let $f : C \to C$ be a $\varrho$-contraction with constant $\varrho \in [0, \frac{1}{q})$, and $\{S_n\}_{n=0}^\infty$ be a countable family of $\ell$-uniformly Lipschitzian pseudocontractive self-mappings on $C$. Let $A : C \to E$ and $B : C \to 2^E$ be a $\sigma$-inverse-strongly accretive mapping of order $q$ and an $m$-accretive operator, respectively. Assume that $\Omega := \bigcap_{n=0}^\infty \mathrm{Fix}(S_n) \cap \mathrm{Fix}(G) \cap (A+B)^{-1}0 \neq \emptyset$ where $G : C \to C$ is the same as defined in Lemma 2.11.

**Algorithm 3.1.** *Parallel composite-type extragradient implicit method for the GSVI (1.4) with the VI and CFPP constraints.*

**Initial Step:** *Given* $\zeta \in (0,1)$ *and* $x_0 \in C$ *arbitrarily.*
**Iteration Steps:** *Given the current iterate* $x_n$, *compute* $x_{n+1}$ *as follows:*
Step 1 *Calculate* $w_n = s_n x_n + (1-s_n)Gx_n$;
Step 2 *Calculate* $v_n = J_{\zeta_2}^{B_2}(w_n - \zeta_2 A_2 w_n)$;
Step 3 *Calculate* $u_n = J_{\zeta_1}^{B_1}(v_n - \zeta_1 A_1 v_n)$;
Step 4 *Calculate* $x_{n+1} = (1-\zeta)S_n x_{n+1} + \zeta J_{\lambda_n}^{B}(\alpha_n f(u_n) + (1-\alpha_n)u_n - \lambda_n A u_n)$, *where* $\{s_n\}, \{\alpha_n\} \subset (0,1)$ *and* $\{\lambda_n\} \subset (0,\infty)$.
*Set* $n := n+1$ *and go to Step 1.*

**Lemma 3.2.** *If* $\{x_n\}$ *is the sequence generated by Algorithm 3.1, then it is bounded.*

*Proof.* Take an element $p \in \Omega := \bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n) \cap \mathrm{Fix}(G) \cap (A+B)^{-1}0$ arbitrarily. Then we have

$$p = Gp = S_n p = J_{\lambda_n}^{B}(p - \lambda_n Ap) = J_{\lambda_n}^{B}(\alpha_n p + (1-\alpha_n)(p - \frac{\lambda_n}{1-\alpha_n}Ap)).$$

By Proposition 2.10 and Lemma 2.11, we deduce that $I - \zeta_1 A_1,\ I - \zeta_2 A_2$ and $G := J_{\zeta_1}^{B_1}(I - \zeta_1 A_1)J_{\zeta_2}^{B_2}(I - \zeta_2 A_2)$ are nonexpansive mappings. Moreover, it can be readily seen that for each $n \geq 0$, there is only an element $x_{n+1} \in C$ s.t.

$$x_{n+1} = (1-\zeta)S_n x_{n+1} + \zeta J_{\lambda_n}^{B}(\alpha_n f(u_n) + (1-\alpha_n)u_n - \lambda_n A u_n). \tag{3.1}$$

In fact, consider the mapping $F_n x = (1-\zeta)S_n x + \zeta J_{\lambda_n}^{B}(\alpha_n f(u_n) + (1-\alpha_n)u_n - \lambda_n A u_n)\ \forall x \in C$. Note that $S_n : C \to C$ is a continuous pseudocontraction. Hence we obtain that for each $x, y \in C$,

$$\langle F_n x - F_n y, J(x-y)\rangle = (1-\zeta)\langle S_n x - S_n y, J(x-y)\rangle \leq (1-\zeta)\|x-y\|^2.$$

Also, from $\zeta \in (0,1)$, we get $0 < 1-\zeta < 1$. Thus, $F_n$ is a continuous and strong pseudocontraction self-mapping on $C$. By Proposition 2.6, we deduce that for each $n \geq 0$, there is only an element $x_{n+1} \in C$, satisfying (3.1). Since $G : C \to C$ is a nonexpansive mapping, by Lemma 2.4 (a) we get

$$\begin{aligned}\|w_n - p\|^q &\leq s_n\|x_n - p\|^q + (1-s_n)\|Gx_n - p\|^q - s_n(1-s_n)\tilde{g}(\|x_n - Gx_n\|)\\ &\leq \|x_n - p\|^q - s_n(1-s_n)\tilde{g}(\|x_n - Gx_n\|).\end{aligned} \tag{3.2}$$

Using the nonexpansivity of $G$ again, we obtain from $u_n = Gw_n$ that

$$\|u_n - p\| \leq \|w_n - p\| \leq \|x_n - p\| \quad \forall n \geq 0. \tag{3.3}$$

Put $y_n := J_{\lambda_n}^{B}(\alpha_n f(u_n) + (1-\alpha_n)u_n - \lambda_n A u_n)\ \forall n \geq 0$. Since $J_{\lambda_n}^{B}$ and $I - \frac{\lambda_n}{1-\alpha_n}A$ are nonexpansive for all $n \geq 0$, we obtain from (3.3) that

$$\begin{aligned}&\|y_n - p\| \qquad (3.4)\\
&= \|J_{\lambda_n}^{B}(\alpha_n f(u_n) + (1-\alpha_n)u_n - \lambda_n A u_n) - p\|\\
&= \|J_{\lambda_n}^{B}(\alpha_n f(u_n) + (1-\alpha_n)(u_n - \frac{\lambda_n}{1-\alpha_n}Au_n)) - J_{\lambda_n}^{B}(\alpha_n p + (1-\alpha_n)(p - \frac{\lambda_n}{1-\alpha_n}Ap))\|\\
&\leq \|(\alpha_n f(u_n) + (1-\alpha_n)(u_n - \frac{\lambda_n}{1-\alpha_n}Au_n)) - (\alpha_n p + (1-\alpha_n)(p - \frac{\lambda_n}{1-\alpha_n}Ap))\|\\
&= \|(1-\alpha_n)((u_n - \frac{\lambda_n}{1-\alpha_n}Au_n) - (p - \frac{\lambda_n}{1-\alpha_n}Ap)) + \alpha_n(f(u_n) - p)\|\\
&\leq (1-\alpha_n)\|u_n - p\| + \alpha_n\|f(u_n) - f(p)\| + \alpha_n\|f(p) - p\|\\
&\leq (1-\alpha_n(1-\varrho))\|u_n - p\| + \alpha_n\|f(p) - p\|\\
&\leq (1-\alpha_n(1-\varrho))\|x_n - p\| + \alpha_n\|f(p) - p\|\\
&= (1-\alpha_n(1-\varrho))\|x_n - p\| + \alpha_n(1-\varrho)\frac{\|f(p) - p\|}{1-\varrho}\end{aligned}$$

$$\leq \max\{\|x_n - p\|, \frac{\|f(p) - p\|}{1-\varrho}\}.$$

Noticing that $S_n$ is a pseudocontraction mapping, we conclude from (3.1) that

$$\begin{aligned}\|x_{n+1} - p\|^q &= (1-\zeta)\langle S_n x_{n+1} - p, J_q(x_{n+1} - p)\rangle + \zeta\langle y_n - p, J_q(x_{n+1} - p)\rangle \\ &\leq (1-\zeta)\|x_{n+1} - p\|^q + \zeta\langle y_n - p, J_q(x_{n+1} - p)\rangle,\end{aligned}$$

which together with Lemma 2.4 (b), implies that

$$\begin{aligned}\|x_{n+1} - p\|^q &\leq \langle y_n - p, J_q(x_{n+1} - p)\rangle \\ &\leq \frac{1}{q}[\|y_n - p\|^q + (q-1)\|x_{n+1} - p\|^q - \tilde{h}(\|y_n - x_{n+1}\|)].\end{aligned}$$

This ensures that

$$\|x_{n+1} - p\|^q \leq \|y_n - p\|^q - \tilde{h}(\|y_n - x_{n+1}\|). \tag{3.5}$$

So it follows from (3.4) that

$$\|x_{n+1} - p\| \leq \|y_n - p\| \leq \max\{\|x_n - p\|, \frac{\|f(p) - p\|}{1-\varrho}\}.$$

By induction, we get $\|x_n - p\| \leq \max\{\|x_0 - p\|, \frac{\|p - f(p)\|}{1-\varrho}\}\ \forall n \geq 0$. Consequently, $\{x_n\}$ is bounded, and so are $\{u_n\}, \{w_n\}, \{y_n\}, \{S_n x_{n+1}\}, \{Au_n\}$. This completes the proof. □

**Theorem 3.3.** *Let $\{x_n\}$ be the sequence generalized by Algorithm 3.1. Suppose that the following conditions hold:*

- (C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;
- (C2) $0 < a \leq \frac{\lambda_n}{1-\alpha_n} \leq b < (\frac{\sigma q}{\kappa_q})^{\frac{1}{q-1}}$ *and* $0 < c \leq s_n \leq d < 1$;
- (C3) $0 < \zeta_i < (\frac{\sigma_i q}{\kappa_q})^{\frac{1}{q-1}}$ *for* $i = 1, 2$.

*Assume that $\sum_{n=0}^{\infty} \sup_{x\in D} \|S_{n+1}x - S_n x\| < \infty$ for any bounded subset $D$ of $C$. Let $S : C \to C$ be a mapping defined by $Sx = \lim_{n\to\infty} S_n x\ \forall x \in C$, and suppose that $\mathrm{Fix}(S) = \bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n)$. Then $x_n \to x^* \in \Omega$, which is the unique solution to the VIP: $\langle (I - f)x^*, J(x^* - p)\rangle \leq 0\ \forall p \in \Omega$, i.e., the fixed point equation $x^* = \Pi_\Omega f(x^*)$.*

*Proof.* First of all, let $x^* \in \Omega$ and $y^* = J_{\zeta_2}^{B_2}(x^* - \zeta_2 A_2 x^*)$. Since $v_n = J_{\zeta_2}^{B_2}(w_n - \zeta_2 A_2 w_n)$ and $u_n = J_{\zeta_1}^{B_1}(v_n - \zeta_1 A_1 v_n)$, we get $u_n = G w_n$. From Proposition 2.10 we have

$$\begin{aligned}\|v_n - y^*\|^q &= \|J_{\zeta_2}^{B_2}(w_n - \zeta_2 A_2 w_n) - J_{\zeta_2}^{B_2}(x^* - \zeta_2 A_2 x^*)\|^q \\ &\leq \|w_n - x^*\|^q - \zeta_2(\sigma_2 q - \kappa_q \zeta_2^{q-1})\|A_2 w_n - A_2 x^*\|^q,\end{aligned}$$

and

$$\begin{aligned}\|u_n - x^*\|^q &= \|J_{\zeta_1}^{B_1}(v_n - \zeta_1 A_1 v_n) - J_{\zeta_1}^{B_1}(y^* - \zeta_1 A_1 y^*)\|^q \\ &\leq \|v_n - y^*\|^q - \zeta_1(\sigma_1 q - \kappa_q \zeta_1^{q-1})\|A_1 v_n - A_1 y^*\|^q.\end{aligned}$$

Combining the last two inequalities, we have

$$\|u_n - x^*\|^q \leq \|w_n - x^*\|^q - \zeta_2(\sigma_2 q - \kappa_q \zeta_2^{q-1})\|A_2 w_n - A_2 x^*\|^q - \zeta_1(\sigma_1 q - \kappa_q \zeta_1^{q-1})\|A_1 v_n - A_1 y^*\|^q. \tag{3.6}$$

Also, using Propositions 2.1 and 2.10 and the convexity of $\|\cdot\|^q$, from (3.4) and (3.6) we get

$$\begin{aligned}&\|y_n - x^*\|^q \\ &\leq \|(1-\alpha_n)((u_n - \frac{\lambda_n}{1-\alpha_n} A u_n) - (x^* - \frac{\lambda_n}{1-\alpha_n} A x^*)) + \alpha_n(f(u_n) - x^*)\|^q\end{aligned}$$

$$
\begin{aligned}
&\le (1-\alpha_n)^q\|(u_n-\frac{\lambda_n}{1-\alpha_n}Au_n)-(x^*-\frac{\lambda_n}{1-\alpha_n}Ax^*)\|^q \\
&\quad +q\alpha_n(1-\alpha_n)^{q-1}\langle f(u_n)-x^*,J_q(u_n-x^*-\frac{\lambda_n}{1-\alpha_n}(Au_n-Ax^*))\rangle+\kappa_q\alpha_n^q\|f(u_n)-x^*\|^q \\
&\le (1-\alpha_n(1-q\varrho))\|u_n-x^*\|^q-\lambda_n(\sigma q-\kappa_q(\frac{\lambda_n}{1-\alpha_n})^{q-1})\|Au_n-Ax^*\|^q \\
&\quad +q\alpha_n(1-\alpha_n)^{q-1}\langle f(x^*)-x^*,J_q(u_n-x^*-\frac{\lambda_n}{1-\alpha_n}(Au_n-Ax^*))\rangle+\kappa_q\alpha_n^q\|f(u_n)-x^*\|^q \\
&\le (1-\alpha_n(1-q\varrho))[\|w_n-x^*\|^q-\zeta_2(\sigma_2q-\kappa_q\zeta_2^{q-1})\|A_2w_n-A_2x^*\|^q-\zeta_1(\sigma_1q-\kappa_q\zeta_1^{q-1}) \\
&\quad \times\|A_1v_n-A_1y^*\|^q]-\lambda_n(\sigma q-\kappa_q(\frac{\lambda_n}{1-\alpha_n})^{q-1})\|Au_n-Ax^*\|^q \\
&\quad +q\alpha_n(1-\alpha_n)^{q-1}\langle f(x^*)-x^*,J_q(u_n-x^*-\frac{\lambda_n}{1-\alpha_n}(Au_n-Ax^*))\rangle+\kappa_q\alpha_n^q\|f(u_n)-x^*\|^q.
\end{aligned}
$$

This together with (3.2) and (3.5), leads to

$$
\begin{aligned}
&\|x_{n+1}-x^*\|^q\le\|y_n-x^*\|^q-\tilde{h}(\|y_n-x_{n+1}\|) \qquad (3.7)\\
&\le (1-\alpha_n(1-q\varrho))[\|w_n-x^*\|^q-\zeta_2(\sigma_2q-\kappa_q\zeta_2^{q-1})\|A_2w_n-A_2x^*\|^q \\
&\quad -\zeta_1(\sigma_1q-\kappa_q\zeta_1^{q-1})\|A_1v_n-A_1y^*\|^q]-\lambda_n(\sigma q-\kappa_q(\frac{\lambda_n}{1-\alpha_n})^{q-1})\|Au_n-Ax^*\|^q \\
&\quad +q\alpha_n(1-\alpha_n)^{q-1}\langle f(x^*)-x^*,J_q(u_n-x^*-\frac{\lambda_n}{1-\alpha_n}(Au_n-Ax^*))\rangle \\
&\quad +\kappa_q\alpha_n^q\|f(u_n)-x^*\|^q-\tilde{h}(\|y_n-x_{n+1}\|) \\
&\le (1-\alpha_n(1-q\varrho))[\|x_n-x^*\|^q-s_n(1-s_n)\tilde{g}(\|x_n-Gx_n\|) \\
&\quad -\zeta_2(\sigma_2q-\kappa_q\zeta_2^{q-1})\|A_2w_n-A_2x^*\|^q-\zeta_1(\sigma_1q-\kappa_q\zeta_1^{q-1}) \\
&\quad \times\|A_1v_n-A_1y^*\|^q]-\lambda_n(\sigma q-\kappa_q(\frac{\lambda_n}{1-\alpha_n})^{q-1})\|Au_n-Ax^*\|^q \\
&\quad +q\alpha_n(1-\alpha_n)^{q-1}\langle f(x^*)-x^*,J_q(u_n-x^*-\frac{\lambda_n}{1-\alpha_n}(Au_n-Ax^*))\rangle \\
&\quad +\kappa_q\alpha_n^q\|f(u_n)-x^*\|^q-\tilde{h}(\|y_n-x_{n+1}\|) \\
&= (1-\alpha_n(1-q\varrho))\|x_n-x^*\|^q-\{(1-\alpha_n(1-q\varrho))[s_n(1-s_n)\tilde{g}(\|x_n-Gx_n\|) \\
&\quad +\zeta_2(\sigma_2q-\kappa_q\zeta_2^{q-1})\|A_2w_n-A_2x^*\|^q+\zeta_1(\sigma_1q-\kappa_q\zeta_1^{q-1})\|A_1v_n-A_1y^*\|^q] \\
&\quad +\lambda_n(\sigma q-\kappa_q(\frac{\lambda_n}{1-\alpha_n})^{q-1})\|Au_n-Ax^*\|^q+\tilde{h}(\|y_n-x_{n+1}\|)\}+q\alpha_n(1-\alpha_n)^{q-1} \\
&\quad \times\langle f(x^*)-x^*,J_q(u_n-x^*-\frac{\lambda_n}{1-\alpha_n}(Au_n-Ax^*))\rangle+\kappa_q\alpha_n^q\|f(u_n)-x^*\|^q.
\end{aligned}
$$

For each $n\ge 0$, we set

$$
\begin{aligned}
\varGamma_n &= \|x_n-x^*\|^q, \\
\varepsilon_n &= \alpha_n(1-q\varrho), \\
\eta_n &= (1-\alpha_n(1-q\varrho))[s_n(1-s_n)\tilde{g}(\|x_n-Gx_n\|)+\zeta_2(\sigma_2q-\kappa_q\zeta_2^{q-1})\|A_2w_n-A_2x^*\|^q \\
&\quad +\zeta_1(\sigma_1q-\kappa_q\zeta_1^{q-1})\|A_1v_n-A_1y^*\|^q]+\lambda_n(\sigma q-\kappa_q(\frac{\lambda_n}{1-\alpha_n})^{q-1})\|Au_n-Ax^*\|^q \\
&\quad +\tilde{h}(\|y_n-x_{n+1}\|),
\end{aligned}
$$

$$\vartheta_n = q\alpha_n(1-\alpha_n)^{q-1}\langle f(x^*)-x^*, J_q(u_n-x^*-\frac{\lambda_n}{1-\alpha_n}(Au_n-Ax^*))\rangle + \kappa_q\alpha_n^q\|f(u_n)-x^*\|^q.$$

Then (3.7) can be rewritten as the following formula:

$$\Gamma_{n+1} \le (1-\varepsilon_n)\Gamma_n - \eta_n + \vartheta_n \quad \forall n \ge 0, \tag{3.8}$$

and hence

$$\Gamma_{n+1} \le (1-\varepsilon_n)\Gamma_n + \vartheta_n \quad \forall n \ge 0. \tag{3.9}$$

We next show the strong convergence of $\{\Gamma_n\}$ by the following two cases:

**Case 1.** Suppose that there exists an integer $n_0 \ge 1$ such that $\{\Gamma_n\}$ is non-increasing. Then

$$\Gamma_n - \Gamma_{n+1} \to 0.$$

From (3.9), we get

$$0 \le \eta_n \le \Gamma_n - \Gamma_{n+1} + \vartheta_n - \varepsilon_n\Gamma_n.$$

Since $\alpha_n \to 0,\ \varepsilon_n \to 0$ and $\vartheta_n \to 0$, we have $\eta_n \to 0$. This ensures that $\lim_{n\to\infty}\tilde{g}(\|x_n - Gx_n\|) = \lim_{n\to\infty}\tilde{h}(\|y_n - x_{n+1}\|) = 0$,

$$\lim_{n\to\infty}\|A_2w_n - A_2x^*\| = \lim_{n\to\infty}\|A_1v_n - A_1y^*\| = 0, \tag{3.10}$$

and

$$\lim_{n\to\infty}\|Au_n - Ax^*\| = 0. \tag{3.11}$$

Note that $\tilde{g}$ and $\tilde{h}$ are strictly increasing, continuous and convex functions with $\tilde{g}(0) = \tilde{h}(0) = 0$. So it follows that

$$\lim_{n\to\infty}\|y_n - x_{n+1}\| = \lim_{n\to\infty}\|x_n - Gx_n\| = 0. \tag{3.12}$$

Thus, from (3.1) we get

$$\lim_{n\to\infty}\|S_nx_{n+1} - x_{n+1}\| = \frac{\zeta}{1-\zeta}\lim_{n\to\infty}\|y_n - x_{n+1}\| = 0. \tag{3.13}$$

On the other hand, using Lemma 2.4 (b) and Lemma 2.7 (ii), we get

$$\begin{aligned}
\|v_n - y^*\|^q &= \|J_{\zeta_2}^{B_2}(w_n - \zeta_2A_2w_n) - J_{\zeta_2}^{B_2}(x^* - \zeta_2A_2x^*)\|^q \\
&\le \langle w_n - \zeta_2A_2w_n - (x^* - \zeta_2A_2x^*), J_q(v_n - y^*)\rangle \\
&= \langle w_n - x^*, J_q(v_n - y^*)\rangle + \zeta_2\langle A_2x^* - A_2w_n, J_q(v_n - y^*)\rangle \\
&\le \frac{1}{q}[\|w_n - x^*\|^q + (q-1)\|v_n - y^*\|^q - \tilde{h}_1(\|w_n - x^* - v_n + y^*\|)] \\
&\quad +\zeta_2\langle A_2x^* - A_2w_n, J_q(v_n - y^*)\rangle,
\end{aligned}$$

which hence attains

$$\|v_n - y^*\|^q \le \|w_n - x^*\|^q - \tilde{h}_1(\|w_n - v_n - x^* + y^*\|) + q\zeta_2\|A_2x^* - A_2w_n\|\|v_n - y^*\|^{q-1}.$$

In a similar way, we get

$$\begin{aligned}
\|u_n - x^*\|^q &= \|J_{\zeta_1}^{B_1}(v_n - \zeta_1A_1v_n) - J_{\zeta_1}^{B_1}(y^* - \zeta_1A_1y^*)\|^q \\
&\le \langle v_n - \zeta_1A_1v_n - (y^* - \zeta_1A_1y^*), J_q(u_n - x^*)\rangle \\
&= \langle v_n - y^*, J_q(u_n - x^*)\rangle + \zeta_1\langle A_1y^* - A_1v_n, J_q(u_n - x^*)\rangle \\
&\le \frac{1}{q}[\|v_n - y^*\|^q + (q-1)\|u_n - x^*\|^q - \tilde{h}_2(\|v_n - y^* - u_n + x^*\|)] \\
&\quad +\zeta_1\langle A_1y^* - A_1v_n, J_q(u_n - x^*)\rangle,
\end{aligned}$$

which hence attains

$$\begin{aligned}\|u_n - x^*\|^q &\le \|v_n - y^*\|^q - \tilde{h}_2(\|v_n - y^* - u_n + x^*\|) + q\zeta_1\|A_1y^* - A_1v_n\|\|u_n - x^*\|^{q-1} \\ &\le \|x_n - x^*\|^q - \tilde{h}_1(\|w_n - v_n - x^* + y^*\|) + q\zeta_2\|A_2x^* - A_2w_n\|\|v_n - y^*\|^{q-1} \\ &\quad -\tilde{h}_2(\|v_n - u_n + x^* - y^*\|) + q\zeta_1\|A_1y^* - A_1v_n\|\|u_n - x^*\|^{q-1}. \end{aligned} \tag{3.14}$$

Since $J^B_{\lambda_n}$ is firmly nonexpansive (due to Lemma 2.7 (ii)), by Lemma2.4 (b) we get

$$\begin{aligned}\|y_n - x^*\|^q &= \|J^B_{\lambda_n}(\alpha_n f(u_n) + (1-\alpha_n)u_n - \lambda_n Au_n) - J^B_{\lambda_n}(x^* - \lambda_n Ax^*)\|^q \\ &\le \langle(\alpha_n f(u_n) + (1-\alpha_n)u_n - \lambda_n Au_n) - (x^* - \lambda_n Ax^*), J_q(y_n - x^*)\rangle \\ &\le \frac{1}{q}[\|(\alpha_n f(u_n) + (1-\alpha_n)u_n - \lambda_n Au_n) - (x^* - \lambda_n Ax^*)\|^q + (q-1)\|y_n - x^*\|^q \\ &\quad -h_1(\|\alpha_n f(u_n) + (1-\alpha_n)u_n - \lambda_n(Au_n - Ax^*) - y_n\|)],\end{aligned}$$

which together with the convexity of $\|\cdot\|^q$ and the nonexpansivity of $I - \frac{\lambda_n}{1-\alpha_n}A$, implies that

$$\begin{aligned}\|y_n - x^*\|^q &\le \|(\alpha_n f(u_n) + (1-\alpha_n)u_n - \lambda_n Au_n) - (x^* - \lambda_n Ax^*)\|^q \\ &\quad -h_1(\|\alpha_n f(u_n) + (1-\alpha_n)u_n - \lambda_n(Au_n - Ax^*) - y_n\|) \\ &= \|(1-\alpha_n)((u_n - \frac{\lambda_n}{1-\alpha_n}Au_n) - (x^* - \frac{\lambda_n}{1-\alpha_n}Ax^*)) + \alpha_n(f(u_n) - x^*)\|^q \\ &\quad -h_1(\|\alpha_n f(u_n) + (1-\alpha_n)u_n - \lambda_n(Au_n - Ax^*) - y_n\|) \\ &\le (1-\alpha_n)\|(u_n - \frac{\lambda_n}{1-\alpha_n}Au_n) - (x^* - \frac{\lambda_n}{1-\alpha_n}Ax^*)\|^q + \alpha_n\|f(u_n) - x^*\|^q \\ &\quad -h_1(\|\alpha_n f(u_n) + (1-\alpha_n)u_n - \lambda_n(Au_n - Ax^*) - y_n\|) \\ &\le (1-\alpha_n)\|u_n - x^*\|^q + \alpha_n\|f(u_n) - x^*\|^q \\ &\quad -h_1(\|\alpha_n f(u_n) + (1-\alpha_n)u_n - \lambda_n(Au_n - Ax^*) - y_n\|).\end{aligned}$$

This together with (3.5) and (3.14), implies that

$$\begin{aligned}&\|x_{n+1} - x^*\|^q \le \|y_n - x^*\|^q \\ \le\ & (1-\alpha_n)\|u_n - x^*\|^q + \alpha_n\|f(u_n) - x^*\|^q \\ & -h_1(\|\alpha_n f(u_n) + (1-\alpha_n)u_n - \lambda_n(Au_n - Ax^*) - y_n\|) \\ \le\ & (1-\alpha_n)[\|x_n - x^*\|^q - \tilde{h}_1(\|w_n - v_n - x^* + y^*\|) + q\zeta_2\|A_2x^* - A_2w_n\|\|v_n - y^*\|^{q-1} \\ & -\tilde{h}_2(\|v_n - u_n + x^* - y^*\|) + q\zeta_1\|A_1y^* - A_1v_n\|\|u_n - x^*\|^{q-1}] + \alpha_n\|f(u_n) - x^*\|^q \\ & -h_1(\|\alpha_n f(u_n) + (1-\alpha_n)u_n - \lambda_n(Au_n - Ax^*) - y_n\|) \\ \le\ & \alpha_n\|f(u_n) - x^*\|^q + \|x_n - x^*\|^q - \{(1-\alpha_n)[\tilde{h}_1(\|w_n - v_n - x^* + y^*\|) \\ & +\tilde{h}_2(\|v_n - u_n + x^* - y^*\|)] + h_1(\|\alpha_n f(u_n) + (1-\alpha_n)u_n - \lambda_n(Au_n - Ax^*) - y_n\|)\} \\ & +q\zeta_2\|A_2x^* - A_2w_n\|\|v_n - y^*\|^{q-1} + q\zeta_1\|A_1y^* - A_1v_n\|\|u_n - x^*\|^{q-1},\end{aligned}$$

which immediately yields

$$\begin{aligned}&(1-\alpha_n)[\tilde{h}_1(\|w_n - v_n - x^* + y^*\|) + \tilde{h}_2(\|v_n - u_n + x^* - y^*\|)] \\ &\quad +h_1(\|\alpha_n f(u_n) + (1-\alpha_n)u_n - \lambda_n(Au_n - Ax^*) - y_n\|) \\ &\le \alpha_n\|f(u_n) - x^*\|^q + \Gamma_n - \Gamma_{n+1} + q\zeta_1\|A_1y^* - A_1v_n\|\|u_n - x^*\|^{q-1} \\ &\quad +q\zeta_2\|A_2x^* - A_2w_n\|\|v_n - y^*\|^{q-1}.\end{aligned}$$

Since $\tilde{h}_1, \tilde{h}_2$ and $h_1$ are strictly increasing, continuous and convex functions with $\tilde{h}_1(0) = \tilde{h}_2(0) = h_1(0) = 0$, from (3.10) we conclude that $\|w_n - v_n - x^* + y^*\| \to 0,\ \|v_n - u_n + x^* - y^*\| \to 0$ and

$\|\alpha_n f(u_n) + (1-\alpha_n)u_n - \lambda_n(Au_n - Ax^*) - y_n\| \to 0$ as $n \to \infty$. Note that

$$\|w_n - u_n\| \leq \|w_n - v_n - x^* + y^*\| + \|v_n - u_n + x^* - y^*\|,$$

and

$$\begin{aligned} & \|u_n - y_n\| \\ = \; & \|\alpha_n f(u_n) + (1-\alpha_n)u_n - \lambda_n(Au_n - Ax^*) - y_n + \alpha_n(u_n - f(u_n)) + \lambda_n(Au_n - Ax^*)\| \\ \leq \; & \|\alpha_n f(u_n) + (1-\alpha_n)u_n - \lambda_n(Au_n - Ax^*) - y_n\| + \alpha_n\|u_n - f(u_n)\| + \lambda_n\|Au_n - Ax^*\|. \end{aligned}$$

So it follows from (3.11) that

$$\lim_{n\to\infty} \|w_n - u_n\| = \lim_{n\to\infty} \|u_n - y_n\| = 0. \tag{3.15}$$

Also, since $w_n = s_n x_n + (1-s_n)Gx_n$, from (3.12) and (3.15) we infer that

$$\|w_n - x_n\| = (1-s_n)\|Gx_n - x_n\| \leq \|Gx_n - x_n\| \to 0 \quad (n \to \infty),$$

$$\|x_n - u_n\| \leq \|x_n - w_n\| + \|w_n - u_n\| \to 0 \quad (n \to \infty), \tag{3.16}$$

and hence

$$\|x_n - x_{n+1}\| \leq \|x_n - u_n\| + \|u_n - y_n\| + \|y_n - x_{n+1}\| \to 0 \quad (n \to \infty).$$

Also, using (3.13) and the $\ell$-Lipschitz continuity of $S_n$, we have

$$\begin{aligned} \|S_n x_n - x_n\| & \leq \|S_n x_n - S_n x_{n+1}\| + \|S_n x_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\| \\ & \leq (\ell+1)\|x_n - x_{n+1}\| + \|S_n x_{n+1} - x_{n+1}\| \to 0 \quad (n \to \infty). \end{aligned}$$

We next claim that $\|x_n - \overline{S}x_n\| \to 0$ as $n \to \infty$ where $\overline{S} := (2I - S)^{-1}$. In fact, it is first clear that $S : C \to C$ is pseudocontractive and $\ell$-Lipschitzian where $Sx = \lim_{n\to\infty} S_n x \; \forall x \in C$. We claim that $\lim_{n\to\infty} \|Sx_n - x_n\| = 0$. Using the boundedness of $\{x_n\}$ and setting $D = \overline{\text{conv}}\{x_n : n \geq 0\}$ (the closed convex hull of the set $\{x_n : n \geq 0\}$), by the assumption we have $\sum_{n=1}^{\infty} \sup_{x\in D} \|S_n x - S_{n-1}x\| < \infty$. Hence, by Lemma 2.3 we get $\lim_{n\to\infty} \sup_{x\in D} \|S_n x - Sx\| = 0$, which immediately arrives at

$$\lim_{n\to\infty} \|S_n x_n - Sx_n\| = 0.$$

Therefore, we have

$$\|x_n - Sx_n\| \leq \|x_n - S_n x_n\| + \|S_n x_n - Sx_n\| \to 0 \quad (n \to \infty). \tag{3.17}$$

Now, let us show that if we define $\overline{S} := (2I - S)^{-1}$, then $\overline{S} : C \to C$ is nonexpansive, $\text{Fix}(\overline{S}) = \text{Fix}(S) = \bigcap_{n=0}^{\infty} \text{Fix}(S_n)$ and $\lim_{n\to\infty} \|x_n - \overline{S}x_n\| = 0$. As a matter of fact, put $\overline{S} := (2I - S)^{-1}$, where $I$ is the identity operator of $E$. Then it is known that $\overline{S}$ is nonexpansive and $\text{Fix}(\overline{S}) = \text{Fix}(S) = \bigcap_{n=0}^{\infty} \text{Fix}(S_n)$ as a consequence of [21, Theorem 6]. From (3.17) it follows that

$$\begin{aligned} \|x_n - \overline{S}x_n\| & = \|\overline{S}\,\overline{S}^{-1} x_n - \overline{S}x_n\| \\ & \leq \|\overline{S}^{-1} x_n - x_n\| = \|(2I - S)x_n - x_n\| = \|x_n - Sx_n\| \to 0 \quad (n \to \infty). \end{aligned} \tag{3.18}$$

In addition, for each $n \geq 0$, we put $T_{\lambda_n} := J_{\lambda_n}^B(I - \lambda_n A)$. Then from (3.15) and $\alpha_n \to 0$, we get

$$\begin{aligned} \|u_n - T_{\lambda_n} u_n\| & \leq \|u_n - J_{\lambda_n}^B(\alpha_n f(u_n) + (1-\alpha_n)u_n - \lambda_n Au_n)\| \\ & \quad + \|J_{\lambda_n}^B(\alpha_n f(u_n) + (1-\alpha_n)u_n - \lambda_n Au_n) - J_{\lambda_n}^B(u_n - \lambda_n Au_n)\| \\ & \leq \|u_n - y_n\| + \|(\alpha_n f(u_n) + (1-\alpha_n)u_n - \lambda_n Au_n) - (u_n - \lambda_n Au_n)\| \\ & = \|u_n - y_n\| + \alpha_n\|f(u_n) - u_n\| \to 0 \quad (n \to \infty). \end{aligned}$$

Since $\lim_{n\to\infty} a(1-\alpha_n) = a > 0$, without loss of generality, we may assume that $\exists\lambda > 0$ s.t. $\lambda \le a(1-\alpha_n) \le \lambda_n \ \forall n \ge 0$. Using Proposition 2.8 (ii), we obtain from (3.16) that

$$\begin{aligned} \|T_\lambda x_n - x_n\| &\le \|T_\lambda x_n - T_\lambda u_n\| + \|T_\lambda u_n - u_n\| + \|u_n - x_n\| \\ &\le 2\|x_n - u_n\| + \|T_\lambda u_n - u_n\| \\ &\le 2\|x_n - u_n\| + 2\|T_{\lambda_n} u_n - u_n\| \to 0 \quad (n\to\infty). \end{aligned} \tag{3.19}$$

We now define the mapping $\Phi : C \to C$ by $\Phi x := \nu_1 \overline{S}x + \nu_2 Gx + (1-\nu_1-\nu_2)T_\lambda x \ \forall x \in C$ with $\nu_1+\nu_2 < 1$ for constants $\nu_1, \nu_2 \in (0,1)$. Then by Lemma 2.14 and Proposition 2.8 (i), we know that $\Phi$ is nonexpansive and

$$\mathrm{Fix}(\Phi) = \mathrm{Fix}(\overline{S}) \cap \mathrm{Fix}(G) \cap \mathrm{Fix}(T_\lambda) = \bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n) \cap \mathrm{Fix}(G) \cap (A+B)^{-1}0 \ (=: \Omega).$$

Taking into account that

$$\|\Phi x_n - x_n\| \le \nu_1\|\overline{S}x_n - x_n\| + \nu_2\|Gx_n - x_n\| + (1-\nu_1-\nu_2)\|T_\lambda x_n - x_n\|,$$

we deduce from (3.12), (3.18) and (3.19) that

$$\lim_{n\to\infty} \|\Phi x_n - x_n\| = 0. \tag{3.20}$$

Let $z_t = tf(z_t) + (1-t)\Phi z_t \ \forall t \in (0,1)$. Then it follows from Proposition 2.9 that $\{z_t\}$ converges strongly to a point $x^* \in \mathrm{Fix}(\Phi) = \Omega$, which solves the VIP:

$$\langle (I-f)x^*, J(x^*-p)\rangle \le 0 \quad \forall p \in \Omega.$$

Also, from (2.3) we get

$$\begin{aligned} &\|z_t - x_n\|^q = \|t(f(z_t)-x_n) + (1-t)(\Phi z_t - x_n)\|^q \\ \le\ & (1-t)^q\|\Phi z_t - x_n\|^q + qt\langle f(z_t) - x_n, J_q(z_t-x_n)\rangle \\ =\ & (1-t)^q\|\Phi z_t - x_n\|^q + qt\langle f(z_t) - z_t, J_q(z_t-x_n)\rangle + qt\langle z_t - x_n, J_q(z_t-x_n)\rangle \\ \le\ & (1-t)^q(\|\Phi z_t - \Phi x_n\| + \|\Phi x_n - x_n\|)^q + qt\langle f(z_t)-z_t, J_q(z_t-x_n)\rangle + qt\|z_t-x_n\|^q \\ \le\ & (1-t)^q(\|z_t - x_n\| + \|\Phi x_n - x_n\|)^q + qt\langle f(z_t)-z_t, J_q(z_t-x_n)\rangle + qt\|z_t-x_n\|^q, \end{aligned}$$

which immediately attains

$$\langle f(z_t) - z_t, J_q(x_n - z_t)\rangle \le \frac{(1-t)^q}{qt}(\|z_t-x_n\| + \|\Phi x_n - x_n\|)^q + \frac{qt-1}{qt}\|z_t - x_n\|^q.$$

From (3.20), we have

$$\limsup_{n\to\infty}\langle f(z_t) - z_t, J_q(x_n - z_t)\rangle \le \frac{(1-t)^q}{qt}M + \frac{qt-1}{qt}M = (\frac{(1-t)^q + qt - 1}{qt})M, \tag{3.21}$$

where $M$ is a constant such that $\|z_t - x_n\|^q \le M$ for all $n \ge 0$ and $t \in (0,1)$. It is clear that $((1-t)^q + qt - 1)/qt \to 0$ as $t \to 0$. Since $J_q$ is norm-to-norm uniformly continuous on bounded subsets of $E$ and $z_t \to x^*$, we get

$$\|J_q(x_n - z_t) - J_q(x_n - x^*)\| \to 0 \quad (t \to 0).$$

So we obtain

$$\begin{aligned} &|\langle f(z_t) - z_t, J_q(x_n - z_t)\rangle - \langle f(x^*) - x^*, J_q(x_n - x^*)\rangle| \\ =\ & |\langle f(z_t) - f(x^*), J_q(x_n - z_t)\rangle + \langle f(x^*) - x^*, J_q(x_n - z_t)\rangle + \langle x^* - z_t, J_q(x_n - z_t)\rangle \\ & -\langle f(x^*) - x^*, J_q(x_n - x^*)\rangle| \\ \le\ & |\langle f(x^*) - x^*, J_q(x_n - z_t) - J_q(x_n - x^*)\rangle| + |\langle f(z_t) - f(x^*), J_q(x_n - z_t)\rangle| \\ & +|\langle x^* - z_t, J_q(x_n - z_t)\rangle| \end{aligned}$$

$$\leq \quad \|f(x^*) - x^*\| \|J_q(x_n - z_t) - J_q(x_n - x^*)\| + (1+\varrho)\|z_t - x^*\| \|x_n - z_t\|^{q-1}.$$

Thus, for each $n \geq 0$, we have

$$\lim_{t\to 0}\langle f(z_t) - z_t, J_q(x_n - z_t)\rangle = \langle f(x^*) - x^*, J_q(x_n - x^*)\rangle.$$

From (3.21), as $t \to 0$, it follows that

$$\limsup_{n\to\infty}\langle f(x^*) - x^*, J_q(x_n - x^*)\rangle \leq 0. \tag{3.22}$$

By (C2), (3.11) and (3.16), we get

$$\begin{aligned} &\|u_n - x^* - \frac{\lambda_n}{1-\alpha_n}(Au_n - Ax^*) - (x_n - x^*)\| \\ \leq &\quad \|u_n - x_n\| + \frac{\lambda_n}{1-\alpha_n}\|Au_n - Ax^*\| \\ \leq &\quad \|u_n - x_n\| + b\|Au_n - Ax^*\| \to 0 \quad (n\to\infty). \end{aligned} \tag{3.23}$$

Using (3.22) and (3.23), we have

$$\limsup_{n\to\infty}\langle f(x^*) - x^*, J_q(u_n - x^* - \frac{\lambda_n}{1-\alpha_n}(Au_n - Ax^*))\rangle \leq 0. \tag{3.24}$$

Now, from (3.7) it is easy to see that

$$\begin{aligned} &\|x_{n+1} - x^*\|^q \\ \leq &\quad (1 - \alpha_n(1 - q\varrho))\|x_n - x^*\|^q \\ &\quad + q\alpha_n(1-\alpha_n)^{q-1}\langle f(x^*) - x^*, J_q(u_n - x^* - \frac{\lambda_n}{1-\alpha_n}(Au_n - Ax^*))\rangle + \kappa_q\alpha_n^q\|f(u_n) - x^*\|^q \\ = &\quad (1 - \alpha_n(1 - q\varrho))\|x_n - x^*\|^q \\ &\quad + \alpha_n(1 - q\varrho)[\frac{q(1-\alpha_n)^{q-1}\langle f(x^*) - x^*, J_q(u_n - x^* - \frac{\lambda_n}{1-\alpha_n}(Au_n - Ax^*))\rangle}{1 - q\varrho} \\ &\quad + \frac{\kappa_q\alpha_n^{q-1}\|f(u_n) - x^*\|^q}{1 - q\varrho}]. \end{aligned} \tag{3.25}$$

Note that $\{\alpha_n(1 - q\varrho)\} \subset [0,1]$, $\sum_{n=0}^{\infty}\alpha_n(1 - q\varrho) = \infty$ and

$$\limsup_{n\to\infty}[\frac{q(1-\alpha_n)^{q-1}\langle f(x^*) - x^*, J_q(u_n - x^* - \frac{\lambda_n}{1-\alpha_n}(Au_n - Ax^*))\rangle}{1 - q\varrho} + \frac{\kappa_q\alpha_n^{q-1}\|f(u_n) - x^*\|^q}{1 - q\varrho}] \leq 0.$$

Applying Lemma 2.15 to (3.25), we deduce that $\Gamma_n \to 0$ as $n \to \infty$. Thus, $x_n \to x^*$ as $n \to \infty$.

**Case 2.** Suppose that $\exists\{\Gamma_{m_l}\} \subset \{\Gamma_m\}$ s.t. $\Gamma_{m_l} < \Gamma_{m_l+1}$ $\forall l \in \mathbf{N}$, where $\mathbf{N}$ is the set of all positive integers. Define the mapping $\tau : \mathbf{N} \to \mathbf{N}$ by

$$\tau(m) := \max\{l \leq m : \Gamma_l < \Gamma_{l+1}\}.$$

Using Lemma 2.13, we have

$$\Gamma_{\tau(m)} \leq \Gamma_{\tau(m)+1} \quad \text{and} \quad \Gamma_m \leq \Gamma_{\tau(m)+1}.$$

Putting $\Gamma_m = \|x_m - x^*\|^q$ $\forall m \in \mathbf{N}$ and using the same inference as in Case 1, we can obtain

$$\lim_{m\to\infty}\|x_{\tau(m)+1} - x_{\tau(m)}\| = 0 \tag{3.26}$$

and

$$\limsup_{m\to\infty}\langle f(x^*) - x^*, J_q(u_{\tau(m)} - x^* - \frac{\lambda_{\tau(m)}}{1-\alpha_{\tau(m)}}(Au_{\tau(m)} - Ax^*))\rangle \leq 0. \tag{3.27}$$

Because of $\Gamma_{\tau(m)} \le \Gamma_{\tau(m)+1}$ and $\alpha_{\tau(m)} > 0$, we conclude from (3.7) that

$$\begin{aligned}\|x_{\tau(m)} - x^*\|^q \quad \le \quad & \frac{q(1-\alpha_{\tau(m)})^{q-1}}{1-q\varrho}\langle f(x^*) - x^*, J_q(u_{\tau(m)} - x^* - \frac{\lambda_{\tau(m)}}{1-\alpha_{\tau(m)}}(Au_{\tau(m)} - Ax^*))\rangle \\ & + \frac{\kappa_q \alpha_{\tau(m)}^{q-1}}{1-q\varrho}\|f(u_{\tau(m)}) - x^*\|^q,\end{aligned}$$

and hence

$$\limsup_{m\to\infty} \|x_{\tau(m)} - x^*\|^q \le 0.$$

Thus, we have

$$\lim_{m\to\infty} \|x_{\tau(m)} - x^*\|^q = 0.$$

Using Proposition 2.1 and (3.26), we obtain

$$\begin{aligned} & \|x_{\tau(m)+1} - x^*\|^q - \|x_{\tau(m)} - x^*\|^q \\ \le \quad & q\langle x_{\tau(m)+1} - x_{\tau(m)}, J_q(x_{\tau(m)} - x^*)\rangle + \kappa_q \|x_{\tau(m)+1} - x_{\tau(m)}\|^q \\ \le \quad & q\|x_{\tau(m)+1} - x_{\tau(m)}\|\|x_{\tau(m)} - x^*\|^{q-1} + \kappa_q\|x_{\tau(m)+1} - x_{\tau(m)}\|^q \to 0 \quad (m \to \infty).\end{aligned}$$

Taking into account $\Gamma_m \le \Gamma_{\tau(m)+1}$, we have

$$\begin{aligned} & \|x_m - x^*\|^q \le \|x_{\tau(m)+1} - x^*\|^q \\ \le \quad & \|x_{\tau(m)} - x^*\|^q + q\|x_{\tau(m)+1} - x_{\tau(m)}\|\|x_{\tau(m)} - x^*\|^{q-1} + \kappa_q\|x_{\tau(m)+1} - x_{\tau(m)}\|^q.\end{aligned}$$

It is easy to see from (3.26) that $x_m \to x^*$ as $m \to \infty$. This completes the proof. □

We also obtain the strong convergence result for the parallel composite-type extragradient implicit method in a real Hilbert space $H$. It is well known that $\kappa_2 = 1$ [33]. Hence, by Theorem 3.3 we derive the following conclusion.

**Corollary 3.4.** *Let $\emptyset \ne C \subset H$ be a closed convex set. Let $f : C \to C$ be a $\varrho$-contraction with constant $\varrho \in [0, 1)$, and $\{S_n\}_{n=0}^\infty$ be a countable family of $\ell$-uniformly Lipschitzian pseudocontractive self-mappings on $C$. Suppose that $B_1, B_2 : C \to 2^H$ are both maximal monotone operators and $A_i : C \to H$ is $\sigma_i$-inverse-strongly monotone mapping for $i = 1, 2$. Let $A : C \to H$ and $B : C \to 2^H$ be a $\sigma$-inverse-strongly monotone mapping and a maximal monotone operator, respectively. Assume that $\Omega := \bigcap_{n=0}^\infty \mathrm{Fix}(S_n) \cap \mathrm{Fix}(G) \cap (A+B)^{-1}0 \ne \emptyset$ where $G : C \to C$ is the same as defined in Lemma 2.11. For any given $x_0 \in C$ and $\zeta \in (0, 1)$, let $\{x_n\}_{n=0}^\infty$ be the sequence generated by*

$$\begin{cases} w_n = s_n x_n + (1-s_n)Gx_n, \\ v_n = J_{\zeta_2}^{B_2}(w_n - \zeta_2 A_2 w_n), \\ u_n = J_{\zeta_1}^{B_1}(v_n - \zeta_1 A_1 v_n), \\ x_{n+1} = (1-\zeta)S_n x_{n+1} + \zeta J_{\lambda_n}^B(\alpha_n f(u_n) + (1-\alpha_n)u_n - \lambda_n A u_n) \quad \forall n \ge 0, \end{cases} \tag{3.28}$$

*where the sequences $\{s_n\}, \{\alpha_n\} \subset (0, 1)$ and $\{\lambda_n\} \subset (0, \infty)$ are such that the following conditions hold:*

- (C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^\infty \alpha_n = \infty$*;*
- (C2) $0 < a \le \frac{\lambda_n}{1-\alpha_n} \le b < 2\sigma$ *and* $0 < c \le s_n \le d < 1$*;*
- (C3) $0 < \zeta_i < 2\sigma_i$ *for* $i = 1, 2$.

*Assume that $\sum_{n=0}^\infty \sup_{x\in D} \|S_{n+1}x - S_n x\| < \infty$ for any bounded subset $D$ of $C$. Let $S : C \to C$ be a mapping defined by $Sx = \lim_{n\to\infty} S_n x \; \forall x \in C$, and suppose that $\mathrm{Fix}(S) = \bigcap_{n=0}^\infty \mathrm{Fix}(S_n)$. Then $x_n \to x^* \in \Omega$, which is the unique solution to the VIP: $\langle (I-f)x^*, p - x^*\rangle \ge 0 \; \forall p \in \Omega$, i.e., the fixed point equation $x^* = P_\Omega f(x^*)$.*

*Remark* 3.5. Compared with the corresponding results in Abdou et al. [22], Sunthrayuth and Cholamjiak [15], and Ceng et al. [13], our results improve and extend them in the following aspects.

(i) The problem of solving the VI for both monotone operators $A, B$ with the FPP constraint of a nonexpansive mapping $S$ in [22, Theorem 3.2] is extended to develop our problem of solving the GSVI (1.4) with the constraints of the VI for both accretive operators $A, B$ and the CFPP of $\{S_n\}_{n=0}^{\infty}$ a countable family of $\ell$-uniformly Lipschitzian pseudocontractions. The parallel iterative algorithm in [22, Theorem 3.2] is extended to develop our parallel composite-type extragradient implicit method.

(ii) The problem of solving the GSVI (1.4) with the FPP constraint of a strict pseudocontraction $T$ in [13, Theorem 1], is extended to develop our problem of solving the GSVI (1.4) with the constraints of the VI for two accretive operators $A, B$ and the CFPP of $\{S_n\}_{n=0}^{\infty}$ a countable family of $\ell$-uniformly Lipschitzian pseudocontractions. The composite viscosity implicit rule in [13, Theorem 3.1] is extended to develop our parallel composite-type extragradient implicit method.

(iii) The problem of solving the VI for both accretive operators $A, B$ with the FPP constraint of a nonexpansive mapping $S$ in [15, Theorem 3.3] is extended to develop our problem of solving the GSVI (1.4) with the constraints of the VI for both accretive operators $A, B$ and the CFPP of $\{S_n\}_{n=0}^{\infty}$ a countable family of $\ell$-uniformly Lipschitzian pseudocontractions. The modified viscosity-type extragradient method in [15, Theorem 3.3] is extended to develop our parallel composite-type extragradient implicit method.

## 4. Some Applications

In this section, we give some applications of Corollary 3.4 to important mathematical problems in the setting of Hilbert spaces.

4.1. **Application to variational inequality problem.** Given a nonempty closed convex subset $C \subset H$ and a nonlinear monotone operator $A : C \to H$. Consider the classical VIP of finding $u^* \in C$ s.t.

$$\langle Au^*, v - u^* \rangle \geq 0 \quad \forall v \in C. \tag{4.1}$$

The solution set of problem (4.1) is denoted by $\mathrm{VI}(C, A)$. It is clear that $u^* \in C$ solves VIP (4.1) if and only if it solves the fixed point equation $u^* = P_C(u^* - \lambda Au^*)$ with $\lambda > 0$. Let $i_C$ be the indicator function of $C$ defined by

$$i_C(u) = \begin{cases} 0 & \text{if } u \in C, \\ \infty & \text{if } u \notin C. \end{cases}$$

We use $N_C(u)$ to indicate the normal cone of $C$ at $u \in H$, i.e., $N_C(u) = \{w \in H : \langle w, v-u \rangle \leq 0 \, \forall v \in C\}$. It is known that $i_C$ is a proper, convex and lower semicontinuous function and its subdifferential $\partial i_C$ is a maximal monotone mapping [11]. We define the resolvent operator $J_\lambda^{\partial i_C}$ of $\partial i_C$ for $\lambda > 0$ by

$$J_\lambda^{\partial i_C}(x) = (I + \lambda \partial i_C)^{-1}(x) \quad \forall x \in H,$$

where

$$\begin{aligned} \partial i_C(u) &= \{w \in H : i_C(u) + \langle w, v - u \rangle \leq i_C(v) \, \forall v \in H\} \\ &= \{w \in H : \langle w, v - u \rangle \leq 0 \, \forall v \in C\} = N_C(u) \quad \forall u \in C. \end{aligned}$$

Hence, we get

$$\begin{aligned} u = J_\lambda^{\partial i_C}(x) \quad &\Leftrightarrow \quad x - u \in \lambda N_C(u) \\ &\Leftrightarrow \quad \langle x - u, v - u \rangle \leq 0 \quad \forall v \in C \\ &\Leftrightarrow \quad u = P_C(x), \end{aligned}$$

where $P_C$ is the metric projection of $H$ onto $C$. Moreover, we also have $(A + \partial i_C)^{-1}0 = \mathrm{VI}(C, A)$ [11].

Thus, putting $B = \partial i_C$ in Corollary 3.4, we obtain the following result:

**Theorem 4.1.** *Let $f, A, A_i, B_i$ $(i = 1, 2)$ and $\{S_n\}_{n=0}^{\infty}$ be the same as in Corollary 3.4. Suppose that $\Omega := \bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n) \cap \mathrm{Fix}(G) \cap \mathrm{VI}(C, A) \neq \emptyset$. For any given $x_0 \in C$ and $\zeta \in (0, 1)$, let $\{x_n\}_{n=0}^{\infty}$ be the sequence generated by*

$$\begin{cases} w_n = s_n x_n + (1 - s_n) G x_n, \\ v_n = J_{\zeta_2}^{B_2}(w_n - \zeta_2 A_2 w_n), \\ u_n = J_{\zeta_1}^{B_1}(v_n - \zeta_1 A_1 v_n), \\ x_{n+1} = (1 - \zeta) S_n x_{n+1} + \zeta P_C(\alpha_n f(u_n) + (1 - \alpha_n) u_n - \lambda_n A u_n) \quad \forall n \geq 0, \end{cases} \tag{4.2}$$

*where the sequences $\{s_n\}, \{\alpha_n\} \subset (0, 1)$ and $\{\lambda_n\} \subset (0, \infty)$ are such that the conditions (C1)-(C3) in Corollary 3.4 hold. Then $x_n \to x^* \in \Omega$, which is the unique solution to the VIP: $\langle (I - f)x^*, p - x^* \rangle \geq 0\ \forall p \in \Omega$, i.e., the fixed point equation $x^* = P_\Omega f(x^*)$.*

4.2. **Application to split feasibility problem.** Let $H_1$ and $H_2$ be two real Hilbert spaces. Consider the following split feasibility problem (SFP) of finding

$$u \in C \text{ s.t. } \mathcal{T} u \in Q, \tag{4.3}$$

where $C$ and $Q$ are closed convex subsets of $H_1$ and $H_2$, respectively, and $\mathcal{T} : H_1 \to H_2$ is a bounded linear operator with its adjoint $\mathcal{T}^*$. The solution set of SFP is denoted by $\mho := C \cap \mathcal{T}^{-1} Q = \{u \in C : \mathcal{T} u \in Q\}$. In 1994, Censor and Elfving [3] first introduced the SFP for modelling inverse problems of radiation therapy treatment planning in a finite dimensional Hilbert space, which arise from phase retrieval and in medical image reconstruction.

It is known that $z \in C$ solves the SFP (4.3) if and only if $z$ is a solution of the minimization problem: $\min_{y \in C} g(y) := \frac{1}{2} \|\mathcal{T} y - P_Q \mathcal{T} y\|^2$. Note that the function $g$ is differentiable convex and has the Lipschitzian gradient defined by $\nabla g = \mathcal{T}^*(I - P_Q)\mathcal{T}$. Moreover, $\nabla g$ is $\frac{1}{\|\mathcal{T}\|^2}$-inverse-strongly monotone, where $\|\mathcal{T}\|^2$ is the spectral radius of $\mathcal{T}^* \mathcal{T}$ [5]. So, $z \in C$ solves the SFP if and only if it solves the variational inclusion problem of finding $z \in H_1$ s.t.

$$\begin{aligned} 0 \in \nabla g(z) + \partial i_C(z) \quad &\Leftrightarrow \quad 0 \in z + \lambda \partial i_C(z) - (z - \lambda \nabla g(z)) \\ &\Leftrightarrow \quad z - \lambda \nabla g(z) \in z + \lambda \partial i_C(z) \\ &\Leftrightarrow \quad z = (I + \lambda \partial i_C)^{-1}(z - \lambda \nabla g(z)) \\ &\Leftrightarrow \quad z = P_C(z - \lambda \nabla g(z)). \end{aligned}$$

Now, setting $A = \nabla g,\ B = \partial i_C$ and $\sigma = \frac{1}{\|\mathcal{T}\|^2}$ in Corollary 3.4, we obtain the following result:

**Theorem 4.2.** *Let $f, A_i, B_i$ $(i = 1, 2)$ and $\{S_n\}_{n=0}^{\infty}$ be the same as in Corollary 3.4. Assume that $\Omega := \bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n) \cap \mathrm{Fix}(G) \cap \mho \neq \emptyset$. For any given $x_0 \in C$ and $\zeta \in (0, 1)$, let $\{x_n\}_{n=0}^{\infty}$ be the sequence generated by*

$$\begin{cases} w_n = s_n x_n + (1 - s_n) G x_n, \\ v_n = J_{\zeta_2}^{B_2}(w_n - \zeta_2 A_2 w_n), \\ u_n = J_{\zeta_1}^{B_1}(v_n - \zeta_1 A_1 v_n), \\ x_{n+1} = (1 - \zeta) S_n x_{n+1} + \zeta P_C(\alpha_n f(u_n) + (1 - \alpha_n) u_n - \lambda_n \mathcal{T}^*(I - P_Q)\mathcal{T} u_n) \quad \forall n \geq 0, \end{cases} \tag{4.4}$$

*where the sequences $\{s_n\}, \{\alpha_n\} \subset (0, 1)$ and $\{\lambda_n\} \subset (0, \infty)$ are such that the conditions (C1)-(C3) in Corollary 3.4 hold where $\sigma = \frac{1}{\|\mathcal{T}\|^2}$. Then $x_n \to x^* \in \Omega$, which is the unique solution to the VIP: $\langle (I - f)x^*, p - x^* \rangle \geq 0\ \forall p \in \Omega$, i.e., the fixed point equation $x^* = P_\Omega f(x^*)$.*

4.3. **Application to LASSO problem.** In this subsection, we first recall the least absolute shrinkage and selection operator (LASSO) [9], which can be formulated as a convex constrained optimization problem:

$$\min_{y\in H}\frac{1}{2}\|\mathcal{T}y-b\|_2^2 \quad \text{subject to } \|y\|_1\le s, \tag{4.5}$$

where $\mathcal{T}: H\to H$ is a bounded operator on $H$, $b$ is a fixed vector in $H$ and $s>0$. Let $\mho$ be the solution set of LASSO (4.5). The LASSO has received much attention because of the involvement of the $\ell_1$ norm which promotes sparsity, phenomenon of many practical problems arising in statics model, image compression, compressed sensing and signal processing theory.

In terms of the optimization theory, ones know that the solution to the LASSO problem (4.5) is a minimizer of the following convex unconstrained minimization problem so-called Basis Pursuit denoising problem:

$$\min_{y\in H} g(y)+h(y), \tag{4.6}$$

where $g(y):=\frac{1}{2}\|\mathcal{T}y-b\|_2^2,\ h(y):=\lambda\|y\|_1$ and $\lambda\ge 0$ is a regularization parameter. It is known that $\nabla g(y)=\mathcal{T}^*(\mathcal{T}y-b)$ is $\frac{1}{\|\mathcal{T}^*\mathcal{T}\|}$-inverse-strongly monotone. Hence, we have that $z$ solves the LASSO if and only if $z$ solves the variational inclusion problem of finding $z\in H$ s.t.

$$\begin{aligned} ll0\in\nabla g(z)+\partial h(z) &\Leftrightarrow 0\in z+\lambda\partial h(z)-(z-\lambda\nabla g(z))\\ &\Leftrightarrow z-\lambda\nabla g(z)\in z+\lambda\partial h(z)\\ &\Leftrightarrow z=(I+\lambda\partial h)^{-1}(z-\lambda\nabla g(z))\\ &\Leftrightarrow z=\operatorname{prox}_h(z-\lambda\nabla g(z)), \end{aligned}$$

where $\operatorname{prox}_h(y)$ is the proximal of $h(y):=\lambda\|y\|_1$ given by

$$\operatorname{prox}_h(y)=\operatorname{argmin}_{u\in H}\{\lambda\|u\|_1+\frac{1}{2}\|u-y\|_2^2\} \quad \forall y\in H,$$

which is separable in indices. Then, for $y\in H$,

$$\begin{aligned}\operatorname{prox}_h(y) &= \operatorname{prox}_{\lambda\|\cdot\|_1}(y)\\ &= (\operatorname{prox}_{\lambda|\cdot|}(y_1),\operatorname{prox}_{\lambda|\cdot|}(y_2),\dots,\operatorname{prox}_{\lambda|\cdot|}(y_n)),\end{aligned}$$

where $\operatorname{prox}_{\lambda|\cdot|}(y_i)=\operatorname{sgn}(y_i)\max\{|y_i|-\lambda,0\}$ for $i=1,2,\dots,n$.

In 2014, Xu [12] suggested the following proximal-gradient algorithm (PGA):

$$x_{k+1}=\operatorname{prox}_h(x_k-\lambda_k\mathcal{T}^*(\mathcal{T}x_k-b)).$$

He proved the weak convergence of the PGA to a solution of the LASSO problem (4.5).

Next, putting $C=H,\ A=\nabla g,\ B=\partial h$ and $\sigma=\frac{1}{\|\mathcal{T}^*\mathcal{T}\|}$ in Corollary 3.4, we obtain the following result:

**Theorem 4.3.** *Let $f, A_i, B_i$ $(i=1,2)$ and $\{S_n\}_{n=0}^\infty$ be the same as in Corollary 3.4 with $C=H$. Assume that $\Omega:=\bigcap_{n=0}^\infty \operatorname{Fix}(S_n)\cap\operatorname{Fix}(G)\cap\mho\ne\emptyset$. For any given $x_0\in H$ and $\zeta\in(0,1)$, let $\{x_n\}_{n=0}^\infty$ be the sequence generated by*

$$\begin{cases} w_n=s_nx_n+(1-s_n)Gx_n,\\ v_n=J_{\zeta_2}^{B_2}(w_n-\zeta_2A_2w_n),\\ u_n=J_{\zeta_1}^{B_1}(v_n-\zeta_1A_1v_n),\\ x_{n+1}=(1-\zeta)S_nx_{n+1}+\zeta\operatorname{prox}_h(\alpha_nf(u_n)+(1-\alpha_n)u_n-\lambda_n\mathcal{T}^*(\mathcal{T}u_n-b)) \quad \forall n\ge 0, \end{cases} \tag{4.7}$$

*where the sequences $\{s_n\},\{\alpha_n\}\subset(0,1)$ and $\{\lambda_n\}\subset(0,\infty)$ are such that the conditions (C1)-(C3) in Corollary 3.4 hold where $\sigma=\frac{1}{\|\mathcal{T}^*\mathcal{T}\|}$. Then $x_n\to x^*\in\Omega$, which is the unique solution to the VIP: $\langle (I-f)x^*, p-x^*\rangle\ge 0\ \forall p\in\Omega$, i.e., the fixed point equation $x^*=P_\Omega f(x^*)$.*

## Statements and Declarations

The authors declare that they have no conflict of interest, and the manuscript has no associated data.

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