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MODIFIED DOUBLE INERTIAL SUBGRADIENT EXTRGRADIENT METHOD FOR SOLVING VARIATIONAL INEQUALITY PROBLEMS IN BANACH SPACES

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ABSTRACT. We present a new method for solving variational inequality problems (VIPs) based on double inertial terms with a modified subgradient extragradient self-adaptive step size method. Our proposal utilizes double inertial acceleration to improve convergence behavior and stability in solving VIPs. Strong convergence theorems for the proposed algorithms are established under some mild assumptions on the control parameters in 2-uniformly convex and smooth real Banach spaces. Numerical experiments demonstrate the efficacy and advantages of our technique compared to existing methods in the literature.

Keywords. Inertial terms, extragradient subgradient method, variational inequality problem, step size, strong convergence, Banach space.

1. INTRODUCTION

The history of variational inequality theory is extensive, having emerged from equilibrium problems and finding use in many different domains. The first variational inequality problem, or Signorini problem, was proposed by Antonio Signorini [31] in 1959. Through his work, Guido Stampacchia expanded the theory and introduced the concept of "variational inequality." Moreover, Stampacchia demonstrated a generalization of the Lax-Milgram theorem, which became fundamental. Due to the efforts of Jacques-Louis Lions and Gaetano Fichera, the idea became well-known in France, see [32, 11, 20] for further information. Early in the 1960s, scientists discovered that the virtual work or power principle in mechanics inevitably leads to variational inequalities. The study of variational inequalities was greatly advanced by Fichera, Lions, and Stampacchia [32, 11, 20]. The critical significance that variational inequality plays beyond its original definition in a variety of domains. For example, they assist in analyzing market competition among enterprises, addressing patterns of population migration, and modeling traffic flow and congestion in transportation networks. Variational inequalities are used in spatial price equilibrium models in economics to identify stable price distributions. These inequalities have applications in finance, including option pricing, portfolio optimization, and equilibrium price modeling. Furthermore, variational inequality modeling can be used to evaluate information flow in knowledge networks and environmental fluxes, such as pollutant dispersion (see [29, 26]).

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The mathematical concept of the variational inequality problem, according to Antman [6] is defined as follows: Let E be a Banach space, C a subset of E, and a functional $A : C \to E^*$, where E^* is the dual space of the space E. The variational inequality problem is the problem of finding a point x in C such that the following inequality holds:

$$\langle y - x, Ax\rangle \ \geq 0 \forall\, y \in C \tag{1.1}$$

where $\langle\cdot,\cdot\rangle : E \times E^* \to \mathbb{R}$ is the duality pairing. The solution set of (1.1) is denoted by $VI(C, A)$. Many iterative techniques have been devised by researchers to address these problems that arise from variational inequalities; among these, the subgradient extragradient method introduced by Censor et al. [9] has demonstrated impressive results which was found to be more effective than the extragradient method of Korpelevich [18]. It has become famous for its effectiveness in handling nonsmooth and nonconvex functions (see [23, 22, 16, 34, 37, 40]). The algorithm iteratively generates points as follows:

$$\begin{cases} x_0 \in H, \\ y_n = P_C(x_n - \lambda A(x_n)), \\ T_n = \{z \in H : \langle z - y_n, x_n - \lambda A(x_n) - y_n\rangle \leq 0\}, \\ x_{n+1} = P_{T_n}(x_n - \lambda A(y_n)), \qquad \forall\ \ n \geq 0. \end{cases} \tag{1.2}$$

where $\lambda \in (0, 1/L)$, C is a nonempty closed convex subset of a real Hilbert space H and L is the Lipschitz constant of A. They obtained weak convergence of the sequence $\{x_n\}$ generated by (1.2) under some suitable conditions. On the other hand, research on variational inequality problems (VIPs) in Banach spaces is extremely important. Despite the fact that Hilbert spaces are a unique kind of Banach space, there are compelling grounds for examining VIPs within the larger framework of Banach spaces. Beyond Hilbert spaces, Banach spaces offer generality and applicability that make modeling physical systems, economic phenomena, and optimization problems more flexible. In addition, the lack of orthogonality in Banach spaces forces scholars to explore geometry and orthogonality ideas in novel approaches. Applications of VIPs in Banach spaces come from a variety of disciplines, such as equilibrium problems, mechanics, and economics, see [1, 5, 12, 14, 17, 19] for further information.
The study of extragradient and subgradient methods in Banach space was recently introduced by Cai et al. [7], where E is a 2-uniformly convex Banach space as follows:

$$\begin{cases} y_n = \Pi_C J^{-1}(Jx_n - \lambda_n A(x_n)), \\ T_n = \{z \in E : \langle z - y_n, Jx_n - \lambda_n A(x_n) - Jy_n\rangle \leq 0\}, \\ w_n = \Pi_{T_n} J^{-1}(Jx_n - \lambda_n A(y_n)), \\ x_{n+1} = J^{-1}(\alpha_n Jx_1 + (1 - \alpha_n)Jw_n) \quad \ \forall\ \ n \geq 0. \end{cases} \tag{1.3}$$

The operator Π_C is the generalized projection operator and J is the normalized duality mapping, $\lambda_n \in (0, \frac{1}{L})$ and $\alpha_n \subset (0, 1)$ satisfying some conditions. They obtained strong convergence of (1.3) to the solution in $VI(A, C)$. Note that the algorithm (1.3) may fail when the estimated Lipschitz constant of the mapping A is unknown. To overcome these challenges, Ma [23] introduced a modified extragradient method for approximating solution of pseudomonotone VIP in $2-$uniformly smooth and uniform convex real Banach spaces. Under suitable condition He establish strong convergence theorem of the sequence $\{x_n\}$ generated as follows:

$$\begin{cases} y_n = \Pi_C J^{-1}(Jx_n - \lambda_n A(x_n)), \\ T_n = \{z \in E : \langle z - y_n, Jx_n - \lambda_n A(x_n) - Jy_n\rangle \leq 0\}, \\ w_n = \Pi_{T_n} J^{-1}(Jx_n - \lambda_n A(y_n)), \\ x_{n+1} = J^{-1}(\alpha_n Jx_1 + (1 - \alpha_n)Jw_n) \quad \ \forall\ \ n \geq 0. \end{cases} \tag{1.4}$$

where the step size λ_n is chosen as follows: for some $\lambda_0 > 0$, $\delta \in (0,1)$, for all $n \in \mathbb{N}$,

$$\lambda_{n+1} = \begin{cases} \min\left\{\lambda_n, \frac{\delta(||x_n-y_n||^2+||w_n-y_n||^2)}{2\langle w_n-y_n, A(x_n)-A(y_n)\rangle}\right\}, & if\ \langle w_n - y_n, A(x_n) - A(y_n)\rangle > 0, \\ \lambda_n, & otherwise \end{cases} \tag{1.5}$$

On the other hand, in solving variational inequality problems, the inclusion of inertial terms plays a pivotal role in enhancing both the stability and convergence properties of algorithms. These terms introduce momentum, which not only mitigates oscillations but also enables the algorithm to escape saddle points and local minima more effectively. The acceleration of convergence is another key benefit: by incorporating historical gradient information, inertial techniques facilitate faster progress toward optimal solutions, particularly in ill-conditioned or noisy problems. Beyond theoretical insights, the practical relevance of inertial methods is evident across diverse fields, including machine learning, image processing, and control systems, where their ability to strike a balance between exploration and exploitation proves invaluable in various spaces see for example [13, 21, 23, 16, 15, 34, 37]. Very recently, Yao et al. [40] and Thong et al. [35] introduced double inertial subgradient extragradient method with adaptive step size in real Hilbert spaces H where step size used in Thong et al. [35] was non-monotonic step size. Hence, the two results are different. The algorithm studied by Yao et al. [40] is as follows:

Step 1. Choose $\mu \in (0,1)$ and $\lambda_1 > 0$. Let $x_0, x_1 \in H$ be given starting points. Set $n = 1$

Step 2. Compute

$$\begin{cases} z_n = x_n + \delta(x_n - x_{n-1}); \\ w_n = x_n + \theta_n(x_n - x_{n-1}); \\ y_n = P_C(w_n - \lambda_n A(w_n)); \end{cases} \tag{1.6}$$

$$\lambda_{n+1} = \begin{cases} \min\left\{\lambda_n, \frac{\mu(||w_n-y_n||}{||A(w_n)-A(y_n)||}\right\}, & A(w_n) \neq A(y_n), \\ \lambda_n, & otherwise \end{cases} \tag{1.7}$$

If $x_n = w_n = y_n$, STOP. Otherwise,

Step 3. Compute

$$x_{n+1} = (1-\alpha_n)z_n + \alpha_n P_{T_n}(w_n - Ay_n),\ n \geq 1 \tag{1.8}$$

where T_n is given by

$$T_n = \{w \in H : \langle w_n - \lambda_n A w_n - y_n, w - y_n\rangle \leq 0\} \tag{1.9}$$

Step 4. Set $n \leftarrow n+1$ and go to **Step 2**.

Inspired and motivated by the findings above as well as the continued investigation into these areas. In order to solve variational inequalities in Banach spaces, we provide in this study a modified subgradient extragradient method that combines double inertial terms with the Mann-type approach. We contributed the following to this study:.

- Our result extend many results in the literature [9, 8, 30, 35, 37, 40] from Hilbert spaces to Banach spaces.
- The step size in our algorithm works without prior knowledge of the Lipschitz constant of the pseudomonotone mapping.
- To accelerate the convergence speed of the proposed algorithms, the double inertial terms are also embedded in our algorithms. Numerical experimental results demonstrate that the proposed algorithms converge faster than the methods without inertial or with single inertial term in [7, 9, 23].

- The strong convergence theorems of the proposed algorithms are proved under some suitable conditions in $2-$uniformly convex and smooth real Banach spaces.

This is the format for the remaining parts of the paper. The next section provides certain necessary definitions and technical lemmas. In the Section 3, we study the convergence of double inertial subgradient extragradient algorithms with Mann-type. Section 4 include a numerical evaluation that confirm our theoretical findings.

2. Preliminaries

In this section, we present some preliminary definitions and concepts which are needed for the establishment of the main result of this paper. Let E be a real Banach space with dual E^*, let $S_E(x) := \{x \in E : ||x|| = 1\}$ denote the unit sphere of E. We denote the value of $x^* \in E^*$ at $x \in E$ by $\langle x, x^* \rangle$. Also, we denote the strong (resp. weak) convergence of a sequence $\{x_n\} \subset E$ to a point $x \in E$ by $x_n \to x$ (resp. $x_n \rightharpoonup x$). A Banach space E is said to be *smooth* if for each $x, y \in S_E$, $\lim\limits_{t\to 0} \frac{||x+ty||-||x||}{t}$ exists. If for all $x, y \in S_E$ with $x \neq y$, for any $\lambda \in (0,1)$, $||\lambda x + (1-\lambda)y|| < 1$, then E is called *strictly convex*. The space E is said to be *uniformly convex* if for any $\epsilon \in (0,2]$ there exists $\delta = \delta(\epsilon) > 0$ such that for all $x, y \in S_E$, $||x-y|| \geq \epsilon$, we have $\frac{||x+y||}{2} \leq 1-\delta$. The *modulus of convexity* of E is the function $\delta_E : [0,2] \to [0,1]$ defined for all $\epsilon \in [0,2]$ by

$$\delta_E(\epsilon) = \begin{cases} \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| = 1, \|y\| = 1, \|x-y\| \geq \epsilon\right\}, \text{ if } 0 < \epsilon \leq 2; \\ 0 \text{ if } \epsilon = 0. \end{cases}$$

In terms of modulus of convexity, the space E is said to be uniformly convex if and only if for all $\epsilon \in (0,2]$, we have that $\delta_E(\epsilon) > 0$; and for $p \in (1, +\infty)$, the space E is said to be p-uniformly convex if and only if there exists a constant $c_p > 0$ such that $\epsilon \in (0,2]$, $\delta_E(\epsilon) \geq c_p \epsilon^p$. It is obvious that every p-uniformly convex real normed space is uniformly convex.

The *modulus of smoothness* of E is the function $\rho_E : [0,\infty) \to [0,\infty)$ for all $\tau > 0$ defined by

$$\rho_E(\tau) = \sup\left\{\frac{||x+\tau y|| + ||x-\tau y||}{2} - 1 : x, y \in S_E\right\}.$$

The space E is called *uniformly smooth* if $\lim\limits_{\tau\to 0} \frac{\rho_E(\tau)}{\tau} = 0$; and E is called *q-uniformly smooth* if there exits a positive real number C_q such that for any $\tau > 0$, $\rho_E(\tau) \leq C_q \tau^q$. Hence, every q-uniformly smooth Banach space is uniformly smooth. We know that the space L_p, ℓ_p and W_p^m for $1 \leq p < 2$ are 2-uniformly convex and uniformly smooth (see [38] for more details).
The normalized duality mapping $J : E \to 2^{E^*}$ is defined by

$$J(x) := \{f \in E^* : \langle x, f \rangle = ||f||^2 = ||x||^2\}.$$

It is known that J has the following properties (for more details see [10, 28, 33]):

(a) If E is smooth, then J is single-valued.
(b) If E is uniformly smooth, then J is norm to norm uniformly continuous on bounded subset of E.
(c) If E is uniformly smooth, then the dual space E^* is uniformly convex; and if E is and uniformly convex, then the dual space E^* is uniformly smooth. Furthermore, J and J^{-1} are both uniformly continuous on bounded subsets of E and E^*, respectively.
(d) If E is a reflexive, strictly convex and smooth Banach space, then J^{-1} (the duality mapping from E^* into E) is single-valued, one to one and onto.

Let $\phi : E \times E \to [0, \infty)$ denote the Lyapunov functional in sense of Alber [2] defined $\forall\, x, y \in E$ by

$$\phi(x, y) = ||x||^2 - 2\langle x, Jy\rangle + ||y||^2. \tag{2.1}$$

The functional ϕ satisfies the following properties (see [27]): $\forall\, x, y, z \in E$

(P1) $(||x|| - ||y||)^2 \leq \phi(x, y) \leq (||x|| + ||y||)^2$,
(P2) $\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy\rangle$,
(P3) $\phi(x, y) = \langle x, Jx - Jy\rangle + \langle y - x, Jy\rangle \leq ||x||||Jx - Jy|| + ||y - x||||y||$,;
(P4) $\phi(z, J^{-1}(\alpha Jx + (1 - \alpha)Jy)) \leq \alpha\phi(z, x) + (1 - \alpha)\phi(z, y)$, where $\alpha \in (0, 1)$ and $x, y \in E$.

Now, we introduce another functional $V : E \times E^* \to [0, \infty)$ by [2], which is a mild modification and have a relationship with Lyapunov functional in (2.1) as follows: for all $x \in E$ and $x^* \in E^*$

$$V(x, x^*) = ||x||^2 - 2\langle x, x^*\rangle + ||x^*||^2 \tag{2.2}$$

From the Definition of ϕ in (2.1), we get for all $x \in E$ and $x^* \in E^*$

$$V(x, x^*) = \phi(x, J^{-1}(x^*)). \tag{2.3}$$

For each $x \in E$, the mapping g defined by $g(x^*) = V(x, x^*)$ for all $x^* \in E^*$ is a continuous, convex function from E^* into $\mathbb{R}$.
Let E a be reflexive, strictly convex and smooth Banach space and C a nonempty closed and convex subset of E. Then by [2], for each $x \in E$, there exists a unique element $u \in C$ (denoted by $\Pi_C x$) such that

$$\phi(u, x) = \min_{y \in C} \phi(y, x).$$

The mapping $\Pi_C : E \to C$, defined by $\Pi_C x = u$ is called the generalized projection operator (see [3]), which have the following important characteristic.

Lemma 2.1. [4] *Let C be a nonempty, closed and convex subset of a smooth Banach space E, then $u = \Pi_C x$ if and only if*

$$\langle u - w, Jx - Ju\rangle \geq 0,\ \forall w \in C.$$

Lemma 2.2. [25] Let E be a reflexive, strictly convex and smooth Banach space and C be a nonempty closed and convex subset of E. Then

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x),\ \forall\, y \in C.$$

Lemma 2.3. [2] *Let E be a reflexive, strictly convex and smooth Banach space and let V be as in* (2.2). *Then, for all $x \in E$ and $x^*, y^* \in E^*$*

$$V(x, x^*) + 2\langle J^{-1}x^* - x, y^*\rangle \leq V(x, x^* + y^*).$$

Lemma 2.4. [27] *Let E be a uniformly smooth real Banach space and $r > 0$. Then there exists a continuous, strictly increasing and convex function $g : [0, 2r] \to [0, \infty]$ such that $g(0) = 0$ and*

$$\phi(u, J^{-1}(tJv + (1 - t)Jw)) \leq t\phi(u, v) + (1 - t)\phi(u, w) - t(1 - t)g(||Jv - Jw||)$$

for all $t \in [0, 1]$, $u \in E$ and $v, w \in B_r := \{z \in E : ||z|| \leq r\}$.

Lemma 2.5. [33] *Let C be a nonempty, closed and convex subset of X and $F : C \to X^*$ be monotone and continuous mapping. For any $y \in C$, we have*

$$y \in VI(C, F)\ \Leftrightarrow\ \langle F(z), z - y\rangle \geq 0\, \forall\, z \in C.$$

Lemma 2.6. [25] *Let E be a uniformly convex and smooth real Banach space and $\{u_n\}$ and $\{v_n\}$ be two sequences in E. If $\lim_{n\to\infty} \phi(u_n, v_n) = 0$ and either $\{u_n\}$ or $\{v_n\}$ is bounded, then $\lim_{n\to\infty} ||u_n - v_n|| = 0$.*

Lemma 2.7. [38] *Let E be a 2-uniformly convex and smooth real Banach space. Then, there exists a positive real-valued constant α such that*

$$\alpha||x-y||^2 \leq \phi(x,y),\ \forall\, x,y \in E.$$

Lemma 2.8. [38] *Let E be a 2-uniformly smooth real Banach space, then there exists $s_0 > 0$ such that for all $x, y \in E$, the following holds*

$$||x+y||^2 \leq ||x||^2 + 2\langle x, Jy\rangle + 2s_0^2||y||^2.$$

Lemma 2.9. [39] *If $\{a_n\}$ is a sequence of nonnegative real numbers satisfying the following inequality:*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\sigma_n + \gamma_n, n \geq 0,$$

where, (i) $\{\alpha_n\} \subset [0,1]$, $\sum \alpha_n = \infty$; (ii) $\lim sup\, \sigma_n \leq 0$; (iii) $\gamma_n \geq 0$; $(n \geq 0)$ and $\sum \gamma_n < \infty$. Then, $a_n \to 0$ as $n \to \infty$.

Lemma 2.10. [24] *Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{a_{n_i}\}$ of $\{a_n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$.*

$$a_{m_k} \leq a_{m_k+1} \text{ and } a_k \leq a_{m_k+1}.$$

In fact, $m_k = \max\{j \leq k : a_j < a_{j+1}\}$.

3. Main Results

In this section, we introduce the subgradient extragradient method involving double inertial extrapolation terms with modified Mann for solving the pseudomonotone variational inequality problem. First, we make the following assumptions:

Assumption 3.1.

- (A1) *C is a nonempty closed and convex subset of 2-uniformly convex and smooth real Banach space E with dual E^*.*
- (A2) *$A : C \to E^*$ is a pseudomonotone and L-Lipschitz continuous mapping with $L > 0$.*
- (A3) *The mapping A is weakly sequentially continuous, i.e., for each sequence $\{x_n\} \subset C$, we have $A(x_n) \rightharpoonup A(x)$ whenever $x_n \rightharpoonup x$ as $n \to \infty$.*
- (A4) *$\{\alpha_n\} \subset (0,1)$ is a sequence with $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$; $\{\tau_n\}$ a positive sequence in $(0, \alpha/2)$ such that $\tau_n = \circ(\alpha_n)$, where α is defined in Lemma 2.7 and $\alpha > \mu$ for some $\mu \in (0,1)$.*
- (A5) *J is a normalized duality mapping on E and $V(C,A)$ is nonempty set.*

The following lemmas are very helpful in analyzing the convergence of our method.

Lemma 3.2. [22] *Assume that (A1)–(A5) holds, then the step size (3.2) is well defined. In addition, we have $\lambda_n \leq \gamma$.*

Lemma 3.3. [22] *Suppose that Assumption (A1)–(A3) holds. Let $\{w_n\}$ and $\{y_n\}$ be two sequences generated by Algorithm 3. If there exists a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ such that $\{w_{n_k}\}$ converges weakly to $z \in E$ and $\lim\limits_{k\to\infty} ||w_{n_k} - y_{n_k}|| = 0$, then $z \in VI(C,A)$.*

Next, we investigate the boundedness of our method.

Lemma 3.4. *Suppose that Assumption 3.1 (A1)–(A5) holds. Let $\{y_n\}$, $\{y_n\}$ and $\{w_n\}$ be sequences generated by Algorithm 3. Then, for all $a^* \in VI(C,A)$*

$$\phi(a^*, y_n) \leq \phi(a^*, w_n) - \Big(1 - \frac{\mu\lambda_n}{\alpha\lambda_{n+1}}\Big)\Big(\phi(u_n, y_n) + \phi(w_n, y_n)\Big). \tag{3.4}$$

Initialization: Choose $x_0, x_1 \in C$ to be arbitrary.
Iterative Steps: Calculate x_{n+1} as follows:

Step 1. Given the iterates x_{n-1} and x_n for each $n \geq 1$, choose δ_n, θ_n in $(0,1)$. Let $x_0, x_1 \in E$ be given starting points. Set $n = 1$

Step 2. Compute

$$\begin{cases} z_n = J^{-1}[J(x_n) + \delta_n(J(x_n) - J(x_{n-1}))], \\ w_n = J^{-1}[J(x_n) + \theta_n(J(x_n) - J(x_{n-1}))], \\ y_n = \Pi_C[J^{-1}(J(w_n) - \lambda_n A(w_n))], \end{cases} \tag{3.1}$$

where the step size λ_n is chosen as follows: for some $\lambda_0 > 0$, for all $n \in \mathbb{N}$,

$$\lambda_{n+1} = \begin{cases} \min\left\{\lambda_n, \frac{\mu(||w_n - y_n||^2 + ||y_n - u_n||^2)}{2\langle y_n - u_n, A(w_n) - A(y_n)\rangle}\right\}, & if \;\; \langle y_n - u_n, A(w_n) - A(y_n)\rangle > 0, \\ \\ \lambda_n, \quad otherwise \end{cases} \tag{3.2}$$

If $x_n = w_n = y_n$, then stop for x_n is a solution. Otherwise,

Step 3. Compute

$$\begin{cases} u_n = \Pi_{T_n}(J^{-1}(Jw_n - \lambda_n Ay_n)), \\ \Pi_{T_n} = \{w \in E : \langle w - y_n, Jw_n - \lambda_n Aw_n - Jy_n\rangle \leq 0\}, \\ v_n = J^{-1}((1-\alpha_n)J(u_n)), \\ x_{n+1} = J^{-1}((1-\beta_n)J(z_n) + \beta_n J(v_n)), \; n \in \mathbb{N}. \end{cases} \tag{3.3}$$

Set $n := n+1$ and return to **Step 1**.

Proof. Let $a^* \in VI(C,A)$, using Lemma 2.2, (2.2) and (2.3), we obtain

$$\begin{aligned} \phi(a^*, u_n) &\leq \phi(a^*, J^{-1}(Jw_n - \lambda_n Ay_n)) - \phi(u_n, J^{-1}(Jw_n - \lambda_n Ay_n)) \\ &= V(a^*, (Jw_n - \lambda_n Ay_n)) - V(u_n, (Jw_n - \lambda_n Ay_n)) \\ &= ||a^*||^2 - 2\langle a^*, Jw_n - \lambda_n Ay_n\rangle + ||Jw_n - \lambda_n Ay_n||^2 \\ &\quad -||u_n||^2 + 2\langle u_n, Jw_n - \lambda_n Ay_n\rangle - ||Jw_n - \lambda_n Ay_n||^2 \\ &= ||a^*||^2 - 2\langle a^*, Jw_n\rangle + 2\lambda_n\langle a^*, Ay_n\rangle \\ &\quad -||u_n||^2 + 2\langle u_n, Jw_n\rangle - 2\lambda_n\langle u_n, Ay_n\rangle \\ &= \phi(a^*, w_n) - \phi(u_n, w_n) + 2\lambda_n\langle a^* - u_n, Ay_n\rangle. \end{aligned} \tag{3.5}$$

Since $y_n = \Pi_C(J^{-1}(Jw_n - \lambda_n Aw_n))$ is in C and $a^* \in VI(C,A)$, then by definition of VIP, we get $\langle y_n - a^*, A(a^*)\rangle \geq 0$ it follows by pseudomonotonicity of A that

$$\langle y_n - a^*, A(y_n)\rangle \geq 0 \tag{3.6}$$

Thus

$$\begin{aligned} \langle a^* - u_n, A(y_n)\rangle &= \langle a^* - y_n, A(y_n)\rangle + \langle y_n - u_n, A(y_n)\rangle \\ &\leq \langle y_n - u_n, A(y_n)\rangle. \end{aligned} \tag{3.7}$$

Furthermore, by definition of T_n, we have

$$\langle u_n - y_n, Jw_n - \lambda_n Aw_n - Jy_n\rangle \leq 0$$

which implies

$$\begin{aligned} \langle u_n - y_n, Jw_n - \lambda_n Ay_n - Jy_n\rangle &= \langle u_n - y_n, Jw_n - \lambda_n Aw_n - Jy_n\rangle \\ &\quad +\lambda_n\langle u_n - y_n, Aw_n - Ay_n\rangle \\ &\leq \lambda_n\langle u_n - y_n, Aw_n - Ay_n\rangle. \end{aligned} \tag{3.8}$$

Next, combining (3.6), (3.7), (3.8), (P2), (3.2) and Lemma 2.7, we get

$$\begin{aligned}
\phi(a^*, u_n) &\leq \phi(a^*, w_n) - \phi(u_n, w_n) + 2\lambda_n\langle u_n - y_n, Ay_n\rangle \\
&= \phi(a^*, w_n) - \phi(u_n, y_n) - \phi(y_n, w_n) + \langle u_n - y_n, Jw_n - Jy_n\rangle \\
&\quad -2\lambda_n\langle u_n - y_n, Ay_n\rangle \\
&= \phi(a^*, w_n) - \phi(u_n, y_n) - \phi(y_n, w_n) + 2\lambda_n\langle u_n - y_n, Jw_n - Ay_n - Jy_n\rangle \\
&= \phi(a^*, w_n) - \phi(u_n, y_n) - \phi(y_n, w_n) + 2\lambda_n\langle u_n - y_n, Aw_n - Ay_n\rangle \\
&\leq \phi(a^*, w_n) - \phi(u_n, y_n) - \phi(y_n, w_n) + \frac{\mu\lambda_n}{\lambda_{n+1}}\Big(||u_n - y_n||^2 + ||w_n - y_n||^2\Big) \\
&\leq \phi(a^*, w_n) - \Big(1 - \frac{\mu\lambda_n}{\alpha\lambda_{n+1}}\Big)\Big(\phi(u_n, y_n) + \phi(w_n, y_n)\Big)
\end{aligned}$$

□

Lemma 3.5. *Suppose that in Assumption 3.1, (A1) - (A5) holds and for $\theta_0, \delta_0 >, \theta_n, \delta_n \in (0,1)$ are chosen such that*

$$\theta_n \leq \bar{\theta}_n = \begin{cases} \min\Big\{\theta_0, \frac{\tau_n}{||Jx_n - Jx_{n-1}||}\Big\}, & if \;\; x_n \neq x_{n-1}, \\ \theta_0, & otherwise \end{cases} \tag{3.9}$$

and

$$\delta_n \leq \bar{\delta}_n = \begin{cases} \min\Big\{\delta_0, \frac{\tau_n}{||Jx_n - Jx_{n-1}||}\Big\}, & if \;\; x_n \neq x_{n-1}, \\ \delta_0, & otherwise \end{cases} \tag{3.10}$$

Then the sequence $\{x_n\}$ generated by Algorithm 3 is bounded.

Proof. Let $a^* \in VI(C, A)$, using (P2), (3.1), (3.9) and Lemma 2.7, we compute as follows

$$\begin{aligned}
\phi(a^*, w_n) &= \phi(a^*, x_n) - \phi(w_n, x_n) + 2\langle w_n - a^*, Jw_n - Jx_n\rangle \\
&\leq \phi(a^*, x_n) - \phi(w_n, x_n) + 2||w_n - a^*||||Jw_n - Jx_n|| \\
&\leq \phi(a^*, x_n) - \phi(w_n, x_n) + 2\theta_n||a^* - w_n||||Jx_n - Jx_{n-1}|| \\
&\leq \phi(a^*, x_n) - \phi(w_n, x_n) + \theta_n[||a^* - w_n||^2 + 1]||Jx_n - Jx_{n-1}|| \\
&\leq \phi(a^*, x_n) - \phi(w_n, x_n) + 2\theta_n\Big(||a^* - x_n||^2 + ||x_n - w_n||^2\Big)||Jx_n - Jx_{n-1}|| \\
&\quad +\theta_n||Jx_n - Jx_{n-1}|| \\
&\leq \phi(a^*, x_n) - \phi(w_n, x_n) + 2\tau_n\Big(||a^* - x_n||^2 + ||w_n - x_n||^2\Big) + \tau_n \\
&\leq \phi(a^*, x_n) - \phi(w_n, x_n) + \frac{2\tau_n}{\alpha}\Big(\phi(a^*, x_n) + \phi(w_n, x_n)\Big) + \tau_n \\
&= \Big(1 + \frac{2\tau_n}{\alpha}\Big)\phi(a^*, x_n) - \Big(1 - \frac{2\tau_n}{\alpha}\Big)\phi(w_n, x_n) + \tau_n.
\end{aligned} \tag{3.11}$$

Using the same process of argument of arriving at (3.11), we get that

$$\phi(a^*, z_n) \leq \Big(1 + \frac{2\tau_n}{\alpha}\Big)\phi(a^*, x_n) - \Big(1 - \frac{2\tau_n}{\alpha}\Big)\phi(z_n, x_n) + \tau_n. \tag{3.12}$$

Next from (3.3), with Lemma 2.4, and using (3.4), (3.11) and (3.12), by letting $\sigma \in E$ to be a zero point, we obtain

$$\begin{aligned}
\phi(a^*, x_{n+1}) &= \phi(a^*, J^{-1}((1-\beta_n)Jz_n + \beta_n Jv_n) \\
&\leq (1-\beta_n)\phi(a^*, z_n) + \beta_n\phi(a^*, v_n) - (1-\beta_n)\beta_n g(||Jz_n - Jv_n||) \\
&= (1-\beta_n)\phi(a^*, z_n) + \beta_n\phi(a^*, J^{-1}(1-\alpha_n)Ju_n) - (1-\beta_n)\beta_n g(||Jz_n - Jv_n||) \\
&\leq (1-\beta_n)\phi(a^*, z_n) + \beta_n(1-\alpha_n)\phi(a^*, u_n) + \beta_n\alpha_n\phi(a^*, \sigma) \\
&\quad -(1-\beta_n)\beta_n g(||Jz_n - Jv_n||) \qquad (3.13) \\
&\leq (1-\beta_n)\Big[\Big(1+\frac{2\tau_n}{\alpha}\Big)\phi(a^*, x_n) - \Big(1-\frac{2\tau_n}{\alpha}\Big)\phi(z_n, x_n) + \tau_n\Big] \\
&\quad +\beta_n(1-\alpha_n)\Big[\Big(1+\frac{2\tau_n}{\alpha}\Big)\phi(a^*, x_n) - \Big(1-\frac{2\tau_n}{\alpha}\Big)\phi(w_n, x_n) + \tau_n \\
&\quad -\Big(1-\frac{\mu\lambda_n}{\alpha\lambda_{n+1}}\Big)\Big(\phi(u_n, y_n) + \phi(w_n, y_n)\Big)\Big] + \beta_n\alpha_n\phi(a^*, \sigma) \\
&\quad -(1-\beta_n)\beta_n g(||Jz_n - Jv_n||) \qquad (3.14) \\
&= \Big[(1-\alpha_n\beta_n) + (1-\alpha_n\beta_n)\frac{2\tau_n}{\alpha}\Big]\phi(a^*, x_n) + (1-\alpha_n\beta_n)\tau_n + \beta_n\alpha_n\phi(a^*, \sigma) \\
&\quad -\Big(1-\frac{\mu\lambda_n}{\alpha\lambda_{n+1}}\Big)\beta_n(1-\alpha_n)\Big(\phi(u_n, y_n) + \phi(w_n, y_n)\Big) - (1-\beta_n)\beta_n g(||Jz_n - Jv_n||) \\
&\quad -\Big(1-\frac{2\tau_n}{\alpha}\Big)\Big[(1-\beta_n)\phi(z_n, x_n) + \beta_n(1-\alpha_n)\phi(w_n, x_n)\Big] \\
&\leq \Big[1-\alpha_n\beta_n + \frac{2\tau_n}{\alpha}\Big]\phi(a^*, x_n) + \tau_n + \beta_n\alpha_n\phi(a^*, \sigma) \\
&\quad -\Big(1-\frac{\mu\lambda_n}{\alpha\lambda_{n+1}}\Big)\beta_n(1-\alpha_n)\Big(\phi(u_n, y_n) + \phi(w_n, y_n)\Big) - (1-\beta_n)\beta_n g(||Jz_n - Jv_n||) \\
&\quad -\Big(1-\frac{2\tau_n}{\alpha}\Big)\Big[(1-\beta_n)\phi(z_n, x_n) + \beta_n(1-\alpha_n)\phi(w_n, x_n)\Big]. \qquad (3.15)
\end{aligned}$$

Furthermore, for any $\tau_0 \in \Big(0, \frac{\alpha}{2}\Big)$, there exists a natural number N_0 such that for all $n \geq N_0$, we have $\frac{2\tau_n}{\alpha} < \alpha_n\tau_0$. Also, since $\lim_{n\to\infty}\lambda_n$ exists, then $\lim_{n\to\infty}\frac{\lambda_n}{\lambda_{n+1}} = 1$, which implies that $\lim_{n\to\infty}\Big(1-\frac{\mu\lambda_n}{\alpha\lambda_{n+1}}\Big) = 1-\frac{\mu}{\alpha} > 0$ (since $\alpha > \mu$), thus there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, we get $\lim_{n\to\infty}\Big(1-\frac{\mu\lambda_n}{\alpha\lambda_{n+1}}\Big) > 0$. Hence, for all $n \geq \max\{N_0, N_1\} := N$, from (3.13), we obtain

$$\begin{aligned}
\phi(a^*, x_{n+1}) &\leq \Big[1-\alpha_n\beta_n\Big(1-\frac{\tau_0}{\beta_1}\Big)\Big]\phi(a^*, x_n) + \alpha_n\beta_n\frac{\tau_0}{\beta_1} + \beta_n\alpha_n\phi(a^*, \sigma) \\
&\quad -\Big(1-\frac{\mu}{\alpha}\Big)\beta_n(1-\alpha_n)\Big(\phi(u_n, y_n) + \phi(w_n, y_n)\Big) - (1-\beta_n)\beta_n g(||Jz_n - Jv_n||) \\
&\quad -\Big(1-\alpha_n\tau_0\Big)\Big[(1-\beta_n)\phi(z_n, x_n) + \beta_n(1-\alpha_n)\phi(w_n, x_n)\Big] \qquad (3.16) \\
&\leq \Big[1-\alpha_n\beta_n\Big(1-\frac{\tau_0}{\beta_1}\Big)\Big]\phi(a^*, x_n) + \alpha_n\beta_n\Big(\frac{\tau_0}{\beta_1} + \phi(a^*, \sigma)\Big) \\
&\leq \max\Big\{\phi(a^*, x_n), \frac{(\tau_0/\beta_n + \phi(a^*, \sigma))}{(1-\tau_0/\beta_1)}\Big\} \\
&\leq \vdots \\
&\leq \max\Big\{\phi(a^*, x_N), \frac{(\tau_0/\beta_n + \phi(a^*, \sigma))}{(1-\tau_0/\beta_1)}\Big\}.
\end{aligned}$$

Therefore, by induction, we get

$$\phi(a^*,x_n) \leq \max\left\{\phi(a^*,x_N), \frac{(\tau_0/\beta_n + \phi(a^*,\sigma))}{(1-\tau_0/\beta_1)}\right\} \forall\, n \geq N.$$

Hence $\phi(a^*,x_n)$ is bounded and by Lemma 2.7, we get that $\{x_n\}$ is bounded, so $\{w_n\}$, $\{z_n\}$, $\{y_n\}$, $\{u_n\}$ and $\{v_n\}$ are also bounded. □

We now state and prove the following theorem:

Theorem 3.6. *Suppose conditions in Assumption 3.1 are satisfied. Let $\{x_n\}_{n=1}^{\infty}$ be sequence generated by Algorithm 3, then $\{x_n\}_{n=1}^{\infty}$ converges strongly to some point in $VI(C,A)$.*

Proof. Let $a^* \in VI(C,A)$, where $a^* := \Pi_{VI(C,A)}\sigma$ and σ is the zero point in E. Then, we will divide the proof into two cases.

Case 1. If the sequence $\{\phi(a^*,x_n)\}$ is non-increasing, then by the boundedness of $\{\phi(a^*,x_n)\}$, we obtain that limit of $\phi(a^*,x_n)$ exists, which follows by uniqueness of limit of real sequence that $\lim_{n\to\infty}\Big(\phi(a^*,x_n)-\phi(a^*,x_{n+1})\Big)=0$. Thus, from (3.16), we get

$$\begin{aligned}
0 \;&\leq\; \Big(1-\frac{\mu}{\alpha}\Big)\beta_n(1-\alpha_n)\Big(\phi(u_n,y_n)+\phi(w_n,y_n)\Big)+(1-\beta_n)\beta_n g(||Jz_n-Jv_n||)\\
&\quad +\Big(1-\alpha_n\tau_0\Big)\Big[(1-\beta_n)\phi(z_n,x_n)+\beta_n(1-\alpha_n)\phi(w_n,x_n)\Big]\\
&\leq\; \alpha_n\beta_n\left[\Big(\frac{\tau_0}{\beta_1}-1\Big)\phi(a^*,x_n)+\frac{\tau_0}{\beta_1}+\phi(a^*,\sigma)\right]+\phi(a^*,x_n)-\phi(a^*,x_{n+1}) \qquad (3.17)
\end{aligned}$$

Taking limit on both sided of (3.16), we obtain

$$\lim_{n\to\infty}\Big(\Big(1-\frac{\mu}{\alpha}\Big)\beta_n(1-\alpha_n)\Big(\phi(u_n,y_n)+\phi(w_n,y_n)\Big)+(1-\beta_n)\beta_n g(||Jz_n-Jv_n||)$$
$$+\Big(1-\alpha_n\tau_0\Big)\Big[(1-\beta_n)\phi(z_n,x_n)+\beta_n(1-\alpha_n)\phi(w_n,x_n)\Big]\Big)=0.$$

Thus

$$\begin{aligned}
\lim_{n\to\infty}\phi(w_n,x_n) &= \lim_{n\to\infty}\phi(z_n,x_n)=\lim_{n\to\infty}\phi(w_n,y_n)\\
&= \lim_{n\to\infty}\phi(u_n,y_n)=\lim_{n\to\infty}g(||Jz_n-Jv_n||)=0. \qquad (3.18)
\end{aligned}$$

Using the property of g in Lemma 2.4 and by Lemma 2.6, we get from (3.18) that

$$\begin{aligned}
\lim_{n\to\infty}||w_n-x_n|| &= \lim_{n\to\infty}||z_n-x_n||=\lim_{n\to\infty}||w_n-y_n||\\
&= \lim_{n\to\infty}||u_n-y_n||=\lim_{n\to\infty}||Jz_n-Jv_n||=0. \qquad (3.19)
\end{aligned}$$

Also from the definition of (v_n), we obtain that

$$||Jv_n-Ju_n||=\alpha_n||Ju_n||\to 0 \text{ as } \to\infty \qquad (3.20)$$

and using (3.19), we get

$$||Jx_{n+1}-Jv_n||=(1-\beta_n)||Jv_n-z_n||\to 0 \qquad (3.21)$$

as $n\to\infty$. Since J and J^{-1} are norm-to-norm uniformly continuous on bounded sets, then from (3.19), (3.20) and (3.21), we respectively obtain

$$\lim_{n\to\infty}||Jz_n-Jx_n||=\lim_{n\to\infty}||v_n-u_n||=\lim_{n\to\infty}||x_{n+1}-v_n||=0 \qquad (3.22)$$

thus, combining the last two limits in (3.22), we get

$$\lim_{n\to\infty}||x_{n+1}-u_n||=0. \qquad (3.23)$$

We know that

$$\begin{aligned} ||Jx_{n+1} - Jx_n|| &\leq ||Jx_{n+1} - Jz_n|| + ||Jz_z - Jv_n|| \\ &= \beta_n ||Jv_n - Jz_n|| + ||Jz_z - Jx_n|| \end{aligned}$$

it follows from (3.19), (3.22) and norm-to-norm uniformly continuity of J^{-1} on bounded sets that

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0. \tag{3.24}$$

Furthermore, since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup p \in E$. From (3.19), we get $w_{n_k} \rightharpoonup p$, and $\lim_{k\to\infty} ||w_{n_k} - y_{n_k}|| = 0$, then by Lemma 3.3 that $p \in VI(C, A)$. For any $z^* \in VI(C, A)$ which means $z^* = P_{VI(C,A)}\sigma$, where σ is a zero point(or vector) in E, then since $p \in VI(C, A)$ from Lemma 2.1, we get

$$\begin{aligned} \limsup_{n\to\infty} \langle x_n - z^*, -Jz^* \rangle &= \lim_{k\to\infty} \langle x_{n_k} - z^*, -Jz^* \rangle \\ &= \langle p - z^*, -Jz^* \rangle = \langle p - z^*, \sigma - Jz^* \rangle \leq 0 \end{aligned} \tag{3.25}$$

Since

$$\langle u_n - z^*, -Jz^* \rangle = \langle u_n - x_{n+1}, -Jz^* \rangle + \langle x_{n+1} - x_n, -Jz^* \rangle + \langle x_n - z^*, -Jz^* \rangle$$

it follows from (3.23), (3.24) and (3.25) that

$$\limsup_{n\to\infty} \langle u_n - z^*, -Jz^* \rangle \leq 0 \tag{3.26}$$

Finally, we show that $\{x_n\}$ converges strongly to $z^* \in VI(C, A)$, from (P4), (2.3), Lemma 2.3, (3.11) and (3.12), we get

$$\begin{aligned} \phi(z^*, x_{n+1}) &= \phi(z^*, J^{-1}((1-\beta_n)Jz_n + \beta_n Jv_n)) \\ &\leq (1-\beta_n)\phi(z^*, z_n) + \beta_n \phi(z^*, v_n) \\ &= (1-\beta_n)\phi(z^*, z_n) + \beta_n V(z^*, (1-\alpha_n)Ju_n) \\ &\leq (1-\beta_n)\phi(z^*, z_n) + \beta_n [V(z^*, (1-\alpha_n)Ju_n + \alpha_n Jz^*) + 2\alpha_n \langle u_n - z^*, -Jz^* \rangle] \\ &\leq (1-\beta_n)\phi(z^*, z_n) + \beta_n(1-\alpha_n)\phi(z^*, u_n) + 2\alpha_n\beta_n \langle u_n - z^*, -Jz^* \rangle \\ &\leq (1-\beta_n)\Big[\Big(1 + \frac{2\tau_n}{\alpha}\Big)\phi(z^*, x_n) + \tau_n\Big] + \beta_n(1-\alpha_n)\Big[\Big(1 + \frac{2\tau_n}{\alpha}\Big)\phi(z^*, x_n) + \tau_n\Big] \\ &\quad + 2\alpha_n\beta_n \langle u_n - z^*, -Jz^* \rangle \\ &\leq \Big(1 - \alpha_n\beta_n(1 - \tau_0/\beta_1)\Big)\phi(z^*, x_n) + \alpha_n\beta_n\Big[2\langle u_n - z^*, -Jz^* \rangle + \frac{\tau_n}{\alpha_n\beta_1}\Big] \end{aligned}$$

Thus

$$\begin{aligned} \phi(z^*, x_{n+1}) &\leq \Big(1 - \alpha_n\beta_n(1 - \tau_0/\beta_1)\Big)\phi(z^*, x_n) \\ &\quad + \alpha_n\beta_n(1 - \tau_0/\beta_1)\Big[\frac{2\langle u_n - z^*, -Jz^* \rangle + \frac{\tau_n}{\alpha_n\beta_1}}{(1 - \tau_0/\beta_1)}\Big] \end{aligned} \tag{3.27}$$

Hence, it follows from (3.27), (3.26) and Lemma 2.9 that $\phi(z^*, x_n)$ converges strongly to 0 and by Lemma 2.7, we get that $\{x_n\}$ converges strongly to $z^* \in VI(C, A)$.

Case 2. Assume that $\{\phi(a^*, x_n)\}_{n=1}^{\infty}$ is non-decreasing sequence of real numbers. As in Lemma 2.10, set $\Psi_n := \phi(a^*, x_n)$ and let $r : \mathbb{N} \to \mathbb{N}$ be a mapping for all $n \geq n_0$ (for some n_0 large enough), defined by

$$r(n) := \max\{k \in \mathbb{N} : k \leq n, \Psi_k \leq \Psi_{k+1}\}.$$

Then, r is a non-decreasing sequence such that $r(n) \to \infty$ as $n \to \infty$. Thus

$$0 \leq \Psi_{r(n)} \leq \Psi_{r(n)+1}, \; \forall \, n \geq n_0$$

which means that $\phi(a^*, x_{r(n)}) \leq \phi(a^*, x_{r(n)+1})$, for all $n \geq n_0$. Since $\{\phi(a^*, x_{r(n)})\}$ is bounded, therefore $\lim_{n\to\infty} \phi(a^*, x_{r(n)})$ exists. Thus following the same line of action as in Case 1, we can show that the following hold:

$$\lim_{n\to\infty} ||w_{r(n)} - x_{r(n)}|| = \lim_{n\to\infty} ||w_{r(n)} - y_{r(n)}|| = 0$$

and

$$\lim_{n\to\infty} ||x_{r(n+1)} - u_{r(n)}|| = \lim_{n\to\infty} ||x_{r(n)+1} - x_{r(r)}|| = 0.$$

Since $\{x_{r(n)}\}$ is bounded, there exists a subsequence of $\{x_{r(n)}\}$, still denoted by $\{x_{r(n)}\}$ such that $x_{r(n)}$ converges weakly to $p \in E$ as $n \to \infty$. By an argument similar to that in Case 1, we can show that $p \in VI(C, A)$ and for any $z^* \in VI(C, A)$ following similar method of logic in Case 1, we get

$$\lim_{n\to\infty} \langle u_{r(n)} - z^*, -Jz^* \rangle \leq 0. \tag{3.28}$$

Also, by (3.27), and $\Psi_{r(n)} \leq \Psi_{r(n)+1}$, we get

$$\begin{aligned} \phi(z^*, x_{r(n)}) &\leq \phi(z^*, x_{r(n)+1}) \\ &\leq \Big(1 - \alpha_{r(n)}\beta_{r(n)}(1 - \tau_0/\beta_1)\Big)\phi(z^*, x_{r(n)}) \\ &\quad + \alpha_{r(n)}\beta_{r(n)}(1 - \tau_0/\beta_1)\Big[\frac{2\langle u_{r(n)} - z^*, -Jz^* \rangle + \frac{\tau_{r(n)}}{\alpha_{r(n)}\beta_1}}{(1 - \tau_0/\beta_1)}\Big] \end{aligned}$$

Therefore, since $(1 - \alpha_{r(n)}\beta_{r(n)}(1 - \tau_0/\beta_1)) > 0$ for all $n \in \mathbb{N}$, then

$$\phi(z^*, x_{r(n)}) \leq \frac{2\langle u_{r(n)} - z^*, -Jz^* \rangle + \frac{\tau_{r(n)}}{\alpha_{r(n)}\beta_1}}{(1 - \tau_0/\beta_1)}. \tag{3.29}$$

which implies by (3.28)

$$\limsup_{n\to\infty} \phi(z^*, x_{r(n)}) \leq 0.$$

Thus

$$\lim_{n\to\infty} \phi(z^*, x_{r(n)}) = 0.$$

Thus

$$\lim_{n\to\infty} \phi(z^*, x_{r(n)+1}) = 0.$$

Therefore

$$\lim_{n\to\infty} \Psi_{r(n)} = \lim_{n\to\infty} \Psi_{r(n)+1} = 0.$$

For all $n \geq n_0$, we have that $\Psi_{r(n)} \leq \Psi_{r(n)+1}$ if $n \neq r(n)$ (that is, $r(n) < n$), because $\Psi_{k+1} \leq \Psi_k$ for $r(n) \leq k \leq n$. As a consequence, we get for all $n \geq n_0$

$$0 \leq \Psi_n \leq \max\{\Psi_{r(n)}, \Psi_{r(n)+1}\} = \Psi_{r(n)+1}.$$

So $\lim_{n\to\infty} \Psi_n = 0$ gives that $\lim_{n\to\infty} \phi(z^*, x_n) = 0$, which implies that $\lim_{n\to\infty} ||z^* - x_n|| = 0$. Thus $x_n \to z^*$ in $VI(C, A)$. ∎

4. Numerical Example

In this section, we provide a numerical experiment to demonstrate the advantages of the suggested method and compare it with some known convergent algorithms, including the algorithm (1.2) introduced by Censor et al. [9] (shortly, Algorithm (1.2)), Algorithms presented by Cai [7](shortly, Algorithm (1.3)) and Ma [23](shortly, Algorithm (1.4)). All the programs are implemented in MATLAB R2023b on a personal computer.

Example 4.1. Let $E = L_2[0,1]$ and $C = \{x \in L_2[0,1] : \langle a, x\rangle \leq b\}$, where $a = t^2 + 1$ and $b = 1$, with norm $||x|| = \sqrt{\int_0^1 |x(t)|^2 dt}$ and inner product $\langle x, y\rangle = \int_0^t x(t)y(t)dt$, for all $x, y \in L_2([0,1])$, $t \in [0,1]$. Define metric projection P_C as follows:

$$P_C(x) = \begin{cases} x, & \text{if } x \in C \\ \frac{b-\langle a,x\rangle}{||a||_{L_2}} a + x, & \text{otherwise.} \end{cases} \tag{4.1}$$

Let $A : L_2[0,1] \to L_2[0,1]$ be defined by $A(x(t)) = e^{-||x||} \int_0^t x(s)ds$, for all $x \in L_2[0,1]$, $t, s \in [0,1]$, then, A is pseudomonotone and uniformly continuous mapping (see [36]) and let $T(x(t)) = \int_0^t x(s)ds$, for all $x \in L_2[0,1]$, $t \in [0,1]$, then T is nonexpansive mapping. For the control parameters, we use $\alpha_n = \frac{1}{5n+2}$, $\delta_n = \frac{1}{2} - \alpha_n$, $\mu_n = \frac{\alpha_n}{n^{0.01}}$ and $\theta_n = \bar{\theta}_n$. We define the sequence $\text{TOL}_n := ||x_{n+1} - x_n||^2$ and apply the stopping criterion $\text{TOL}_n < \varepsilon$ for the iterative processes because the solution to the problem is unknown. ε is the predetermined error. Here, the terminating condition is set to $\varepsilon = 10^{-5}$. The numerical experiments are listed on Table 1. Also, we illustrate the efficiency of strong convergence of the proposed Algorithm (1.2) introduced by Censor et al. [9] (shortly, Algorithm (1.2)), Algorithms presented by Cai [7](shortly, Algorithm (1.3)) and Ma [23](shortly, Algorithm (1.4)) in Figure 1.

For the numerical experiments illustrated in Figure 1 and Table 1 below, we take into consideration the resulting cases.

Case 1: $x_0 = t^3$ and $x_1 = t^2 + 1$.
Case 2: $x_0 = t^2$ and $x_1 = t^4 + t$.
Case 3: $x_0 = \frac{t^2}{2} + t$ and $x_1 = 2t^3 + t$.
Case 4: $x_0 = t^2$ and $x_1 = (t/5)^3 + t$.

Table 1. Comparison of Algorithm (3), Algorithm (1.2), Algorithm (1.3) and Algorithm (1.4).

	Algorithm 3		Algorithm (1.2)		Algorithm (1.3)		Algorithm (1.4)	
	Iter.	CPU (sec)	Iter.	CPU (sec)	Iter.	CPU (sec)	Iter.	CPU (sec)
Case 1	79	2.8295	245	7.7275	81	20.1548	375	28.4084
Case 2	78	2.6678	102	3.4574	423	10.4600	374	8.4675
Case 3	84	2.9675	128	3.5662	564	15.6135	515	13.4206
Case 4	74	2.6625	102	3.2013	423	10.2072	374	8.5089

5. Conclusion

In this work, we propose a novel double modified subgradient extragradient self-adaptive step size method for solving variational inequality problems (VIPs). Our technique combines double inertial acceleration to enhance convergence behavior and improve stability concerns in solving variational inequality problems. Through rigorous convergence analysis, we establish that our method converges

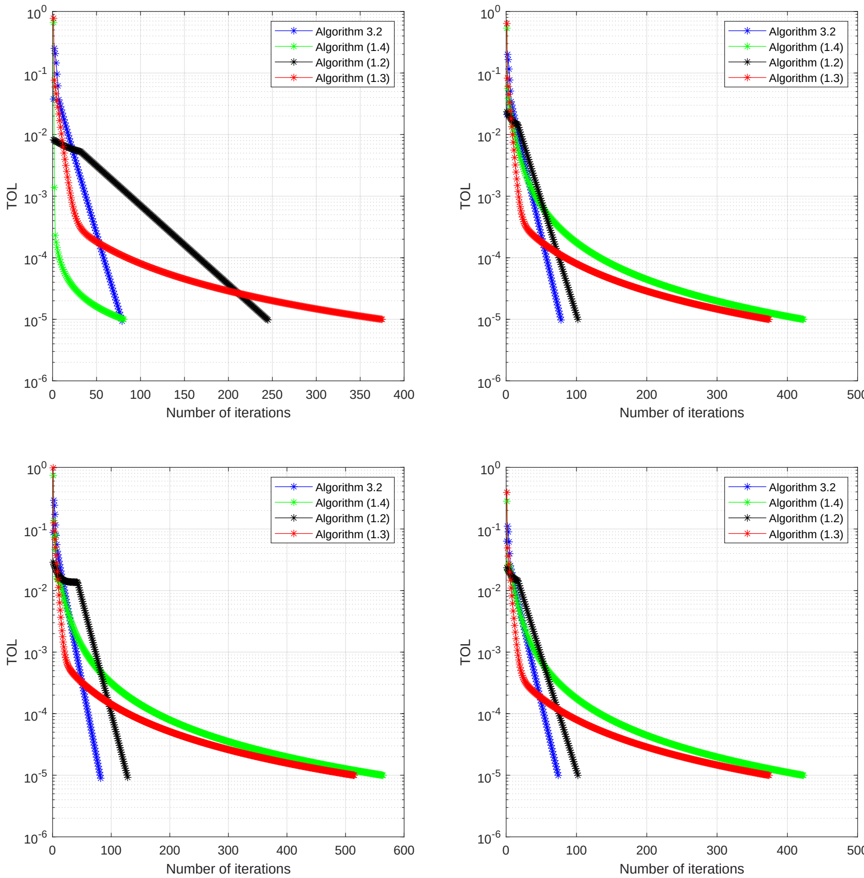

FIGURE 1. The error plotting of Comparison of Algorithm 3, Algorithm (1.2), Algorithm (1.3) and Algorithm (1.4) for Example 4.1.

strongly to the unique solution of the pseudomonotone variational inequality problem in 2−uniformly convex and smooth real Banach spaces.

STATEMENTS AND DECLARATIONS

The authors declare that they have no conflict of interest, and the manuscript has no associated data.

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