FIXED POINT METHODS AND OPTIMIZATION
Volume 1 (2024), No. 2, 145–160
https://doi.org/10.69829/fpmo-024-0102-ta04
Tulipa Opera Scholarum

GENERALIZED SPLIT ZERO POINT PROBLEMS INVOLVING COMONOTONE AND QUASI-COCOERCIVE OPERATORS

MOHAMMAD ESLAMIAN[1,*]

[1] *Department of Mathematics, University of Science and Technology of Mazandaran, Behshahr, Iran*

ABSTRACT. In this paper, we investigate generalized split zero point problems involving a finite family of maximally comonotone operators and a finite family of quasi-cocoercive operators within Hilbert spaces. We propose a novel algorithm that leverages both inertial methods and a self-adaptive step size strategy. By imposing suitable control conditions on the associated parameters, we establish the strong convergence of the iterative sequence to the unique solution of a variational inequality problem. Furthermore, we demonstrate the applicability of our results to various problems, including the multiple-sets split feasibility problem and the split monotone variational inclusion problem.

Keywords. Comonotone operator, Quasi-cocoercive operators, Zero point, Inclusion problem.

1. INTRODUCTION

Let $\mathcal{H}$ be a real Hilbert space, and consider a set-valued operator $B : \mathcal{H} \rightrightarrows \mathcal{H}$. The zero point problem involves finding a point $z \in \mathcal{H}$ such that $0 \in B(z)$. Such a point z is termed a zero of B, and the set of all zeros of B is denoted by $B^{-1}(0)$. This problem is closely tied to various domains in nonlinear analysis and optimization, including convex minimization, variational inequality problems, equilibrium problems, monotone inclusions, fixed point problems, and saddle point problems (see [2, 4]). One of the most prominent and widely used methods for solving the zero point problem is the proximal point algorithm, initially introduced by Martinet and subsequently studied in depth by Rockafellar [22] within the context of Hilbert spaces. Over the past few decades, the zero point problem has been the focus of extensive research (see, e.g., [15, 10, 7]).

A fundamental problem in nonlinear analysis and optimization is to find a zero of the sum of two monotone operators, formulated as:

$$\text{Find } x^* \in \mathcal{H} \text{ such that } 0 \in A(x^*) + B(x^*), \tag{1.1}$$

where $A : \mathcal{H} \to \mathcal{H}$ is a monotone, single-valued operator, and $B : \mathcal{H} \rightrightarrows \mathcal{H}$ is a maximally monotone, set-valued operator defined on the Hilbert space $\mathcal{H}$. This problem 1.1 has applications in a variety of areas, such as convex optimization, image processing, and signal processing. A key special case of the monotone inclusion problem (1.1) is the variational inequality problem (VIP):

$$\text{Find } x^* \in C \text{ such that } 0 \in A(x^*) + N_C(x^*), \tag{1.2}$$

where C is a nonempty closed convex subset of $\mathcal{H}$ and $N_C(x^*)$ denotes the normal cone to C at x^*. The VIP (1.2) is equivalent to identifying a point $x^* \in C$ such that:

$$\langle A(x^*), y - x^* \rangle \geq 0, \quad \forall y \in C.$$

*Corresponding author.
E-mail address: eslamian@mazust.ac.ir
2020 Mathematics Subject Classification: 47H05; 49J53; 47J20; 65K15.
Accepted November 27, 2024.

The solution set for this problem is denoted by $VI(C, A)$. The theory of variational inequalities has played a critical role in advancing research across various fields, including partial differential equations, optimal control, mathematical programming, and general optimization (see, e.g., [17]). Its versatility has made it a cornerstone for solving practical problems in engineering, economics, and applied sciences.

Linear inverse problems are frequently encountered in numerous real-world applications, such as signal and image processing and medical image reconstruction. In 2005, Censor et al. [8] introduced the multiple-sets split feasibility problem (MSSFP), which was initially motivated by the inverse problem of intensity-modulated radiation therapy (IMRT). The MSSFP aims to find a point that is closest to the intersection of a family of closed convex sets in one space such that its image, under a linear transformation, is closest to the intersection of another family of closed convex sets in an image space. Byrne et al. [6] extended this to the split common null point problem, which can be stated as follows: given set-valued operators $B_i : \mathcal{H} \rightrightarrows \mathcal{H}$ for $1 \leq i \leq m$, $G_j : \mathcal{K} \rightrightarrows \mathcal{K}$ for $1 \leq j \leq n$, and bounded linear operators $\mathcal{L}_j : \mathcal{H} \to \mathcal{K}$ for $1 \leq j \leq n$, the goal is to find a point $x^* \in \mathcal{H}$ such that:

$$x^* \in \left(\bigcap_{i=1}^{m} B_i^{-1} 0 \right) \bigcap \left(\bigcap_{j=1}^{n} \mathcal{L}_j{}^{-1} (G_j^{-1} 0) \right).$$

This problem has been the focus of several studies (see [9, 13, 21, 19] and references therein).

There exists a deep connection between the monotonicity of operators and the convexity of functions. A classical result establishes that the convexity of a function f is linked to the monotonicity of its gradient ∇f (see [2]). However, to address functions that lack convexity, it is necessary to relax the monotonicity requirement. In 2020, Bauschke et al. [3] introduced the concept of ρ-comonotonicity, a generalized notion of monotonicity for set-valued operators in Hilbert spaces. Building on this framework, Kohlenbach [18] subsequently developed a Halpern-type proximal point algorithm designed to approximate zeros of comonotone operators. In recent work [12], the author extended these ideas to study the split common null point problem involving a finite collection of maximally comonotone operators.

In recent years, inertial techniques have garnered significant attention due to their ability to accelerate convergence and enhance algorithmic performance (see [20, 1, 14, 26] and references therein). These methods incorporate momentum terms to improve the speed of iterative algorithms.

In this paper, we address the generalized split zero point problems involving a finite family of maximally comonotone operators and a finite family of quasi-cocoercive operators in Hilbert spaces. We introduce a novel algorithm that incorporates an inertial approach to enhance the convergence speed of the iterative process. Additionally, our method employs a self-adaptive step size strategy that can be implemented efficiently without requiring prior knowledge of the operators' norms. We establish the strong convergence of the proposed method under suitable conditions on the control parameters, ensuring that the iterative sequence converges to the unique solution of an associated variational inequality problem. Additionally, we demonstrate the practical applicability of our results by studying, the multiple-sets split feasibility problem and the split monotone variational inclusion problem. This work not only generalizes existing results in the literature but also provides a framework for solving a broader class of optimization and split feasibility problems in Hilbert spaces.

2. Preliminaries

Throughout the present paper, $\mathcal{H}$ denotes a real Hilbert space with inner product $\langle ., . \rangle$ and induced norm $\|.\|$. The identity operator is denoted by I, namely, $I(x) = x$ for all $x \in \mathcal{H}$. Strong convergence of a sequence $\{x_n\}$ in $\mathcal{H}$ to x is denoted by $x_n \to x$ and weak convergence by $x_n \rightharpoonup x$. Let $T : \mathcal{H} \to \mathcal{H}$ be an operator. A point $x \in \mathcal{H}$ such that $Tx = x$ is called a fixed point of T. The set of fixed points of operator T shall be denoted by $Fix(T)$. We recall the following definitions concerning an operator $T : \mathcal{H} \to \mathcal{H}$.

Definition 2.1. The operator $T : \mathcal{H} \to \mathcal{H}$ is called:
• L-Lipschitz continuous if $L > 0$ and

$$\|T(x) - T(y)\| \leq L\|x - y\|, \qquad \forall x, y \in \mathcal{H}.$$

If $L = 1$, then T called a nonexpansive mapping.
• β-strongly monotone if $\beta > 0$ and

$$\langle T(x) - T(y), x - y\rangle \geq \beta\|x - y\|^2, \quad \forall x, y \in \mathcal{H}.$$

• β-cocoercive if $\beta > 0$ and

$$\langle T(x) - T(y), x - y\rangle \geq \beta\|T(x) - T(y)\|^2, \quad \forall x, y \in \mathcal{H}.$$

• β-quasi-cocoercive [5] if it satisfies cocoercivity relative to its set of zeros, i.e.,

$$\langle T(x), x - z\rangle \geq \beta\|T(x)\|^2, \quad \forall x \in \mathcal{H},\ z \in T^{-1}0.$$

Definition 2.2. Let $\mathcal{H}$ be a Hilbert space. A mapping $T : \mathcal{H} \to \mathcal{H}$ is said to be demi-closed at 0 if, for any sequence $\{x_n\}$ in $\mathcal{H}$, the conditions $x_n \rightharpoonup z$ and $T(x_n) \to 0$, imply $Tz = 0$.

Definition 2.3. Let $\mathcal{H}$ be a Hilbert space, let $T : \mathcal{H} \to \mathcal{H}$, and let $\alpha \in [0, \infty)$. Then T is called α-conically nonexpansive if there exists a nonexpansive operator $S : \mathcal{H} \to \mathcal{H}$ such that $T = (1 - \alpha)I + \alpha S$. Given an α-conically nonexpansive operator, it is α-averaged when $\alpha \in (0, 1)$ and nonexpansive when $\alpha = 1$.

Based on Lemmas 3.1, 3.2, 3.3, and 3.4 from reference [12], the following properties of α-conically nonexpansive mappings can be established:

Lemma 2.4. *Let $\mathcal{H}$ be a Hilbert space, $\alpha \in (0, \infty)$ and let $T : \mathcal{H} \to \mathcal{H}$ be α-conically nonexpansive. Then, the following properties hold:*

(i) The operator $I - T$ is demiclosed in 0.
(ii) The set of fixed points, $Fix(T)$, is closed and convex.
(iii) The operator T is Lipschitz continuous.
(iv) For any $x^* \in Fix(T)$ and $x \in \mathcal{H}$, the following inequality holds:

$$\langle x - x^*, x - Tx\rangle \geq \frac{1}{2\alpha}\|x - Tx\|^2.$$

Definition 2.5. Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. For any $x \in \mathcal{H}$, the metric projection $P_C(x)$ of x onto the set C is defined as:

$$P_C(x) = \arg\min_{y \in C} \|y - x\|.$$

Since C is nonempty, closed, and convex, the projection $P_C(x)$ is guaranteed to exist and be unique. The metric projection is fundamental in optimization and variational inequality problems, as it identifies the closest point in C to a given point x.

For a set- valued operator $B : \mathcal{H} \rightrightarrows \mathcal{H}$, we define its domain, range, and graph as follows:

$$D(B) := \{x \in \mathcal{H} : B(x) \neq \emptyset\}, \quad R(B) := \bigcup\{B(z) : z \in D(B)\},$$

$$G(B) := \{(x, y) \in \mathcal{H} \times \mathcal{H} : x \in D(B), y \in B(x)\}.$$

The inverse of B, denoted by B^{-1}, is defined such that:

$$x \in B^{-1}(y) \iff y \in B(x).$$

An operator B is said to be monotone if, for all $x, y \in D(B)$, we have:

$$\langle x-y, u-v\rangle \geq 0 \quad \text{for all } u \in B(x) \text{ and } v \in B(y).$$

A monotone operator B is considered maximally monotone if it has no proper monotone extension, or equivalently (by Minty's theorem), if:

$$R(I+\lambda B)=\mathcal{H} \quad \text{for all } \lambda>0.$$

A more general concept than monotonicity is ρ-comonotonicity [3]. For a given $\rho \in \mathbb{R}$, an operator $B : \mathcal{H} \rightrightarrows \mathcal{H}$ is said to be ρ-comonotone if:

$$\langle x-y, u-v\rangle \geq \rho\|u-v\|^2 \quad \forall(x, u),(y, v) \in G(B).$$

Note that if B is 0-comonotone, then B is monotone. When B is ρ-comonotone with $\rho>0$, it is also known as ρ-cocoercive, which is a stronger condition than monotonicity. For $\rho<0$, the concept of ρ-comonotonicity is referred to as $|\rho|$-cohypomonotonicity (see [[10], Definition 2.2]).

An operator B is termed maximally ρ-comonotone if it is ρ-comonotone and there exists no other ρ-comonotone operator $\mathbf{D} : \mathcal{H} \rightrightarrows \mathcal{H}$ such that $G(\mathbf{D})$ properly contains $G(B)$.

For ρ-comonotone operators, the resolvent plays a crucial role in approximating zero points. Given $\lambda>0$, the resolvent J_λ^B is defined as:

$$J_\lambda^B:=(I+\lambda B)^{-1},$$

which maps $R(I+\lambda B)$ to $D(B)$. The resolvent operator is a fundamental tool in the approximation theory for zero points of maximally comonotone operators.

The following results are derived from [[3], Propositions 2.10 and 2.13] and [[18], Lemma 2.3].

Lemma 2.6. *Let $B : \mathcal{H} \rightrightarrows \mathcal{H}$ be maximally ρ-comonotone with $\rho \in \mathbb{R}$ and let $\lambda>0$. If $\rho>-\lambda$, then λB is maximally $\frac{\rho}{\lambda}$-comonotone with $\frac{\rho}{\lambda}>-1$, also J_λ^B is single-valued and $D(J_\lambda^B)=R(I+\lambda B)=\mathcal{H}$.*

There is a close relationship between the resolvent of a maximally comonotone operator and a conically nonexpansive operator, as stated in the following lemma.

Lemma 2.7. [3] *Let $\mathcal{H}$ be a Hilbert space and let $T : \mathcal{H} \to \mathcal{H}$ be an operator.*
(i) T is nonexpansive if and only if it is the resolvent of a maximally $(-\frac{1}{2})$-comonotone operator $B : \mathcal{H} \rightrightarrows \mathcal{H}$.
(ii) Let $\alpha \in (0, \infty)$. Then T is α-conically nonexpansive if and only if it is the resolvent of a maximally ρ-comonotone operator $B : \mathcal{H} \rightrightarrows \mathcal{H}$, where $\rho=\frac{1}{2\alpha}-1>-1$.
(iii) Let $\alpha \in (0,1)$. Then T is α-averaged if and only if it is the resolvent of a maximally ρ-comonotone operator $B : \mathcal{H} \rightrightarrows \mathcal{H}$, where $\rho=\frac{1}{2\alpha}-1>-\frac{1}{2}$.

Lemma 2.8. [12] *Let $\mathcal{H}$ be a Hilbert space, $B : \mathcal{H} \rightrightarrows \mathcal{H}$ be ρ-comonotone with $\rho \in \mathbb{R}$ and let $\lambda, \mu>0$. If $\rho>-\lambda,-\mu$, then there exists a constant $L>0$, such that*

$$\|x-J_\mu^B x\| \leq (L+1+\frac{L\mu}{\lambda})\|x-J_\lambda^B x\|, \quad \forall x \in R(I+\lambda B) \cap R(I+\mu B).$$

Lemma 2.9. [12] *Let $B : \mathcal{H} \rightrightarrows \mathcal{H}$ be a ρ-comonotone operator with $\rho>-1$. Then, the set $B^{-1}(0)= Fix(J_1^B)$, and consequently, $B^{-1}(0)$ is closed and convex.*

Definition 2.10. For a nonempty closed and convex subset C of $\mathcal{H}$, the indicator function i_C of C is given by:

$$i_C:= \begin{cases}0, & \text{if } x \in C, \\ \infty, & \text{if } x \notin C.\end{cases}$$

Furthermore, the normal cone of C at $u \in C$, $N_C(u)$ is given as:

$$N_C(u)=\{f \in \mathcal{H} : \langle u-y, f\rangle \geq 0, \quad \forall y \in C\}.$$

Remark 2.11. The indicator function i_C is proper, lower semicontinuous, and convex. Consequently, its subdifferential ∂i_C is a maximal monotone operator. It is well-known that the subdifferential of the indicator function is the normal cone to the set, i.e., $\partial i_C(u) = N_C(u)$ for any $u \in \mathcal{H}$. Moreover, the resolvent of ∂i_C satisfies

$$J_r^{\partial i_C}(x) = P_C x, \quad \forall x \in \mathcal{H},\, r > 0,$$

where P_C denotes the projection onto the closed convex set C. For further details, see [2, 22, 23].

Lemma 2.12. [17] *Let C be a nonempty closed and convex subset of a real Hilbert space $\mathcal{H}$, and let $T : C \to \mathcal{H}$ be a strongly monotone and Lipschitz continuous mapping. Then $VI(C,T)$ consists of only one point.*

Lemma 2.13. [25] *Let the operator $T : \mathcal{H} \to \mathcal{H}$ be l-Lipschitz continuous and δ-strongly monotone with constants $l > 0$, $\delta > 0$. Assume that $\gamma \in (0, \frac{2\delta}{l^2})$. For $\alpha \in (0,1)$ define $T_\alpha = I - \alpha\gamma T$. Then for all $x, y \in \mathcal{H}$,*

$$\|T_\alpha x - T_\alpha y\| \le (1 - \alpha\eta)\|x - y\|$$

holds, where $\eta = 1 - \sqrt{1 - \gamma(2\delta - \gamma l^2)} \in (0,1)$.

Lemma 2.14. *([16]) Assume $\{\Gamma_n\}$ is a sequence of nonnegative real numbers such that*

$$\begin{cases} \Gamma_{n+1} \le (1 - \Xi_n)\Gamma_n + \Xi_n \vartheta_n, & n \ge 0, \\ \Gamma_{n+1} \le \Gamma_n - \xi_n + \zeta_n, & n \ge 0, \end{cases}$$

where $\{\Xi_n\}$ is a sequence in $(0,1)$, $\{\xi_n\}$ is a sequence of nonnegative real numbers and $\{\vartheta_n\}$ and $\{\zeta_n\}$ are two sequences in $\mathbb{R}$ such that

(i) $\sum_{n=1}^{\infty} \Xi_n = \infty$,
(ii) $\lim_{n\to\infty} \zeta_n = 0$
(iii) $\lim_{k\to\infty} \xi_{n_k} = 0$, *implies* $\limsup_{k\to\infty} \vartheta_{n_k} \le 0$ *for any subsequence* $\{n_k\} \subset \{n\}$.

Then $\lim_{n\to\infty} \Gamma_n = 0$.

3. The Algorithm and its Convergence

In this section, we present our algorithm and show its convergence analysis. We begin with the following assumptions under which our strong convergence is obtained.

Assumption 3.1. Assume that the following hold:

(C1) $\mathcal{H}_0$ and $\mathcal{H}_i, \mathcal{K}_i$, $i = 1, 2, ..., M$, are real Hilbert spaces.
(C2) The operator $F : \mathcal{H}_0 \to \mathcal{H}_0$ is l-Lipschitz continuous and δ-strongly monotone with constants $l > 0, \delta > 0$.
(C3) For each $i \in \{1, 2, ..., M\}$, $(s_{n,i}) \subset (0, \infty)$ with $s_{n,i} \ge s_i > 0$ for all $n \in \mathbb{N}$ and that $B_i : \mathcal{H}_i \rightrightarrows \mathcal{H}_i$ is (set-valued) maximally ς_i-comonotone operator with $\varsigma_i \in (-s_i, 0]$.
(C4) For each $i \in \{1, 2, ..., M\}$, $A_i : \mathcal{K}_i \to \mathcal{K}_i$ is a σ_i- quasi-cocoercive operator and A_i is demiclosed at 0.
(C5) For each $i \in \{1, 2, ..., M\}$, $\mathcal{L}_i : \mathcal{H}_0 \to \mathcal{H}_i$, is a bounded linear operator such that $\mathcal{L}_i \ne 0$.
(C6) For each $i \in \{1, 2, ..., M\}$, $\mathcal{J}_i : \mathcal{H}_0 \to \mathcal{K}_i$, is a bounded linear operator such that $\mathcal{J}_i \ne 0$.
(C7) $\Omega = \bigcap_{i=1}^{M} \left(\mathcal{L}_i^{-1}(B_i^{-1}(0)) \cap \mathcal{J}_i^{-1}(A_i^{-1}(0))\right) \ne \emptyset$.
(C8) For $i \in \{1, 2, ..., M\}$, $\{a_i\}, \{b_i\} \subset (0, 1]$, $\sum_{i=1}^{M} a_i = \sum_{i=1}^{M} b_i = 1$.
(C9) $\{d_{n,i}\}$ and $\{e_{n,i}\}$ are bounded sequences in $(0, \infty)$.
(C10) $\{\varepsilon_n\}$ is a nonnegative sequence such that $\lim_{n\to\infty} \frac{\varepsilon_n}{\beta_n} = 0$ where $\{\beta_n\} \subset (0,1)$ satisfies $\lim_{n\to\infty} \beta_n = 0$ and $\sum_{n=0}^{\infty} \beta_n = \infty$.

We now present the proposed method of this paper.

Algorithm 1

Initialization Take $\alpha > 0$, $\mu_i \in (0, 2\sigma_i)$ and $\rho_i \in (0, 2(\frac{\varsigma_i}{s_i} + 1))$, $i = 1, 2, .., M$. Choose sequences $\{a_i\}$, $\{b_i\}$, $\{d_{n,i}\}$, $\{e_{n,i}\}$, $\{\beta_n\}$ and $\{\varepsilon_n\}$ such that the Assumption 3.1 hold. Let $x_1, x_0 \in \mathcal{H}_0$ be two initial points.

Iterative Steps: Given the iterates x_{n-1} and x_n, $(n \geq 1)$. Calculate x_{n+1} as follows:

Step 1: Compute $w_n = x_n + \alpha_n(x_n - x_{n-1})$, where $0 \leq \alpha_n \leq \overline{\alpha}_n$ such that

$$\overline{\alpha}_n = \begin{cases} \min\left\{\frac{\varepsilon_n}{\|x_n - x_{n-1}\|}, \alpha\right\}, & x_n \neq x_{n-1}, \\ \alpha, & \text{otherwise.} \end{cases} \tag{3.1}$$

Step 2: Compute

$$z_n = w_n - \sum_{i=1}^{M} a_i\, \theta_{n,i}\, \mathcal{J}_i{}^*(A_i \mathcal{J}_i w_n),$$

where the stepsizes are chosen in such a way that

$$\theta_{n,i} = \frac{\mu_i \|A_i \mathcal{J}_i w_n\|^2}{\|\mathcal{J}_i{}^*(A_i \mathcal{J}_i w_n)\|^2 + d_{n,i}}, \quad i = 1, 2, ..., M. \tag{3.2}$$

Step 3: Compute

$$y_n = z_n - \sum_{i=1}^{M} b_i\, \tau_{n,i}\, \mathcal{L}_i{}^*(\mathcal{L}_i z_n - J^{B_i}_{s_{n,i}}(\mathcal{L}_i z_n)),$$

where the stepsizes are chosen in such a way that

$$\tau_{n,i} = \frac{\rho_i \|\mathcal{L}_i z_n - J^{B_i}_{s_{n,i}}(\mathcal{L}_i z_n)\|^2}{\|\mathcal{L}_i{}^*(\mathcal{L}_i z_n - J^{B_i}_{s_{n,i}}(\mathcal{L}_i z_n))\|^2 + e_{n,i}}, \quad i = 1, 2, ..., M. \tag{3.3}$$

Step 4: Compute

$$x_{n+1} = (I - \beta_n F) y_n.$$

Set $n := n + 1$ and go to step 1.

Now we are in position to state our main convergence result.

Theorem 3.2. *Let $\{x_n\}$ be a sequence generated by Algorithm 1 under Assumption 3.1. Then, the sequence $\{x_n\}$ converges strongly to the unique solution $x^\star \in VI(\Omega, F)$.*

Proof. Considering Condition (C3) and applying Lemma 2.6, we conclude that for each $i \in \{1, 2, \ldots, M\}$, the operator $s_{n,i}B_i$ is $(\frac{\varsigma_i}{s_{n,i}})$-comonotone. Since

$$\frac{\varsigma_i}{s_{n,i}} \geq \frac{\varsigma_i}{s_i} > -1,$$

it follows that $s_{n,i}B_i$ is also $(\frac{\varsigma_i}{s_i})$-comonotone. By Lemma 2.7, the operator $J^{B_i}_{s_{n,i}}$ is $\frac{1}{2\left(\frac{\varsigma_i}{s_i}+1\right)}$-conically nonexpansive. From Lemma 2.9, we know that for each $i \in \{1, 2, \ldots, M\}$, the set $B_i^{-1}(0)$ is closed and convex. Additionally, since the operator $\mathcal{L}_i : \mathcal{H}_0 \to \mathcal{H}_i$ is bounded and linear, it follows that $\mathcal{L}_i{}^{-1}(B_i^{-1}(0))$ is also a closed convex set. Moreover, by leveraging Assumptions 3.1 (C4) and (C6), we can further deduce that, for each $i \in \{1, 2, \ldots, M\}$, the set $\mathcal{J}_i{}^{-1}(A_i^{-1}(0))$ is closed and convex as well. Consequently, the set Ω is closed and convex. Finally, under Assumption 3.1 (C2), the variational inequality problem $VIP(\Omega, F)$ admits a unique solution. We denote this unique solution by $x^\star \in \mathcal{H}_0$. Next, we demonstrate that the sequence $\{x_n\}$ is bounded. By leveraging the convexity of the norm squared function $\|\cdot\|^2$, Assumption 3.1 (C4), and the definition of $\theta_{n,i}$, we can derive the following:

$$\begin{aligned}
\|z_n - x^\star\|^2 &= \|w_n - \sum_{i=1}^{M} a_i\theta_{n,i}\,\mathcal{J}_i{}^*(A_i\mathcal{J}_i w_n) - x^\star\|^2 \\
&\le \sum_{i=1}^{M} a_i\|(w_n - x^\star) - \theta_{n,i}\,\mathcal{J}_i{}^*(A_i\mathcal{J}_i w_n)\|^2 \\
&= \sum_{i=1}^{M} a_i\big(\|w_n - x^\star\|^2 - 2\langle w_n - x^\star, \theta_{n,i}\mathcal{J}_i{}^*(A_i\mathcal{J}_i w_n)\rangle \\
&\quad + \|\theta_{n,i}\mathcal{J}_i{}^*(A_i\mathcal{J}_i w_n)\|^2\big) \\
&= \sum_{i=1}^{M} a_i\big(\|w_n - x^\star\|^2 - 2\theta_{n,i}\langle \mathcal{J}_i w_n - \mathcal{J}_i x^\star, A_i\mathcal{J}_i w_n\rangle \\
&\quad + (\theta_{n,i})^2\|\mathcal{J}_i{}^*(A_i\mathcal{J}_i w_n)\|^2\big) \\
&\le \sum_{i=1}^{M} a_i\big(\|w_n - x^\star\|^2 - 2(\frac{\mu_i\|A_i\mathcal{J}_i w_n\|^2}{\|\mathcal{J}_i{}^*(A_i\mathcal{J}_i w_n)\|^2 + d_{n,i}})(\sigma_i)\|A_i\mathcal{J}_i w_n\|^2 \\
&\quad + (\frac{\mu_i\|A_i\mathcal{J}_i w_n\|^2}{\|\mathcal{L}_i{}^*(A_i\mathcal{J}_i w_n)\|^2 + d_{n,i}})^2\|\mathcal{J}_i{}^*(A_i\mathcal{J}_i w_n)\|^2\big) \\
&\le \|w_n - x^\star\|^2 - \sum_{i=1}^{M} a_i(2\sigma_i - \mu_i)\frac{\mu_i\|A_i\mathcal{J}_i w_n\|^4}{\|\mathcal{J}_i{}^*(A_i\mathcal{J}_i w_n)\|^2 + d_{n,i}}.
\end{aligned} \tag{3.4}$$

Since $x^\star \in \Omega$, it follows that $x^\star \in \mathcal{L}_i^{-1}(B_i^{-1}(0))$ for each $i \in \{1, 2, ..., M\}$. Thus, we have $\mathcal{L}_i x^\star \in B_i^{-1}(0) = \mathrm{Fix}(J_{s_{n,i}}^{B_i})$. Now, by applying Lemma 2.4 (iv), along with the convexity of $\|\cdot\|^2$ and the definition of $\tau_{n,i}$, we can derive the following:

$$\begin{aligned}
\|y_n - x^\star\|^2 &= \|z_n - \sum_{i=1}^{M} b_i\tau_{n,i}\,\mathcal{L}_i{}^*(\mathcal{L}_i z_n - J_{s_{n,i}}^{B_i}\mathcal{L}_i z_n) - x^\star\|^2 \\
&\le \sum_{i=1}^{M} b_i\|(z_n - x^\star) - \tau_{n,i}\,\mathcal{L}_i{}^*(\mathcal{L}_i z_n - J_{s_{n,i}}^{B_i}\mathcal{L}_i z_n)\|^2 \\
&= \sum_{i=1}^{M} b_i\big(\|z_n - x^\star\|^2 - 2\langle z_n - x^\star, \tau_{n,i}\mathcal{L}_i{}^*(\mathcal{L}_i z_n - J_{s_{n,i}}^{B_i}\mathcal{L}_i z_n)\rangle \\
&\quad + \|\tau_{n,i}\mathcal{L}_i{}^*(\mathcal{L}_i z_n - J_{s_{n,i}}^{B_i}\mathcal{L}_i z_n)\|^2\big) \\
&= \sum_{i=1}^{M} b_i\big(\|z_n - x^\star\|^2 - 2\tau_{n,i}\langle \mathcal{L}_i z_n - \mathcal{L}_i x^\star, \mathcal{L}_i z_n - J_{s_{n,i}}^{B_i}\mathcal{L}_i z_n\rangle \\
&\quad + (\tau_{n,i})^2\|\mathcal{L}_i{}^*(\mathcal{L}_i z_n - J_{s_{n,i}}^{B_i}\mathcal{L}_i z_n)\|^2\big) \\
&\le \sum_{i=1}^{M} b_i\big(\|z_n - x^\star\|^2 - 2(\frac{\rho_i\|\mathcal{L}_i z_n - J_{s_{n,i}}^{B_i}\mathcal{L}_i z_n\|^2}{\|\mathcal{L}_i{}^*(\mathcal{L}_i z_n - J_{s_{n,i}}^{B_i}\mathcal{L}_i z_n)\|^2 + e_{n,i}})(\frac{\zeta_i}{s_i} + 1)\|\mathcal{L}_i z_n - J_{s_{n,i}}^{B_i}\mathcal{L}_i z_n\|^2
\end{aligned}$$

$$+ (\frac{\rho_i\|\mathcal{L}_i z_n - J^{B_i}_{s_{n,i}}\mathcal{L}_i z_n\|^2}{\|\mathcal{L}_i{}^*(\mathcal{L}_i z_n - J^{B_i}_{s_{n,i}}\mathcal{L}_i z_n)\|^2 + e_{n,i}})^2\|\mathcal{L}_i{}^*(\mathcal{L}_i z_n - J^{B_i}_{s_{n,i}}\mathcal{L}_i z_n)\|^2)$$

$$\leq \|z_n - x^\star\|^2 - \sum_{i=1}^{M} b_i(2(\frac{\zeta_i}{s_i}+1) - \rho_i)\frac{\rho_i\|\mathcal{L}_i z_n - J^{B_i}_{s_{n,i}}\mathcal{L}_i z_n\|^4}{\|\mathcal{L}_i{}^*(\mathcal{L}_i z_n - J^{B_i}_{s_{n,i}}\mathcal{L}_i z_n)\|^2 + e_{n,i}}. \tag{3.5}$$

From (3.1), we know that $\alpha_n\|x_n - x_{n-1}\| \leq \varepsilon_n$ for all n. Moreover, since $\lim_{n\to\infty}\frac{\varepsilon_n}{\beta_n} = 0$, it follows that:

$$\lim_{n\to\infty}\frac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\| = 0.$$

Consequently, there exists a constant $\mathbb{M}_1 > 0$ such that

$$\frac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\| \leq \mathbb{M}_1.$$

From the definition of w_n, we obtain:

$$\begin{aligned}\|w_n - x^\star\| &= \|x_n + \alpha_n(x_n - x_{n-1}) - x^\star\|\\ &\leq \|x_n - x^\star\| + \alpha_n\|x_n - x_{n-1}\|\\ &= \|x_n - x^\star\| + \beta_n\tfrac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\|\\ &= \|x_n - x^\star\| + \beta_n\mathbb{M}_1.\end{aligned} \tag{3.6}$$

By our assumptions and using equations (3.4), (3.5), and (3.6), we deduce:

$$\|y_n - x^\star\| \leq \|w_n - x^\star\| \leq \|x_n - x^\star\| + \beta_n\mathbb{M}_1. \tag{3.7}$$

Now, let $\nu \in (0, \frac{2\delta}{l^2})$. Since $\lim_{n\to\infty}\beta_n = 0$, there exists an index $n_0 \in \mathbb{N}$ such that for all $n > n_0$, we have $\beta_n < \nu$. Thus, $\frac{\beta_n}{\nu} \in (0,1)$. Applying Lemma 2.13 for all $n > n_0$, we obtain:

$$\begin{aligned}\|(I-\beta_n F)y_n - (I-\beta_n F)x^\star\| &= \|(I - \tfrac{\beta_n}{\nu}\nu F)y_n - (I - \tfrac{\beta_n}{\nu}\nu F)x^\star\|\\ &\leq (1 - \tfrac{\beta_n}{\nu}\eta)\|y_n - x^\star\|,\end{aligned} \tag{3.8}$$

where $\eta = 1 - \sqrt{1-\nu(2\delta - \nu l^2)} \in (0,1)$. Utilizing the inequalities (3.7) and (3.8), we obtain that:

$$\begin{aligned}\|x_{n+1} - x^\star\| &= \|y_n - \beta_n F y_n - x^\star\|\\ &= \|(I-\beta_n F)y_n - (I-\beta_n F)x^\star - \beta_n F x^\star\|\\ &\leq \|(I-\beta_n F)y_n - (I-\beta_n F)x^\star\| + \beta_n\|Fx^\star\|\\ &\leq (1-\tfrac{\beta_n}{\nu}\eta)\|y_n - x^\star\| + \beta_n\|Fx^\star\|\\ &\leq (1-\tfrac{\beta_n}{\nu}\eta)\|x_n - x^\star\| + \beta_n\mathbb{M}_1 + \beta_n\|Fx^\star\|\\ &\leq (1-\tfrac{\beta_n}{\nu}\eta)\|x_n - x^\star\| + \tfrac{\beta_n}{\nu}\eta[\tfrac{\nu(\mathbb{M}_1 + \|Fx^\star\|)}{\eta}]\\ &\leq \max\{\|x_n - x^\star\|, \tfrac{\nu(\mathbb{M}_1+\|Fx^\star\|)}{\eta}\}\\ &\leq \cdots \leq \max\{\|x_{n_0} - x^\star\|, \tfrac{\nu(\mathbb{M}_1+\|Fx^\star\|)}{\eta}\}.\end{aligned}$$

This implies that the sequence $\{x_n\}$ is bounded. Additionally, we can conclude that the sequences $\{y_n\}$ and $\{w_n\}$ are also bounded. We have:

$$\begin{aligned}\|w_n - x^\star\|^2 &= \|x_n + \alpha_n(x_n - x_{n-1}) - x^\star\|^2 \\ &\le \|x_n - x^\star\|^2 + (\alpha_n)^2\|x_n - x_{n-1}\|^2 + 2\alpha_n\langle x_n - x^\star, x_n - x_{n-1}\rangle \\ &\le \|x_n - x^\star\|^2 + (\alpha_n)^2\|x_n - x_{n-1}\|^2 + 2\alpha_n\|x_n - x^\star\|\|x_n - x_{n-1}\|.\end{aligned}$$

By utilizing inequality (3.8) and the inequality $\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle, \forall x, y \in \mathcal{H}$, we obtain:

$$\begin{aligned}\|x_{n+1} - x^\star\|^2 &= \|(I - \beta_n F)y_n - (I - \beta_n F)x^\star - \beta_n F x^\star\|^2 \\ &\le \|(I - \beta_n F)y_n - (I - \beta_n F)x^\star\|^2 - 2\beta_n\langle Fx^\star, x_{n+1} - x^\star\rangle \\ &\le (1 - \tfrac{\beta_n}{\nu}\eta)^2\|y_n - x^\star\|^2 + 2\beta_n\langle Fx^\star, x^\star - x_{n+1}\rangle \\ &\le (1 - \tfrac{\beta_n}{\nu}\eta)\|w_n - x^\star\|^2 + 2\beta_n\langle Fx^\star, x^\star - x_{n+1}\rangle \\ &\le (1 - \tfrac{\beta_n}{\nu}\eta)\|x_n - x^\star\|^2 + (\tfrac{\beta_n}{\nu}\eta)(\tfrac{2\nu}{\eta})\langle Fx^\star, x^\star - x_{n+1}\rangle \\ &+ \alpha_n\|x_n - x_{n-1}\|(\alpha_n\|x_n - x_{n-1}\| + 2\|x_n - x^\star\|) \\ &\le (1 - \tfrac{\beta_n}{\nu}\eta)\|x_n - x^\star\|^2 + (\tfrac{\beta_n}{\nu}\eta)(\tfrac{2\nu}{\eta})\langle Fx^\star, x^\star - x_{n+1}\rangle \\ &+ 3\alpha_n\|x_n - x_{n-1}\|\mathbb{M}_2 \\ &= (1 - \tfrac{\beta_n}{\nu}\eta)\|x_n - x^\star\|^2 \\ &+ \tfrac{\beta_n}{\nu}\eta[\tfrac{2\nu}{\eta}\langle Fx^\star, x^\star - x_{n+1}\rangle + \tfrac{3\nu\alpha_n}{\beta_n}\tfrac{\mathbb{M}_2}{\eta}\|x_n - x_{n-1}\|] \\ &= (1 - \Xi_n)\|x_n - x^\star\|^2 + \Xi_n\vartheta_n, \quad \forall n > n_0,\end{aligned} \tag{3.9}$$

where $\mathbb{M}_2 = \sup_{n\in\mathbb{N}}\{\|x_n - x^\star\|, \alpha_n\|x_n - x_{n-1}\|\}$ and

$$\Xi_n = \frac{\beta_n}{\nu}\eta, \quad \vartheta_n = \frac{2\nu}{\eta}\langle Fx^\star, x^\star - x_{n+1}\rangle + \frac{3\nu\alpha_n}{\beta_n}\frac{\mathbb{M}_2}{\eta}\|x_n - x_{n-1}\|.$$

It is straightforward to observe that $\Xi_n \to 0$ and $\sum_{n=1}^{\infty}\Xi_n = \infty$. Since $\{x_n\}$ is bounded, there exists a constant $\mathbb{M}_3 > 0$ such that

$$2\langle Fx^\star, x^\star - x_{n+1}\rangle \le \mathbb{M}_3.$$

From the definition of $\{x_{n+1}\}$ and inequality (3.8), we deduce that

$$\begin{aligned}\|x_{n+1} - x^\star\|^2 &= \|y_n - \beta_n F y_n - x^\star\|^2 \\ &= \|(I - \beta_n F)y_n - (I - \beta_n F)x^\star - \beta_n F x^\star\|^2 \\ &\le \|(I - \beta_n F)y_n - (I - \beta_n F)x^\star\|^2 - 2\beta_n\langle Fx^\star, x_{n+1} - x^\star\rangle \\ &\le (1 - \tfrac{\beta_n}{\nu}\eta)^2\|y_n - x^\star\|^2 + 2\beta_n\langle Fx^\star, x^\star - x_{n+1}\rangle \\ &\le \|y_n - x^\star\|^2 + \beta_n\mathbb{M}_3 \quad \forall n > n_0.\end{aligned} \tag{3.10}$$

From (3.6) we have

$$\begin{aligned}\|w_n - x^\star\|^2 &\le (\|x_n - x^\star\| + \beta_n\mathbb{M}_1)^2 \\ &= \|x_n - x^\star\| + \beta_n(2\mathbb{M}_1\|x_n - x^\star\| + \beta_n(\mathbb{M}_1)^2) \\ &\le \|x_n - x^\star\|^2 + \beta_n\mathbb{M}_4.\end{aligned} \tag{3.11}$$

for some constant $\mathbb{M}_4 > 0$. From inequalities (3.4),(3.5),(3.10) and (3.11), for all $n > n_0$, we get

$$\begin{aligned} \|x_{n+1} - x^\star\|^2 &\leq \|x_n - x^\star\|^2 + \beta_n \mathbb{M}_4 - \textstyle\sum_{i=1}^M a_i(2\sigma_i - \mu_i) \frac{\mu_i \|A_i \mathcal{J}_i w_n\|^4}{\|\mathcal{J}_i{}^*(A_i \mathcal{J}_i w_n)\|^2 + d_{n,i}} \\ &\quad - \textstyle\sum_{i=1}^M b_i(2(\frac{\zeta_i}{s_i} + 1) - \rho_i) \frac{\rho_i \|\mathcal{L}_i z_n - J^{B_i}_{s_{n,i}} \mathcal{L}_i z_n\|^4}{\|\mathcal{L}_i{}^*(\mathcal{L}_i z_n - J^{B_i}_{s_{n,i}} \mathcal{L}_i z_n)\|^2 + e_{n,i}} + \beta_n \mathbb{M}_3. \end{aligned} \tag{3.12}$$

Now we set

$$\begin{aligned} \xi_n &= \textstyle\sum_{i=1}^M a_i(2\sigma_i - \mu_i) \frac{\mu_i \|A_i \mathcal{J}_i w_n\|^4}{\|\mathcal{J}_i{}^*(A_i \mathcal{J}_i w_n)\|^2 + d_{n,i}} \\ &\quad + \textstyle\sum_{i=1}^M b_i(2(\frac{\zeta_i}{s_i} + 1) - \rho_i) \frac{\rho_i \|\mathcal{L}_i z_n - J^{B_i}_{s_{n,i}} \mathcal{L}_i z_n\|^4}{\|\mathcal{L}_i{}^*(\mathcal{L}_i z_n - J^{B_i}_{s_{n,i}} \mathcal{L}_i z_n)\|^2 + e_{n,i}}, \end{aligned}$$

and

$$\eta_n = \beta_n(\mathbb{M}_3 + \mathbb{M}_4), \quad \Gamma_n = \|x_n - x^\star\|^2. \tag{3.13}$$

Hence, inequality (3.12) can be rewritten as:

$$\Gamma_{n+1} \leq \Gamma_n - \xi_n + \eta_n. \tag{3.14}$$

To prove that $\Gamma_n \to 0$, by Lemma 2.14 (considering inequalities (3.9) and (3.14)), it is sufficient to show that for any subsequence $\{n_k\} \subset \{n\}$, if $\lim_{k\to\infty} \xi_{n_k} = 0$, then

$$\limsup_{k\to\infty} \vartheta_{n_k} \leq 0.$$

Assuming $\lim_{k\to\infty} \xi_{n_k} = 0$, we deduce the following:

$$\lim_{k\to\infty} \|\mathcal{L}_i z_{n_k} - J^{B_i}_{s_{n_k},i} \mathcal{L}_i z_{n_k}\| = \lim_{k\to\infty} \|A_i \mathcal{J}_i w_{n_k}\| = 0, \qquad i = 1, 2, ..., M. \tag{3.15}$$

This implies that

$$\lim_{k\to\infty} \|y_{n_k} - z_{n_k}\| = \lim_{k\to\infty} \|z_{n_k} - w_{n_k}\| = 0. \tag{3.16}$$

Note that

$$\|x_n - w_n\| = \alpha_n \|x_n - x_{n-1}\| = \beta_n \frac{\alpha_n}{\beta_n} \|x_n - x_{n-1}\| \to 0.$$

Hence

$$\lim_{k\to\infty} \|z_{n_k} - x_{n_k}\| = \lim_{k\to\infty} \|w_{n_k} - x_{n_k}\| = 0. \tag{3.17}$$

Also we have

$$\lim_{k\to\infty} \|x_{n_k+1} - y_{n_k}\| = \lim_{k\to\infty} \beta_{n_k} \|F(y_{n_k})\| = 0. \tag{3.18}$$

From above inequalities we arrive at

$$\begin{aligned} \|x_{n_k+1} - x_{n_k}\| &\leq \|x_{n_k+1} - y_{n_k}\| + \|y_{n_k} - z_{n_k}\| \\ &\quad + \|z_{n_k} - w_{n_k}\| + \|w_{n_k} - x_{n_k}\| \to 0, \ k \to \infty. \end{aligned} \tag{3.19}$$

Since $\{x_{n_k}\}$ is bounded, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ that converges weakly to $\widehat{x}$. Without loss of generality, we assume that $x_{n_k} \rightharpoonup \widehat{x}$. Given that $\lim_{k\to\infty} \|w_{n_k} - x_{n_k}\| = 0$, it follows that $w_{n_k} \rightharpoonup \widehat{x}$. Since $\mathcal{J}_i$ is a bounded linear operator, we have $\mathcal{J}_i(w_{n_k}) \rightharpoonup \mathcal{J}_i\widehat{x}$. Additionally, since $\lim_{k\to\infty} \|A_i \mathcal{J}_i w_{n_k}\| = 0$, by the demiclosedness of A_i, we obtain $A_i \mathcal{J}_i \widehat{x} = 0$. Therefore, $\mathcal{J}_i \widehat{x} \in A_i^{-1}(0)$, for $i = 1, 2, \ldots, M$. Next, since $\lim_{k\to\infty} \|z_{n_k} - x_{n_k}\| = 0$, we conclude that $z_{n_k} \rightharpoonup \widehat{x}$. Again, since $\mathcal{L}_i$ is a bounded linear operator, it follows that $\mathcal{L}_i(z_{n_k}) \rightharpoonup \mathcal{L}_i\widehat{x}$. From Lemma 2.8, for each $i \in \{1, 2, \ldots, M\}$, there exists a constant $L_i > 0$ such that

$$\|\mathcal{L}_i z_{n_k} - J^{B_i}_{s_i} \mathcal{L}_i z_{n_k}\| \leq (L_i + 1 + \frac{L_i\, s_i}{s_{n_k,i}}) \|\mathcal{L}_i z_{n_k} - J^{B_i}_{s_{n_k},i} \mathcal{L}_i z_{n_k}\|. \tag{3.20}$$

This implies that

$$\lim_{k\to\infty} \|\mathcal{L}_i z_{n_k} - J^{B_i}_{s_i} \mathcal{L}_i z_{n_k}\| = 0, \quad i = 1, 2, ..., M. \tag{3.21}$$

Using Lemma 2.4, we conclude that $\mathcal{L}_i\widehat{x} \in \text{Fix}(J_{s_i}^{B_i}) = B_i^{-1}(0)$. Therefore, $\widehat{x} \in \Omega$. Now we show that

$$\limsup_{k\to\infty}\langle Fx^\star, x^\star - x_{n_k}\rangle \leq 0. \tag{3.22}$$

To prove this inequality, we select a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that

$$\lim_{j\to\infty}\langle Fx^\star, x^\star - x_{n_{k_j}}\rangle = \limsup_{k\to\infty}\langle Fx^\star, x^\star - x_{n_k}\rangle.$$

Since $x^\star$ is the unique solution of the variational inequality $VIP(\Omega, F)$ and $\{x_{n_{k_j}}\}$ converges weakly to $\widehat{x} \in \Omega$, we conclude that

$$\limsup_{k\to\infty}\langle Fx^\star, x^\star - x_{n_k}\rangle = \lim_{j\to\infty}\langle Fx^\star, x^\star - x_{n_{k_j}}\rangle = \langle Fx^\star, x^\star - \widehat{x}\rangle \leq 0.$$

Therefore

$$\limsup_{k\to\infty}\vartheta_{n_k} \leq 0.$$

Since all the conditions of Lemma 2.14 are satisfied, we immediately deduce that $\lim_{n\to\infty}\Gamma_n = \lim_{n\to\infty}\|x_n - x^\star\|^2 = 0$, which implies that the sequence $\{x_n\}$ converges strongly to $x^\star$, the unique solution of the variational inequality $VIP(\Omega, F)$. □

4. Application

In this section, we present applications of our main theoretical results to specific problem classes, including the split monotone variational inclusion problem and the multiple-set split feasibility problem.

4.1. **Monotone inclusion problem.**

We begin by considering the following monotone inclusion problem:

$$\text{Find } x^* \in \mathcal{H} \text{ such that } 0 \in \mathbf{D}(x^*) + \mathbf{G}(x^*),$$

where $\mathbf{D} : \mathcal{H} \to \mathcal{H}$ is a monotone and Lipschitz continuous single-valued operator, and $\mathbf{G} : \mathcal{H} \rightrightarrows \mathcal{H}$ is a maximally monotone set-valued operator defined on the Hilbert space $\mathcal{H}$. To address this problem, we utilize the forward-backward-forward (FBF) operator $\mathbf{U} : \mathcal{H} \to \mathcal{H}$, originally proposed by Tseng in [24]. The FBF operator is defined as follows:

$$\mathbf{U} := I - J_\gamma^{\mathbf{G}}(I - \gamma\mathbf{D}) - \gamma\left[\mathbf{D}x - \mathbf{D}\circ J_\gamma^{\mathbf{G}}(I - \gamma\mathbf{D})\right],$$

where $\gamma > 0$ and $J_\gamma^{\mathbf{G}} := (I + \gamma\mathbf{G})^{-1}$ denotes the resolvent of the operator $\mathbf{G}$. We now present the following key results:

Lemma 4.1. *[Adapted from [5], Proposition 1] Assume $\mathbf{D}$ is monotone and L-Lipschitz continuous, and $\mathbf{G}$ is maximally monotone. Then the operator $\mathbf{U}$ satisfies:*

(i) The set of zeros of the sum $\mathbf{D} + \mathbf{G}$ coincides with the set of zeros of $\mathbf{U}$.
(ii) The operator $\mathbf{U}$ is Lipschitz continuous.
(iii) If $\gamma < \frac{1}{L}$, then $\mathbf{U}$ is quasi-cocoercive with modulus $\omega = \frac{1-\gamma L}{(1+\gamma L)^2}$.

We will now apply Algorithm 1 to solve the split monotone variational inclusion problem. To proceed, we first consider the following conditions.

Assumption 4.2. Assume that the following hold:

(C1) $\mathcal{H}_0$ is a real Hilbert space and $\mathcal{K}_i$, $i = 1, 2, \ldots, M$, are finite dimensional real Hilbert spaces.
(C2) The operator $F : \mathcal{H}_0 \to \mathcal{H}_0$ is l-Lipschitz continuous and δ-strongly monotone with constants $l > 0, \delta > 0$.
(C3) For each $i \in \{1, 2, \ldots, M\}$, $\mathbf{G}_i : \mathcal{K}_i \rightrightarrows \mathcal{K}_i$ is a maximal monotone set-valued operator and $\mathbf{D}_i : \mathcal{K}_i \to \mathcal{K}_i$ is a monotone and L_i-Lipschitz continuous operator.

(C4) For each $i \in \{1,2,...,M\}$, $\mathcal{J}_i : \mathcal{H}_0 \to \mathcal{K}_i$, is a bounded linear operator such that $\mathcal{J}_i \neq 0$.
(C5) $\Omega = \bigcap_{i=1}^{M} \mathcal{J}_i^{-1}\big((\mathbf{D}_i + \mathbf{G}_i)^{-1}0\big) \neq \emptyset$.
(C6) For $i \in \{1,2,...,M\}$, $\{a_i\} \subset (0,1]$, $\sum_{i=1}^{M} a_i = 1$.
(C7) $\{d_{n,i}\}$ is a bounded sequence in $(0,\infty)$.
(C8) $\{\varepsilon_n\}$ is a nonnegative sequence such that $\lim_{n\to\infty} \frac{\varepsilon_n}{\beta_n} = 0$ where $\{\beta_n\} \subset (0,1)$ satisfies $\lim_{n\to\infty} \beta_n = 0$ and $\sum_{n=0}^{\infty} \beta_n = \infty$.

We introduce the following algorithm designed to solve the split monotone variational inclusion problem.

Algorithm 2

Initialization Take $\alpha > 0$, $\gamma_i < \frac{1}{L_i}$ and $\mu_i \in (0, \frac{2(1-\gamma_i L_i)}{(1+\gamma_i L_i)^2})$, $i = 1,2,..,M$. Choose sequences $\{a_i\}$, $\{d_{n,i}\}$, $\{\beta_n\}$ and $\{\varepsilon_n\}$ such that the Assumption 4.2 hold. Let $x_1, x_0 \in \mathcal{H}_0$ be two initial points.
Iterative Steps: Given the iterates x_{n-1} and x_n, $(n \geq 1)$. Calculate x_{n+1} as follows:
Step 1: Compute $w_n = x_n + \alpha_n(x_n - x_{n-1})$, where $\{\alpha_n\}$ is defined in (3.1).
Step 2: Compute

$$y_n = w_n - \sum_{i=1}^{M} a_i\, \theta_{n,i}\, \mathcal{J}_i^{*}(\mathbf{U}_i \mathcal{J}_i w_n)$$

where

$$\mathbf{U}_i := I - J_{\gamma_i}^{\mathbf{G}_i}(I - \gamma_i \mathbf{D}_i) - \gamma_i \left[\mathbf{D}_i x - \mathbf{D}_i \circ J_{\gamma_i}^{\mathbf{G}_i}(I - \gamma_i \mathbf{D}_i)\right],$$

and the stepsizes are chosen in such a way that

$$\theta_{n,i} = \frac{\mu_i \|\mathbf{U}_i \mathcal{J}_i w_n\|^2}{\|\mathcal{J}_i^{*}(\mathbf{U}_i \mathcal{J}_i w_n)\|^2 + d_{n,i}}, \quad i = 1,2,...,M. \tag{4.1}$$

Step 3: Compute

$$x_{n+1} = (I - \beta_n F) y_n.$$

Set $n := n+1$ and go to step 1.

In a finite-dimensional real Hilbert space, every continuous mapping is demiclosed at 0. Utilizing Theorem 3.2 and Lemma 4.1, we derive the following strong convergence result for solving the split monotone variational inclusion problem.

Theorem 4.3. *Let $\{x_n\}$ be a sequence generated by Algorithm 2 under Assumption 4.2. Then, the sequence $\{x_n\}$ converges strongly to the unique solution $x^\star \in VI(\Omega, F)$.*

Remark 4.4. Let $\mathbf{G}_i : \mathcal{K}_i \rightrightarrows \mathcal{K}_i$ be a maximally monotone set-valued operator and $\mathbf{f}_i : \mathcal{K}_i \to \mathcal{K}_i$ be an η_i-cocoercive operator for each $i \in \{1,2,...,M\}$. Consider the operator defined by $\mathcal{T}_i = J_{\lambda_i}^{\mathbf{G}_i}(I - \lambda_i \mathbf{f}_i)$ with $\lambda_i \in (0, 2\eta_i)$. It is known that $\mathcal{T}_i$ is an $\left(\frac{2\eta_i}{4\eta_i - \lambda_i}\right)$-averaged operator (see [11] for details). Consequently, $I - \mathcal{T}_i$ is $\frac{4\eta_i - \lambda_i}{4\eta_i}$-cocoercive and demiclosed at zero. By setting $A_i = I - \mathcal{T}_i$ and $B_i = 0$ in Algorithm 1, we derive a new algorithm tailored for solving the split monotone variational inclusion problem.

4.2. **Multiple-set split feasibility problem.** We now demonstrate the application of Algorithm 1 to solve the multiple-set split feasibility problem in Hilbert spaces. To establish the strong convergence of our approach, we first outline the following assumptions that are essential for ensuring convergence.

Assumption 4.5. Assume that the following hold:

(C1) $\mathcal{H}_0$ and $\mathcal{K}_i$, $i = 1, 2, ..., M$, are real Hilbert spaces.
(C2) The operator $F : \mathcal{H}_0 \to \mathcal{H}_0$ is l-Lipschitz continuous and δ-strongly monotone with constants $l > 0, \delta > 0$.
(C3) For each $i \in \{1, 2, ..., M\}$, Q_i, is nonempty closed and convex subset of $\mathcal{K}_i$ and $\{C_i\}_{i=1}^M$ is a finite family of nonempty closed and convex subsets of $\mathcal{H}_0$.
(C4) For each $i \in \{1, 2, ..., M\}$, $\mathcal{J}_i : \mathcal{H}_0 \to \mathcal{K}_i$, is a bounded linear operator such that $\mathcal{J}_i \neq 0$.
(C5) $\Omega = \{x \in \bigcap_{i=1}^M C_i : \quad \mathcal{J}_i x \in Q_i, i = 1, 2, ..., M\} \neq \emptyset$.
(C6) For $i \in \{1, 2, ..., M\}$, $\{a_i\}, \{b_i\} \subset (0, 1]$, $\sum_{i=1}^M a_i = \sum_{i=1}^M b_i = 1$.
(C7) $\{d_{n,i}\}$ is a bounded sequence in $(0, \infty)$.
(C8) $\{\varepsilon_n\}$ is a nonnegative sequence such that $\lim_{n\to\infty} \frac{\varepsilon_n}{\beta_n} = 0$ where $\{\beta_n\} \subset (0, 1)$ satisfies $\lim_{n\to\infty} \beta_n = 0$ and $\sum_{n=0}^{\infty} \beta_n = \infty$.

We now propose the following algorithm to address the multiple-set split feasibility problem .

Algorithm 3

Initialization Take $\alpha > 0$ and $\mu_i \in (0, 2)$ $i = 1, 2, .., M$. Choose sequences $\{a_i\}$, $\{b_i\}$, $\{d_{n,i}\}$, $\{\beta_n\}$ and $\{\varepsilon_n\}$ such that the Assumption 4.5 hold. Let $x_1, x_0 \in \mathcal{H}_0$ be two initial points.
Iterative Steps: Given the iterates x_{n-1} and x_n, $(n \geq 1)$. Calculate x_{n+1} as follows:
Step 1: Compute $w_n = x_n + \alpha_n(x_n - x_{n-1})$, where $\{\alpha_n\}$ is defined in (3.1).
Step 2: Compute

$$z_n = w_n - \sum_{i=1}^{M} a_i\, \theta_{n,i}\, \mathcal{J}_i^*(\mathcal{J}_i w_n - P_{Q_i}\mathcal{J}_i w_n)$$

where the stepsizes are chosen in such a way that

$$\theta_{n,i} = \frac{\mu_i \|\mathcal{J}_i w_n - P_{Q_i}\mathcal{J}_i w_n\|^2}{\|\mathcal{J}_i^*(\mathcal{J}_i w_n - P_{Q_i}\mathcal{J}_i w_n)\|^2 + d_{n,i}}, \quad i = 1, 2, ..., M. \tag{4.2}$$

Step 3: Compute

$$y_n = \sum_{i=1}^{M} b_i\, P_{C_i}(z_n).$$

Step 4: Compute

$$x_{n+1} = (I - \beta_n F)y_n.$$

Set $n := n + 1$ and go to step 1.

Theorem 4.6. *Let $\{x_n\}$ be a sequence generated by Algorithm 3 under Assumption 4.5. Then, the sequence $\{x_n\}$ converges strongly to the unique solution $x^\star \in VI(\Omega, F)$.*

Proof. For $i = 1, 2, \ldots, M$, let us define $B_i = \partial i_{C_i}$, which is known to be a maximal monotone operator. According to Remark 2.11, we have $J_r^{\partial i_{C_i}}(x) = P_{C_i}x$ for all $x \in \mathcal{H}_0$ and any $r > 0$. Additionally, it follows that $B_i^{-1}(0) = \text{Fix}(P_{C_i}) = C_i$. Similarly, for $i = 1, 2, \ldots, M$, let us set $A_i = I - P_{Q_i}$. It is straightforward to verify that A_i is a 1-cocoercive operator. Furthermore, we have $A_i^{-1}(0) = \text{Fix}(P_{Q_i}) = Q_i$. Now, by setting $\mathcal{H}_i = \mathcal{H}_0$, $\mathcal{J}_i = I$, $(i = 1, 2, \ldots, M)$ and $\theta_{n,i} = 1$ in Algorithm 1, we obtain the desired result directly from Theorem 3.2. ☐

5. Numerical Experiment

In this section, we present a computational experiment to demonstrate the effectiveness of our proposed algorithm. Specifically, we consider a minimization problem defined over the solution set of a multiple-set split feasibility problem. The algorithm was implemented in MATLAB R2014b and executed on a laptop equipped with an Intel Core i7 processor and 12 GB of RAM.

Example 5.1. We address the following constrained optimization problem:

$$\text{Minimize} \quad f(x) = \frac{1}{2}\|x - \mathfrak{p}\|^2 \quad \text{subject to} \quad x \in \Omega = \bigcap_{i=1}^{4} \left(C_i \cap \mathcal{L}_i^{-1}(Q_i)\right),$$

where $\mathfrak{p} = (0.2, 0, 0, 0, 0) \in \mathbb{R}^5$, $C_i \subset \mathbb{R}^5$, and $Q_i \subset \mathbb{R}^{10}$ are defined as follows:

$$\begin{aligned} C_i &= \{x \in \mathbb{R}^5 : \langle z_i, x\rangle \leq r_i\}, \quad i = 1, 2, 3, 4,\\ Q_1 &= \{x \in \mathbb{R}^{10} : \|x - (q_1, 0, 0, \ldots, 0)\| \leq 1\},\\ Q_2 &= \{x \in \mathbb{R}^{10} : \|x - (0, q_2, 0, \ldots, 0)\| \leq 1\},\\ Q_3 &= \{x \in \mathbb{R}^{10} : \|x - (0, 0, q_3, 0, \ldots, 0)\| \leq 1\},\\ Q_4 &= \{x \in \mathbb{R}^{10} : \|x - (0, 0, 0, q_4, 0, \ldots, 0)\| \leq 1\}. \end{aligned}$$

Here, $\mathcal{L}_i : \mathbb{R}^5 \to \mathbb{R}^{10}$ are bounded linear operators, with the elements of their representation matrices randomly generated within the closed interval $[-2, 2]$. This problem can be equivalently reformulated as a variational inequality problem of the form:

$$VI(\Omega, F), \quad \text{where} \quad F(x) = \nabla f(x) = \nabla\left(\frac{1}{2}\|x - \mathfrak{p}\|^2\right) = x - \mathfrak{p}.$$

We analyze the convergence behavior of the sequence $\{x_n\}$ generated by Algorithm 3. For this experiment, the coordinates of the vectors $z_i (i = 1, 2, 3, 4)$, were randomly generated within the interval $[1, 4]$, while the scalar values $r_i (i = 1, 2, 3, 4)$ were drawn from the interval $[1, 2]$. Similarly, the vectors $q_i (i = 1, 2, 3, 4)$ were randomly generated within $[0, 1]$. The coordinates of the initial approximations x_0 and x_1 were also chosen randomly from the interval $(0, 1)$. The stopping criterion was set as $E_n = \|x_n - x_{n-1}\| < 10^{-5}$. The algorithm parameters were configured as follows: $a_i = b_i = \frac{1}{4}$, $\alpha_n = 0.6$, $\beta_n = \frac{2}{n+3}$, $\varepsilon_n = \frac{1}{(n+2)^{1.2}}$, and $\theta_{n,i} = \frac{1.5\|(I-P_{Q_i})\mathcal{L}_i w_n\|^2}{\|\mathcal{L}_i^*((I-P_{Q_i})\mathcal{L}_i w_n)\|^2+0.001}$. The numerical results are depicted in Figures 1 and 2, illustrating the convergence performance of our algorithm.

Statements and Declarations

The author declare that he has no conflict of interest, and the manuscript has no associated data.

Funding

This research is supported by University of Science and Technology of Mazandaran, under grant No.241541172 (2023/12/9).

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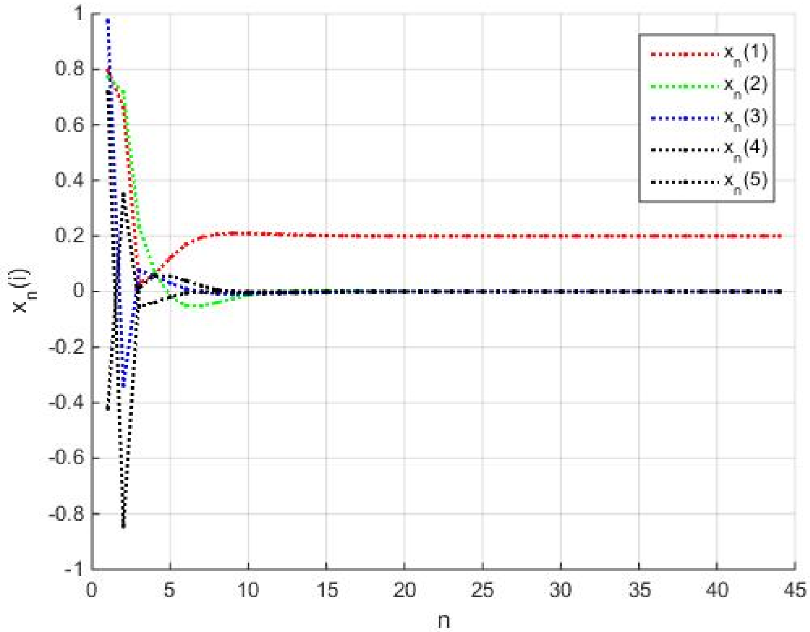

FIGURE 1. The graph of x_n for the Example 5.1.

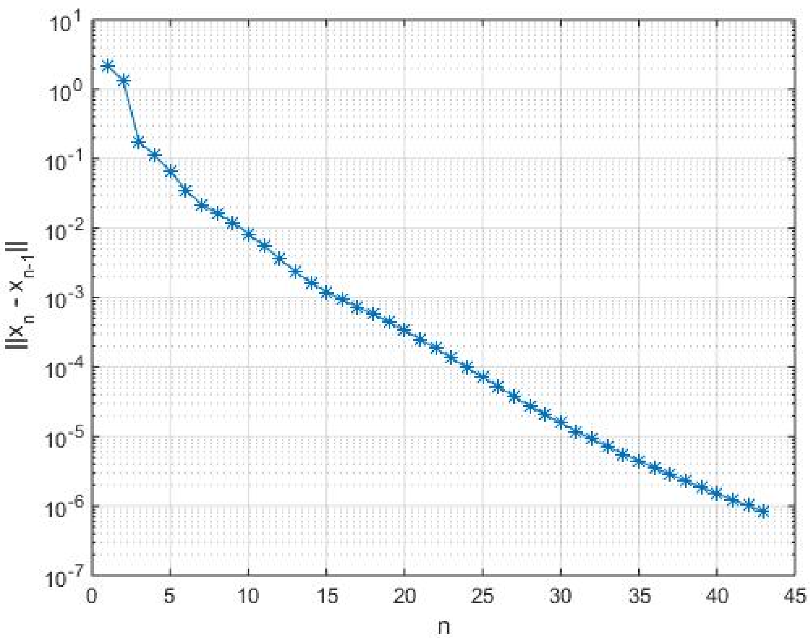

FIGURE 2. The graph of the error $\|x_n - x_{n-1}\|$ for the Example 5.1.

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